The geometry of cubic hypersurfaces

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Preface

Algebraic geometry starts with cubic polynomial equations! Everything of smaller degree, like linear maps or quadratic forms, fall in the realm of linear algebra. An important body of work, from the beginning of algebraic geometry to our days, has been devoted to cubics. In fact, cubics of dimension one, so elliptic curves, have been playing a central role in algebraic and arithmetic geometry and cubic surfaces with their 27 lines form one of the most studied classes of geometric objects.

This is a first set of notes of an ongoing lecture course at the University of Bonn in the winter term 2017 - 2018. They are in very rough form and will most certainly contain mistakes, typos, inaccuracies, and oversights. I will be most grateful for comments of any sort on these notes and will try to update them regularly on my webpage.

I have tried to give accurate references. If you spot any omissions, wrong attributions or simply want to point out references that have not been mentioned, please get in contact with me.

Be aware that 'Proofs' do not necessarily contain complete arguments. Often, I try to convey the basic idea of a proof, sometimes only in the special case of cubics, but (have to) refer for details to the literature.

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Basic facts

This first chapter collects general results on smooth hypersurfaces, especially those of relevance to cubic hypersurfaces. Results that are particular to any special dimension, cubic curves, surfaces, threefolds, etc. behave all very differently, will be dealt with in subsequent parts of these notes.

1 Numerical and cohomological invariants

The goal of this first section is to compute the standard invariants, numerical and cohomological, of smooth cubic hypersurfaces $X \subset \mathbb{P}^{n+1}$. Essentially all results and arguments are valid for arbitrary degree, but specializing to the case of cubics often simplifies the formulae. We will also record explicit values for low dimensional cubics for later use. We shall usually work over $\mathbb{C}$, but will point out how to argue over arbitrary fields in Section 1.6.

1.1 Let us begin with recalling the Lefschetz hyperplane theorem, see e.g. [149, V.13] or, for the ℓ-adic versions over arbitrary fields, [74, Exp. XIII], [1, Exp. XI], [48, IV]:

Assume $X \subset Y$ is a smooth ample divisor of a smooth projective variety Y of dimension $n+1$. Then pull-back and push-forward yield natural maps between (co)homology and homotopy groups. They satisfy:

(i) $H^k(Y,\mathbb{Z}) \longrightarrow H^k(X,\mathbb{Z})$ is bijective for $k < n$ and injective for $k \leq n$.
(ii) $H_k(X,\mathbb{Z}) \longrightarrow H_k(Y,\mathbb{Z})$ is bijective for $k < n$ and surjective for $k \leq n$.
(iii) $\pi_k(X) \longrightarrow \pi_k(Y)$ is bijective for $k < n$ and surjective for $k \leq n$.

Combined with Poincaré duality $H^k(X,\mathbb{Z}) \simeq H_{2n-k}(X,\mathbb{Z})$, these results provide information about the cohomology groups of X in all degrees.

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Corollary 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and dimension $n > 1$. Then X is simply connected and for $k \neq n$ one has*

$$H^k(X,\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$ □

Remark 1.2. According to the universal coefficient theorem, see e.g. [51, p. 186], there exist short exact sequences

$$0 \longrightarrow \mathrm{Ext}^1(H_{k-1}(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow H^k(X,\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_k(X,\mathbb{Z}),\mathbb{Z}) \longrightarrow 0.$$

We apply this to the hypersurface $X \subset \mathbb{P}^{n+1}$ and $k = n$. Using that $H_{n-1}(X,\mathbb{Z}) \simeq H_{n-1}(\mathbb{P}^{n+1},\mathbb{Z})$ is trivial or isomorphic to $\mathbb{Z}$, one finds that $H^n(X,\mathbb{Z}) \simeq \mathrm{Hom}(H_n(X,\mathbb{Z}),\mathbb{Z}) \simeq \mathbb{Z}^{\oplus b_n(X)}$, i.e. $H^n(X,\mathbb{Z})$ is torsion free.

1.2 The Lefschetz hyperplane theorem (with coefficients in a field) has also an algebraic proof. For hypersurfaces the argument can be combined with Bott's vanishing results to gain control over certain (twisted) Hodge numbers. As those will be frequently used in the sequel, we record them here.

We start with the classical *Bott vanishing* for $\mathbb{P} := \mathbb{P}^{n+1}$, which can be deduced from the (dual of the) *Euler sequence*

$$0 \longrightarrow \Omega_{\mathbb{P}} \longrightarrow \mathcal{O}(-1)^{\oplus n+2} \longrightarrow \mathcal{O} \longrightarrow 0 \tag{1.1}$$

and the short exact sequences obtained by taking exterior products

$$0 \longrightarrow \Omega_{\mathbb{P}}^p \longrightarrow \bigwedge^p \left(\mathcal{O}(-1)^{\oplus n+2}\right) \longrightarrow \Omega_{\mathbb{P}}^{p-1} \longrightarrow 0.$$

A closer inspection of the associated long exact sequences reveals that

$$H^q(\mathbb{P},\Omega_{\mathbb{P}}^p(k)) = 0$$

unless:

(i) $0 \le p = q \le n$, $k = 0$, in which case $h^{p,p}(\mathbb{P}) = 1$,
(ii) $q = 0$, $k > p$, in which case $h^0(\mathbb{P},\Omega_{\mathbb{P}}^p(k)) = \binom{n+1+k-p}{k} \cdot \binom{k-1}{p}$, or
(iii) $q = n+1$, $k < p-(n+1)$, in which case $h^{n+1}(\mathbb{P},\Omega_{\mathbb{P}}^p(k)) = \binom{-k+p}{-k} \cdot \binom{-k-1}{n+1-p}$.

The last two cases are Serre dual to each other. The well known formula

$$h^0(\mathbb{P},\mathcal{O}(k)) = \binom{n+1+k}{k} \tag{1.2}$$

is a special case of (ii).

To deduce vanishings for X one then uses the standard short exact sequences

$$0 \longrightarrow \Omega_{\mathbb{P}}^p(-d) \longrightarrow \Omega_{\mathbb{P}}^p \longrightarrow \Omega_{\mathbb{P}}^p|_X \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \Omega_{\mathbb{P}}|_X \longrightarrow \Omega_X \longrightarrow 0 \tag{1.3}$$

and the exterior powers of the latter

$$0 \longrightarrow \Omega_X^{p-1}(-d) \longrightarrow \Omega_{\mathbb{P}}^p|_X \longrightarrow \Omega_X^p \longrightarrow 0. \tag{1.4}$$

Note that as a special case of (1.4) one obtains the *adjunction formula*:

Lemma 1.3. *The canonical bundle of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d is*

$$\omega_X \simeq \mathcal{O}_X(d-(n+2)). \tag{1.5}$$

It is ample for $d > n+2$, trivial for $d = n+2$, and anti-ample (i.e. ω_X^* is ample) in all other cases. □

Applying cohomology and Bott vanishing then yields

Corollary 1.4. *The natural map*

$$H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) \longrightarrow H^q(X, \Omega_X^p(k))$$

is injective (bijective) if $p + q \leq n$ ($p + q < n$) and $k < d$. □

Note that in particular, *Kodaira vanishing* holds (over any field!):

$$H^q(X, \Omega_X^p(k)) = 0$$

for $k > 0$ and $p + q > n$, which is Serre dual to the vanishing for $p + q < n$ and $k < 0$.

Remark 1.5. For $d = 3$ and $n > 1$, the vanishing of $H^0(X, \Omega_X^p) = 0$, $p > 0$, can also be deduced (at least in characteristic zero) from the fact that cubic hypersurfaces are unirational, see Section 1.2.

Corollary 1.6. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Then*

$$\mathrm{Pic}(X) \simeq \mathbb{Z} \cdot \mathcal{O}_X(1)$$

for $n > 2$. If $n = 2$ and $d \leq n + 1 = 3$ and $k = \mathbb{C}$, then one still has $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$.

Proof Over $\mathbb{C}$, the proof is a consequence of the exponential sequence (in the analytic topology) $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$, which yields the exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \xrightarrow{\sim} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X).$$

Now, by Lefschetz hyperplane theorem or Corollary 1.4, $H^1(X, \mathcal{O}_X) = 0$ for $n > 1$ and $H^2(X, \mathcal{O}_X) = 0$ for $n > 2$ or $d \leq n + 1$ (using Serre duality).

See [74, XII. Cor 3.6] for a proof over arbitrary field. The vanishing $H^2(X, \mathcal{O}_X) = 0$ is there used to extend any line bundle on X to a formal neighbourhood and then to $\mathbb{P}^{n+1}$ by algebraization. □

Remark 1.7. For the motivated reader, the results shall be translated into motivic language, cf. [5, 123] for basic facts. For the pure motive $\mathfrak{h}(X)$ of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d in the category of rational Chow motives $\mathrm{Mot}(k)$ there exists a decomposition, cf. [126],

$$\mathfrak{h}(X) \simeq \mathfrak{h}^n(X)_{\mathrm{pr}} \oplus \bigoplus_{i=0}^{n} \mathbb{Q}(-i).$$

Here, $\mathbb{Q}(1)$ is the Tate motive $(\mathrm{Spec}(k), \mathrm{id}, 1)$ and the primitive part $\mathfrak{h}^n(X)_{\mathrm{pr}}$ has cohomology concentrated in degree n. Moreover, $\mathrm{CH}^*(\mathfrak{h}^n(X)_{\mathrm{pr}})$ is the homological trivial part of $\mathrm{CH}^*(X)$. Note that not much more is known about the Chow ring of (cubic) hypersurfaces. However, according to Paranjape [125], see also [139], one knows $\mathrm{CH}^{n-1}(X)\otimes\mathbb{Q} \simeq \mathbb{Q}$ for smooth cubic hypersurfaces of dimension $n \geq 5$. The expectation however is that $\mathrm{CH}^i(X) \otimes \mathbb{Q} \simeq \mathbb{Q}$ for $i > (2n-1)/3$. See also Section **5.??**.

1.3 Next, we would like to compute the remaining Betti number $b_n(X)$ of a smooth hypersurface $X \subset \mathbb{P} = \mathbb{P}^{n+1}$ and we approach this via the Euler number

$$e(X) = \sum_{i=0}^{2n} (-1)^i\, b_i(X) = \sum_{i=0,\neq n}^{2n} (-1)^i\, b_i(X) + (-1)^n\, b_n(X).$$

Using $b_i(X) = b_i(\mathbb{P})$ for $i = 0, \ldots, 2n, \neq n$, one finds

$$e(X) = \begin{cases} n + b_n(X) & \text{if } n \text{ is even} \\ n + 1 - b_n(X) & \text{if } n \text{ is odd.} \end{cases}$$

Rephrasing this in terms of the primitive Betti number $b_n(X)_{\mathrm{pr}} \coloneqq \dim_{\mathbb{Q}} H^n(X,\mathbb{Q})_{\mathrm{pr}}$, which equals $b_n(X) - 1$ for even $n > 0$ and $b_n(X)$ for n odd (use $b_{n-2}(X) = 1$ or 0, respectively), yields

$$b_n(X)_{\mathrm{pr}} = (-1)^n(e(X) - (n+1)).$$

This reduces our task to the computation of $e(X) = \int_X \mathrm{c}_n(X)$. Now, the total Chern character of X can be computed by using the restriction of the Euler sequence (1.1) and the dual (1.3) of the normal bundle sequence:

$$\begin{aligned} \mathrm{c}(X) \coloneqq \sum \mathrm{c}_i(X) &= \mathrm{c}(\mathcal{T}_{\mathbb{P}}|_X) \cdot \mathrm{c}(\mathcal{O}_X(d))^{-1} = \mathrm{c}(\mathcal{O}_X(1))^{n+2} \cdot \mathrm{c}(\mathcal{O}_X(d))^{-1} \\ &= \frac{(1+h)^{n+2}}{(1+dh)} = \left(1 - dh + (dh)^2 \pm \cdots\right) \cdot \sum \binom{n+2}{i} h^i, \end{aligned}$$

where $h \coloneqq \mathrm{c}_1(\mathcal{O}_X(1))$. Hence,

$$\begin{aligned} \mathrm{c}_n(X) &= \frac{1}{d^2} \cdot \left((-1)^{n+2} \cdot d^{n+2} + \cdots + \binom{n+2}{n} \cdot d^2\right) \cdot h^n \\ &= \frac{1}{d^2} \cdot \left((1-d)^{n+2} + d \cdot (n+2) - 1\right) \cdot h^n, \end{aligned}$$

which combined with $\int_X h^n = d$ leads to

$$e(X) = \frac{1}{d} \cdot \Big((1-d)^{n+2} + d \cdot (n+2) - 1\Big).$$

For $d = 3$ the right hand side becomes

$$e(X) = \frac{1}{3}\Big((-2)^{n+2} + 3n + 5\Big). \tag{1.6}$$

Corollary 1.8. *The primitive middle Betti number of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d and dimension $n > 0$ is given by*

$$b_n(X)_{\mathrm{pr}} = \frac{(-1)^n}{d}\Big(d - 1 + (1-d)^{n+2}\Big),$$

which for $d = 3$ becomes $(-1)^n \cdot (2/3) \cdot \Big(1 + (-1)^n \cdot 2^{n+1}\Big)$. □

Exercise 1.9. For a smooth cubic hypersurface, the n-th Betti number itself can then be expressed as

$$b_n(X) = \frac{1}{6}\Big(2^{n+3} + 3 + (-1)^n \cdot 7\Big).$$

We record the result for cubics and small dimensions in the following table, where we include further information about the intersection form to be discussed later.

n	$e(X)$	$b_n(X)_{\mathrm{pr}}$	$b_n(X)$	$\tau(X)$	$(b_n^+(X), b_n^-(X))$
0	3	3	3	3	(3, 0)
1	0	2	2		
2	9	6	7	−5	(1, 6)
3	−6	10	10		
4	27	22	23	19	(21, 2)
5	−36	42	42		
6	93	86	87	−53	(17, 70)
7	−162	170	170		
8	351	342	343	163	(253, 90)
9	−672	682	682		
10	1377	1366	1367	−485	(441, 926)

1.4 After having computed all Betti numbers $b_i(X)$ of smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$, we now aim at determining their Hodge numbers $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$.

They are encoded by the *Hirzebruch χ_y-genus*, which for an arbitrary smooth projective variety X of dimension n is defined as the polynomial

$$\chi_y(X) \coloneqq \sum_{p=0}^{n} \chi^p(X)\, y^p$$

with coefficients

$$\chi^p(X) \coloneqq \chi(X, \Omega_X^p) = \sum_{q=0}^{n} (-1)^q\, h^{p,q}(X).$$

Corollary 1.10. *For a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ one has*

$$\chi^p(X) = (-1)^{n-p}\, h^{p,n-p}(X) + \begin{cases} (-1)^p & \text{if } 2p \neq n \\ 0 & \text{if } 2p = n \end{cases}$$

and, therefore,

$$\begin{array}{lll} h^{p,n-p}(X) \neq 0 & \text{if and only if} & \chi^p(X) \neq (-1)^p \\ h^{p,n-p}(X) = 1 & \text{if and only if} & \chi^p(X) = (-1)^{n-p} + (-1)^p. \end{array}$$

□

This can be pictured by the Hodge diamonds for $n \equiv 0\,(2)$ and $n \equiv 1\,(2)$.

$$\begin{array}{ccccc} & & 1 & & \\ & & \vdots & & \\ & & 1 & & \\ h^{n,0} & \dots & h^{n/2,n/2} & \dots & h^{0,n} \\ & & 1 & & \\ & & \vdots & & \\ & & 1 & & \end{array}$$

$$\begin{array}{cccccc} & & & 1 & & \\ & & & \vdots & & \\ & & & 1 & & \\ h^{n,0} & \dots & h^{(n+1)/2,(n-1)/2} & h^{(n-1)/2,(n+1)/2} & \dots & h^{0,n} \\ & & & 1 & & \\ & & & \vdots & & \\ & & & 1 & & \end{array}$$

This prompts certain natural questions: For which d and n is $h^{n,0} \neq 0$? Or, how to compute $\max\{p \mid h^{p,n-p} \neq 0\}$, which encodes the level of the Hodge structure of X? By Corollary 1.4 one knows that $h^{n,0} = 0$ for cubic hypersurfaces of dimension $n > 1$.

In principle, $\chi_y(X)$ can be computed by the Hirzebruch–Riemann–Roch formula. Indeed,

$$\chi^p(X) = \int_X \operatorname{ch}(\Omega_X^p) \cdot \operatorname{td}(X),$$

which using Chern roots γ_i of $\mathcal{T}_X$ leads to

$$\chi_y(X) = \int_X \prod_{i=1}^n \frac{(1 - ye^{-\gamma_i})\,\gamma_i}{1 - e^{-\gamma_i}},$$

cf. [85, Cor. 5.1.4]. The characteristic classes of Ω_X^p and of $\mathcal{T}_X$, the latter is needed for the computation of $\operatorname{td}(X)$, can all be explicitly determined by using the Euler sequence and the conormal sequence. However, the computation is not particularly enlightening until everything is put in a generating series, cf. [82, Thm. 22.1.1].

Theorem 1.11 (Hirzebruch)**.** *For smooth hypersurfaces $X_n \subset \mathbb{P}^{n+1}$ of degree d one has*

$$\sum_{n=0}^{\infty} \chi_y(X_n)\, z^{n+1} = \frac{1}{(1+yz)\,(1-z)} \cdot \frac{(1+yz)^d - (1-z)^d}{(1+yz)^d + y\,(1-z)^d}. \tag{1.7}$$

A variant of this formula for the primitive Hodge numbers $h^{p,q}(X)_{\rm pr} \coloneqq \dim H^{p,q}(X)_{\rm pr} = h^{p,q}(X) - \delta_{p,q}$ has been worked out in [1, XI]:

$$\sum_{p,q\geq 0, n\geq 0} h^{p,q}(X_n)_{\rm pr}\, y^p z^q = \frac{1}{(1+y)\,(1+z)} \cdot \left[\frac{(1+y)^d - (1+z)^d}{(1+z)^d\, y + (1+y)^d\, z} - 1\right].$$

We consider the usual specializations of the χ_y-genus for cubic hypersurfaces ($d = 3$):

(i) $y = 0$. So, we consider $\chi_{y=0}(X) = \chi^0(X) = \chi(X, \mathcal{O}_X)$. The left hand side of (1.7) can be readily computed as

$$\sum_{n=0}^{\infty} \chi(X_n, \mathcal{O}_{X_n})\, z^{n+1} = 3\,z + 0\,z^2 + z^3 + z^4 + \cdots.$$

Indeed, the first coefficient is $\chi(X_0 = \{x_1, x_2, x_3\}, \mathcal{O}_{X_0}) = 3$ and the second $\chi(X_1 = E, \mathcal{O}_E) = 0$ with E an elliptic curve. For $n > 1$ use Bott vanishing and the short exact sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_X \longrightarrow 0$ to compute $\chi(X, \mathcal{O}_X) = \chi(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) - \chi(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-3)) = 1$.

To confirm (1.7) in this case, we compute its right hand side and find

$$\begin{aligned}\frac{1}{1-z}\left(1-(1-z)^3\right) &= \frac{1}{1-z}-(1-z)^2\\ &= (1+z+z^2+\cdots)-(1-2z+z^2)\\ &= 3\,z+0\,z^2+z^3+z^4+\cdots.\end{aligned}$$

(ii) $y=-1$. Observe that $\chi_{y=-1}(X)=e(X)$. In this case (1.7) taken literally yields

$$\sum_{n=0}^{\infty} e(X_n)\,z^{n+1} = \frac{1}{(1-z)^2}\cdot\frac{(1-z)^3-(1-z)^3}{(1-z)^3-(1-z)^3},$$

which is of course not very useful. Only when the right hand side of (1.7) for $y=-1$ is computed as the limit $y \to -1$ via L'Hôpital's rule, one obtains the useful formula

$$\begin{aligned}\sum_{n=0}^{\infty} e(X_n)\,z^{n+1} &= \frac{3\,z}{(1-z)^2\,(1+2\,z)}\\ &= 3\,z\cdot(1+z+z^2+\cdots)\cdot(1-2\,z+(2\,z)^2-(2\,z)^3\pm\cdots)\\ &= 3\,z+0\,z^2+9\,z^3+\cdots,\end{aligned}$$

which sheds a new light on (1.6). The reader may want to check that one indeed gets the same answer.

(iii) $y=1$. This is the most interesting case. According to the Hirzebruch signature theorem [82, Thm. 15.8.2]

$$\chi_{y=1}(X)=\tau(X).$$

Recall that for $n\equiv 0\,(2)$ the intersection pairing

$$H^n(X,\mathbb{R})\times H^n(X,\mathbb{R})\longrightarrow\mathbb{R}$$

is a non-degenerate symmetric bilinear form which, of course, can be diagonalized to become $\mathrm{diag}(+1,\ldots,+1,-1,\ldots,-1)$. Now, by definition,

$$\tau(X)=b_n^+(X)-b_n^-(X), \tag{1.8}$$

where $b_n^\pm(X)$ is the number of ± 1. Then the Hodge–Riemann bilinear relations imply $\tau(X)=\sum_{p,q}(-1)^p\,h^{p,q}(X)$, cf. [85, Cor. 3.3.18]. Note that, although the definition of the signature only involves the middle cohomology, indeed all Hodge numbers $h^{p,q}(X)$, also for $p+q\neq n$, enter the sum.

As a side remark, observe that the right hand side of (1.7) for $y=1$ reads

$$\frac{1}{(1-z^2)}\cdot\frac{(1+z)^d-(1-z)^d}{(1+z)^d+(1-z)^d},$$

which is anti-symmetric in z. Hence, only X_n with $n\equiv 0\,(2)$ enter the computation, so that one need not worry about defining an analogue of the signature for alternating intersection forms. In any case, (1.7) yields for $d = 3$ the intriguing formula

$$\begin{aligned}\sum_{n=0}^{\infty} \tau(X_n)\, z^{n+1} &= \frac{6\,z + 2\,z^3}{(1-z)^2\,(2+6\,z^2)} \\ &= z \cdot (3 + z^2) \cdot (1 + z^2 + z^4 + \cdots) \cdot (1 - 3\,z^2 + (3\,z^2)^2 - (3\,z^2)^3 \pm \cdots) \\ &= z \cdot (3 - 5\,z^2 + 19\,z^4 - 53\,z^6 + 163\,z^8 - 485\,z^{10} \pm \cdots).\end{aligned}$$

Maybe more instructive is the closed formula for the signature of an even dimensional smooth cubic hypersurface $X \subset \mathbb{P}^{2m+1}$:

$$\tau(X_{2m}) = (-1)^m \cdot 2 \cdot 3^m + 1. \tag{1.9}$$

In principle, we have now computed all Hodge numbers of smooth (cubic) hypersurfaces, but to decode (1.7) is not always easy. For later use, we record the middle Hodge numbers of smooth cubic hypersurfaces of dimension ≤ 10.

n	$b_n(X)_{\rm pr}$	$H^n_{\rm pr}$	$h^{p,q}_{\rm pr}$
1	2	$H^{1,0} \oplus H^{0,1}$	1 1
2	6	$H^{1,1}_{\rm pr}$	6
3	10	$H^{2,1} \oplus H^{1,2}$	5 5
4	22	$H^{2,1} \oplus H^{2,2}_{\rm pr} \oplus H^{1,3}$	1 20 1
5	42	$H^{3,2} \oplus H^{2,3}$	21 21
6	86	$H^{4,2} \oplus H^{3,3}_{\rm pr} \oplus H^{2,4}$	8 70 8
7	170	$H^{5,2} \oplus H^{4,3} \oplus H^{3,4} \oplus H^{2,5}$	1 84 84 1
8	342	$H^{5,3} \oplus H^{4,4}_{\rm pr} \oplus H^{3,5}$	45 252 45
9	682	$H^{6,3} \oplus H^{5,4} \oplus H^{4,5} \oplus H^{3,6}$	11 330 330 11
10	1366	$H^{7,3} \oplus H^{6,4} \oplus H^{5,5}_{\rm pr} \oplus H^{4,6} \oplus H^{3,7}$	1 220 924 220 1

Remark 1.12. Later, in Section 4.4, we will see that for a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ the Hodge numbers are given by

$$h^{p,n-p}(X)_{\rm pr} = \binom{n+2}{2n+1-3p}.$$

These numbers are reasonable in the sense that they satisfy complex conjugation $h^{p,n-p} = h^{n-p,p}$, but the combinatorial consequence of combining $\sum_{p=0}^{n} h^{p,n-p}(X)_{\rm pr} = b_n(X)_{\rm pr}$ with Corollary 1.8 seems less clear, see Exercise 4.11. From this description we will eventually be able to read off easily properties of Hodge numbers. For example, one finds:

(i) $h^{p,n-p}(X)_{\rm pr} \neq 0$ if and only if $n-1 \leq 3p \leq 2n+1$ and
(ii) $h^{p,n-p}(X)_{\rm pr} = 1$ if and only if $3p = 2n+1$ or $3p = n-1$.
(iii) The level of the Hodge structure

$$\ell = \ell(H^n(X)) \coloneqq \max\{\, |p-q| \mid H^{p,q}(X) \neq 0 \,\}$$

satisfies $\ell > 1$ for $n > 5$ and $\ell > 2$ for $n > 8$. The first computations of this sort were done in [129].

Note that the two cases in (ii) are Serre dual to each other.

1.5 Our next goal is to determine the intersection form on $H^n(X,\mathbb{Z})$ for a smooth cubic hypersurface $X \subset \mathbb{P} = \mathbb{P}^{n+1}$. Recall from Section 1.1 that $H^n(X,\mathbb{Z})$ is torsion free, i.e. $H^n(X,\mathbb{Z}) \simeq \mathbb{Z}^{\oplus b_n(X)}$. The non-degenerate intersection pairing

$$H^n(X,\mathbb{Z}) \times H^n(X,\mathbb{Z}) \longrightarrow H^{2n}(X,\mathbb{Z}) \simeq \mathbb{Z}$$

is symplectic for $n \equiv 1\,(2)$ and symmetric for $n \equiv 0\,(2)$. In the first case, $H^n(X,\mathbb{Z})$ admits a basis $\gamma_1, \ldots, \gamma_{b_n=2m}$ for which the intersection matrix has the standard form

$$\begin{pmatrix} 0 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 0 & & & 1 \\ -1 & & & 0 & & \\ & \ddots & & & \ddots & \\ & & -1 & & & 0 \end{pmatrix}.$$

For $n \equiv 0\,(2)$ the intersection pairing on $H^n(X,\mathbb{Z})$ defines a unimodular lattice. The determinant of the intersection matrix (with respect to any integral basis), i.e. the discriminant of the lattice, is ± 1. The classification of unimodular lattices is a classical topic. It distinguishes between *even lattices* Λ, i.e. those for which $(\alpha)^2 = (\alpha.\alpha) \equiv 0\,(2)$ for all $\alpha \in \Lambda$, and *odd lattices*.

Assume that Λ is an odd lattice, i.e. that there exists $\alpha \in \Lambda$ with $(\alpha)^2 \equiv 1\,(2)$, unimodular, and indefinite, then

$$\Lambda \simeq \mathrm{I}_{r,s} \coloneqq \mathbb{Z}(1)^{\oplus r} \oplus \mathbb{Z}(-1)^{\oplus s},$$

where $\mathbb{Z}(a)$ is the lattice of rank one with intersection form given by $(1)^2 = a$, see [136, V. Thm. 4]. This can be applied to any even-dimensional hypersurface of odd degree, as $(h^{n/2}.h^{n/2}) = \int_X h^{n/2} \cdot h^{n/2} = d$. That the intersection pairing on $H^n(X,\mathbb{Z})$ is indeed indefinite can be deduced easily (at least for cubic hypersurfaces) from a comparison of $\tau(X)$ and $b_n(X)$, cf. Corollary 1.8 and (1.9).

Corollary 1.13. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface of even dimension. Then the intersection form on its middle cohomology yields a lattice isomorphic to*

$$H^n(X,\mathbb{Z}) \simeq \mathbb{Z}(1)^{\oplus b_n^+} \oplus \mathbb{Z}(-1)^{\oplus b_n^-} \simeq \mathrm{I}_{b_n^+,b_n^-}.$$

Here, $b_n^\pm := b_n^\pm(X)$ are determined by $b_n^+ + b_n^- = b_n(X) = (1/3)\,(2^{n+2}+5)$, see Corollary 1.8, and $b_n^+ - b_n^- = \tau(X) = (-1)^{n/2}\cdot 2\cdot 3^{n/2}+1$, see (1.9). □

More interesting, however, is the primitive cohomology $H^n(X,\mathbb{Z})_{\mathrm{pr}}$. The intersection form is still non-degenerate there, but not necessarily unimodular, and, as it turns out, not odd. By definition and using that $b_{n-2} = 1$ for $n > 0$, it is the orthogonal complement $(h^{n/2})^\perp \subset H^n(X,\mathbb{Z})$. However, note that

$$H^n(X,\mathbb{Z})_{\mathrm{pr}} \oplus \mathbb{Z}\cdot h^{n/2} \subset H^n(X,\mathbb{Z}) \tag{1.10}$$

is not an equality. It describes a finite index subgroup. The square of the index is

$$\mathrm{ind}^2 = \pm\mathrm{disc}\,(\mathbb{Z}\cdot h^{n/2})\cdot \mathrm{disc}\,\left(H^n(X,\mathbb{Z})_{\mathrm{pr}}\right) = \pm 3\cdot \mathrm{disc}\,\left(H^n(X,\mathbb{Z})_{\mathrm{pr}}\right),$$

where we use that $H^n(X,\mathbb{Z})$ is unimodular and $\mathbb{Z}\cdot h^{n/2} \simeq \mathbb{Z}(3)$, see [86, Ch. 14.0.2] for the general statement and references. This also shows that disc $\left(H^n(X,\mathbb{Z})_{\mathrm{pr}}\right)$ is at least divisible by three and, therefore, $H^n(X,\mathbb{Z})_{\mathrm{pr}}$ is not unimodular.

The following is a folklore result for cubics (in dimension four, cf. [80]) and has been generalized to other degrees and complete intersections in [15]. For the definition of the lattices A_2, E_6, E_8, and U see [86, Ch. 15] and the references therein.

Proposition 1.14. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface of even, positive dimension. Then the intersection form on its middle primitive cohomology $H^n(X,\mathbb{Z})_{\mathrm{pr}}$ is described as follows:*

(i) *For $n = 2$ one has $H^n(X,\mathbb{Z})_{\mathrm{pr}} \simeq E_6(-1)$.*
(ii) *For $n > 2$ one has $H^n(X,\mathbb{Z})_{\mathrm{pr}} \simeq A_2 \oplus E_8^{\oplus a} \oplus U^{\oplus b}$. Here, $b := \min\{b_n^+(X)-3, b_n^-(X)\}$ and*

$$a := \begin{cases} \dfrac{b_n^+ - b_n^- - 3}{8} = \dfrac{\tau(X)-3}{8} = \dfrac{3^{n/2}-1}{4} & \text{if } n \equiv 0\,(4) \\[2ex] \dfrac{b_n^- - b_n^+ + 3}{8} = \dfrac{3-\tau(X)}{8} = \dfrac{3^{n/2}+1}{4} & \text{if } n \equiv 2\,(4). \end{cases}$$

In particular, disc $\left(H^n(X,\mathbb{Z})_{\mathrm{pr}}\right) = 3$ *and the inclusion (1.10) has index three.*

Note that $n \equiv 0\,(4)$ if and only if $b_n^+ \geq b_n^-$, see (1.9). Also, observe that $b \geq (1/3)(2^{n+1} - 3^{n/2+1} - 1)$, which is rather large for $n \geq 4$, i.e. $H^n(X,\mathbb{Z})_{\mathrm{pr}}$ contains many copies of the hyperbolic plane U. This often simplifies lattice theoretic arguments.

Proof Assume $n > 2$, so that $b_n^+(X) > 3$, and consider the odd, unimodular lattice $\Lambda \coloneqq \mathbb{Z}^{\oplus 3} \oplus E_8^{\oplus a} \oplus U^{\oplus b}$. It has rank $\mathrm{rk}(\Lambda) = b_n(X)$ and signature $\tau(\Lambda) = \tau(X)$.[1] Therefore, Λ and $H^n(X,\mathbb{Z})$ are odd, indefinite, unimodular lattices of the same rank and signature and hence isomorphic to each other (and to $\mathrm{I}_{b_n^+,b_n^-}$), cf. [136, V. Thm. 6].

Recall that a primitive vector $\alpha \in \Lambda$ in an odd unimodular lattice Λ is called *characteristic* if $(\alpha.\beta) \equiv (\beta)^2\,(2)$ for all $\beta \in \Lambda$. Obviously, the orthogonal complement $\alpha^\perp \subset \Lambda$ of a characteristic class is always even. The converse also holds, cf. [108, Lem. 3.3]. Indeed, for any primitive $\alpha \in \Lambda$ in the unimodular lattice Λ there exists $\beta_0 \in \Lambda$ with $(\alpha.\beta_0) = 1$. Then for all $\beta \in \Lambda$ the class $\beta - (\alpha.\beta)\beta_0$ is contained in $\alpha^\perp$ and in particular of even square if $\alpha^\perp$ is assumed to be even. Hence, $(\beta)^2 \equiv (\alpha.\beta)^2\,(\beta_0)^2\,(2)$. As Λ is odd, there exists a β with $(\beta)^2$ odd and hence $(\beta_0)^2$ must be odd. Altogether this proves $(\beta)^2 \equiv (\alpha.\beta)\,(2)$ for all β, i.e. α is characteristic.

For example, $(1,1,1) \in \mathbb{Z}^{\oplus 3}$ is characteristic, for its orthogonal complement is A_2. In this case it can also be checked directly by observing that $((1,1,1).(x_1,x_2,x_3)) = x_1+x_2+x_3 \equiv x_1^2+x_2^2+x_3^2 = ((x_1,x_2,x_3))^2\,(2)$. But then $(1,1,1) \in \Lambda$ is also characteristic and its orthogonal complement is the lattice in (ii).

One now applies a general result for unimodular lattices from [152, Thm. 3]: Two primitive vectors $\alpha,\beta \in \Lambda$ are in the same $\mathrm{O}(\Lambda)$-orbit if and only if $(\alpha)^2 = (\beta)^2$ and are either both characteristic or both not.

Therefore, to prove the assertion, it suffices to show that $h^{n/2} \in H^n(X,\mathbb{Z})$ is characteristic or, equivalently, that $H^n(X,\mathbb{Z})_{\mathrm{pr}}$ is even. We postpone the proof of this statement to Corollary 2.11, where it fits more naturally in the discussion of Picard–Lefschetz theory and of the monodromy action of the universal family of hypersurfaces. A more topological argument is given in [108].

It remains to deal with the case $n = 2$, where we have $H^2(X,\mathbb{Z}) \simeq \mathrm{I}_{1,6}$. It is easy to check that $\alpha \coloneqq (3,1,\ldots,1) \in \mathrm{I}_{1,6}$ is characteristic with $(\alpha)^2 = 3$ and its orthogonal complement turns out to be $E_6(-1) \simeq \alpha^\perp \subset \mathrm{I}_{1,6}$.[2] Now consider the class of the hyperplane section $h \in H^2(X,\mathbb{Z})$. As in this case $\mathrm{Pic}(X) \simeq H^2(X,\mathbb{Z})$, one can argue algebraically, using the Hirzebruch–Riemann–Roch formula, to prove that h is characteristic. Indeed,

$$\chi(X,L) = (1/2)\ ((L.L) + (L.h)) + 1$$

implies $(L.h) \equiv (LL) \equiv 0\,(2)$. Hence, using [152, Thm. 3] again, $H^2(X,\mathbb{Z})_{\mathrm{pr}} \simeq \alpha^\perp \simeq E_6(-1)$.

Later we will describe the isomorphisms $H^2(X,\mathbb{Z})_{\mathrm{pr}} \simeq E_6(-1)$ from a more geometric perspective and, in particular, write down bases of both lattices in terms of lines, see Sections **4**.1-3. □

[1] Note that $\tau \equiv 3\,(8)$ is a general fact for unimodular lattices with characteristic element α with $(\alpha)^2 \equiv 3\,(8)$, cf. [136, V. Thm. 2]). In our situation, $\tau(X) \equiv 3\,(8)$ can be deduced from (1.9).

[2] Let $e_1 \coloneqq (0,1,-1,0,0,0,0)$, $e_2 \coloneqq (0,0,1,-1,0,0,0)$, $e_3 \coloneqq (0,0,0,1,-1,0,0)$, $e_4 \coloneqq (1,0,0,0,1,1,1)$, $e_5 \coloneqq (0,0,0,0,1,-1,0)$, $e_7 \coloneqq (0,0,0,0,0,1,-1)$. They span $\alpha^\perp$ and their intersection matrix is just $E_6(-1)$.

Remark 1.15. In [100, Thm. 11.1] it is shown that the purely lattice theoretic description in Corollary 1.13 of the intersection product on $H^n(X,\mathbb{Z})$ can be realized geometrically in the following sense: For $n \equiv 0\,(4)$ a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ is diffeomorphic to a connected sum of the form $M\#k(S^n \times S^n)$ with $k = b_n^-(X)$ and, therefore, $b_n(M) = \tau(X)$. For $n \equiv 2\,(4)$, $n \geq 4$ the hypersurface is diffeomorphic to a connected sum of the form $M\#k(S^n \times S^n)$ with $k = b_n^+(X) - 1$ and, therefore, $b_n(M) = -\tau(X) + 2$.

For $n \equiv 1\,(2)$, a smooth cubic hypersurface X is diffeomorphic to $M\#k(S^n \times S^n)$, with $k = b_n(X)/2 - 1$ and, hence, $b_n(M) = 2$. For $n = 1, 3$, or 7 this can be improved to $k = b_n(X)/2$ and $b_n(M) = 0$.

The remaining case of smooth cubic surfaces $X \subset \mathbb{P}^3$ is slightly different. Viewing X as the blow-up of $\mathbb{P}^2$ in six points (see Section **4**.2.2) reveals that it is diffeomorphic to the connected sum $\mathbb{P}^2\#6\overline{\mathbb{P}^2}$.

1.6 We conclude with a number of comments on (cubic) hypersurfaces over arbitrary fields and notably in positive characteristics. Most of the subtleties and pathologies that usually occur for varieties over fields of positive characteristic can safely be ignored for hypersurfaces. In the following, let $X \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface over an arbitrary field k.

(i) The Hodge–de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H_{\mathrm{dR}}^{p+q}(X/k) \tag{1.11}$$

degenerates, cf. Section 4.6. For $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > \dim(X)$, this follows from [49]. Indeed, smooth hypersurfaces over fields of positive characteristic can of course be lifted to characteristic zero.

More directly and avoiding the assumption $\mathrm{char}(k) > \dim(X)$, one can argue as follows. The computations above show in particular that the Hodge numbers $h^{p,q}(X) = \dim(E_1^{p,q})$ of smooth hypersurfaces only depend on d and n, but not on $\mathrm{char}(k)$. From (1.11) one deduces that $\sum_{p+q=k} h^{p,q}(X) \geq \dim H_{\mathrm{dR}}^{p+q}(X/k)$. Moreover, equality holds if and only if the spectral sequence degenerates. On the other hand, $\dim H_{\mathrm{dR}}^{p+q}(X/k)$ is upper semi-continuous. Hence, the degeneration of the spectral sequence in characteristic zero implies the degeneration in positive characteristic.

(ii) The Kodaira vanishing $H^q(X, \Omega_X^p \otimes L) = 0$ for $p + q > n$ and $L \in \mathrm{Pic}(X)$ ample holds. This can either be seen as a consequence of [49] for large enough characteristic or read off from Corollaries 1.4 and 1.6. In particular, all numerical assertions on Hodge numbers remain valid over arbitrary fields. Also, for algebraically closed fields, the étale Betti numbers equal the ones computed in characteristic zero. The only case not covered by these comments is the case of cubic surfaces in characteristic two.

(iii) Assume $k = \mathbb{F}_q$. Then the Weil conjectures show that

$$Z(X,t) \coloneqq \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r}\right) = \frac{P(t)^{(-1)^{n+1}}}{\prod_{i=0}^{n}(1-q^i t)}$$

with $P(t) = \prod(1-\alpha_i t)$ of degree $b_n(X)_{\rm pr}$ and α_i algebraic integers of absolute value $|\alpha_i| = q^{n/2}$. In fact, this was proved prior to the proof of the general Weil conjectures by Bombieri and Swinnerton-Dyer [22] for cubic threefolds and by Dwork [56] for arbitrary hypersurfaces. Of course, as cubic surfaces are rational, the Weil conjectures follow from the Weil conjectures for $\mathbb{P}^2$ and for curves.

2 Linear system and Lefschetz pencils

This section discusses the linear system of (cubic) hypersurfaces. Basic facts concerning the discriminant divisor are reviewed and, in particular, its degree is computed. We describe the monodromy group of the family of smooth hypersurfaces as a subgroup of the orthogonal group of the middle cohomology and complement the results with a comparison of the action of the group of diffeomorphisms.

Hypersurfaces $X \subset \mathbb{P} = \mathbb{P}^{n+1}$ of degree d are parametrized by the projective space

$$|\mathcal{O}(d)| \simeq \mathbb{P}^{N(d,n)},$$

where $N = N(d,n) = h^0(\mathbb{P}^{n+1}, \mathcal{O}(d)) - 1 = \binom{n+1+d}{d} - 1$. The *universal hypersurface* shall be denoted

$$\mathcal{X} \subset \mathbb{P}^N \times \mathbb{P}. \tag{2.1}$$

It is of bidegree $(1,d)$, i.e. a divisor contained in the linear system $|\mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{O}_{\mathbb{P}}(1)|$, and the fibre of the (flat) first projection $\mathcal{X} \longrightarrow \mathbb{P}^N$ over the point corresponding to $X \subset \mathbb{P}$ is indeed just X.

More explicitly, $\mathcal{X}$ can be described as the zero set of the universal equation $G = \sum a_I x^I$, where $a_I \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ are the linear coordinates corresponding to the monomials $x^I \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$. In other words, if one writes $\mathbb{P}$ as $\mathbb{P} = \mathbb{P}(V)$ for some vector space V of dimension $n+2$, then $H^0(\mathbb{P}, \mathcal{O}(d)) = S^d(V^*)$ and $\mathbb{P}^N = \mathbb{P}(S^d(V^*))$. Hence, $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) = S^d(V)$ and then G corresponds to the identity in $\mathrm{End}(S^d(V)) \simeq S^d(V) \otimes S^d(V^*) = H^0(\mathbb{P}^N \times \mathbb{P}, \mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{O}_{\mathbb{P}}(d))$.

The universal hypersurface $\mathcal{X}$ is smooth. For this observe that the second projection $\mathcal{X} \longrightarrow \mathbb{P}$ is in fact the projective bundle $\mathbb{P}(\mathrm{Ker(ev)}) \longrightarrow \mathbb{P}$, where

$$\mathrm{ev}\colon H^0(\mathbb{P}, \mathcal{O}(d)) \otimes \mathcal{O} \twoheadrightarrow \mathcal{O}(d)$$

is the evaluation map.

2.1 The natural $\mathrm{SL}(n+2)$-action on $H^0(\mathbb{P}, \mathcal{O}(d))$ descends to an action of $\mathrm{SL}(n+2)$ and $\mathrm{PGL}(n+2)$ on $|\mathcal{O}_{\mathbb{P}}(d)|$. Both are linearized in the sense that they are obtained by composing homomorphisms $\mathrm{SL}(n+2) \longrightarrow \mathrm{SL}(N+1)$ and $\mathrm{PGL}(n+2) \longrightarrow \mathrm{PGL}(N+1)$ with the natural actions of $\mathrm{SL}(N+1)$ and $\mathrm{PGL}(N+1)$ on $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ and $|\mathcal{O}_{\mathbb{P}^N}(1)|$.

The following table records the dimensions of the linear system of cubic hypersurfaces of small dimensions. It also contains information about the moduli space

$$M_n \coloneqq |\mathcal{O}_{\mathbb{P}}(3)|_{\mathrm{sm}}/\mathrm{PGL}(n+2)$$

and the discriminant divisor $D(n) \coloneqq D(3,n) \subset |\mathcal{O}_{\mathbb{P}}(3)|$, both to be discussed below. We write $N(n) \coloneqq N(3,n)$.

n	$N(n)$	$\dim(\mathrm{PGL}(n+2))$	$\dim(M_n)$	$\deg(D(n))$
0	3	3	0	4
1	9	8	1	12
2	19	15	4	32
3	34	24	10	80
4	55	35	20	192
5	83	48	35	448

The closed formula for the dimension of the moduli space is

$$\dim(M_n) = \binom{n+2}{3} = \frac{n^3 + 3\,n^2 + 2\,n}{6}. \tag{2.2}$$

2.2 We are mostly interested in smooth hypersurfaces. They are parametrized by a Zariski open subset which shall be denoted

$$U(n,d) \coloneqq |\mathcal{O}_{\mathbb{P}}(d)|_{\mathrm{sm}} \coloneqq \{\, X \in |\mathcal{O}_{\mathbb{P}}(d)| \mid X \text{ smooth} \,\} \subset |\mathcal{O}_{\mathbb{P}}(d)|.$$

For an algebraically closed ground field k, Bertini's theorem shows that $U(n,d)$ is non-empty and hence dense. In fact, if $\mathrm{char}(k) = 0$ or at least $\mathrm{char}(k) \nmid d$, then the Fermat hypersurface

$$X = V\left(\sum_{i=0}^{n+1} x_i^d\right) \subset \mathbb{P}$$

is always smooth, as it is easy to check using the Jacobian criterion. In [94, p. 333] one finds the following explicit equations for smooth hypersurfaces over arbitrary fields

$$\begin{cases} \sum_{i=0}^{m-1} x_i\, x_{i+m} & \text{if } d=2, n+2=2m \\ \sum_{i=0}^{m-1} x_i\, x_{i+m} + x_{n+1}^2 & \text{if } d=2, n+1=2m \\ \sum_{i=0}^{n+1} x_i^d & \text{if } d\geq 3, \text{char}(k)\nmid d \\ \sum_{i=0}^{n} x_i\, x_{i+1}^{d-1} + x_0^d & \text{if } d\geq 3, \text{char}(k)\mid d. \end{cases} \tag{2.3}$$

Hence, $U(n,d) = |\mathcal{O}_{\mathbb{P}}(d)|_{\text{sm}} \neq \emptyset$ always holds.

Definition 2.1. The *discriminant divisor* $D(d,n) \subset |\mathcal{O}_{\mathbb{P}}(d)|$ is the complement of the Zariski open (and dense) subset $U(d,n) \subset |\mathcal{O}_{\mathbb{P}}(d)|$ of smooth hypersurfaces endowed with the reduced induced scheme structure.

Theorem 2.2. *The discriminant divisor $D(d,n)$ is an irreducible divisor. Its degree is $(d-1)^{n+1}\,(n+2)$, which for $d=3$ reads*

$$\deg(D(3,n)) = 2^{n+1}\,(n+2).$$

Proof Consider the universal hypersurface $\mathcal{X} \subset \mathbb{P}^N \times \mathbb{P}$ as above and define

$$\mathcal{X}_{\text{sing}} \coloneqq \mathcal{X} \cap \bigcap_{i=0}^{n+1} V_i,$$

where $V_i \coloneqq V(\partial_i G)$ are the hypersurfaces of bidegree $(1, d-1)$ defined by

$$\partial_i G \coloneqq \sum a_I \frac{\partial x^I}{\partial x_i} \in H^0\left(\mathbb{P}^N \times \mathbb{P}, \mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{O}_{\mathbb{P}}(d-1)\right).$$

By the Jacobian criterion, $\mathcal{X}_{\text{sing}} \subset \mathcal{X} \longrightarrow \mathbb{P}^N$ is the family of singular loci of the fibres $\mathcal{X}_t$, i.e. $(\mathcal{X}_{\text{sing}})_t = (\mathcal{X}_t)_{\text{sing}}$.

As the *Euler equation* (see [24, Ch. 4]) holds in its universal form $\sum x_i\, \partial_i G = d \cdot G$, one has $\mathcal{X} \subset \bigcap V_i$ if $\text{char}(k) \nmid d$ (which we will tacitly assume, but see Remark 2.3). Hence, $\mathcal{X}_{\text{sing}} = \bigcap V_i$ and, therefore, $\text{codim}(\mathcal{X}_{\text{sing}}) \leq n+2$. To prove that equality holds, consider the other projection $\mathcal{X}_{\text{sing}} \longrightarrow \mathbb{P}$ which is a $\mathbb{P}^k$-bundle with $k = N-n-2$. To see this, observe that the homomorphism of sheaves on $\mathbb{P}$

$$\varphi\colon H^0\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)\right) \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}(d-1)^{\oplus n+2},\ F \longmapsto (\partial_i F)$$

is surjective, which can be checked e.g. at the point $z = [1:1:\cdots:1]$ by using that $(\partial_i x_j^d)(z) = d \cdot \delta_{ij}$, and

$$\mathcal{X}_{\text{sing}} \simeq \mathbb{P}(\text{Ker}(\varphi)) \longrightarrow \mathbb{P}.$$

This clearly proves $\text{codim}(\mathcal{X}_{\text{sing}}) = n+2$, but also that $\mathcal{X}_{\text{sing}}$ is smooth and irreducible. To be precise, one needs to verify that $\mathcal{X}_{\text{sing}} \simeq \mathbb{P}(\text{Ker}(\varphi))$ as schemes and not only as sets, which is left to the reader.

Next, $D \coloneqq D(d, n)$ is by definition the image of $\mathcal{X}_{\text{sing}}$ under the projection

$$\mathcal{X}_{\text{sing}} \subset \mathcal{X} \subset \mathbb{P}^N \times \mathbb{P} \longrightarrow \mathbb{P}^N.$$

Let us denote the pull-backs of the hyperplane sections on $\mathbb{P}^N$ and $\mathbb{P}$ (both denoted by h) to $\mathbb{P}^N \times \mathbb{P}$ by h_1 and h_2. Then compute the relevant intersection number as the coefficient of $h_1^N\, h_2^{n+1}$

$$(h_1^{N-1}.\mathcal{X}_{\text{sing}}) = (h_1^{N-1}.(h_1 + (d-1)\,h_2)^{n+2}) = (n+2)\,(d-1)^{n+1}.$$

On the other hand, if D were of codimension > 1, then $(h^{N-1}.D) = 0$, leading to the contradiction $(h_1^{N-1}.\mathcal{X}_{\text{sing}}) = 0$. Hence, $D \subset \mathbb{P}^N$ really is a divisor. The computation also shows that in order to prove the claimed degree formula for D, it suffices to prove that $\mathcal{X}_{\text{sing}} \twoheadrightarrow D$ is generically injective or, in other words, that the generic singular hypersurface $X \in |\mathcal{O}_{\mathbb{P}}(d)|$ has exactly one singular point (which is in fact an ordinary double point). (One needs to assume $\text{char}(k) = 0$ for the set-theoretic injectivity to imply that the morphism is of degree one.) One way of doing it would be to write down examples of hypersurface in each degree with exactly one ordinary double point[3] or to argue geometrically (assuming $\text{char}(k) = 0$) by considering again the projective bundle $\mathcal{X}_{\text{sing}} \longrightarrow \mathbb{P}$. The fibre over a point z can be thought of as a linear system with z as its only base point. By Bertini's theorem with base points, see e.g. [78, III. Rem. 10.9.2], the generic element will then be singular exactly at z. To see that generically it has to be an ordinary double point, just write down one hypersurface with such a singular point at z (but possibly other singular points), e.g. the union of $(d-2)$ generic hyperplanes $\mathbb{P}^n \subset \mathbb{P}$ and of a cone with vertex z over a quadric in some hyperplane. □

Remark 2.3. In [1, Exp. XVII] the discriminant divisor is viewed as the dual variety of the Veronese embedding $v_d \colon \mathbb{P} \hookrightarrow \mathbb{P}^{N\vee}$, i.e. as the locus of hyperplanes (parametrized by $\mathbb{P}^N$) that are tangent to $v_d(\mathbb{P})$. It is also proved that the smooth locus of $D(d, n)$ is the maximal open subset over which $\mathcal{X}_{\text{sing}} \twoheadrightarrow D(d, n)$ is an isomorphism and that it coincides with the set of those singular hypersurfaces with one ordinary double point as only singularity.

2.3 There is a classical and more algebraic approach to the discriminant divisor using resultants, cf. [27, 38, 50, 65]. Here are some general facts. Consider homogeneous polynomials in $k[x_0, \ldots, x_{n+1}]$ of degree $d_i > 0$, $i = 0, \ldots, m$. Then there exists a unique polynomial, the *resultant*, $R(y_{i,I}) = R_{d,n}(y_{i,I}) \in k[y_{i,I}]$, $i = 0, \ldots, m$, $|I| = d_i$, such that:

(i) For all $f_i \in k[x_0, \ldots, x_{n+1}]_{d_i}$, $i = 0, \ldots, m$, the intersection $\bigcap V(f_i) \subset \mathbb{P}^{n+1}_{\bar{k}}$ is non-empty if and only if $R(f_0, \ldots, f_m) = 0$.

[3] Duco van Straten has provided me with examples in certain degrees. Note that writing down examples with just one singular point is easy, e.g. the cone over the smooth examples in (2.3) has only one singular point, which however is an ordinary double point only for $d = 2$.

(ii) $R(x_0^{d_0}, \ldots, x_m^{d_m}) = 1$ (normalization).
(iii) $R \in k[y_{i,I}]$ is irreducible.

In (i), $R(f_0, \ldots, f_m)$ is the shorthand for applying R to the coefficients of the polynomials f_i. Moreover, R is homogeneous of degree $\prod_{j \neq i} d_j$ in the variables $y_{i,I}$ and so of total degree $\prod d_i \cdot \sum(1/d_i)$.

Consider $F \in k[x_0, \ldots, x_{n+1}]$ and apply the above to $f_i = \partial_i F$, $i = 0, \ldots, m = n+2$, which are all homogeneous of degree $d_i = d - 1$. Then $X = V(F)$ is singular, i.e. $\bigcap V(f_i) \neq \emptyset$, if and only if $X \in |\mathcal{O}(d)|$ is in the zero locus of R. Strictly speaking, R defines a hypersurface in $\mathbb{P}^{N'} = \mathrm{Proj}(k[y_{i,I}])$, $i = 0, \ldots, n+1$, $|I| = d-1$, so $N' = (n+2)\binom{n+d}{d-1} - 1$. Its pull-back via the linear embedding $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$ that maps x^I to $\left[i_j x^{I_j}\right]_{j=0,\ldots,n+1}$, where for $I = (i_0, \ldots, i_{n+1})$ one sets $I_j := (i_0, \ldots, (i_j - 1), \ldots, i_{n+1})$, describes the image of $\mathcal{X}_{\mathrm{sing}}$, i.e. the discriminant divisor. The irreducibility still holds, cf. [50, Sec. 5&6].

Remark 2.4. The resultant is usually normalized to yield the *discriminant*

$$\Delta_{d,n} := d^{c_{d,n}} \cdot R_{d,n}(\partial_i G) \in H^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}((d-1)^{n+1}(n+2))\right), \tag{2.4}$$

where $c_{d,n} = (1/d)((-1)^{n+2} - (d-1)^{n+2})$. With this normalization, $\Delta_{d,n}$ becomes an irreducible polynomial in $\mathbb{Z}[y_I]$, which makes it unique up to ± 1.

Example 2.5. The case $n = 0$ and $d = 3$, so three points in $\mathbb{P}^1$, leads to the classical discriminant for cubic polynomials $f(X)$. If $\alpha_1, \alpha_2, \alpha_3$ are the zeros of $f(X)$, then by definition $\Delta(f(X)) = ((\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3))^2$. For $f(X) = X^3 + aX + b$ one has $\Delta(f(X)) = -4a^3 - 27b^2$.

The discriminant of a general polynomial $a_0 x_0^3 + a_1 x_0^2 x_1 + a_2 x_0 x_1^2 + a_3 x_1^3$ is the rather complicated polynomial of degree four

$$\Delta_{3,0} = a_1^2 a_2^2 - 4a_3 a_1^3 - 4a_2^3 a_0 - 27 a_3^2 a_0^2 + 18 a_0 a_1 a_2 a_3.$$

As an exercise, the reader may want to compare this with the normalization $R(x_0^2, x_1^2) = 1$, which in (2.4), using $c_{3,0} = -1$, yields $\Delta_{3,0}\left((1/3)(x_0^3 + x_1^3)\right) = (1/3)$. Also, confirming Remark 2.3, the singular set of $D(3,0) = V(\Delta_{3,0})$ is indeed the curve of triple points.

Example 2.6. For $n = 1$ and $d = 3$ the discriminant divisor is of degree 12 or, equivalently, the discriminant is an element of the vector space $H^0(\mathbb{P}^9, \mathcal{O}_{\mathbb{P}^9}(12))$ which is of dimension 293.930. Written as a linear combination of monomials, 12.894 of the coefficients are non-trivial, cf. [38, p. 99]. If the partial derivatives $\partial_i F$ are written as $\partial_0 F = a_{11} x_0^2 + a_{12} x_1^2 + a_{13} x_2^2 + a_{14} x_0 x_1 + a_{15} x_0 x_2 + a_{16} x_1 x_2$, etc., and one defines $[\ell_1 \ell_2 \ell_3] := \det(a_{i,\ell_j}) \in H^0(\mathbb{P}^9, \mathcal{O}(3))$, with pairwise distinct ℓ_i, then Δ is a polynomial of degree four in the $[\ell_1 \ell_2 \ell_3]$ involving only 68 terms. In short, the discriminant is complicated.

Maybe just one word on the comparison between the discriminant introduced here and the discriminant of a plane cubic $E \subset \mathbb{P}^2$ in Weierstrass form $y^2 = 4x^3 - g_2 x - g_3$ which is classically defined as $\Delta(E) \coloneqq g_2^3 - 27g_3^2$. This is a rather simple polynomial of degree three in the coefficients, whereas the full discriminant of cubic plane curves is a polynomial of degree 12. The reason for this is that bringing a cubic polynomial in x_0, x_1, x_2 into Weierstrass form involves non-linear transformations. More concretely, the coefficients g_2 and g_3 of the Weierstrass form are of degree four and six, respectively, in the coefficients of the original cubic equation, see e.g. [95, Ch. 3].

Corollary 2.7. *Assume $k = \bar{k}$. Then for the generic line $\mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ the induced family $\mathcal{X}_{\mathbb{P}^1} \twoheadrightarrow \mathbb{P}^1$ has exactly $(d-1)^{n+2}\,(n+2)$ singular fibres $\mathcal{X}_1, \mathcal{X}_2, \ldots$, each with exactly one singular point $x_i \in \mathcal{X}_i$. Moreover, the x_i are all ordinary double points.* □

A pencil with these properties is called a *Lefschetz pencil*. Note that by Bertini's theorem [78, III. Cor. 10.9], at least when $\mathrm{char}(k) = 0$, the total space $\mathcal{X}_{\mathbb{P}^1}$ is still smooth. See [1, Exp. XVII].

In more concrete terms, for generic choice of polynomials $F_0, F_1 \in H^0(\mathbb{P}, \mathcal{O}(d))$ for exactly $(d-1)^{n+2}\,(n+2)$ values $t = [t_0 : t_1]$ the hypersurface $\mathcal{X}_t = V(t_0\,F_0 + t_1\,F_1)$ is singular. Each singular fibre $\mathcal{X}_i$ has exactly one singular point x_i, which, moreover, is an ordinary double point. Note that $x_i \neq x_j$ for $i \neq j$, as otherwise x_i would be a singular point of all the fibres.

Example 2.8. There are, of course, pencils $\mathcal{X}_{\mathbb{P}^1} \twoheadrightarrow \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$, the fibres of which have more or worse singularities. The *Hesse pencil* of plane cubics $\mathcal{X}_t \subset \mathbb{P}^2$ given by

$$t_0\,(x_0^3 + x_1^3 + x_2^3) + t_1\,x_0x_1x_2$$

is such an example. Here, the fibre $\mathcal{X}_{[0:1]}$ consists of three lines yielding three singular points. The Hesse pencil is a special instance of the *Dwork pencil* (or *Fermat pencil*), see [19], defined by the equation

$$t_0 \left(\sum_{i=0}^{n+1} x_i^{n+2} \right) - t_1\,d \prod_{i=0}^{n+1} x_i$$

of hypersurfaces of degree $d = n+2$.

Clearly, the number of singular fibres of any pencil is bounded by $(d-1)^{n+1}\,(n+2)$, unless all fibres are singular. Note that for general pencils the total space $\mathcal{X}_{\mathbb{P}^1}$ need not be smooth.

2.4 We now assume $k = \mathbb{C}$. Let us consider the universal family of smooth hypersurfaces

$$\pi\colon \mathcal{X} \twoheadrightarrow U(d,n) \subset |\mathcal{O}_{\mathbb{P}^{n+1}}(d)|.$$

Note the change in notation. If needed later, we will denote the universal family of all hypersurfaces by $\bar{\mathcal{X}}$. It is smooth and projective and contains $\mathcal{X}$ as a dense open subset. Fix a point $0 \in U(d,n)$ and denote the fibre over it by $X := \mathcal{X}_0$. The *monodromy representation*

$$\rho\colon \pi_1(U(d,n),0) \longrightarrow \mathrm{GL}(H^n(X,\mathbb{Z})) \tag{2.5}$$

is the homomorphism obtained by parallel transport with respect to the Gauss–Manin connection. Equivalently, $R^n\pi_*\mathbb{Z}$ is a locally constant system on $U(n,d)$ and (2.5) is the corresponding representation of the fundamental group. Note that the image

$$\Gamma(d,n) := \mathrm{Im}\,(\rho\colon \pi_1(U(d,n),0) \longrightarrow \mathrm{GL}(H^n(X,\mathbb{Z})))\,,$$

called the *monodromy group*, depends on $0 \in U(d,n)$ only up to conjugation.

The image has been determined in [12] in complete generality. We state the result for $d = 3$ and use the shorthand

$$\Gamma_n := \Gamma(3,n) \subset \mathrm{GL}(H^n(X,\mathbb{Z})).$$

Theorem 2.9. *The monodromy group Γ_n of the universal smooth cubic hypersurface $\mathcal{X} \longrightarrow |\mathcal{O}_{\mathbb{P}^{n+1}}(3)|_{\rm sm}$ is the group*

$$\Gamma_n \simeq \begin{cases} \tilde{\mathrm{O}}^+(H^n(X,\mathbb{Z})) & \text{if } n \equiv 0\,(2) \\ \mathrm{SpO}(H^n(X,\mathbb{Z}),q) & \text{if } n \equiv 1\,(2). \end{cases}$$

Before sketching the main steps of the proof, let us explain the notation: For n even, one defines $\tilde{\mathrm{O}}(H^n(X,\mathbb{Z})) \subset \mathrm{O}(H^n(X,\mathbb{Z}))$ as the subgroup of all orthogonal transformations $g\colon H^n(X,\mathbb{Z}) \xrightarrow{\sim} H^n(X,\mathbb{Z})$ with $g(h^{n/2}) = h^{n/2}$. Via the induced action on $H^n(X,\mathbb{Z})_{\rm pr}$ it can be identified with the subgroup (cf. [86, Prop. 14.2.6])

$$\tilde{\mathrm{O}}(H^n(X,\mathbb{Z})) \simeq \{\, g \in \mathrm{O}(H^n(X,\mathbb{Z})_{\rm pr}) \mid \mathrm{id} = \bar{g} \in \mathrm{O}(A_{H^n(X,\mathbb{Z})_{\rm pr}})\,\}.$$

Here, we use the notation $A_\Lambda := \Lambda^*/\Lambda$ for the discriminant of a lattice Λ, which for the primitive cohomology of a smooth cubic hypersurface is just $\mathbb{Z}/3\mathbb{Z}$.

Another subgroup is given as $\mathrm{O}^+(H^n(X,\mathbb{Z})_{\rm pr}) := \mathrm{Ker}(\mathrm{sn}_b\colon \mathrm{O}(H^n(X,\mathbb{Z})_{\rm pr}) \longrightarrow \{\pm 1\})$. Here, the spinor norm $\mathrm{sn}_n(s_\delta)$ of a reflection in $\delta^\perp$ is defined as $(-1)^{n/2} \cdot (\delta)^2/|(\delta)^2|$. In other words, if g is written as a product $\prod s_{\delta_i}$ of reflections with $\delta_i \in H^n(X,\mathbb{R})_{\rm pr}$ (using the Cartan–Dieudonné theorem), then $\mathrm{sn}_n(g) = 1$ if the number of δ_i with $(\delta_i)^2 < 0$ for $n \equiv 0\,(4)$ (respectively, of δ_i with $(\delta_i)^2 > 0$ for $n \equiv 2\,(4)$) is even.

The orthogonal group in the theorem is the finite index subgroup of $\mathrm{O}(H^n(X,\mathbb{Z})_{\rm pr})$:

$$\tilde{\mathrm{O}}^+(H^n(X,\mathbb{Z})) := \tilde{\mathrm{O}}(H^n(X,\mathbb{Z})) \cap \mathrm{O}^+(H^n(X,\mathbb{Z})_{\rm pr}).$$

For n odd, the intersection product on $H^n(X,\mathbb{Z}) = H^n(X,\mathbb{Z})_{\rm pr}$ is alternating and can be put in the standard normal form. However, there exists an auxiliary rather subtle topological invariant, which is the quadratic form $q\colon H^n(X,\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ (see [26, Sec. 1]) that enters the definition of the group $\mathrm{SpO}(H^n(X,\mathbb{Z}),q)$ in the above theorem. Using that any $\alpha \in H^n(X,\mathbb{Z})$ can be represented by an embedded sphere $S^n \hookrightarrow X$, one has for $n \neq 1, 3, 7$ that $q(\alpha) = 0$ if and only if the topological normal bundle of $S^n \hookrightarrow X$ is trivial.[4] Then $\mathrm{SpO}(H^n(X, \mathbb{Z}), q)$ is defined as the group of isomorphisms $g \colon H^n(X, \mathbb{Z}) \xrightarrow{\sim} H^n(X, \mathbb{Z})$ compatible with the alternating intersection form (.) and the quadratic form q.

Remark 2.10. The appearance of the primitive cohomology in Theorem 2.9 is not a surprise. Indeed, the restriction of $\mathrm{c}_1(\mathcal{O}(1))^{n/2} \in H^n(\mathbb{P}^{n+1}, \mathbb{Z})$ to any of the fibres $\mathcal{X}_t$ defines a section of the locally constant system $R^n\pi_*\mathbb{Z}$. Hence, the primitive cohomology groups $H^n(\mathcal{X}_t, \mathbb{Z})_{\mathrm{pr}}$ of the fibres glue to a locally constant subsheaf $R^n_{\mathrm{pr}}\pi_*\mathbb{Z} \subset R^n\pi_*\mathbb{Z}$. Equivalently, the monodromy representation (2.5) satisfies $\rho(\gamma)(h^{n/2}) = h^{n/2}$ for all $\gamma \in \pi_1(U(d, n))$, i.e. $h^{n/2}$ is monodromy invariant.

In fact, $h^{n/2}$ is the only monodromy invariant class up to scaling. Indeed, Deligne's invariant cycle theorem [149, V. Thm. 16.24] shows that the monodromy invariant part of $H^n(X, \mathbb{Q})^\rho \subset H^n(X, \mathbb{Q})$ is the image of the restriction $H^n(\bar{\mathcal{X}}, \mathbb{Q}) \twoheadrightarrow H^n(X, \mathbb{Q})$, where $\bar{\mathcal{X}} \subset \mathbb{P}^N \times \mathbb{P}$ denotes the universal family of all hypersurfaces. But writing $\bar{\mathcal{X}}$ as a projective bundle over $\mathbb{P}$ shows that $H^n(\bar{\mathcal{X}}, \mathbb{Q}) \simeq \bigoplus H^{n-2i}(\mathbb{P}, \mathbb{Q}) \cdot \mathrm{c}_1(\mathcal{O}_{\mathbb{P}^N}(1))^i$. As $\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^N}(1))$ restricts trivially to the fibres of the first projection $\bar{\mathcal{X}} \twoheadrightarrow \mathbb{P}^N$, only $H^n(\mathbb{P}, \mathbb{Q})$ survives the map $H^n(\bar{\mathcal{X}}, \mathbb{Q}) \twoheadrightarrow H^n(X, \mathbb{Q})$ and, therefore, its image is spanned by $h^{n/2}$.

Similarly, the monodromy representation preserves the intersection form on $H^n(X, \mathbb{Z})$. Therefore, $\mathrm{Im}(\rho) \subset \mathrm{O}(H^n(X, \mathbb{Z}))$ for n even and $\mathrm{Im}(\rho) \subset \mathrm{Sp}(H^n(X, \mathbb{Z}))$ for n odd.

Note that one can deduce from the theorem the well-known fact [149, V. Thm. 15.27] that $H^n(X, \mathbb{Q})_{\mathrm{pr}}$ is an irreducible $\Gamma(d, n)$-module or, equivalently, that $R^n_{\mathrm{pr}}\pi_*\mathbb{Q}$ cannot be written as a direct sum of non-trivial locally constant systems.

2.5 The computation of the monodromy group $\Gamma(d, n)$ proceeds in three steps.

(i) Show that $\Gamma(d, n)$ equals the monodromy group of the smooth part of a Lefschetz pencil $\mathcal{X}_{\mathbb{P}^1} \twoheadrightarrow \mathbb{P}^1$.

(ii) Assume $\mathcal{X} \twoheadrightarrow \Delta$ is a family of hypersurfaces over a disk with $\mathcal{X}$ and $\mathcal{X}_{t \neq 0}$ smooth and such that the central fibre $\mathcal{X}_0$ has one ordinary double point as its only singularity. Let γ be the simple loop around $0 \in \Delta$. Describe the induced monodromy operation $\rho(\gamma) \colon H^n(X, \mathbb{Z}) \xrightarrow{\sim} H^n(X, \mathbb{Z})$, where $X = \mathcal{X}_\varepsilon$ is a distinguished smooth fibre, as a reflection s_δ.

(iii) Let $\mathcal{X}_{\mathbb{P}^1} \twoheadrightarrow \mathbb{P}^1$ be a Lefschetz pencil with nodal singular fibres over $t_1, \ldots, t_\ell \in \mathbb{P}^1 \setminus \infty$. Describe the sub-group $\langle s_{\delta_i} \rangle \subset \mathrm{GL}(H^n(X, \mathbb{Z}))$ generated by the monodromy operations around all the nodal fibres $\mathcal{X}_{t_1}, \ldots, \mathcal{X}_{t_\ell}$.

[4] The *Arf invariant* of q, also called the *Kervaire invariant* of X, is often viewed as the analogue of the discrminant of the symmetric intersection form for n even. Recall that the Arf invariant $A(q) \in \mathbb{F}_2$ of the binary quadratic form $q = ax^2 + xy + by^2$ is ab. For arbitrary q, which can be written as a direct sum of those, it is defined by additive extension. Due to [100, Prop. 12.1], the Kervaire invariant is non-trivial for cubic hypersurfaces

To give at least a rough idea, here are a few more details for all three steps. For details of the statements and proofs we have to refer to the literature, cf. [149].

(i) Similar to the Lefschetz hyperplane theorem for smooth hyperplane sections of smooth projective varieties, cf. Section 1.1, a result of Zariski, see [75] or [149, V. Thm. 15.22], shows that for a very general line $\mathbb{P}^1 \subset \mathbb{P}^N = |\mathcal{O}(d)|$ the natural map

$$\pi_1(\mathbb{P}^1 \setminus D) \twoheadrightarrow \pi_1(\mathbb{P}^N \setminus D) \tag{2.6}$$

is surjective. (From now on we suppress the base point in $\mathbb{P}^1 \subset \mathbb{P}^N$ in the notation.) The restriction of $R^n\pi_*\mathbb{Z}$ to $\mathbb{P}^1 \setminus D$, which is isomorphic to the higher direct image for the restriction of the family to $\mathbb{P}^1 \setminus D$, corresponds to the representation

$$\pi_1(\mathbb{P}^1 \setminus D) \twoheadrightarrow \pi_1(\mathbb{P}^N \setminus D) \longrightarrow \mathrm{GL}(H^n(X,\mathbb{Z}))$$

obtained by composing (2.5) with (2.6). Hence, $\Gamma(d,n)$ can be computed as the monodromy group of an arbitrary Lefschetz pencil $\mathcal{X}_{\mathbb{P}^1} \longrightarrow \mathbb{P}^1$, i.e. as the image of

$$\rho_{\mathbb{P}^1} : \pi_1(\mathbb{P}^1 \setminus D) \longrightarrow \mathrm{GL}(H^n(X,\mathbb{Z})). \tag{2.7}$$

By Theorem 2.2, $\mathbb{P}^1 \setminus D \simeq \mathbb{P}^1 \setminus \{t_1,\ldots,t_\ell\}$ with $\ell = (d-1)^{n+1}(n+2)$. Therefore, $\pi_1(\mathbb{P}^1 \setminus D)$ is isomorphic to a quotient of the free group $\pi_1(\mathbb{C} \setminus \{t_1,\ldots,t_\ell\}) \simeq \mathbb{Z}^{*\ell}$ with free generators given by the simple loops γ_i around the points $t_i \in \mathbb{C}$. Thus, in order to describe the image of (2.7), we need to compute the monodromy operators $\rho_{\mathbb{P}^1}(\gamma_i)$ and the group they generate. (In our discussion the details concerning the base point and the dependence on the path connecting it to circles around the critical values are suppressed.)

(ii) Let $x \in \mathcal{X}_0$ be the ordinary double point of the central fibre of a family $\mathcal{X} \longrightarrow \Delta$ obtained from the above by restriction to a small disk $\Delta \hookrightarrow \mathbb{P}^1$, $0 \longmapsto t_i$. Intersecting a ball $B(x) \subset \mathcal{X}$ around p with the nearby smooth fibre $X = \mathcal{X}_\varepsilon$ retracts to a sphere $S^n \subset B(x) \cap X \subset X$. It is called the *vanishing sphere* and its cohomology class $\delta = [S^n] \in H^n(X,\mathbb{Z})$ is the *vanishing class*. Its main property, responsible for its name and verified by a local computation, is that it generates the kernel of the push-forward map (cf. [149, V. Cor. 14.17]):

$$H^n(X,\mathbb{Z}) \simeq H_n(X,\mathbb{Z}) \longrightarrow H_n(\mathcal{X},\mathbb{Z}).$$

The self intersection $(\delta)^2$, determined by the normal bundle of $S^n \subset X$, is given by

$$(\delta)^2 = \begin{cases} 0 & \text{if } n \equiv 1\,(2) \\ -2 & \text{if } n \equiv 2\,(4) \\ 2 & \text{if } n \equiv 0\,(4). \end{cases}$$

Of course, the vanishing for odd n follows from the fact that in this case the intersection pairing on the middle cohomology is alternating. The other two cases are obtained by an explicit computation, see [149, IV.15.2].

The crucial input is the description of the monodromy operation $\rho(\gamma)$ induced by a simple loop around $0 \in \Delta$. It is described by the *Picard–Lefschetz formula*:

$$\rho(\gamma) = s_\delta \colon \alpha \longmapsto \alpha + \varepsilon_n \, (\alpha.\delta) \, \delta, \tag{2.8}$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv 1, 2 \, (4) \\ -1 & \text{if } n \equiv 0, 3 \, (4), \end{cases}$$

i.e. $\varepsilon_n = -(-1)^{n(n-1)/2}$. Note that the sign is such that for n even s_δ is a reflection in $\delta^\perp$ and so, in particular, $s_\delta(\delta) = -1$ and $s_\delta^2 = \mathrm{id}$. For n odd, the monodromy is not of finite order, as $s_\delta^k(\alpha) = \alpha + \varepsilon_n \, k \, (\alpha.\delta) \, \delta$.[5]

(iii) We have computed the images $\rho_{\mathbb{P}^1}(\gamma_i)$ of the free loops around the singular fibres $\mathcal{X}_{t_i}$, $i = 1, \ldots, \ell = \deg D(d, n)$, as the operators s_{δ_i} (which are reflections for even n and of infinite order if n is odd).

Consider now families $\mathcal{X}^i \longrightarrow \Delta^i$ around each $t_i \in \mathbb{P}^1$ as in (ii). We may assume that the smooth reference fibre is X for all of them. Note that all vanishing classes $\delta_i \in H^n(X, \mathbb{Z})$ are contained in the primitive cohomology. To see this, consider the composition

$$H^n(X, \mathbb{Z}) \simeq H_n(X, \mathbb{Z}) \longrightarrow H_n(\mathcal{X}^i, \mathbb{Z}) \longrightarrow H_n(\mathbb{P}, \mathbb{Z}) \simeq H^{n+2}(\mathbb{P}, \mathbb{Z}) \longrightarrow H^{n+2}(X, \mathbb{Z}),$$

which is the product with the hyperplane class. In fact, the *vanishing cohomology* $H^n(X, \mathbb{Z})_{\mathrm{van}} := \mathrm{Ker}(H^n(X, \mathbb{Z}) \longrightarrow H^{n+2}(\mathbb{P}, \mathbb{Z}))$, which in our situation coincides with the primitive cohomology, is generated over $\mathbb{Z}$ by the vanishing classes, see [149, V. Lem. 14.26] or for the algebraic treatment [48, Sec. 4.3]. This leads to the following consequence.

Corollary 2.11. *The primitive cohomology $H^n(X, \mathbb{Z})_{\mathrm{pr}}$ of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ is generated by classes δ with $(\delta)^2$ even and, in fact, $(\delta)^2 = -2, 0, 2$ for $n \equiv 2\,(4)$, $n \equiv 1\,(2)$, and $n \equiv 0\,(4)$, respectively. In particular, for $n \equiv 0\,(2)$, the lattice $H^n(X, \mathbb{Z})_{\mathrm{pr}}$ is even.* □

Note that the fact that $H^n(X, \mathbb{Z})_{\mathrm{pr}}$ is generated by the vanishing classes δ_i in particular shows that $b_n(X)_{\mathrm{pr}} \leq \deg D(d, n)$, which is confirmed by a quick comparison of Corollary 1.8 with Theorem 2.2.

For even n the *Weyl group* $W \subset \mathrm{O}(H^n(X, \mathbb{Z})_{\mathrm{pr}})$ is by definition the subgroup generated by the reflections s_{δ_i}. For n odd $W \subset \mathrm{Sp}(H^n(X, \mathbb{Z}))$ is defined analogously. It turns out,

[5] In [12] the sign of the intersection form is changed for $n \equiv 3\,(4)$, so that in this case as well $s_\delta(\alpha) = \alpha + (\alpha.\delta)\,\delta$.

that in both cases the Weyl group acts transitively on $\Delta \coloneqq \{\delta_i\}$, cf. [109, Prop. 7.5] or [149, Prop. 15.23].

A lattice Λ (symmetric or alternating) with a class of vectors $\Delta \subset \Lambda$ generating Λ and with the associated Weyl group acting transitively on Δ is called a *vanishing lattice*, see [57, 93]. By our discussion so far we have

$$\Gamma(d,n) = \mathrm{Im}(\rho) = \mathrm{Im}(\rho_{\mathbb{P}^1}) = W.$$

The proof of Theorem 2.9 for even $n > 2$ is in [12] reduced to a purely lattice theoretic result in [57] describing the Weyl group of a *complete* vanishing lattice as this particular subgroup of the orthogonal group of the lattice. The lattice $H^n(X,\mathbb{Z})_{\mathrm{pr}}$ is complete, which means that Δ contains a certain configuration of six vanishing classes. The fact that according to Proposition 1.14 for $n > 2$ the lattice contains $A_2 \oplus U^{\oplus 2}$ is part of the picture. The case of cubic surfaces is well known classically and is usually stated as

$$\Gamma(3,2) \simeq W(E_6).$$

This is the only case of cubic hypersurfaces in which the monodromy group is actually finite. Indeed, it is an index two subgroup of the finite orthogonal group $\mathrm{O}(H^2(X,\mathbb{Z})_{\mathrm{pr}})$ of the definite lattice $H^2(X,\mathbb{Z})_{\mathrm{pr}}$. We shall come back to it in Section **4**.1.3. For n odd the result is deduced from [93].

Remark 2.12. In the algebraic setting, say over an algebraically closed field k, the geometric monodromy group, i.e. the Zariski closure of $\pi_1^{\text{ét}}(U) \longrightarrow \mathrm{GL}(H^n(X,\bar{\mathbb{Q}}_\ell))$, has been determined in [48, Sec. 4.4]. It is either a finite group for $n = 2$, the full orthogonal group $\mathrm{O}(H^n(X,\bar{\mathbb{Q}}_\ell)_{\mathrm{pr}})$ for n even and $n > 2$, or the full symplectic group $\mathrm{Sp}(H^n(X,\bar{\mathbb{Q}}_\ell))$. For n even the proof comes down to the fact that an algebraic subgroup $G \subset \mathrm{O}(V)$ of a complex vector space V with a non-degenerate symmetric bilinear form is either finite or $\mathrm{O}(V)$. The latter case holds as soon as there exists a G-orbit of classes δ with $(\delta)^2 = 2$ generating V and such that G contains all reflections s_δ induced by classes in that orbit, cf. [48, Lem. 4.4.2].

Exercise 2.13. Consider the case $n = 0$ for which the universal smooth hypersurface $\mathcal{X} \longrightarrow \mathbb{P}^3 \setminus S$, defined over the complement of a quartic surface S with explicit equation given in Example 2.5, is in fact an étale cover of degree three. Show that the monodromy group $\Gamma(0,3)$ is in fact $\mathfrak{S}_3$. It equals the Galois group of the field extension $K(\mathbb{P}^3) \subset K(\mathcal{X})$. See [76].

2.6 As any monodromy transformation is induced by a diffeomorphism, the monodromy group Γ_n is a subgroup of the image of the natural representation

$$\tau \colon \mathrm{Diff}^+(X) \longrightarrow \mathrm{O}(H^n(X,\mathbb{Z}))$$

of the group of orientation preserving diffeomorphisms. It turns out that $\mathrm{Im}(\tau)$ is slightly larger than Γ_n. Details have been worked out in [12]. Here are the main steps.

Let us first consider the case that n is even and $n > 2$. Clearly, $\mathrm{Diff}^+(X)$ also acts on $H^2(X,\mathbb{Z}) \simeq \mathbb{Z}\cdot h$ and, therefore, sends h to h or to $-h$, where the latter is realized by complex conjugation defined on any X defined by an equation with coefficients in $\mathbb{R}$. As a consequence, $\mathrm{Diff}^+(X)$ respects the direct sum decomposition $H^n(X,\mathbb{Q}) = H^n(X,\mathbb{Q})_{\mathrm{pr}}\oplus \mathbb{Q}\cdot h^{n/2}$. This eventually leads to

$$\mathrm{Im}(\tau) \simeq \begin{cases} \tilde{\mathrm{O}}(H^n(X,\mathbb{Z})) & \text{for } n \equiv 0\,(4) \\ \mathrm{O}(H^n(X,\mathbb{Z})_{\mathrm{pr}}) & \text{for } n \equiv 2\,(4). \end{cases}$$

To prove this note that $\tilde{\mathrm{O}}^+(H^n(X,\mathbb{Z})) \subset \mathrm{Im}(\tau)$ and that for $n \equiv 2\,(4)$ complex conjugation induces an element in $\mathrm{Im}(\tau)$ the restriction of which to $H^n(X,\mathbb{Z})_{\mathrm{pr}}$ acts non-trivially on the discriminant $A_{H^n_{\mathrm{pr}}} \simeq A_{\mathbb{Z}\cdot h^{n/2}} \simeq \mathbb{Z}/3\mathbb{Z}$. Hence, it is enough to find an orientation preserving diffeomorphism g which acts with spinor norm $\mathrm{sn}_n(\tau(g)) = -1$ on $H^n(X,\mathbb{Z})$. For this, one uses the connected sum decomposition of X as $M'\#(S^n\times S^n)$, cf. Remark 1.15, and the diffeomorphism g obtained by gluing the identity on M' with the product $\iota\times\iota$ of the diffeomorphism $\iota\colon S^n \longrightarrow S^n$, $(x_1,\ldots,x_{n+1}) \longmapsto (x_1,\ldots,x_n,-x_{n+1})$. It acts on the induced orthogonal decomposition $H^n(X,\mathbb{Z}) \simeq H^n(M',\mathbb{Z})\oplus H^n(S^n\times S^n,\mathbb{Z})$ by id on $U^\perp = H^n(M',\mathbb{Z})$ and by $-\mathrm{id}$ on $H^n(S^n\times S^n,\mathbb{Z}) \simeq U$. Write $-\mathrm{id}_U = s_{e-f}\circ s_{e+f}$ (with $e, f \in U$ the standard basis) to see that indeed $\mathrm{sn}_n(\tau(g)) = -1$.[6]

For cubic surfaces, there is no reason for a diffeomorphism to respect the hyperplane class (up to sign) and indeed $\mathrm{Im}(\tau) = \mathrm{O}(H^2(X,\mathbb{Z}))$, cf. [153] and Section **4**.1.3.

For n odd the result reads

$$\mathrm{Im}(\tau) = \begin{cases} \mathrm{SpO}(H^n(X,\mathbb{Z})) & \text{if } n \neq 1,3,7 \\ \mathrm{Sp}(H^n(X,\mathbb{Z})) & \text{if } n = 1,3,7. \end{cases}$$

Indeed, the description of $q([S^n])$ for $n \neq 1,3,7$ in terms of the normal bundle of $S^n \subset X$ is invariant under diffeomorphisms. In the other cases one proves that s_δ is realized by a diffeomorphism for any primitive $\delta \in H^n(X,\mathbb{Z})$. As those generate the symplectic group, this is enough to prove the claim for $n = 1,3,7$. For a given δ, there exist δ' with $(\delta.\delta') = 1$ and a decomposition $X \simeq M'\#(S^n\times S^n)$ with $H^n(S^n\times S^n,\mathbb{Z})$ spanned by δ,δ'. In [12] it is then observed that the reflection s_δ is realized by gluing the identity on M' with the diffeomorphism $(x,y) \longmapsto (x, x\cdot y)$, where $x\cdot y$ is the multiplication in $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ for the three cases $n = 1,3,7$.

6 In [12] the result for $n \equiv 2\,(4)$ is stated as $\mathrm{Im}(\tau) = \tilde{\mathrm{O}}(H^n(X,\mathbb{Z}))\times\{\pm 1\}$. Indeed, complex conjugation defines an element of order two in $\mathrm{Im}(\tau)$ that acts non-trivially on the discriminant of $H^n(X,\mathbb{Z})_{\mathrm{pr}}$. Moreover, it commutes with the index two subgroup $\Gamma_n = \tilde{\mathrm{O}}^+(H^n(X,\mathbb{Z}))$, as the universal family is defined over $\mathbb{R}$ and hence monodromy commutes with complex conjugation. However, that complex conjugation also commutes with the additional diffeomorphism g seems to need an extra argument.

3 Automorphisms and deformations

Smooth hypersurfaces behave nicely in many respects. For example, for most of them the deformation theory is easy to understand, not showing any of the pathological features to be reckoned with for arbitrary smooth projective varieties. Similarly, their group of automorphisms are usually finite and generically even trivial. We will assume $d \geq 3$ throughout this section. The only slightly exotic cases that need special care are $(n, d) = (1, 3)$ and $(n, d) = (2, 4)$, i.e. plane cubic (elliptic) curves and quartic K3 surfaces.

3.1 First order information about the group of automorphisms of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ and of its deformations are encoded by the cohomology groups $H^0(X, \mathcal{T}_X)$ and $H^1(X, \mathcal{T}_X)$, respectively. Those can be computed in terms of the standard exact sequences. We begin, however, with the following well-known fact.

Lemma 3.1. *Assume* char$(k) \nmid d$. *Then the hypersurface $X \subset \mathbb{P}^{n+1}$ defined by $F \in k[x_0, \ldots, x_{n+1}]_d$ is smooth if and only if the partial derivatives $\partial_i F$ form a regular sequence in $k[x_0, \ldots, x_{n+1}]$.*

Proof A standard result in commutative algebra shows that a sequence $a_i \in A$, $i = 1, \ldots, \dim(A)$, in a regular local ring A is regular if and only if $\mathrm{ht}((a_i)) = \dim(A)$, cf. [114, Thm. 16.B]. Hence, $(\partial_i F)$ is a regular sequence if and only if the affine intersection $V((\partial_i F)) = \bigcap V(\partial_i F) \subset \mathbb{A}^{n+2}$ is zero-dimensional. However, as the polynomials $\partial_i F$ are homogeneous, $V((\partial_i F))$ is $\mathbb{G}_m$-invariant. Hence, $(\partial_i F)$ is a regular sequence if and only if the projective intersection $V((\partial_i F)) \subset \mathbb{P}^{n+1}$ is empty. This implies that also $X \cap V((\partial_i F))$ is empty and, by the Jacobian criterion, that X is smooth. Conversely, if X is smooth and char$(k) \nmid d$, the Euler equation:

$$d \cdot F = \sum_{i=0}^{n+1} x_i \, \partial_i F. \tag{3.1}$$

shows that $V((\partial_i F)) = X_{\mathrm{sing}} = \emptyset$, i.e. $(\partial_i F)$ is a regular sequence. □

Example 3.2. The assumption on the characteristic is needed, as shown by the example $F = x_0^2 x_1 - x_0 x_1^2$ with char$(k) = 3$. Indeed, in this case $X = \{0, \infty, [1 : 1]\}$ is smooth, but $\partial_0 F = -x_1 \, (x_0 + x_1)$ and $\partial_1 F = x_0 \, (x_0 + x_1)$ have a common zero in $[1 : -1]$.

Corollary 3.3. *Let $X \subset \mathbb{P} = \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d.*

(i) *If $n \geq 0$ and $d \geq 3$ but $(n, d) \neq (1, 3)$, then $H^0(X, \mathcal{T}_X) = 0$.*
(ii) *If $n > 2$ or $d \leq 3$, then $H^1(X, \mathcal{T}_{\mathbb{P}}|_X) = 0$ and the normal bundle sequence induces a surjection*

$$H^0(X, \mathcal{O}_X(d)) \twoheadrightarrow H^1(X, \mathcal{T}_X).$$

Proof We shall give a proof under the additional assumption that $\mathrm{char}(k) \nmid d$ and refer to [94, Sec. 11.7] for the general case.

Combining the Euler sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(1)^{\oplus n+2} \longrightarrow \mathcal{T}_{\mathbb{P}} \longrightarrow 0$ and the normal bundle sequence $0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{P}}|_X \longrightarrow \mathcal{O}_X(d) \longrightarrow 0$, we obtain a diagram

$$\begin{array}{ccccccc} H^0(\mathcal{T}_{\mathbb{P}}|_X) & \longrightarrow & H^0(\mathcal{O}_X(d)) & \longrightarrow & H^1(\mathcal{T}_X) & \longrightarrow & H^1(\mathcal{T}_{\mathbb{P}}|_X) \\ \uparrow & \nearrow_{(\partial_i F)} & & & & & \\ H^0(\mathcal{O}_X(1))^{\oplus n+2}. & & & & & & \end{array} \tag{3.2}$$

Here, as before, $\partial_i F \in H^0(\mathbb{P}, \mathcal{O}(d-1))$ are the $n+2$ partial derivatives of the homogeneous polynomial $F \in H^0(\mathbb{P}, \mathcal{O}(d))$ defining X. The quotient of the vertical map is contained in $H^1(\mathcal{O}_X)$, which is trivial for $n \geq 2$ or $n = 0$.

Now, for the first assertion, observe that $H^0(X, \mathcal{T}_X) = 0$ if and only if the kernel of the composition

$$(\partial_i F)\colon H^0(X, \mathcal{O}_X(1))^{\oplus n+2} \longrightarrow H^0(X, \mathcal{O}_X(d))$$

is spanned by $(x_0, \ldots, x_{n+1})$. Assume $\sum h_i\, \partial_i F$ vanishes on X for some $h_i \in H^0(\mathbb{P}, \mathcal{O}(1))$. Then, after rescaling, $\sum h_i\, \partial_i F = d \cdot F = \sum x_i\, \partial_i F$ and, therefore, $\sum (h_i - x_i)\, \partial_i F = 0$. Using that $(\partial_i F)$ is a regular sequence, see Lemma 3.1, and $d \geq 3$, this yields $h_i = x_i$. The remaining case $n = 1$, $d > 3$ follows from the fact that $H^0(C, \omega_C^*) = 0$ for a smooth curve of genus $g(C) > 1$.

For the second assertion observe that $H^1(X, \mathcal{T}_{\mathbb{P}}|_X) = 0$, whenever $H^1(X, \mathcal{O}_X(1)) = 0 = H^2(X, \mathcal{O}_X)$, which holds as soon as $n > 2$, see Corollary 1.4. □

3.2 Let X be a smooth projective variety and assume $\mathcal{O}_X(1)$ is an ample line bundle. We are interested in the two groups:

$$\mathrm{Aut}(X, \mathcal{O}_X(1)) \subset \mathrm{Aut}(X).$$

Here, $\mathrm{Aut}(X)$ is the group of all automorphisms $g\colon X \xrightarrow{\sim} X$ over k. The subgroup $\mathrm{Aut}(X, \mathcal{O}_X(1))$ is the group of all such automorphisms with the additional property that $g^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(1)$. These groups are in fact the groups of k-rational points of group schemes over k, which we shall also denote by $\mathrm{Aut}(X, \mathcal{O}_X(1))$ and $\mathrm{Aut}(X)$.

Remark 3.4. Standard Hilbert scheme theory [61] ensures that $\mathrm{Aut}(X, \mathcal{O}_X(1))$ is a quasi-projective variety and that $\mathrm{Aut}(X)$ is at least locally of finite type. Indeed, there exists an open embedding

$$\mathrm{Aut}(X) \hookrightarrow \mathrm{Hilb}(X \times X),\ g \longmapsto \Gamma_g,$$

mapping an automorphism to its graph. The Hilbert scheme $\mathrm{Hilb}(X \times X)$ of $X \times X$ is locally of finite type and in fact the disjoint union $\coprod \mathrm{Hilb}^P(X \times X)$, $P \in \mathbb{Q}[T]$, of projective varieties $\mathrm{Hilb}^P(X \times X)$ parametrizing subschemes $Z \subset X \times X$ with Hilbert polynomial $\chi(Z, (\mathcal{O}(m) \boxtimes \mathcal{O}_X(m))|_Z) = P(m)$, see [61].

The Hilbert polynomial of the graph Γ_g of an arbitrary isomorphism is $\chi(X, \mathcal{O}_X(m) \otimes g^*\mathcal{O}_X(m))$. Thus, for $P(m) \coloneqq \chi(X, \mathcal{O}_X(2m))$ one has a locally closed embedding

$$\mathrm{Aut}(X, \mathcal{O}_X(1)) \hookrightarrow \mathrm{Hilb}^P(X \times X).$$

Note that it may fail to be open in general, as $\chi(X, \mathcal{O}_X(m) \otimes g^*\mathcal{O}_X(m)) = \chi(X, \mathcal{O}_X(2m))$ may not necessarily imply that $g^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(1)$.

Proposition 3.5. *The Zariski tangent spaces of* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(X, \mathcal{O}_X(1))$ *at the identity are given by*

$$T_{\mathrm{id}}\mathrm{Aut}(X, \mathcal{O}_X(1)) \subset T_{\mathrm{id}}\mathrm{Aut}(X) \simeq H^0(X, \mathcal{T}_X). \tag{3.3}$$

The inclusion is an equality if $H^1(X, \mathcal{O}_X) = 0$.

Proof This follows from the description of the tangent space of the Hilbert scheme of closed subschemes of Y at the point $[Z] \in \mathrm{Hilb}(Y)$ corresponding to $Z \subset Y$ as

$$T_{[Z]}\mathrm{Hilb}(Y) \simeq \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z),$$

cf. [61, Thm. 6.4.9]. For $Z \coloneqq \Gamma_{\mathrm{id}} \subset Y \coloneqq X \times X$ this becomes

$$T_{\mathrm{id}}\mathrm{Aut}(X) \simeq \mathrm{Hom}(\mathcal{I}_\Delta, \mathcal{O}_\Delta) \simeq H^0(\Delta, \mathcal{N}_{\Delta/X\times X}) \simeq H^0(X, \mathcal{T}_X).$$

As for our purposes the inclusion in (3.3) is all we need, we leave the second assertion to the reader. Hint: Use $H^1(X, \mathcal{O}_X) \simeq T_{[\mathcal{O}_X]}\mathrm{Pic}(X)$. In fact, $T_{\mathrm{id}}\mathrm{Aut}(X, \mathcal{O}_X(1))$ is the kernel of the map $H^0(X, \mathcal{T}_X) \longrightarrow H^1(X, \mathcal{O}_X)$ induced by the first Chern class $\mathrm{c}_1(L)$. □

For a smooth hypersurface of dimension $n \geq 2$ and degree $d \geq 3$ the result immediately yields

$$T_{\mathrm{id}}\mathrm{Aut}(X, \mathcal{O}_X(1)) = T_{\mathrm{id}}\mathrm{Aut}(X) \simeq H^0(X, \mathcal{T}_X) \simeq 0,$$

which allows one to prove the following general finiteness result. The original proof in [115] is different. It avoids cohomological methods and relies on techniques from commutative algebra. See [124, Rem. 6] for historical remarks.

Corollary 3.6. *Let* $X \subset \mathbb{P}^{n+1}$ *be a smooth hypersurface of dimension* $n \geq 0$ *and degree* $d \geq 3$*, but* $(n, d) \neq (1, 3)$*. Then* $\mathrm{Aut}(X, \mathcal{O}_X(1))$ *is finite and* $\mathrm{Aut}(X)$ *is discrete. In fact, if* $(n, d) \neq (1, 3), (2, 4)$*, then* $\mathrm{Aut}(X, \mathcal{O}_X(1)) = \mathrm{Aut}(X)$*, and then both groups are finite.*

Proof As $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X, \mathcal{O}_X(1))$ are group schemes, all tangent spaces are isomorphic and in our case trivial by Corollary 3.3. Hence, $\mathrm{Aut}(X)$, which is locally of finite type, is a countable set of reduced isolated points. As $\mathrm{Aut}(X, \mathcal{O}_X(1))$ is quasi-projective, it must be a finite set of reduced isolated points.

The equality $\mathrm{Aut}(X, \mathcal{O}_X(1)) = \mathrm{Aut}(X)$ for $n > 2$ follows from Corollary 1.6. For $n = 2$

and $d \neq 4$, use that $\omega_X \simeq \mathcal{O}(d-(n+2))$ is preserved by all automorphisms and that $\mathrm{Pic}(X)$ is torsion free, see Remark 1.2. □

Remark 3.7. (i) For $n = 1$ and $d = 3$ the result really fails, but not too badly. For a smooth plane cubic $E \subset \mathbb{P}^2$ and $\mathrm{char}(k) \neq 3$, one has:

$$0 = T_{\mathrm{id}}\mathrm{Aut}(E, \mathcal{O}_E(1)) \subset T_{\mathrm{id}}\mathrm{Aut}(E) \simeq H^0(E, \mathcal{T}_E) \simeq H^0(E, \mathcal{O}_E) \simeq k,$$

see [94, Sec. 11.7.5]. So, even in this case, $\mathrm{Aut}(E, \mathcal{O}_E(1))$ is in fact finite, but $\mathrm{Aut}(E)$ certainly is not.

(ii) The finiteness of $\mathrm{Aut}(X)$ also fails for $n = 2$ and $d = 4$ in general. Indeed, there exist quartic K3 surfaces with infinite automorphism groups, see [86] for examples and references.

The groups of automorphisms of the universal smooth hypersurface $\mathcal{X} \longrightarrow U = |\mathcal{O}(d)|_{\mathrm{sm}}$ sit in a relative quasi-projective family

$$\mathbf{Aut} := \mathrm{Aut}(\mathcal{X}/U, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow U = |\mathcal{O}(d)|_{\mathrm{sm}}. \tag{3.4}$$

More precisely, there exist functorial bijections between $\mathrm{Mor}_U(T, \mathbf{Aut})$ and the set of automorphisms $g\colon \mathcal{X}_T \xrightarrow{\sim} \mathcal{X}_T$ over T with $g^*\mathcal{O}_{\mathcal{X}_T}(1) \simeq \mathcal{O}_{\mathcal{X}_T}(1)$ modulo $\mathrm{Pic}(T)$. As in the absolute case, mapping g to its graph, yields a locally closed embedding $\mathbf{Aut} \subset \mathrm{Hilb}(\mathcal{X} \times_U \mathcal{X}/U)$ into the relative Hilbert scheme.

According to Corollary 3.6, the fibres of $\mathbf{Aut} \longrightarrow U$, i.e. the groups $\mathrm{Aut}(X, \mathcal{O}_X(1))$, are finite and, therefore, $\mathbf{Aut} \longrightarrow U$ is a quasi-finite morphism. In fact, it turns out to be finite, cf. Remark 1.7.

3.3 The first order description of the deformation behavior of a smooth projective variety X is similar. Firstly, there is a natural bijection between $H^1(X, \mathcal{T}_X)$ and the set of flat morphisms $\mathcal{X} \longrightarrow \mathrm{Spec}(k[\varepsilon])$ with closed fibre $\mathcal{X}_0 = X$, cf. [78, II. Ex. 9.13.2]. This can be extended to the following picture, cf. [61, Ch. 6]: If $H^0(X, \mathcal{T}_X) = 0$, then the functor

$$F_X\colon (Art/k) \longrightarrow (Sets),$$

mapping a local Artinian k-algebra A to the set of flat morphisms $\mathcal{X} \longrightarrow \mathrm{Spec}(A)$ with the choice of an isomorphism $\mathcal{X}_0 \simeq X$ for the closed fibre $\mathcal{X}_0$ has a pro-representable hull, see [61, Def. 6.3.1]. This means, that there exist a complete local k-algebra R and a flat family $\mathcal{X} \longrightarrow \mathrm{Spf}(R)$, $\mathcal{X}_0 \simeq X$, for which the induced transformation $h_R = \mathrm{Mor}_{k\text{-}alg}(R, \ ) \longrightarrow F_X$ is bijective for $A = k[\varepsilon]$. We shall write $\mathrm{Def}(X) := \mathrm{Spf}(R)$ with the closed point $0 \in \mathrm{Def}(X)$ and the Zariski tangent space $T_0\mathrm{Def}(X) \simeq H^1(X, \mathcal{T}_X)$.

Similarly, one may consider the polarized version

$$F_{X,\mathcal{O}_X(1)}\colon (Art/k) \longrightarrow (Sets)$$

mapping A to the set of flat polarized families $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow \mathrm{Spec}(A)$ with closed fibre $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \simeq (X, \mathcal{O}_X(1))$. Again, the functor $F_{X,\mathcal{O}_X(1)}$ has a pro-representable hull R' with $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow \mathrm{Def}(X, \mathcal{O}_X(1)) := \mathrm{Spf}(R')$. Only if $\mathrm{Aut}(X)$ or $\mathrm{Aut}(X, \mathcal{O}_X(1))$ are trivial, one can expect a universal family to exist, i.e. F_X or $F_{X,\mathcal{O}_X(1)}$, respectively, to being pro-representable. This is the difference between universal and versal families.

The natural forgetful transformation $F_{X,\mathcal{O}_X(1)} \longrightarrow F_X$ yields a morphism

$$\mathrm{Def}(X, \mathcal{O}_X(1)) \longrightarrow \mathrm{Def}(X), \tag{3.5}$$

which in general is neither injective nor surjective. The first Chern class $\mathrm{c}_1(\mathcal{O}_X(1)) \in H^1(X, \Omega_X) \simeq \mathrm{Ext}^1(\mathcal{T}_X, \mathcal{O}_X)$ interpreted as an extension class defines an exact sequence $0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}(\mathcal{O}_X(1)) \longrightarrow \mathcal{T}_X \longrightarrow 0$. Here, the sheaf $\mathcal{D}(\mathcal{O}_X(1))$ can be thought of as the sheaf of differential operators of $\mathcal{O}_X(1)$ of order ≤ 1.

Then $T_0\mathrm{Def}(X, \mathcal{O}_X(1)) \simeq H^1(X, \mathcal{D}(\mathcal{O}_X(1)))$ and the tangent map of (3.5) is part of a long exact sequence (see [135, Sec. 3.3] for more details):

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{D}(\mathcal{O}_X(1))) & \longrightarrow & H^1(X, \mathcal{T}_X) & \longrightarrow & H^2(X, \mathcal{O}_X) & \longrightarrow \cdots . \\ & & & \simeq T_0\mathrm{Def}(X, \mathcal{O}_X(1)) & & \simeq T_0\mathrm{Def}(X) & & & \end{array}$$

In fact, for most hypersurfaces the outer terms are trivial.

Remark 3.8. Over $\mathbb{C}$, the formal spaces $\mathrm{Def}(X)$ and $\mathrm{Def}(X, \mathcal{O}_X(1))$ can alternatively be thought of as germs of complex spaces. Standard deformation theory ensures that the universal families $\mathcal{X} \longrightarrow \mathrm{Def}(X)$ and $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow \mathrm{Def}(X, \mathcal{O}_X(1))$ can in fact be extended from families over formal bases to families over some small complex spaces. While this remains true in the algebraic setting for $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow \mathrm{Def}(X, \mathcal{O}_X(1))$, cf. [61, Thm. 8.4.10], it fails for the unpolarized situation.

The universal family of smooth hypersurfaces $\mathcal{X} \longrightarrow U = |\mathcal{O}(d)|_{\mathrm{sm}}$ induces a morphism $(U, 0) \longrightarrow \mathrm{Def}(X, \mathcal{O}_X(1))$ of the formal neighbourhood of $0 := [X] \in U$. We think of $|\mathcal{O}(d)|$ as a component of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^{n+1})$ and of $\mathcal{O}_X(d)$ as the normal bundle $\mathcal{N}_{X/\mathbb{P}^{n+1}}$. Then $T_0U \simeq H^0(X, \mathcal{O}_X(d))$ and the tangent map of the composition $(U, 0) \longrightarrow \mathrm{Def}(X, \mathcal{O}_X(1)) \longrightarrow \mathrm{Def}(X)$ is the boundary map of the normal bundle sequence $H^0(X, \mathcal{O}_X(d)) \to H^1(X, \mathcal{T}_X)$. Conversely, for an arbitrary deformation $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow \mathrm{Spec}(A)$ of X there exists a relative embedding $\mathcal{X} \hookrightarrow \mathbb{P}^{n+1}_A$ extending the given one $X \subset \mathbb{P}^{n+1}$. Here one uses that $H^1(X, \mathcal{O}_X(1)) = 0$, which ensures that all sections of $\mathcal{O}_X(1)$ on X extend to sections of $\mathcal{O}_{\mathcal{X}}(1)$.

Proposition 3.9. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Assume $n > 2$ or $n = 2$, $d \leq 3$. Then there is a natural surjection*

$$H^0(X, \mathcal{O}_X(d)) \simeq T_0|\mathcal{O}(d)| \twoheadrightarrow T_0\mathrm{Def}(X, \mathcal{O}_X(1)) \xrightarrow{\sim} T_0\mathrm{Def}(X) \simeq H^1(X, \mathcal{T}_X).$$

Furthermore, the forgetful morphism (3.5) is an isomorphism

$$\mathrm{Def}(X, \mathcal{O}_X(1)) \xrightarrow{\sim} \mathrm{Def}(X).$$

Proof Most of the proposition is an immediate consequence of the preceding discussion and the vanishings $H^1(X, \mathcal{O}_X) = 0 = H^2(X, \mathcal{O}_X)$. In order to see that the isomorphism $T_0\mathrm{Def}(X, \mathcal{O}_X(1)) \xrightarrow{\sim} T_0\mathrm{Def}(X)$ between the tangent spaces is induced by an isomorphism $\mathrm{Def}(X, \mathcal{O}_X(1)) \xrightarrow{\sim} \mathrm{Def}(X)$ it suffices to observe that both spaces are smooth and so isomorphic to $\mathrm{Spf}(k[[z_0, \ldots, z_m]])$ with $m = \dim T_0$. This could either be deduced from the vanishing $H^2(X, \mathcal{D}(\mathcal{O}_X(1))) = H^2(X, \mathcal{T}_X) = 0$ for $n > 3$ or, simply, from the fact that $|\mathcal{O}(d)|$ is smooth. □

Remark 3.10. The kernel of $H^0(X, \mathcal{O}_X(d)) \twoheadrightarrow H^1(X, \mathcal{T}_X)$ is a quotient of $H^0(X, \mathcal{T}_{\mathbb{P}}|_X)$ (and in fact equals it for $n \geq 2$), which should be thought of as the tangent space of the orbit through $[X] \in |\mathcal{O}(d)|$ of the natural $\mathrm{GL}(n+2)$-action on $|\mathcal{O}(d)|$, see Section **2**.1.3.

It may be worth pointing out the following consequence, which we will only state for cubic hypersurfaces.

Corollary 3.11. *Any local deformation of a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ as a variety over k is again a cubic hypersurface.* □

For $n = 2$ it is easy to construct smooth projective deformations of cubic surfaces that are not cubic surfaces any longer. However, for $n > 2$ the fact that $\rho(X) = 1$ allows one to prove that global smooth projective deformations of cubic hypersurfaces are again cubic hypersurfaces, cf. [91, Thm. 3.2.5].[7] The situation is more complicated when one is interested in non-projective or, equivalently, non-Kählerian global deformations.

3.4 In [115] it has also been observed that in fact for generic hypersurfaces the automorphism group is trivial. Generalizations for complete intersections have been proved more recently in [18, 32].

Theorem 3.12. *Assume $n > 0$, $d \geq 3$, and $(n, d) \neq (1, 3)$. Then there exists a dense open subset $V \subset |\mathcal{O}(d)|_{\mathrm{sm}}$ such that for all geometric points $[X] \in V$ one has*

$$\mathrm{Aut}(X) = \mathrm{Aut}(X, \mathcal{O}_X(1)) = \{\mathrm{id}\}.$$

Proof There are three proofs in the literature. The original due to Matsumura and Monsky [115] and two more recent ones by Poonen [128] and Chen, Pan, and Zhang [32]. In fact, $\mathrm{Aut}(X, \mathcal{O}_X(1)) = \{\mathrm{id}\}$ also holds for $(n, d) = (1, 3)$ as long as $\mathrm{char}(k) \neq 3$.

[7] Thanks to J. Ottem for the reference. Compare this to the well-known fact that $\mathrm{Def}(\mathbb{P}^1 \times \mathbb{P}^1)$ is a reduced point but yet $\mathbb{P}^1 \times \mathbb{P}^1$ can be deformed to any other Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ with n even.

For simplicity we shall assume that $(n,d) \neq (1,3),(2,4)$ and so $\mathrm{Aut}(X) = \mathrm{Aut}(X, \mathcal{O}_X(1))$, see Corollary 3.6.[8] Of course, one may assume k algebraically closed.

Consider the finite morphism (3.4) and let $V_i \subset U$ be the (possibly empty) open set of hypersurfaces $[X] \in U$ with $|\mathrm{Aut}(X)| \leq i$. The theorem asserts that V_1 is not empty. In [128] this is verified by writing down an explicit equation of one smooth hypersurface without any non-trivial polarized automorphisms. Let us instead adapt the arguments in [32] to our situation.[9] The proof is split in three steps and works in characteristic zero:

(i) As $H^0(X, \mathcal{T}_X) = 0$, the morphism $\mathbf{Aut} \longrightarrow U$ is unramified. After passing to a dense open subset $V \subset U$, we may assume it is in fact étale. Fix $[X] \in V$ and assume there exists $\mathrm{id} \neq g \in \mathrm{Aut}(X)$. After base change to an open neighbourhood of $[g] \in \mathbf{Aut}$, considered as an étale open neighbourhood of $[X] \in V$, we may assume that there exists a relative automorphism $\mathbf{g}\colon \mathcal{X} \xrightarrow{\sim} \mathcal{X}$, so $\pi \circ \mathbf{g} = \mathbf{g}$, with $\mathbf{g}|_X = g$. The base change is suppressed in the notation.

The relative tangent sequence $0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathcal{X}}|_X \longrightarrow T_{[X]}V \otimes \mathcal{O}_X \longrightarrow 0$ induces an exact sequence $H^0(X, \mathcal{T}_{\mathcal{X}}|_X) \longrightarrow T_{[X]}V \twoheadrightarrow H^1(X, \mathcal{T}_X)$. The surjectivity follows from Proposition 3.9. All maps are compatible with the action of $\mathbf{g}$. However, as π is $\mathbf{g}$-invariant, the action on $T_{[X]}V$ is trivial and, therefore, the action of g on $H^1(X, \mathcal{T}_X)$ is trivial as well.

(ii) The automorphism $g \in \mathrm{Aut}(X) \subset \mathrm{PGL}(n+2)$ can be lifted to an element in $\mathrm{SL}(n+2)$ which we shall also call g. It is still of finite order and, after a linear coordinate change, can be assumed to act by $g(x_i) = \lambda_i\, x_i$ for some roots of unities λ_i. This is where one needs $\mathrm{char}(k) = 0$.

Let $F \in H^0(\mathbb{P}, \mathcal{O}(d))$ be a homogeneous polynomial defining X. As $g(X) = X$, the induced action of g on $H^0(\mathbb{P}, \mathcal{O}(d))$ satisfies $g(F) = \mu\, F$ for some root of unity μ. Moreover, changing g by $\mu^{-1/d}$ we may assume that $\mu = 1$ (but possibly g is now only a finite order element in $\mathrm{GL}(n+2)$). For greater clarity, we rewrite (3.2) as the short exact sequence

$$W := (V \otimes V^*)/k \cdot \mathrm{id} \simeq H^0(X, \mathcal{T}_{\mathbb{P}}|_X) \hookrightarrow H^0(X, \mathcal{O}_X(d)) \twoheadrightarrow H^1(X, \mathcal{T}_X),$$

with $V := \langle x_0, \ldots, x_{n+1} \rangle$ and using $H^0(X, \mathcal{T}_X) = 0 = H^1(X, \mathcal{T}_{\mathbb{P}}|_X)$ observed earlier. All maps are compatible with the action of g and, in fact, the isomorphism is GL-equivariant. Note that $H^0(X, \mathcal{O}_X(d))$ is endowed with the action of g by interpreting $\mathcal{O}_X(d)$ as the normal bundle $\mathcal{N}_{X/\mathbb{P}}$. The action is compatible with the one induced by the isomorphism $H^0(X, \mathcal{O}_X(d)) \simeq H^0(\mathbb{P}, \mathcal{O}(d))/k \cdot F$.

(iii) By step (i) one knows that g acts trivially on $H^1(X, \mathcal{T}_X)$. Thus, in order to arrive

[8] There is a technical subtlety for $(n,d) = (2,4)$ in positive characteristic which needs to be checked. The question is whether for the generic quartic K3 surface all automorphisms are polarized. This is clear in vanishing characteristic using transcendental techniques. In [115] the case is excluded but I believe one should be able to settle this one way or the other.

[9] Thanks to O. Benoist for the reference.

at a contradiction, it suffices to show that the g-invariant part $H^0(X, \mathcal{O}_X(d))^g$ cannot map onto $H^1(X, \mathcal{T}_X)$. So it is enough to show that its dimension $h^0(\mathcal{O}_X(d))^g$ satisfies

$$h^0(\mathcal{O}_X(d))^g < h^1(X, \mathcal{T}_X) + \dim(W^g).$$

As $h^1(X, \mathcal{T}_X) = h^0(\mathcal{O}_X(d)) - \dim(W)$, this reduces the task to proving

$$\dim(W) - \dim(W^g) < h^0(\mathcal{O}_X(d)) - h^0(\mathcal{O}_X(d))^g. \tag{3.6}$$

Note that the left hand side equals $\dim(V \otimes V^*) - \dim(V \otimes V^*)^g$ and, as $g(F) = F$, the right hand side is nothing but $h^0(\mathbb{P}, \mathcal{O}(d)) - h^0(\mathbb{P}, \mathcal{O}(d))^g$. The weak inequality in (3.6) follows from the obvious equality $W^g = W \cap H^0(X, \mathcal{O}_X(d))^g$.

The strict equality follows from purely combinatorial considerations for which we refer to [32]. The idea is to write $V = \bigoplus V_\lambda$ with $V_\lambda := \langle x_i \mid \lambda_i = \lambda \rangle$. Then, the left hand side is $\dim(W) - \dim(W^g) = (n+2)^2 - \sum \dim(V_\lambda)^2$. To compute the right hand side, one decomposes $S^d(V) = S^d\left(\bigoplus V_\lambda\right)$ and shows that the non-invariant part grows faster than on the left hand side for $d \geq 3$.[10] □

Remark 3.13. It is worth pointing out that [128] provides equations for smooth hypersurfaces X defined over the prime field k, so $k = \mathbb{F}_p$ or $k = \mathbb{Q}$, such that $\mathrm{Aut}(\bar{X}, \mathcal{O}_{\bar{X}}(1)) = \{\mathrm{id}\}$ for $\bar{X} := X \times_k \bar{k}$. For cubic hypersurfaces the equations are of the form $c\,x_0^3 + \sum_{i=0}^n x_i\,x_{i+1}^2 + x_{n+1}^3$, where $n > 2$ and $\mathrm{char}(k) \neq 3$. The hard part of this approach is then the verifcation that the hypersurface given by this equation has no polarized automorphisms.

Remark 3.14. Assume $n \geq 2$ and $d \geq 3$ as before. Then $|\mathrm{Aut}(X)|$ is universally bounded, i.e. there is a constant $C(d, n)$ such that for all smooth $X \in |\mathcal{O}(d)|$

$$|\mathrm{Aut}(X)| < C(d, n).$$

This follows again from the fact that $\mathrm{Aut}(\mathcal{X}/U, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow U$ is a finite morphism, see Remark **2**.1.7.

The bound $C(d, n)$ can be made effective. In [83] it has been shown that $C(d, n)$ can be chosen of the form $C(d, n) = C(n) \cdot d^n$. The bound is unlikely to be optimal.

Question 3.15. *Is there anything known about the codimension of the closed set of hypersurfaces* $X \in [\mathcal{O}(d)|$ *for which* $\mathrm{Aut}(X, \mathcal{O}_X(1)) \neq \{\mathrm{id}\}$*?*

Corollary 3.16. *Assume* $X \subset \mathbb{P}^{n+1}$ *is a smooth hypersurface of degree* $d \geq 3$ *over* $\mathbb{C}$*. Then the action of* $\mathrm{Aut}(X)$ *on the middle cohomology* $H^n(X, \mathbb{Z})$ *is faithful.*

[10] This part feels a little unsatisfactory. It would be good to find a cleaner more conceptual argument for it. It is interesting to observe that the argument breaks down at this point for $n = 0$. And, indeed, the automorphism group of a cubic $X \subset \mathbb{P}^1$ is never trivial.

Proof First, observe that the arguments in the proof of Theorem 3.12 show that the action of $\mathrm{Aut}(X)$ on $H^1(X, \mathcal{T}_X)$ is faithful. Second, use the injectivity of the equivariant contraction map $H^1(X, \mathcal{T}_X) \longrightarrow \mathrm{End}(H^n(X, \mathbb{C}))$, cf. Corollary 4.21. □

4 Jacobian ring

The Jacobian ring is a finite-dimensional quotient of the coordinate ring of a smooth hypersurface obtained by dividing out the partial derivatives of the defining equation. At first glance, it looks like a rather coarse invariant but it turns out to encode the isomorphism type of the hypersurface as an abstract variety. There are purely algebraic aspects of the Jacobian ring as well as Hodge theoretic ones, which shall be explained or at least sketched in this section.

4.1 We shall assume $\mathrm{char}(k) = 0$ (or, at least, that $\mathrm{char}(k)$ is prime to the degree d of the hypersurfaces and prime to $d-1$) and shall write $S := k[x_0, \dots, x_{n+1}]$, which is naturally graded $S = \bigoplus_{i \geq 0} S_i$. So, $F \in S_d$ for a homogeneous polynomial F of degree d and $\partial_i F \in S_{d-1}$ for its partial derivatives $\partial_i F := \partial F / \partial x_i$. We will also need the *Hessian* of F

$$H(F) := \left(\frac{\partial^2 F}{\partial x_i \, \partial x_j} \right)_{i,j},$$

which is a matrix of homogeneous polynomials of degree $d-2$ and its determinant $\det H(F) \in S_\sigma$. Here, and throughout, we will use the shorthand

$$\sigma := (n+2) \cdot (d-2). \tag{4.1}$$

The reader may want to compare this number to the much larger degree of the discriminant divisor $\deg D(d,n) = (d-1)^{n+1}(n+2)$. For the case of interest to us, $d = 3$, one simply has

$$\sigma = n + 2.$$

Recall that the polynomial F can be recovered from its partial derivatives by means of the Euler equation:

$$d \cdot F = \sum_{i=0}^{n+1} x_i \, \partial_i F. \tag{4.2}$$

Definition 4.1. The *Jacobian ideal* of a homogeneous polynomial $F \in S_d$ of degree d is the homogeneous ideal

$$J(F) := (\partial_i F) \subset S = k[x_0, \dots, x_{n+1}]$$

generated by the partial derivatives of F. The *Jacobian ring* or *Milnor ring* of F is the quotient

$$S \twoheadrightarrow R(F) := S/J(F).$$

An immediate consequence of (4.2) is that the quotient map factors through the coordinate ring of X:

$$S \twoheadrightarrow S/(F) \twoheadrightarrow R(F).$$

If the hypersurface defined by F is denoted $X \subset \mathbb{P} := \mathbb{P}^{n+1}$, then we shall also write $J(X)$ and $R(X)$ instead of $J(F)$ and $R(F)$. As F is determined by X up to scaling, there is no risk of ambiguity. If F or X are understood, we will abbreviate further to $J = J(F)$ and $R = R(F)$. Note that R is naturally graded.

As an immediate consequence of Lemma 3.1, one has

Corollary 4.2. *For a smooth hypersurface $X \subset \mathbb{P}$ defined by a homogeneous polynomial F the Jacobian ring $R(X) = R(F)$ is a zero-dimensional local ring and a finite-dimensional k-algebra.* □

4.2 The content of the next result is to show that R is a Gorenstein ring with its (one-dimensional) socle in degree $\sigma = (n + 2) \cdot (d - 2)$ and to compute the dimensions of its graded pieces.

Proposition 4.3. *Assume that the homogeneous polynomial $F \in S_d$ defines a smooth hypersurface $X \subset \mathbb{P} = \mathbb{P}^{n+1}$. Then the Jacobian ring $R(X) = R(F)$ has the following properties:*

(i) *The Jacobian ring $R = R(X)$ is an Artinian graded ring with $R_i = 0$ for $i > \sigma$ and $R_\sigma \simeq k$. Moreover, R_σ is generated by* $\det H(F)$.

(ii) *Multiplication yields a perfect pairing*

$$R_i \times R_{\sigma-i} \longrightarrow R_\sigma \simeq k.$$

(iii) *The Poincaré polynomial of R is given by*

$$P(R) := \sum_{i=0}^{\sigma} \dim_k(R_i)\, t^i = \left(\frac{1 - t^{d-1}}{1 - t}\right)^{n+2}. \tag{4.3}$$

For $d = 3$ the dimensions of the graded pieces of the Jacobian ring $R(F)$ are simply

$$\dim_k(R_i) = \binom{n+2}{i}.$$

Proof We write $f_i := \partial_i F$. Then, by Lemma 3.1, $f_0, \ldots, f_{n+1} \in S$ is a regular sequence of homogeneous polynomials of degree $d-1$ and this is in fact all we need for the proof.

Let us begin by recalling basic facts about the *Koszul complex* of a regular sequence

$f_0, \ldots, f_{n+1} \in S$. As always, $\mathbb{P}^{n+1} = \mathbb{P}(V)$ and so $V^* = \langle x_0, \ldots, x_{n+1} \rangle$. Then the Koszul complex is the complex (concentrated in (homological) degree $n+2, \ldots, 0$)

$$K_\bullet(f_i)\colon \left(\bigwedge^{n+2} V^* \longrightarrow \cdots \longrightarrow \bigwedge^k V^* \longrightarrow \cdots \longrightarrow \bigwedge^2 V^* \longrightarrow V^* \longrightarrow k \right) \otimes_k S$$

with differentials

$$\partial_p(x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sum (-1)^j f_{i_j} \cdot x_{i_1} \wedge \cdots \wedge \widehat{x_{i_j}} \wedge \cdots \wedge x_{i_p}.$$

Now, for a regular sequence (f_i) the Koszul complex is exact in degree $\neq 0$ with:

$$H_0(K_\bullet(f_i)) \simeq \operatorname{Coker}(S \otimes V^* \longrightarrow S) \simeq R := S/(f_i),$$

see [137]. The exactness of the complex $K_\bullet(f_i) \longrightarrow R$ and the fact that the differentials in the Koszul complex are homogeneous of degree $d-1$ shows

$$\begin{aligned} \dim R_i &= \dim(S_i) - (n+2)\dim(S_{i-(d-1)}) \pm \cdots \\ &= \sum_{j=0}^{n+2} (-1)^j \binom{n+2}{j} \dim(S_{i-j(d-1)}). \end{aligned}$$

Of course, $\dim(S_{i-j(d-1)}) = h^0(\mathbb{P}, \mathcal{O}(i - j(d-1))) = \binom{n+1+i-j(d-1)}{i-j(d-1)}$, see (1.2). This in principle allows one to compute the right hand side. It can be made more explicit by observing that in $K_\bullet(f_i)$ only the differentials depend on the sequence (f_i). Hence, $\dim R_i$ can be computed by choosing a particular sequence, e.g. $f_i = x_i^{d-1}$. In this case, if a monomial $x^I = x_0^{i_0} \cdots x_{n+1}^{i_{n+1}}$ is not contained in the Jacobian ideal $(f_i = x_i^{d-1})$, then all $i_j \leq d-2$ and hence $|I| \leq (n+2)\cdot(d-2) = \sigma$. In other words, $R_i = 0$ for $i > \sigma$, which is not quite so obvious from the above dimension formula. Moreover, if $x^I \notin (f_i)$ for $|I| = \sigma$, then $x^I = \prod x_i^{d-2}$, i.e. R_σ is one-dimensional and generated by the Hessian determinant of $F = \sum x_i^d$. To compute the Poincaré polynomial completely, observe that

$$R\left(\sum x_i^d\right) \simeq k[x_0]/(x_0^{d-1}) \otimes \cdots \otimes k[x_{n+1}]/(x_{n+1}^{d-1})$$

and hence

$$P\left(R\left(\sum x_i^d\right)\right) = P\left(k[x]/(x^{d-1})\right)^{n+2} = (1 + t + \cdots + t^{d-2})^{n+2} = \left(\frac{1-t^{d-1}}{1-t}\right)^{n+2}.$$

One can also argue without specializing to the case of a Fermat (or any other) hypersurface and without relying on the Koszul complex as follows. For an exact sequence $0 \longrightarrow M^m \longrightarrow \cdots \longrightarrow M^0 \longrightarrow 0$ of graded S-modules the additivity of the Poincaré polynomial implies $\sum(-1)^j P(M^j) = 0$. Now, define $R^i := S/(f_0, \ldots, f_i)$ and consider the sequences $0 \longrightarrow R^{i-1} \xrightarrow{\cdot f_i} R^{i-1} \longrightarrow R^i \longrightarrow 0$ which are exact due to the regularity of the sequence (f_i). Then

$$P(R^i) = P(R^{i-1}) - t^{d-1} \cdot P(R^{i-1}) = (1 - t^{d-1}) \cdot P(R^{i-1})$$

and by induction

$$P(R) = (1 - t^{d-1})^{n+2} \cdot P(S).$$

Using $P(S) = 1/(1-t)^{n+2}$, this yields (iii) and, in particular, $R_i = 0$ for $i > \sigma$ and $R_\sigma \simeq k$.

Let us next show that the Hessian determinant $\det\left(\partial_i f_j\right)$ is not contained in the ideal (f_i) and thus generates R_σ. For this consider the dual Koszul complex $K^\bullet(f_i) = \mathrm{Hom}_S(K_\bullet(f_i), S)$, which quite generally satisfies the duality

$$H^p(K^\bullet(f_i)) \simeq H_{n+2-p}(K_\bullet(f_i)),$$

see [101, Ch. 4]. So, for a regular sequence (f_i) the complex $K^\bullet(f_i)$ is exact in degree $\neq n+2$ and $H^{n+2}(K^\bullet) \simeq R$. This can also be checked directly, for example by using that $H^i(K^\bullet(f_i)) \simeq \mathrm{Ext}^i_S(R, S)$. Suppose now that $H = \left(h_{ij}\right)$ is a matrix of homogeneous polynomials of degree $d-2$ such that $H \cdot \left(x_j\right)_j = (f_i)_i$, i.e. $\sum h_{ij}\, x_j = f_i$. Then H induces a morphism of complexes $\bigwedge^\bullet H\colon K_\bullet(f_i) \longrightarrow K_\bullet(x_i)$, the dual of which is a morphism $K^\bullet(x_i) \longrightarrow K^\bullet(f_i)$. The latter induces in degree $n+2$ the map

$$k \simeq S/(x_i) \simeq H^{n+2}(K^\bullet(x_i)) \longrightarrow H^{n+2}(K^\bullet(f_i)) \simeq R,\ 1 \longmapsto \det(H),$$

which can also be interpreted as the map $\eta\colon \mathrm{Ext}^{n+2}_S(k, S) \longrightarrow \mathrm{Ext}^{n+2}_S(R, S)$ induced by the short exact sequence $0 \longrightarrow (x_i)/(f_i) \longrightarrow R \longrightarrow k \longrightarrow 0$. As $(x_i)/(f_i)$ has 0-dimensional support and, thus, $\mathrm{Ext}^{n+1}_S((x_i)/(f_i), S) = 0$, the map η is injective. Therefore, $\det(H) \neq 0$ in R. To relate this to our assertion, observe that the Euler equation (4.2) implies $H(F) \cdot \left(x_j\right)_j = (d-1)\,(\partial_i F)_i$ and set $H := (1/(d-1)) \cdot H(F)$.

It remains to prove that the pairing defined by multiplication is perfect. Evidence comes from the equation $t^\sigma \cdot P(1/t) = P(t)$ for the Poincaré polynomial computed above. This already shows that $\dim R_i = \dim R_{\sigma-i}$. To verify that the pairing is non-degenerate, one has to show that for any homogeneous $g \notin (f_i)$ there exists a homogeneous polynomial h with $0 \neq \bar{g} \cdot \bar{h} \in R_\sigma$ or, equivalently, such that the degree σ part $(\bar{g})_\sigma$ of the homogeneous ideal $(\bar{g})$ in R is not trivial. Let i be maximal with $(\bar{g})_i \neq 0$ and pick $0 \neq \bar{G} \in (\bar{g})_i$. Suppose $i < \sigma$. Then $G \cdot (x_i) \subset (f_i)$, which induces a non-trivial homomorphism of S-modules $k \longrightarrow R$, $1 \longmapsto \bar{G}$. Hence, $\dim_k \mathrm{Hom}_S(k, R) > 1$, but this is absurd. Indeed, splitting the Koszul complex $K_\bullet(f_i)$ into short exact sequences and using that $\mathrm{Ext}^i_S(k, \bigwedge^p V^* \otimes S) = 0$ for $i < n+2$, one finds a sequence of inclusions

$$\mathrm{Hom}_S(k, R) \hookrightarrow \mathrm{Ext}^1_S(k, \mathrm{Ker}(\partial_0)) \hookrightarrow \quad \cdots \quad \hookrightarrow \mathrm{Ext}^{n+2}_S(k, \textstyle\bigwedge^{n+2} V^* \otimes S) \simeq k \quad \square$$

Remark 4.4. Let us add a more analytic argument for the fact that the Hessian determinant generates the socle, cf. [68]. For this we assume $k = \mathbb{C}$ and define the residue of

$g \in \mathbb{C}[x_0, \ldots, x_{n+1}]$ (with respect to F) as

$$\mathrm{Res}(g) := \left(\frac{1}{2\pi i}\right)^{n+2} \int_\Gamma \frac{g\, dx_0 \wedge \ldots \wedge dx_{n+1}}{f_0 \cdot \cdots \cdot f_{n+1}},$$

where as before $f_i = \partial_i F$ and $\Gamma := \{x \in \mathbb{C}^{n+2} \mid |f_i(x)| = \varepsilon_i\}$ with $0 < \varepsilon_i \ll 1$. Then one checks two things:

(i) If $g \in (f_i)$, then $\mathrm{Res}(g) = 0$. This follows from Stokes's theorem. Indeed, for example for $g = hf_0$ one has

$$(2\pi i)^{n+2}\,\mathrm{Res}(g) = \int_\Gamma \frac{h\, dx_0 \wedge \ldots \wedge dx_{n+1}}{f_1 \cdot \cdots \cdot f_{n+1}} = \int_{\Gamma_0} d\left(\frac{h}{f_1 \cdots f_{n+1}}\right) \wedge dx_0 \wedge \ldots \wedge dx_{n+1} = 0,$$

as $h/(f_1 \cdots f_{n+1})$ is holomorphic around $\Gamma_0 := \{z \in \mathbb{C}^{n+2} \mid |f_0(z)| < \varepsilon_0,\ |f_{i>0}(z)| = \varepsilon_i\}$.

(ii) The residue of $g = \det H(F)$ is non-zero. In fact, $\mathrm{Res}(\det H(F)) = \deg(f)$. Here, $f\colon \mathbb{C}^{n+2} \longrightarrow \mathbb{C}^{n+2}$ is the map $x = (x_i) \longmapsto (f_i(x))$, which is of degree $\deg(f) = \dim \mathcal{O}_{\mathbb{C}^{n+2},0}/(f_i)$. Indeed,

$$\left(\frac{1}{2\pi i}\right)^{n+2} \int_\Gamma \frac{\det H(F)\, dx_0 \wedge \ldots \wedge dx_{n+1}}{f_0 \cdot \cdots \cdot f_{n+1}} = \left(\frac{1}{2\pi i}\right)^{n+2} \int_\Gamma \frac{df_0 \wedge \ldots \wedge df_{n+1}}{f_0 \cdot \cdots \cdot f_{n+1}}$$
$$= \left(\frac{1}{2\pi i}\right)^{n+2} \int_\Gamma f^*\left(\frac{dz_0}{z_0} \wedge \ldots \wedge \frac{dz_{n+1}}{z_{n+1}}\right) = \deg(f) \cdot \prod_{j=0}^{n+1} \frac{1}{2\pi i} \int_{|z_j| = \varepsilon_j} \frac{dz_j}{z_j}$$
$$= \deg(f).$$

Clearly, (i) and (ii) together imply $\det H(F) \notin J(F)$.

In [133] one finds a proof of the above proposition that reduces the assertion to statements in local duality theory as in [77]. In [149] the results are deduced from global Serre duality on $\mathbb{P}^{n+1}$.

Here is an immediate consequence of the perfectness of the pairing $R_i \times R_{\sigma-i} \longrightarrow R_\sigma$.

Corollary 4.5. *Assume $i + j \leq \sigma$. Then the natural map*

$$R_i \hookrightarrow \mathrm{Hom}(R_j, R_{i+j})$$

induced by multiplication is injective. □

Remark 4.6. Let $X = V(F) \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Then its group of polarized automorphisms $\mathrm{Aut}(X, \mathcal{O}_X(1))$, which is essentially always finite (see Corollary 3.6), acts on the finite dimensional Jacobian ring $R = R(X)$. The action is graded and faithful. As a generalization of the Poincaré polynomial $P(R)$ one considers for any $g \in \mathrm{Aut}(X, \mathcal{O}_X(1))$ the polynomial

$$P(R, g) := \sum \mathrm{tr}\,(g \mid R_i)\, t^i.$$

Then the Poincaré polynomial is recovered as $P(R) = P(R, \mathrm{id})$. The equation (4.3) has been generalized by Bott–Tate and Orlik–Solomon [124] to

$$P(R, g) = \frac{\det(1 - g\, t^{d-1} \,|\, V)}{\det(1 - g\, t \,|\, V)}, \tag{4.4}$$

where $V = S_1 = \langle x_0, \ldots, x_{n+1}\rangle$. This can then be used to see that $|\mathrm{Aut}(X, \mathcal{O}_X(1))|$ is bounded by a function only depending on d and n, see [124, Cor. 2.7], which we have hinted at already in Remark 3.14.

4.3 As a graded version of a result of Mather and Yau [113], Donagi showed in [54] that the Jacobian ring of a hypersurface determines the hypersurface up to projective equivalence.

Example 4.7. To motivate Donagi's result, let us discuss the case of smooth cubic curves $E = X \subset \mathbb{P} = \mathbb{P}^2$. The interesting information encoded by the Jacobian ring

$$R = R_0 \oplus R_1 \oplus R_2 \oplus R_3$$

is the perfect pairing $R_1 \times R_2 \longrightarrow R_3 \simeq k$. We shall describe this for a plane cubic in Weierstraß form $y^2 = 4x^3 - g_2 x - g_3$, i.e. with $F = x_1^2 x_2 - 4x_0^3 + g_2 x_0 x_2^2 + g_3 x_2^3$. The partial derivatives are

$$\partial_0 F = -12x_0^2 + g_2 x_2^2,\ \partial_1 F = 2x_1 x_2,\ \text{and}\ \partial_2 F = x_1^2 + 2g_2 x_0 x_2 + 3g_3 x_2^2.$$

From this one deduces bases for R_1, R_2, and R_3, namely:

$$R_1 = \langle \bar{x}_0, \bar{x}_1, \bar{x}_2\rangle,\ R_2 = \langle \bar{x}_2^2, \bar{x}_0\bar{x}_1, \bar{x}_0\bar{x}_2\rangle,\ \text{and}\ R_3 = \langle \bar{x}_2^3\rangle.$$

With respect to theses bases, the multiplication $R_1 \times R_2 \longrightarrow R_3$ is described by the matrix

$$\begin{pmatrix} \bar{x}_0\bar{x}_2^2 & \bar{x}_0^2\bar{x}_1 & \bar{x}_0^2\bar{x}_2 \\ \bar{x}_1\bar{x}_2^2 & \bar{x}_1\bar{x}_1^2 & \bar{x}_0\bar{x}_1\bar{x}_2 \\ \bar{x}_2^3 & \bar{x}_0\bar{x}_1\bar{x}_2 & \bar{x}_0\bar{x}_2^2 \end{pmatrix} = \begin{pmatrix} -3g_3/(2g_2) & 0 & g_2/12 \\ 0 & (27g_3^2 - g_2^3)/(6g_2) & 0 \\ 1 & 0 & -3g_3/(2g_2) \end{pmatrix}.$$

Recall that the discriminant of an elliptic curve in Weierstraß form is by definition $\Delta(E) = g_2^3 - 27g_3^2$ and its j-function $j(E) = 1728\,\frac{g_2^3}{\Delta(E)}$, cf. [78, Sec. IV.4]. Hence, the perfect pairing $R_1 \times R_2 \longrightarrow R_3 \simeq k$ determines $j(E)$ and, hence (at least for k algebraically closed) the isomorphism type of E. Note that already the determinant

$$\frac{\Delta(E)^2}{72g_2^3} = 24 \cdot 1728 \cdot g_2^3 \cdot j(E)^{-2}$$

of the above matrix almost remembers the isomorphism type of E.

Proposition 4.8. *Let $X, X' \subset \mathbb{P} = \mathbb{P}^{n+1}$ be two smooth hypersurfaces such that there exists an isomorphism $R(X) \simeq R(X')$ of graded rings. Then the two hypersurfaces are* *equivalent, i.e. there exists an automorphism* $g \in \mathrm{PGL}(n+2)$ *of the ambient* $\mathbb{P}$ *with* $g(X) = X'$.

Proof We follow the proof in [149, Ch. 18]. Denote the polynomials defining X and X' by F and F'. The given graded isomorphism $R(F) \xrightarrow{\sim} R(F')$ can be lifted to an isomorphism $g\colon S \xrightarrow{\sim} S$ with $g(J(F)) = J(F')$ and we can thus reduce to the case $g = \mathrm{id}$, i.e. $J(F) = J(F')$.

Consider the path $F_t := t \cdot F' + (1-t) \cdot F$ connecting F and F'. Deriving with respect to t yields $(d/dt)F_t = F' - F$ which is contained in the ideal $(F) + (F') \subset J(F) = J(F')$.

On the other hand, the tangent space of the $\mathrm{GL}(n+2)$-orbit at F_t is just $J(F_t)_d = J(F)_d$, which can be seen by computing for $A = (a_{ij}) \in M(n+2, \mathbb{C})$

$$\frac{d}{ds} F_t\,((\mathrm{id} + s \cdot A)x)\,|_{s=0} = \sum_i \partial_i F_t \sum_j a_{ij} x_j.$$

Hence, the path F_t is tangent to all orbits and, therefore, stays inside the $\mathrm{GL}(n+2)$-orbit through F. This proves the assertion.

Note that in general the given isomorphism between the Jacobian rings is not induced by any $g \in \mathrm{PGL}(n+2)$ identifying X and X'. □

There exist examples of smooth projective varieties X that can be embedded as hypersurfaces $X \hookrightarrow \mathbb{P}$ in non-equivalent ways. For example, the Femat quartic $X \subset \mathbb{P}^3$ is known to admit exactly three equivalence classes of degree four polarizations [47]. The three Jacobian rings are therefore non-isomorphic.

Remark 4.9. In Proposition 4.8 it is enough to assume that there is a ring isomorphism $R(X) \simeq R(X')$, not necessarily graded. Indeed, any ring isomorphism induces a graded isomorphism $\bigoplus \mathfrak{m}_R^i/\mathfrak{m}_R^{i+1} \simeq \bigoplus \mathfrak{m}_{R'}^i/\mathfrak{m}_{R'}^{i+1}$, where $\mathfrak{m}_R \subset R(X)$ and $\mathfrak{m}_{R'} \subset R(X')$ are the maximal ideals. Then use that $R(X) \simeq \bigoplus \mathfrak{m}_R^i/\mathfrak{m}_R^{i+1}$ as graded k-algebras.[11]

4.4 For later use, we study the part of the Jacobian ring $R(X)$ which only takes into account the degrees

$$t(p) := (n - p + 1) \cdot d - (n+2).$$

Observe that these indices enjoy the symmetry

$$t(p) + t(n-p) = (n+2) \cdot (d-2) = \sigma.$$

Therefore, there exist perfect pairings

$$R_{t(p)} \times R_{t(n-p)} \longrightarrow R_\sigma \simeq k.$$

[11] Thanks to J. Rennemo for explaining this to me.

Also note that Let us first check for which p one finds a non-trivial $R_{t(p)}$. This is the case if and only if $0 \leq t(p) \leq \sigma = (n+2)\cdot(d-2)$, i.e. for

$$n-e \leq p \leq e := \frac{(n+2)\cdot(d-1)}{d} - 1.$$

For $d = 3$ this becomes

$$\frac{n-1}{3} \leq p \leq \frac{2n+1}{3}. \tag{4.5}$$

Observe that $t(p) = \sigma$ if and only if $n - p + 1 = \frac{(n+2)\cdot(d-1)}{d}$, which leads to the next

Lemma 4.10. *For given n and d the following conditions are equivalent:*

(i) $d \mid (n+2)$.
(ii) *There exists* $p \in \mathbb{Z}$ *with* $t(p) = 0$.
(iii) *There exists* $p \in \mathbb{Z}$ *with* $t(p) = \sigma$.
(iv) *There exists* $p \in \mathbb{Z}$ *with* $t(p) = d$.
(v) *There exists* $p \in \mathbb{Z}$ *with* $\dim R_{t(p)} = 1$.
(vi) $\bigoplus R_{t(p)} \simeq \bigoplus R_{md}$. □

We also record that for $d = 3$

$$\dim_k(R_{t(p)}) = \binom{n+2}{3\,(n-p+1)-(n+2)} = \binom{n+2}{2\,n+1-3p}.$$

Exercise 4.11. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Show that

$$\sum \dim_k(R_{t(p)}) = b_n(X)_{\rm pr},$$

where $b_n(X)_{\rm pr}$ was computed in Section 1.3. For $d = 3$ this becomes the mysterious formula, cf. Remark 1.12:

$$\sum_p \binom{n+2}{2\,n+1-3\,p} = (-1)^n \cdot (2/3) \cdot \left(1 + (-1)^n \cdot 2^{n+1}\right). \tag{4.6}$$

A geometric explanation will be given below.

4.5 There is a beautiful technique going back to [54] that, under certain numerical conditions, allows one to recover the full Jacobian ring $R := R(X)$ from just the mutiplications $R_d \times R_{t(p)} \longrightarrow R_{t(p)+d}$. This is useful as R_d and the $R_{t(p)}$ can be described geometrically. We start with the geometric description of $R_d(X)$. We recommend [36] for an instructive brief discussion and [149] for a more detailed one.

Lemma 4.12. *Let* $X \subset \mathbb{P}^{n+1}$ *be a smooth hypersurface of degree d. Assume* $\dim(X) > 2$ *or* $d \leq 3$*. Then there exists a natural isomorphism*

$$R_d(X) \simeq H^1(X, \mathcal{T}_X).$$

Proof This follows from (3.2) in the proof of Corollary 3.3 and $H^1(X, \mathcal{T}_{\mathbb{P}}|_X) = 0$, which holds under the present assumptions. The case $n = 1$, $d = 3$ needs an additional argument which is left to the reader. □

Example 4.13. Observe that the isomorphism confirms the dimension formula (2.2):

$$\dim(M_n) = \dim H^1(X, \mathcal{T}_X) = \dim R_3(X) = \binom{n+2}{3}.$$

Before turning to the geometric interpretation of the $R_{t(p)}(X)$, we present the following purely algebraic result. It was proved by Donagi [54] for generic polynomial and in [55] in general.

Proposition 4.14 (Symmetrizer lemma)**.** *Assume* (i) $i < j$, (ii) $i + j \leq \sigma - 1$, *and* (iii) $d + \max\{i, j\} \leq \sigma + 3$. *Then the image of the injection (see Corollary 4.5)*

$$R_{j-i} \hookrightarrow \mathrm{Hom}(R_i, R_j)$$

is the subspace of all linear maps $\varphi\colon R_i \longrightarrow R_j$ *satisfiying* $g \cdot \varphi(h) = h \cdot \varphi(g) \in R_{i+j}$ *for all* $g, h \in R_i$.

Proof The subspace described by the symmetry condition is the kernel of

$$\mathrm{Hom}(R_i, R_j) \longrightarrow \mathrm{Hom}(\textstyle\bigwedge^2 R_i, R_{i+j}),\ \varphi \longmapsto (g \wedge h \longmapsto g\varphi(h) - h\varphi(g)).$$

As R_{i-j} obviously maps into it, one has to prove the exactness of the sequence

$$R_{j-i} \longrightarrow \mathrm{Hom}(R_i, R_j) \longrightarrow \mathrm{Hom}(\textstyle\bigwedge^2 R_i, R_j).$$

This is done by comparing it to a certain Koszul complex on $\mathbb{P}^{n+1}$. See [149, Prop. 18.21] for details.

Note that for cubic hypersurfaces the discussion below makes use of the symmetrizer lemma only for $i = 1, 2$, but the proof does not seem to become any easier in these cases. □

The proposition can then be applied repeatedly. Suppose $R_i \times R_j \longrightarrow R_{i+j}$ is known. From it one recovers $R_i \times R_{j-i} \longrightarrow R_j$, for which (ii) and (iii) still hold. However, it may happen that (i) no longer holds, i.e. that $i \geq j - i$, but this can be remedied by swapping the factors, which does not effect the symmetric conditions (ii) and (iii). The procedure stops at some $R_\ell \times R_\ell \longrightarrow R_\ell$ and a moment's thought reveals that $\ell = \mathrm{g.c.d.}(i, j)$. Applied to $i = d$ and $j = t(p)$ (or, with reversed order) this becomes the following result.

Proposition 4.15. *Assume* $(2n + 1)/n \leq d$. *Fix p such that* $0 < t(p) \leq \sigma - d - 1$, *and let* $\ell := \mathrm{g.c.d.}(d, n + 2)$. *Then multiplication*

$$R_d \times R_{t(p)} \longrightarrow R_{d+t(p)} = R_{t(p-1)}$$

determines the multiplication $R_\ell \times R_\ell \longrightarrow R_{2\ell}$. *(Note that in general* ℓ *is not of the form* $t(p)$*.)* □

The next result is a special case of a more general one, which beyond the cubic case is known for all smooth hypersurfaces except when $(d, n) = (4, 4m)$ or $d \mid (n + 2)$. The argument is easier for cubic hypersurfaces and so we restrict to this case.

Corollary 4.16. *Assume* $X = V(F) \subset \mathbb{P}^{n+1}$ *is a smooth cubic hypersurface of dimension* $n > 2$ *with* $3 \nmid (n + 2)$. *Then there exist integers* p *with* $0 < t(p) \le n - 1$ *and for each of them the graded algebra* $R = R(X)$, *and hence by Proposition 4.8 also* X, *is uniquely determined by the multiplication* $R_3 \times R_{t(p)} \longrightarrow R_{3+t(p)}$.

Proof The condition $0 < t(p) \le \sigma - d - 1$ in Proposition 4.15 turns for $d = 3$ into (cf. (4.5))

$$n + 3 \le 3p < 2n + 1, \tag{4.7}$$

which has integral solutions for all $n > 2$. For any such p, multiplication $R_3(X) \times R_{t(p)} \longrightarrow R_{3+t(p)}$ determines $R_1 \times R_1 \longrightarrow R_2$, as $3 \nmid (n + 2)$ implies g.c.d.$(3, t(p)) = 1$.

Suppose this multiplication is isomorphic to another one $R'_1 \times R'_1 \longrightarrow R'_2$ associated with a cubic $X' = V(F') \subset \mathbb{P}^{n+1}$. The isomorphism $R_1 \simeq R'_1$ corresponds to a linear coordinate change $g\colon \langle x_0, \dots, x_{n+1}\rangle \xrightarrow{\sim} \langle x_0, \dots, x_{n+1}\rangle$ and the compatibility with the multiplication can be interpreted as an isomorphism

$$\left(k[x_0, \dots, x_{n+1}]_2 \simeq S^2(R_1) \longrightarrow R_2\right) \simeq \left(k[x_0, \dots, x_{n+1}]_2 \simeq S^2(R'_1) \longrightarrow R'_2\right).$$

Hence, under g, their kernels are identified, which are spanned by the partial derivatives $\partial_i F$ and $\partial_i F'$, respectively. Thus, g induces a ring isomorphism $k[x_0, \dots, x_{n+1}] \xrightarrow{\sim} k[x_0, \dots, x_{n+1}]$ that restricts to $J(X) \xrightarrow{\sim} J(X')$ and, hence, $R(X) \simeq R(X')$. □

Note that under the assumptions of the proposition, there always exists p with $t(p) = 1$ or $t(p) = 2$. The proposition covers, for example, cubics of dimension $n = 2, 3, 5, 6, 8, 9$. In these cases the occurring $t(p)$ are the following ones:

n	$0 < t(p) \le \sigma - 4$
2	$t(1) = 2$
3	$t(2) = 1$
5	$t(2) = 5, t(3) = 2$
6	$t(3) = 4, t(4) = 1$
8	$t(3) = 8, t(4) = 5, t(5) = 2$
9	$t(4) = 7, t(5) = 4, t(6) = 1$

Remark 4.17. The result is sharp. For example, for $n = 4$, the only $0 \leq t(p) \leq \sigma = 6$ occurring are $t(1) = 6$, $t(2) = 3$, and $t(3) = 0$. But the pairing $R_3 \times R_3 \longrightarrow R_6 \simeq k$ does certainly not determine the cubic nor does the identity $R_3 \times R_0 \longrightarrow R_3$.

4.6 The next step is to describe the parts $R_{t(p)}(X)$ geometrically. This is the following celebrated result due to Carlson and Griffiths [29].

Theorem 4.18 (Carlson–Griffiths)**.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Assume $n > 2$ or $d \leq 3$. Then for all integers p there exists an isomorphism*

$$H^{p,n-p}(X)_{\rm pr} \simeq R_{t(p)}(X), \tag{4.8}$$

with $t(p) = (n - p + 1) \cdot d - (n + 2)$, compatible with the natural pairings on both sides, i.e. there exist commutative diagrams

$$\begin{array}{ccccc} H^{p,n-p}(X)_{\rm pr} & \times & H^{n-p,p}(X)_{\rm pr} & \longrightarrow & H^{n,n}(X)_{\rm pr} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ R_{t(p)}(X) & \times & R_{t(n-p)}(X) & \longrightarrow & R_\sigma(X). \end{array} \tag{4.9}$$

Moreover, using the isomorphism $H^1(X, \mathcal{T}_X) \simeq R_d(X)$, cf. Lemma 4.12, and the pairing $\mathcal{T}_X \times \Omega_X^p \longrightarrow \Omega_X^{p-1}$, one obtains commutative diagrams

$$\begin{array}{ccccc} H^1(X, \mathcal{T}_X) & \times & H^{p,n-p}(X)_{\rm pr} & \longrightarrow & H^{p-1,n-p+1}(X)_{\rm pr} \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ R_d(X) & \times & R_{t(p)}(X) & \longrightarrow & R_{t(p-1)}(X), \end{array} \tag{4.10}$$

The proof of the theorem is involved and we will not attempt to present it in full. However, we will outline the most important parts of the general theory that enter the proof and, in particular, explain how to establish a link between the Jacobian ring and the primitive cohomology at all. As we will restrict to the case of hypersurfaces in $\mathbb{P}^{n+1}$ throughout, certain aspects simplify. We refer to [28] and [149] for more details and some of the crucial computations.

(i) The de Rham complex of a (smooth) k-variety X of dimension n is the complex

$$\Omega_X^\bullet : \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow \Omega_X^2 \longrightarrow \cdots \longrightarrow \Omega_X^n.$$

The $\Omega_X^i := \bigwedge^i \Omega_X$ are coherent sheaves (here in the Zariski topology), but the differentials $d\colon \Omega_X^i \longrightarrow \Omega_X^{i+1}$ are only k-linear. The *de Rham cohomology* of X is then defined as the hypercohomology of this complex:

$$H^*_{\rm dR}(X/k) := \mathbb{H}^*(X, \Omega_X^\bullet),$$

which can be computed via the *Hodge–de Rham* spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H_{\mathrm{dR}}^{p+q}(X/k). \tag{4.11}$$

Note that the E_1-terms are just cohomology groups of coherent sheaves. The spectral sequence is associated with the *Hodge filtration*, which is induced by the complexes

$$F^p\Omega_X^\bullet : \Omega_X^p \longrightarrow \cdots \longrightarrow \Omega_X^n$$

concentrated in degrees $p, \ldots, n$ and the natural morphism $F^p\Omega_X^\bullet \longrightarrow \Omega_X^\bullet$. Then one defines $F^pH_{\mathrm{dR}}^*(X/k) := \mathrm{Im}\left(\mathbb{H}^*(F^p\Omega_X^\bullet) \longrightarrow \mathbb{H}^*(\Omega_X^\bullet) = H_{\mathrm{dR}}^*(X/k)\right)$.

Remark 4.19. If X is smooth and projective over a field k satisfying $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) = p > \dim(X)$ and if X is liftable to $W_2(k)$, then (4.11) degenerates [49]. This applies to smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$, for which the assumption on the $\mathrm{char}(k)$ can be avoided, cf. Section 1.6.

(ii) For open varieties the Hodge–de Rham spectral sequence does not necessarily degenerate, but a replacement is available. Consider the open complement $j\colon U := \mathbb{P} \setminus X \hookrightarrow \mathbb{P} = \mathbb{P}^{n+1}$ of a smooth hypersurface $X \subset \mathbb{P}$. There are quasi-isomorphisms (see the discussion following (4.18))

$$\Omega_{\mathbb{P}}^\bullet(\log(X)) \xrightarrow{\sim} \Omega_{\mathbb{P}}^\bullet(*X) = j_*\Omega_U^\bullet. \tag{4.12}$$

Here, $\Omega_{\mathbb{P}}^p(*X) := j_*\Omega_U^p$ (Zariski topology!) is the sheaf of meromorphic p-forms on $\mathbb{P}$ with poles (of arbitrary order) along X. Furthermore, $\Omega_{\mathbb{P}}^1(\log(X)) \subset \Omega_{\mathbb{P}}^1(*X)$ is the subsheaf locally generated by $d\log(f) = \frac{df}{f}$, where f is the local equation for X, and $\Omega_{\mathbb{P}}^p(\log(X)) := \bigwedge^p\left(\Omega_X^1(\log(X))\right)$. In our case, $X = V(F)$ and

$$\Omega_{\mathbb{P}}^1(\log(X))|_{U_j} = d\log(F_j)\,\mathcal{O}_{U_j} = \frac{dF_j}{F_j}\,\mathcal{O}_{U_j}$$

on the standard open subset $U_j := \mathbb{P} \setminus V(x_j)$ with $F_j := F(x_0/x_j, \ldots, x_{n+1}/x_j)$. The differentials in both complexes are the usual ones. By construction, $\Omega_{\mathbb{P}}^\bullet(\log(X))$ is the subcomplex of forms α with α and $d\alpha$ having at most simple poles along X. The Hodge filtration in the open case is defined by

$$F^p\Omega_{\mathbb{P}}^\bullet(\log(X))\colon \Omega_{\mathbb{P}}^p(\log(X)) \longrightarrow \cdots \longrightarrow \Omega_{\mathbb{P}}^{n+1}(\log(X))$$

(and not as the direct image of $F^p\Omega_U^\bullet$) in degrees $p, \ldots, n+1$. It induces the spectral sequence

$$E_1^{p,q} = H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(\log(X))) \Rightarrow \mathbb{H}^{p+q}(\mathbb{P}, \Omega_{\mathbb{P}}^\bullet(\log(X))), \tag{4.13}$$

where the right hand side is isomorphic to

$$\mathbb{H}^\bullet(\mathbb{P}, \Omega_{\mathbb{P}}^\bullet(\log(X))) \simeq \mathbb{H}^*(\mathbb{P}, \Omega_{\mathbb{P}}^\bullet(*X)) \simeq H_{\mathrm{dR}}^*(U/k).$$

Again due to [49], this spectral sequence degenerates under the assumptions of Remark 4.19 and so in particular for smooth hypersurfaces.

Observe that the residue

$$\mathrm{res}\colon \frac{df}{f}\,h \longmapsto h|_X$$

yields a short exact sequence $0 \longrightarrow \Omega^1_{\mathbb{P}} \longrightarrow \Omega^1_{\mathbb{P}}(\log(X)) \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$. Taking exterior powers, one obtains an exact sequence of complexes

$$\begin{array}{c} 0 \longrightarrow \Omega^\bullet_{\mathbb{P}} \longrightarrow \Omega^\bullet_{\mathbb{P}}(\log(X)) \xrightarrow{\mathrm{res}} (i_*\Omega^\bullet_X)[-1] \longrightarrow 0 \\ \simeq \Omega^\bullet_{\mathbb{P}}(*X) \end{array}$$

with $\mathrm{res}\left(\frac{df}{f} \wedge \alpha\right) = \alpha|_X$. In fact, the sequence is compatible with the Hodge filtrations of all three complexes, which leads to exact sequences

$$0 \longrightarrow F^p\Omega^\bullet_{\mathbb{P}} \longrightarrow F^p\Omega^\bullet_{\mathbb{P}}(\log(X)) \xrightarrow{\mathrm{res}} (i_*F^{p-1}\Omega^\bullet_X)[-1] \longrightarrow 0.$$

The induced long exact cohomology sequences read

$$\cdots \longrightarrow H^i_{\mathrm{dR}}(\mathbb{P}/k) \longrightarrow H^i_{\mathrm{dR}}(U/k) \longrightarrow H^{i-1}_{\mathrm{dR}}(X/k) \longrightarrow H^{i+1}_{\mathrm{dR}}(\mathbb{P}/k) \longrightarrow \cdots \tag{4.14}$$

and

$$\cdots \longrightarrow F^pH^i_{\mathrm{dR}}(\mathbb{P}/k) \longrightarrow F^pH^i_{\mathrm{dR}}(U/k) \longrightarrow F^{p-1}H^{i-1}_{\mathrm{dR}}(X/k) \longrightarrow F^pH^{i+1}_{\mathrm{dR}}(\mathbb{P}/k) \longrightarrow \tag{4.15}$$

Note that the Hodge filtration $F^pH^*_{\mathrm{dR}}(U/k)$ is defined as the image of

$$\mathbb{H}^*(\mathbb{P}, F^p\Omega^\bullet_{\mathbb{P}}(\log(X))) \longrightarrow \mathbb{H}^*(\mathbb{P}, \Omega^\bullet_{\mathbb{P}}(\log(X))) \simeq H^*_{\mathrm{dR}}(U/k)$$

and not via the Hodge filtration of $\Omega^\bullet_U$.

If X is defined over $k = \mathbb{C}$, there is an analytic version of the above for the associated complex manifold X^{an}. The de Rham complex $\Omega^\bullet_{X^{\mathrm{an}}}$ is defined similarly (now in the analytic topology) and so is the Hodge–de Rham spectral sequence

$$E_1^{p,q} = H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X^{\mathrm{an}}). \tag{4.16}$$

The Poincaré lemma shows that in the analytic topology the inclusion $\mathbb{C} \hookrightarrow \mathcal{O}_{X^{\mathrm{an}}}$ yields a quasi-isomorphism $\mathbb{C} \xrightarrow{\sim} \Omega^\bullet_{X^{\mathrm{an}}}$ and hence an isomorphism

$$H^*(X^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X^{\mathrm{an}}) = \mathbb{H}^*(\Omega^\bullet_{X^{\mathrm{an}}}).$$

The natural morphism $X^{\mathrm{an}} \longrightarrow X$ of ringed spaces provides a comparison map from the algebraic to the analytic de Rham cohomology. For X smooth and projective, GAGA shows that $H^q(X,\Omega_X^p) \xrightarrow{\sim} H^q(X^{\rm an},\Omega_{X^{\rm an}}^p)$. Hence, the left hand sides of (4.11) and (4.16) coincide and, therefore, also the right hand sides do, i.e.

$$H^*_{\rm dR}(X/\mathbb{C}) \xrightarrow{\sim} H^*_{\rm dR}(X^{\rm an}), \tag{4.17}$$

which is compatible with the Hodge filtration. In fact, (4.17) continues to hold for arbitrary smooth varieties without any projectivity assumption, see [71, Thm. 1'].

Also the open case can be cast in the analytic setting, where (4.12) is replaced by

$$\Omega^\bullet_{\mathbb{P}^{\rm an}}(\log(X^{\rm an})) \xrightarrow{\sim} \Omega^\bullet_{\mathbb{P}^{\rm an}}(*X^{\rm an}) \xrightarrow{\sim} j_*\mathcal{A}^\bullet_U \xrightarrow{\sim} Rj_*\mathbb{C}_U. \tag{4.18}$$

Here, the complex $\mathcal{A}_U$ is the standard $\mathcal{C}^\infty$-de Rham complex.

The verification of the quasi-isomorphisms in (4.18), and, similarly, in (4.12), is readily reduced to the case of $U = \mathbb{C}\setminus\{0\} \hookrightarrow \mathbb{C}$. In this case,

$$\begin{array}{rcl} \Omega^\bullet_{\mathbb{C}}(\log(\{0\}))(\mathbb{C}) & : & \mathcal{O}_{\mathbb{C}} \longrightarrow \frac{dz}{z}\mathcal{O}_{\mathbb{C}}, \\ \Omega^\bullet_{\mathbb{C}}(*\{0\})(\mathbb{C}) & : & \sum z^n\,\mathcal{O}_{\mathbb{C}} \longrightarrow \sum z^n dz\,\mathcal{O}_{\mathbb{C}}, \\ j_*\mathcal{A}^\bullet(\mathbb{C}) & : & \mathcal{C}^\infty_U \longrightarrow dx\,\mathcal{C}^\infty_U + dy\,\mathcal{C}^\infty_U \to (dx\wedge dy)\,\mathcal{C}^\infty_U. \end{array}$$

The cohomology of all three satisfies $H^i \simeq \mathbb{C}$ for $i = 0, 1$ and $H^i = 0$ otherwise.

Also, there is an analytic version of (4.13) and there exists a natural isomorphism

$$\mathbb{H}^*(\mathbb{P},\Omega^\bullet_{\mathbb{P}}(\log(X))) \simeq \mathbb{H}^*(\mathbb{P}^{\rm an},\Omega^\bullet_{\mathbb{P}^{\rm an}}(\log(X^{\rm an}))).$$

To simplify the notation, we will from now on also write X, U, and $\mathbb{P}$ for the associated analytic varieties. Then the exact sequence (4.14) becomes the classical Gysin sequence

$$\cdots \to H^i(\mathbb{P},\mathbb{C}) \to H^i(U,\mathbb{C}) \to H^{i-1}(X,\mathbb{C}) \to H^{i+1}(\mathbb{P},\mathbb{C}) \to \cdots.$$

This is interesting only for $i-1 = n$. As the map $H^k(X,\mathbb{C}) \longrightarrow H^{k+2}(\mathbb{P},\mathbb{C})$ is surjective for $k = n-1, n$, one finds

$$\begin{array}{ccc} H^{n+1}(U,\mathbb{C}) & \xrightarrow{\sim} & H^n(X,\mathbb{C})_{\rm pr} \\ \cup & & \cup \\ F^pH^{n+1}(U,\mathbb{C}) & \xrightarrow{\sim} & F^{p-1}H^n(X,\mathbb{C})_{\rm pr}. \end{array}$$

However, as it turns out, the Hodge filtration is difficult to compute and it is preferable to replace it by the *pole filtration* $F^\bullet_{\rm pol}$ of the complex $\Omega^\bullet_{\mathbb{P}}(*X)$. Under the quasi-isomorphisms (4.12) and (4.18) the two compare as follows

$$\begin{array}{ccccccc} F^p\Omega^\bullet_{\mathbb{P}}(\log(X)) & : & \Omega^p_{\mathbb{P}}(\log(X)) & \to & \Omega^{p+1}_{\mathbb{P}}(\log(X)) & \cdots \longrightarrow & \Omega^{n+1}_{\mathbb{P}}(\log(X)) \\ \cap & & \cap & & \cap & & \cap \\ F^p_{\rm pol}\Omega^\bullet_{\mathbb{P}}(*X) & : & \Omega^p_{\mathbb{P}}(X) & \longrightarrow & \Omega^{p+1}_{\mathbb{P}}(2X) & \cdots \to & \Omega^{n+1}_{\mathbb{P}}((n-p+2)X). \end{array}$$

This is usually not a quasi-isomorphism. However, using that $H^*(\mathbb{P},\mathbb{C})_{\rm pr}=0$ and applying Bott vanishing, see Section 1.2, one finds

$$F^p_{\rm pol}H^{n+1}(U,\mathbb{C})\simeq F^pH^{n+1}(U,\mathbb{C}).$$

The advantage of using the pole filtration comes from the following

Lemma 4.20 (Griffiths)**.** *Let $X\subset\mathbb{P}=\mathbb{P}^{n+1}$ be a smooth hypersurface. Then $F^{p+1}H^{n+1}(U,\mathbb{C})\simeq F^pH^n(X,\mathbb{C})_{\rm pr}$ is isomorphic to*

$$F^{p+1}_{\rm pol}H^{n+1}(U,\mathbb{C})\simeq\frac{H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p+1)X))}{dH^0(\mathbb{P},\Omega^n_{\mathbb{P}}((n-p)X))}\tag{4.19}$$

and $F^{p+1}H^{n+1}(U,\mathbb{C})/F^{p+2}H^{n+1}(U,\mathbb{C})\simeq H^{p,n-p}(X)_{\rm pr}$ is isomorphic to

$$\frac{F^{p+1}_{\rm pol}H^{n+1}(U,\mathbb{C})}{F^{p+2}_{\rm pol}H^{n+1}(U,\mathbb{C})}\simeq\frac{H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p+1)X))}{H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p)X))+dH^0(\mathbb{P},\Omega^n_{\mathbb{P}}((n-p)X))}.\tag{4.20}$$

Proof By definition, the left hand side in (4.19) is the image of the map

$$\mathbb{H}^{n+1}(\mathbb{P},F^{p+1}_{\rm pol}\Omega^\bullet_{\mathbb{P}}(*X))\longrightarrow\mathbb{H}^{n+1}(\mathbb{P},\Omega^\bullet_{\mathbb{P}}(*X)).$$

The natural map $\Omega^{n+1}_{\mathbb{P}}((n-p+1)X)[-(n+1)]\longrightarrow F^{p+1}_{\rm pol}\Omega^\bullet_{\mathbb{P}}(*X)$ induces

$$H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p+1)X))\longrightarrow F^{p+1}_{\rm pol}\mathbb{H}^{n+1}(\mathbb{P},\Omega^\bullet_{\mathbb{P}}(*X))\simeq F^{p+1}_{\rm pol}H^{n+1}(U,\mathbb{C}).$$

It is rather straightforward to show that the map is surjective and that its kernel is the image of $d\colon H^0(\mathbb{P},\Omega^n_{\mathbb{P}}((n-p)X))\longrightarrow H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p+1)X))$. The isomorphism in (4.20) follows. □

Observe that

$$\begin{aligned}H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p+1)X))&\simeq H^0(\mathbb{P},\mathcal{O}((n-p+1)\cdot d-(n+2)))\\&\simeq H^0(\mathbb{P},\mathcal{O}(t(p))).\end{aligned}$$

Thus, in order to prove (4.8), it suffices to show that the image of

$$(\partial_iF)\colon H^0(\mathbb{P},\mathcal{O}((n-p)\cdot d-(n+2)))\longrightarrow H^0(\mathbb{P},\mathcal{O}(t(p)))$$

equals $H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p)X))+dH^0(\mathbb{P},\Omega^n_{\mathbb{P}}((n-p)X))$. This is a rather unpleasant computation in terms of rational differential forms on $\mathbb{P}$ and $\mathbb{C}^{n+2}$. We omit this here and refer to [149, Thm. 18.10] or [28, Ch. 3.2].[12]

[12] One could try an alternative argument: Apply $\bigwedge^{n+1}$ to the Euler sequence to obtain $0\longrightarrow\mathcal{O}(-(n+2))\longrightarrow\mathcal{O}(-(n+1))^{\oplus n+2}\longrightarrow\Omega^n_{\mathbb{P}}\longrightarrow 0$. Tensor with $\mathcal{O}((n-p)\cdot d)$ and consider the composition $H^0(\mathbb{P},\mathcal{O}((n-p)\cdot d-(n+1)))^{\oplus n+2}\twoheadrightarrow H^0(\mathbb{P},\Omega^n_{\mathbb{P}}((n-p)\cdot d))\xrightarrow{d}H^0(\mathbb{P},\Omega^{n+1}_{\mathbb{P}}((n-p+1)\cdot d))$, which should be compared to the map given by (∂_iF). But the attempt becomes quickly as technical as the standard approach.

To prove the commutativity of (4.9) one first needs to fix an appropriate isomorphism $H^{n,n}(X)_{\rm pr} \simeq R_\sigma(X)$ which again involves rational forms. Also the commutativity of (4.10) is not straightforward. One has to argue that multiplication

$$H^0(\mathbb{P}, \mathcal{O}(d)) \times H^0(\mathbb{P}, \mathcal{O}(t(p))) \longrightarrow H^0(\mathbb{P}, \mathcal{O}(t(p-1)))$$

is related to the contraction

$$H^1(X, \mathcal{T}_X) \times H^{n-p}(X, \Omega_X^p) \longrightarrow H^{n-p+1}(X, \Omega_X^{p-1})$$

via the surjection $H^0(\mathbb{P}, \mathcal{O}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d)) \longrightarrow H^1(X, \mathcal{T}_X)$. The multiplication takes place in the top degree $n+1$ of $F^{p+1}_{\rm pol}\Omega^\bullet_{\mathbb{P}}(*X)$, whereas the contraction applies to the lowest degree (from degree p to $p-1$).

We conclude this section by a result that will later be used to prove that the period map is unramified. We state the result for cubic hypersurfaces only.

Corollary 4.21 (Infinitesimal Torelli Theorem)**.** *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface of dimension $n > 2$. Then the contraction $\mathcal{T}_X \times \Omega_X^p \longrightarrow \Omega_X^{p-1}$ yields an injection*

$$H^1(X, \mathcal{T}_X) \hookrightarrow \operatorname{Hom}\left(\bigoplus_{p+q=n} H^{p,q}(X)_{\rm pr}, \bigoplus_{p+q=n} H^{p-1,q+1}(X)_{\rm pr}\right)$$

Proof There exists a p such that $0 \le t(p) \le \sigma - 3$. Then by Corollary 4.5 multiplication in the Jacobian ring $R(X)$ yields an injection

$$R_3 \hookrightarrow \operatorname{Hom}(R_{t(p)}, R_{t(p)+3}).$$

Now use $R_{t(p)} \simeq H^{p,n-p}(X)_{\rm pr}$ and the compatibility of the multiplication in $R(X)$ with the contraction map. □

5 Quadric fibrations and ramified covers

This part is devoted to standard constructions for cubic hypersurfaces, for example using linear projection to view them as quadratic fibrations or studying triple cover of the ambient projective space ramified along the cubic.

5.1 Let $\mathrm{P} = \mathbb{P}(W) \subset \mathbb{P} = \mathbb{P}(V)$ be a linear subspace. Here, $\dim(V) = n+2$ and $k = \dim(W)$. Furthermore, pick a generic linear subspace $\mathbb{P}(U) \subset \mathbb{P}(V)$ of codimension k. Here, generic means that the composition $U \subset V \twoheadrightarrow V/W$ is an isomorphism or, equivalently, that $U + W = V$.

The linear projection $\varphi \colon \mathbb{P} \dashrightarrow \mathbb{P}(U) \simeq \mathbb{P}(V/W)$ from P is the rational map that sends $x \in \mathbb{P} \setminus \mathrm{P}$ to the unique point of intersection of the linear subspace $\overline{x\mathrm{P}} \simeq \mathbb{P}^k$ with $\mathbb{P}(U) \simeq \mathbb{P}^{n+1-k}$. It is the rational map associated with the linear system $|\mathcal{I}_{\mathrm{P}} \otimes$

$\mathcal{O}(1)| \subset |\mathcal{O}(1)|$ with base locus $\mathrm{P} \subset \mathbb{P}$, which is resolved by a simple blow-up. The resulting morphism $\varphi\colon \mathrm{Bl}_{\mathrm{P}}(\mathbb{P}) \longrightarrow \mathbb{P}(V/W)$ is associated with the complete linear system $|\tau^*\mathcal{O}(1) \otimes \mathcal{O}(-E)|$:

$$\begin{array}{ccccc} E = \mathbb{P}(\mathcal{N}_{\mathrm{P}/\mathbb{P}}) & \hookrightarrow & \mathrm{Bl}_{\mathrm{P}}(\mathbb{P}) & & \\ \downarrow & & \downarrow{\scriptstyle\tau} & \searrow^{\varphi} & \\ \mathrm{P} & \hookrightarrow & \mathbb{P} & \dashrightarrow & \mathbb{P}(V/W). \end{array}$$

The fibre $\varphi^{-1}(y)$, $y \in \mathbb{P}(U) \simeq \mathbb{P}(V/W)$, is the strict transform of $\mathbb{P}^k \simeq \overline{y\mathrm{P}} \subset \mathbb{P}$ in $\mathrm{Bl}_{\mathrm{P}}(\mathbb{P})$. So, all fibres of $\varphi\colon \mathrm{Bl}_{\mathrm{P}}(\mathbb{P}) \longrightarrow \mathbb{P}(V/W)$ are projective spaces $\mathbb{P}^k$ and, indeed, $\mathrm{Bl}_{\mathrm{P}}(\mathbb{P}) \simeq \mathbb{P}(\mathcal{F}^*)$ for the locally free sheaf $\mathcal{F} := \varphi_*\tau^*\mathcal{O}(1)$ on $\mathbb{P}(V/W)$ which is of rank $k+1$. To determine $\mathcal{F}$ explicitly, tensor the structure sequence of the exceptional divisor $E \subset \mathrm{Bl}_{\mathrm{P}}(\mathbb{P})$ with $\tau^*\mathcal{O}(1)$ to get the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau^*\mathcal{O}(1) \otimes \mathcal{O}(-E) & \longrightarrow & \tau^*\mathcal{O}(1) & \longrightarrow & \tau^*\mathcal{O}(1)|_E & \longrightarrow & 0 \\ & & \simeq \varphi^*\mathcal{O}(1) & & \simeq \mathcal{O}_\varphi(1) & & \simeq \mathcal{O}(1,0) & & \end{array} \tag{5.1}$$

Here, we use that $\mathcal{N}_{\mathrm{P}/\mathbb{P}} \simeq V/W \otimes \mathcal{O}(1)$, which yields an isomorphism

$$E = \mathbb{P}(\mathcal{N}_{\mathrm{P}/\mathbb{P}}) \simeq \mathrm{P} \times \mathbb{P}(V/W)$$

compatible with the two natural projections. This also shows that $\mathcal{O}(E)|_E \simeq \mathcal{O}(1,-1)$ on $E \simeq \mathrm{P} \times \mathbb{P}(V/W)$ and, hence, $\varphi^*\mathcal{O}(1)|_E \simeq (\tau^*\mathcal{O}(1) \otimes \mathcal{O}(-E))|_E \simeq \mathcal{O}(0,1)$. Therefore, φ restricted to E is the projection onto $\mathbb{P}(V/W)$. Thus, $\mathcal{F}$ is described by the direct image under φ of (5.1)

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow H^0(\mathrm{P}, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{P}(V/W)} \longrightarrow 0.$$

The sequence splits, which yields a non-canonical isomorphism $\mathcal{F} \simeq \mathcal{O}(1) \oplus (W^* \otimes \mathcal{O}_{\mathbb{P}(V/W)})$ and, hence, $\det(\mathcal{F}) \simeq \mathcal{O}(1)$, which is all we shall use.

Let now $X \subset \mathbb{P} = \mathbb{P}(V)$ be a cubic hypersurface with equation $F \in H^0(\mathbb{P}, \mathcal{O}(3))$. The pull-back τ^*F is a section of $\tau^*\mathcal{O}(3)$, whose zero divisor $V(\tau^*F)$ is the total transform of X. If $\mathrm{P} \subset \mathbb{P}$ is contained in X (cf. Exercise 2.4), then the total transform has two components, E and the strict transform $\mathrm{Bl}_{\mathrm{P}}(X)$ of $X \subset \mathbb{P}$. More precisely, in this case F is contained in $H^0(\mathbb{P}, \mathcal{O}(3) \otimes \mathcal{I}_{\mathrm{P}}) \subset H^0(\mathbb{P}, \mathcal{O}(3))$ and τ^*F in $H^0(\mathrm{Bl}_{\mathrm{P}}(\mathbb{P}), \tau^*\mathcal{O}(3) \otimes \mathcal{O}(-E)) \subset H^0(\mathrm{Bl}_{\mathrm{P}}(\mathbb{P}), \tau^*\mathcal{O}(3))$. Therefore,

$$\mathrm{Bl}_{\mathrm{P}}(X) = V(\tau^*F) \in |\tau^*\mathcal{O}(3) \otimes \mathcal{O}(-E)|$$

in $\mathrm{Bl}_{\mathrm{P}}(\mathbb{P})$. To describe $\mathrm{Bl}_{\mathrm{P}}(X)$ as a quadric fibration over $\mathbb{P}(V/W)$, we compute the direct image of τ^*F under φ. First, observe that

$$\tau^*\mathcal{O}(3) \otimes \mathcal{O}(-E) \simeq \tau^*\mathcal{O}(2) \otimes (\tau^*\mathcal{O}(1) \otimes \mathcal{O}(-E)) \simeq \mathcal{O}_\varphi(2) \otimes \varphi^*\mathcal{O}(1).$$

Hence,

$$\varphi_*(\tau^*\mathcal{O}(3)\otimes\mathcal{O}(-E))\simeq\varphi_*(\mathcal{O}_\varphi(2)\otimes\varphi^*\mathcal{O}(1))\simeq S^2(\mathcal{F})\otimes\mathcal{O}(1)$$

and, therefore, τ^*F can be thought of as a section $q\in H^0(\mathbb{P}(V/W),S^2(\mathcal{F})\otimes\mathcal{O}(1))$ or as a homomorphism $q\colon\mathcal{F}^*\longrightarrow\mathcal{F}\otimes\mathcal{O}(1)$. Then the fibre of $\mathrm{Bl}_{\mathrm{P}}(X)\subset\mathrm{Bl}_{\mathrm{P}}(\mathbb{P})\simeq\mathbb{P}(\mathcal{F}^*)$ over $t\in\mathbb{P}(V/W)$ is the quadric defined by $q\in S^2(\mathcal{F}(t))$. In particular, the fibre is smooth if and only if this quadric is non-degenerate. Hence, the discriminant divisor D_{P} of $\varphi\colon\mathrm{Bl}_{\mathrm{P}}(X)\longrightarrow\mathbb{P}(V/W)$ is

$$D_{\mathrm{P}}=V(\det(q))\subset\mathbb{P}(V/W).$$

Here, $\det(q)\colon\ \det(\mathcal{F})^*\longrightarrow\det(\mathcal{F})\otimes\mathcal{O}(k+1)$ is viewed as a section of $\det(\mathcal{F})^2\otimes\mathcal{O}(k+1)\simeq\mathcal{O}(k+3)$. The discussion is summarized by the following classicial fact, see e.g. [22, Lem. 2] for $n=3$ and [9, Ch. 4].

Lemma 5.1. *Assume a smooth cubic hypersurface $X\subset\mathbb{P}^{n+1}$ contains a linear subspace $\mathrm{P}=\mathbb{P}^{k-1}$. Then the linear projection from P defines a morphism*

$$\varphi\colon\mathrm{Bl}_{\mathrm{P}}(X)\twoheadrightarrow\mathbb{P}^{n+1-k}$$

with a quadric hypersurface in $\overline{yP}\simeq\mathbb{P}^k$ as the fibre over y. The fibres are singular exactly over the discriminant divisor $D_{\mathrm{P}}\in|\mathcal{O}(k+3)|$. □

We are most interested in the case $k=n-1$.

Corollary 5.2. *Assume $X\subset\mathbb{P}^{n+1}$ is a smooth cubic hypersurface containing a linear $\mathrm{P}=\mathbb{P}^{n-2}\subset X\subset\mathbb{P}^{n+1}$. Linear projection from P yields a quadric fibration $\mathrm{Bl}_{\mathrm{P}}(X)\longrightarrow\mathbb{P}^2$ with discriminant curve $D_P\in|\mathcal{O}(n+2)|$.* □

Remark 5.3. Assume $\mathbb{P}^{n-2}\simeq\mathrm{P}\subset X\subset\mathbb{P}=\mathbb{P}^{n+1}$ are as above. The quadratic fibration $\mathrm{Bl}_{\mathrm{P}}(X)\longrightarrow\mathbb{P}^2$ allows one to reprove the unirationality (of degree two) of X, cf. Corollary 1.12. Indeed, pick a generic $\mathbb{P}^2\subset\mathbb{P}$ and let $\tilde{\mathbb{P}}^2$ be the intersection of $\tau^{-1}(\mathbb{P}^2)\subset\mathrm{Bl}_{\mathrm{P}}(\mathbb{P})$ and $\mathrm{Bl}_{\mathrm{P}}(X)\subset\mathrm{Bl}_{\mathrm{P}}(\mathbb{P})$. The generic fibre of $\tilde{\mathbb{P}}^2\longrightarrow\mathbb{P}^2$ over $y\in\mathbb{P}^2$ is the intersection of the quadric $\varphi^{-1}(y)\subset\overline{y\mathrm{P}}$ with the line $\mathbb{P}^1\simeq\mathbb{P}^2\cap\overline{y\mathrm{P}}$ and, therefore, consists of two points, i.e. $\tilde{\mathbb{P}}^2\longrightarrow\mathbb{P}^2$ is of degree two. The base change $\mathrm{Bl}_{\mathrm{P}}(X)\times_{\mathbb{P}^2}\tilde{\mathbb{P}}^2\longrightarrow\tilde{\mathbb{P}}^2$ is a quadric fibration with a section and hence rational.

In fact, for $n\geq 7$ the existence of $\mathbb{P}^{n-2}\subset X$ implies the rationality of X, see [9, Cor. 4.2]. Indeed, the scheme theoretic generic fibre is a quadric over the C_2-field $K(\mathbb{P}^2)$, cf. [102].

5.2

5.3

2

Moduli spaces

1 GIT-quotient

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To study the geometry of a particular hypersurface $X \subset \mathbb{P} = \mathbb{P}^{n+1}$ or to understand how a certain feature changes when X is deformed, the actual embedding of X into the projective space $\mathbb{P}$ is often of no importance. The embeddings of a fixed X into $\mathbb{P}$ are parametrized by the choice of a basis of $H^0(X, \mathcal{O}_X(1))$ up to scaling.[1] So, instead of the linear system $|\mathcal{O}_{\mathbb{P}}(d)|$ one is really interested in the quotient $|\mathcal{O}_{\mathbb{P}}(d)|/\mathrm{GL}(n+2)$. Ideally, one would like this quotient to exist in the category of varieties or schemes and to come with a universal family. However, as it turns out, this is too much to ask for.

Example 1.1. Consider the easiest case of interest to us: $d = 3$ and $n = 0$, i.e. three points in $\mathbb{P}^1$. Up to a linear coordinate change, there are only three possibilities: $\{x_1, x_2, x_3\}$ (three distinct points), $\{2 \cdot x_1, x_2\}$ (two distinct points of which one with multiplicity two), or $\{3x\}$ (a triple point). Thus, the moduli space parametrizing all hypersurfaces $X \subset \mathbb{P}^1$ of degree three should consists of three points. On the other hand, together with all possible embeddings they are parametrized by the projective space $|\mathcal{O}_{\mathbb{P}^1}(3)|$, which is connected and, therefore, does not admit a morphism onto a disconnected space.

The same phenomenon can be described in terms of orbit closures. For example, the limit of the one-parameter subgroup $\mathrm{diag}(t, 1/t)$ applied to the set $V(x_0^2\,x_1 - x_0\,x_1^2) = \{0, \infty, [1:1]\}$ viewed as a point in $|\mathcal{O}_{\mathbb{P}^1}(3)|$ is $\{2\cdot 0, \infty\}$ for $t \to \infty$ and $\{0, 2\cdot\infty\}$ for $t \to 0$. Hence, all these points should be identified under the quotient map to any moduli space with a reasonable geometric structure.

Similar phenomena occur in higher dimensions and for all $d > 1$. The way out is

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[1] For $n \geq 2$ and $d \neq n+2$ the line bundle $\mathcal{O}_X(1)$ itself does not depend on the embedding, as it is determined by the property that $\mathcal{O}_X(d-(n+2)) \simeq \omega_X$. For $n > 2$ one can alternatively use that $\mathcal{O}_X(1)$ is the ample generator of $\mathrm{Pic}(X)$.

to only allow *stable* hypersurfaces. They are parametrized by an open subset of $|\mathcal{O}_{\mathbb{P}}(d)|$ and include all smooth hypersurfaces. This then leads to a quasi-projective moduli space (without universal family in general) parametrizing orbits of hypersurfaces. To obtain a projective moduli space one has to allow semi-stable hypersurfaces which, however, leads to a moduli space that identifies certain orbits.

We briefly review the main features of GIT needed to understand moduli spaces of (smooth, cubic) hypersurfaces. We recommend [107, Ch. 6] for a quick introduction and the classic [119] or the textbooks [52, 116] for more details and references. Although we definitely want the moduli spaces to be defined over arbitrary fields, we shall assume that k is algebraically closed, just to keep the discussion geometric.

1.1 Let A be a finite type (say integral) k-algebra and G a linear algebraic group over k with an action on $X = \text{Spec}(A)$ or, equivalently, an action on A. If a quotient $X \twoheadrightarrow X/G$ in the geometric sense exists, then $X/G = \text{Spec}(A^G)$, where $A^G \subset A$ is the invariant ring. In order for X/G to be a variety, the ring A^G would need to be again of finite type. This is Hilbert's 14th problem which has been answered by Hilbert himself in characteristic zero for $G = \text{SL}$ and in general by Nagata and Harboush, see [119] or the entertaining [118] for a historic account, references, and proofs:

If G is reductive, then A^G is again a finite type k-algebra.

This seems to settle the question in the affine case by just defining $X/G := \text{Spec}(A^G)$ with the quotient morphism $X \twoheadrightarrow X/G$ induced by the inclusion $A^G \subset A$. However, this is, in general, a quotient only in a weaker sense.

Definition 1.2. A morphism $\pi\colon X \twoheadrightarrow Y$ is a *categorical quotient* for the action of a group G on X if it is G-invariant[2] and if any other G-invariant morphism $\pi'\colon X \twoheadrightarrow Y'$ factors uniquely through a morphism $Y \twoheadrightarrow Y'$.

A G-invariant morphism $\pi\colon X \twoheadrightarrow Y$ is a *good quotient* if it satisfies the following conditions: (i) π is affine and surjective; (ii) $\pi(Z)$ of any closed G-invariant subset $Z \subset X$ is closed; (iii) $\pi(Z_1) \cap \pi(Z_2) = \pi(Z_1 \cap Z_2)$ for all closed G-invariant sets $Z_1, Z_2 \subset X$; and (iv) $\mathcal{O}_Y$ is the sheaf of G-invariant sections of $\mathcal{O}_X$, i.e. $\mathcal{O}_Y \simeq (\pi_*\mathcal{O}_X)^G$ which means that $\pi^*\colon \mathcal{O}_Y(U) \xrightarrow{\sim} \mathcal{O}_X(\pi^{-1}(U))^G$ for all open $U \subset Y$.

A good quotient is *geometric* if in addition the pre-image of any closed point is an orbit.[3]

By definition, any geometric quotient is a good quotient and, as proved in [119], any good quotient is also a categorical quotient:

$$\text{geometric} \ \Rightarrow\ \text{good} \ \Rightarrow\ \text{categorical}.$$

[2] i.e. the two morphisms $G \times X \twoheadrightarrow X$ defined by the second projection and by the group action composed with π coincide

[3] The exact definition of these notions varies from source to source. The subtle differences will be of no importance in our situation.

Note that a good quotient is equipped with the quotient topology and parametrizes the closed orbit of the action. Hence, a good quotient is geometric exactly when all orbits are closed, see [107, Prop. 6.1.7].

The main results on affine quotients is the following, cf. [119] or [107, Prop. 6.3.1]: *If a reductive linear algebraic group G acts on an affine variety* $X = \mathrm{Spec}(A)$*, i.e. A is a finite type k-algebra, then* $X \longrightarrow X/\!/G := \mathrm{Spec}(A^G)$ *is a good quotient.*

In particular, it is a categorical quotient, but usually not a geometric one.

1.2 With certain modifications, the same recipe can be applied to projective varieties. Assume $A = \bigoplus_{i\geq 0} A_i$ is a finite type graded k-algebra generated by A_1 and assume that the projective variety $X = \mathrm{Proj}(A)$ is endowed with the action of a reductive linear algebraic group. Note that, in contrast to the affine case, the action is not necessarily induced by an action of G on A. However, we shall assume it is, in which case it is induced by a G-action on A_1. This is called a *linearization*. Geometrically it is realized by an embedding $X \hookrightarrow \mathrm{Proj}(S^*(A_1)) \simeq \mathbb{P}^m$ such that the action of G on X is the restriction of an action of G on $\mathbb{P}^m$ induced by a linear representation $G \longrightarrow \mathrm{GL}(A_1)$.

One is tempted to imitate the affine case and define the quotient simply as $\mathrm{Proj}(A^G)$. Note that A^G is naturally graded and again of finite type, but possibly not generated in degree one. This can be easily remedied by passing to $\bigoplus_{i\geq 0} A_{mi}$ for some $m > 0$. However, the graded inclusion $A^G \subset A$ does not define a morphism between the associated projective schemes. Indeed, a homogeneous prime or maximal ideal in A may intersect A^G in its inessential ideal $(A^G)_+ = \bigoplus_{i>0} (A^G)_i$. In other words, there exists a morphism

$$X^{\mathrm{ss}} \longrightarrow X^{\mathrm{ss}}/\!/G := \mathrm{Proj}(A^G)$$

only on the open set $X^{\mathrm{ss}} := X \setminus V((A^G)_+) \subset X$. This naturally leads to the central definition in GIT.

Definition 1.3. A point $x \in X$ is *semi-stable* if it is contained in the open subset $X^{\mathrm{ss}} \subset X$, i.e. if there exists a homogeneous G-invariant $f \in A_i$, for some $i > 0$, with $f(x) \neq 0$. A point $x \in X$ is *stable* if x is semi-stable and the induced morphism $G \longrightarrow X^{\mathrm{ss}}$ is proper, i.e. the orbit $G \cdot x$ is closed in X^{ss} and the stabilizer G_x is finite.

Exercise 1.4. For a linearized action of a reductive group G on $\mathbb{P}(V)$, a point $[x] \in \mathbb{P}(V)$ is semi-stable if and only if $0 \notin \overline{G \cdot x} \subset V$. A point $[x] \in \mathbb{P}(V)$ is stable if and only if the morphism $G \longrightarrow V$, $g \longmapsto g \cdot x$ is proper.

Using open affine covers, the problem is reduced to the affine case which eventually leads to the following key result in GIT.

Theorem 1.5 (Mumford)**.** *Assume that a linearization of the action of a reductive linear algebraic group G on* $X = \mathrm{Proj}(A)$ *has been fixed. Then the natural morphism* $X^{\mathrm{ss}} \longrightarrow X^{\mathrm{ss}}/\!/G$ *is a good quotient.*

1.3 Let us turn to the concrete GIT problem that concerns us. Consider $G := \mathrm{SL}(n+2)$ with its natural action on $\mathbb{P}^{n+1}$ and the induced action on all complete linear systems $|\mathcal{O}(d)|$. Instead of $\mathrm{SL}(n+2)$ one often considers $\mathrm{PGL}(n+2)$. Both groups are reductive and the orbits of their actions on $|\mathcal{O}(d)|$ are of course the same. The advantage of working with SL is that its action on $|\mathcal{O}(d)|$ comes with a natural linearization. The relevant result for us is the following, see [94, Sec. 11.8] for the arithmetic version over $\mathrm{Spec}(\mathbb{Z})$.

Corollary 1.6. *Every smooth hypersurface $X \subset \mathbb{P}$ of degree $d \geq 3$ defines a stable point $[X] \in |\mathcal{O}(d)|$ for the action of $G = \mathrm{SL}(n+2)$, i.e.*

$$U(d,n) = |\mathcal{O}(d)|_{\mathrm{sm}} \subset |\mathcal{O}(d)|^{\mathrm{s}}.$$

Proof The semi-stability is an immediate consequence of Theorem **1**.2.2 and holds in fact for $d > 1$. Indeed, the complement of $U(d,n) \subset \mathbb{P}^N = |\mathcal{O}(d)|$ is the discriminant divisor $D = D(d,n)$, which is the zero set $V(\Delta)$ of the discriminant $\Delta = \Delta_{d,n} \in H^0(\mathbb{P}^N, \mathcal{O}(\ell))$, $\ell = (d-1)^{n+1}(n+2)$. As the smoothness of a hypersurface $X \subset \mathbb{P}$ does not depend on the embedding, the discriminant divisor D is invariant under the action of GL. Hence, for all $g \in \mathrm{GL}$ the induced action on $H^0(\mathbb{P}^N, \mathcal{O}(\ell))$, sending Δ to $g^*\Delta$, satisfies $D = V(\Delta) = V(g^*\Delta)$. Therefore, $g^*\Delta = \lambda_g \cdot \Delta$ for some $\lambda_g \in \mathbb{G}_m$. This in fact defines a morphism of algebraic groups $\mathrm{GL} \longrightarrow \mathbb{G}_m$, $g \longmapsto \lambda_g$. However, the only characters of GL are powers of the determinant, which by definition is trivial on $G = \mathrm{SL}$. Hence, Δ is a G-invariant homogeneous polynomial that does not vanish at any point $[X] \in |\mathcal{O}(d)|$ that corresponds to a smooth hypersurface. In other words, $U \subset |\mathcal{O}(d)|^{\mathrm{ss}}$.

In order to show stability, one has to prove that the morphism

$$G \longrightarrow |\mathcal{O}(d)|^{\mathrm{ss}},\ g \longmapsto g[X]$$

is proper for X smooth. Let us first prove that the stabilizer $G_{[X]}$ is finite. Clearly, any $g \in G_{[X]}$ induces an automorphism of the polarized $(X, \mathcal{O}_X(1))$. This yields a morphism $G_{[X]} \longrightarrow \mathrm{Aut}(X, \mathcal{O}_X(1))$, the fibre of which is contained in the finite subgroup $\mu_{n+2} = \mathrm{Ker}(\mathrm{SL}(n+2) \longrightarrow \mathrm{PGL}(n+2))$. Now use Corollary **1**.3.6 and Remark **1**.3.7.

To conclude one needs to show that the orbit $G \cdot [X]$ is closed in $|\mathcal{O}(d)|^{\mathrm{ss}}$. Let us first show it is closed in the open subset $U = |\mathcal{O}(d)|_{\mathrm{sm}} \subset |\mathcal{O}(d)|^{\mathrm{ss}}$. Consider its closure $\overline{G \cdot [X]}$ in U and pick a point $[X'] \in \overline{G \cdot [X]} \setminus G \cdot [X]$. Then $G \cdot [X'] \subset \overline{G \cdot [X]} \setminus G \cdot [X]$ and hence $\dim(G \cdot [X']) < \dim(G \cdot [X])$ which would imply $\dim(G_{[X']}) > 0$ contradicting the above discussion. Now, consider the morphism

$$\pi \colon |\mathcal{O}(d)|^{\mathrm{ss}} \longrightarrow |\mathcal{O}(d)|^{\mathrm{ss}} /\!/ G = \mathrm{Proj}\left(k[H^0(\mathbb{P}^N, \mathcal{O}(1))]^G\right).$$

Clearly, U is the pre-image of the open non-vanishing locus of $\Delta \in H^0(\mathbb{P}^N, \mathcal{O}(\ell))^G \subset k[H^0(\mathbb{P}^N, \mathcal{O}(1))]^G$ and, therefore, $\pi^{-1}(\pi([X])) \subset U$ for all smooth X. As the subset $G \cdot [X]$ of $\pi^{-1}(\pi(x))$ is closed in the bigger set U, it is also closed in $\pi^{-1}(\pi([X]))$. However,

$\pi^{-1}(\pi([X]))$ as the pre-image of a closed point is closed in $|\mathcal{O}(d)|^{\rm ss}$. Altogether this proves that $G \cdot [X] \subset |\mathcal{O}(d)|^{\rm ss}$ is closed. □

Remark 1.7. The techniques of the proof show that the morphism

$$\mathrm{PGL}(n+2) \times U \longrightarrow U \times U,\ (g, [X]) \longmapsto ([X], g[X])$$

is proper. Now, the pre-image of the diagonal $\Delta \subset U \times U$ can be interpreted as the scheme **Aut** $= \mathrm{Aut}(\mathcal{X}/U, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow U$ of polarized automorphisms of the universal family of smooth hypersurfaces $\mathcal{X} \longrightarrow U \subset |\mathcal{O}(d)|$, cf. Section **1**.3.2. So, in particular, the fibre over $[X] \in U$ is the finite group $\mathrm{Aut}(X, \mathcal{O}_X(1))$. Note that as a consequence one finds that $\mathrm{Aut}(\mathcal{X}/U, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow U$ is a finite morphism, cf. [94, Cor. 11.8.4].

Example 1.8. For $d = 1$, i.e. for hyperplanes, no $[X] \in |\mathcal{O}(1)|$ is semi-stable. Indeed, in this case, $U(1, n) = |\mathcal{O}(1)| \simeq \mathbb{P}^{\vee}$ and $k[x_0, \ldots, x_{n+2}]^{\rm SL} = k$.

Smooth quadrics, so $d = 2$, are semi-stable by the above, but they are not stable. Indeed, the stabilizer of a quadric, say of $\sum x_i^2$ and in fact every smooth quadric is of this form after a linear coordinate change, is the special orthogonal group $\mathrm{SO}(n+2) \subset \mathrm{SL}(n+2)$, which is not finite.

Exercise 1.9. The above proof did not cover the case $n = 0, 1$. Verify that stability still holds in these cases. The only problematic case is $n = 1$ and $d = 3$.

Show that for $n = 0$ and $d = 3$ (semi-)stability is equivalent to smoothness.

The next question one should ask is whether the inclusion $|\mathcal{O}(d)|_{\rm sm} \subset |\mathcal{O}(d)|^{\rm s}$ is strict. How can one interpret geometrically its complement? How big is $|\mathcal{O}(d)|^{\rm ss} \setminus |\mathcal{O}(d)|^{\rm s}$?

1.4 For actual computations of stable and semi-stable points, the Hilbert–Mumford criterion is a powerful tool. It roughly says that it suffices to check one-parameter subgroups and gives a numerical criterion for those.

A one-parameter subgroup of a (reductive) group G is a morphism $\lambda\colon \mathbb{G}_m \longrightarrow G$ of algebraic groups. If a linear action $\rho\colon G \longrightarrow \mathrm{GL}(V)$ is given, then the induced action $\rho \circ \lambda\colon \mathbb{G}_m \longrightarrow \mathrm{GL}(V)$ can be diagonalized, i.e. there exists a basis (e_i) of V such that $\lambda(t)(e_i) = t^{r_i} e_i$, $r_i \in \mathbb{Z}$. The *Hilbert–Mumford weight* of a point $x = \sum x_i e_i \in V$ with respect to this one-parameter subgroup is

$$\mu(x, \lambda) := -\min\{\, r_i \mid x_i \neq 0 \,\}.$$

Theorem 1.10 (Hilbert–Mumford criterion)**.** *For a linearized action of a reductive group G on $\mathbb{P}(V)$ a point $[x] \in \mathbb{P}(V)$ is semi-stable if and only if $\mu(x, \lambda) \geq 0$ for all one-parameter subgroups $\lambda\colon \mathbb{G}_m \longrightarrow G$. The point $[x]$ is stable if and only if strict inequality holds for all non-trivial λ.*

Using Exercise 1.4, one direction is easy to prove. The difficulty lies in checking that it suffices to test one-parameter subgroups.

Example 1.11. (i) A plane cubic curve $E \subset \mathbb{P}^2$ is stable if and only if it is smooth. It is semi-stable if and only if it has at most ordinary double points as singularities, cf. [116, Exa. 7.2] or [119].

(ii) The Hilbert–Mumford criterion allows one to prove that cubic surfaces $S \subset \mathbb{P}^3$ with at most ordinary double points as singularities are stable. Assume that $S := V(F) \subset \mathbb{P}^3$ is integral and defines a point $x \in |\mathcal{O}(3)|$ that is not stable, i.e. such that there exists a $\lambda \colon \mathbb{G}_m \longrightarrow \mathrm{SL}(4)$ with $\mu(x, \lambda) \leq 0$. After a linear coordinate change we may assume that the action is diagonal and the induced action on the linear coordinates is given by $\lambda(t)(x_j) = t^{r_j} x_j$ with $r_0 \leq \cdots \leq r_3$ and $\sum r_j = 0$. Then, on a cubic polynomial $F = \sum a_I x^I$ the action is given by $\lambda(t)(F) = \sum a_I t^{rI} x^I$, where $I = (i_0, \ldots, i_3)$, $\sum i_j = 3$, and $rI := \sum r_j i_j$. By an elementary computation, see [14, Prop. 6.5], one shows that $\mu(x, \lambda) \leq 0$ implies that $[1 : 0 : 0 : 0] \in S$ is either a non-ordinary double point or a cusp.

There exist complete classification of (semi-)stable points in other linear systems $|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ for other small values of d and n. But even for $d = 3$, only the cases $n \leq 4$ have been studied in detail, see Sections **??**.**??**.

Remark 1.12. In [63] it is shown that a hypersurface $X \subset \mathbb{P}^{n+1}$ defines a semi-stable point in $|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ if and only if the subspace $\langle \partial_i F \rangle \subset k[x_0, \ldots, x_{n+1}]_d$ defines a semi-stable point in $\mathrm{Gr}(n+2, k[x_0, \ldots, x_{n+1}]_d)$ with respect to the natural $\mathrm{SL}(n+2)$-action on the Grassmannian.

1.5 Ideally, one would like the universal family $\mathcal{X} \longrightarrow |\mathcal{O}(d)|_{\mathrm{sm}}$ of smooth hypersurfaces to descend to a universal family $\bar{\mathcal{X}} \longrightarrow |\mathcal{O}(d)|_{\mathrm{sm}}/\!/G$ (of varieties isomorphic to smooth hypersurfaces). The natural (and only) choice for such a family would be the quotient $\bar{\mathcal{X}} := \mathcal{X}/\!/G$. However, over a point $[X] \in |\mathcal{O}(d)|_{\mathrm{sm}}/\!/G$ the fibre of this family would be the quotient $X/\mathrm{Aut}(X, \mathcal{O}_X(1))$ of X by the finite group $\mathrm{Aut}(X, \mathcal{O}_X(1))$ and not X itself. This is the reason for

$$M_{d,n} := |\mathcal{O}(d)|_{\mathrm{sm}}/\!/G \tag{1.1}$$

not representing the *moduli functor*

$$\mathcal{M}_{d,n} \colon (Sch/k)^o \longrightarrow (Sets).$$

By definition, $\mathcal{M}_{d,n}$ sends a k-scheme T to the set $\mathcal{M}_{d,n}(T)$ of equivalence classes of polarized smooth projective families $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \longrightarrow T$ with $\mathcal{O}_{\mathcal{X}}(1) \in \mathrm{Pic}_{\mathcal{X}/T}(T)$ such that all geometric fibres are isomorphic to a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ (over the appropriate field) of degree d with polarization given by the restriction $\mathcal{O}_{\mathbb{P}^{n+1}}(1)|_X$.

However, $M_{d,n}$ is still a *coarse moduli space* which means the following.

Corollary 1.13. *Assume $d \geq 3$. There exists a natural transformation $\mathcal{M}_{d,n} \longrightarrow \underline{M}_{d,n}$ such that*

(i) *The induced map* $\mathcal{M}_{d,n}(\mathrm{Spec}(k')) \xrightarrow{\sim} \underline{M}_{d,n}(\mathrm{Spec}(k'))$ *is bijective for any algebraically closed field extension* k'/k.
(ii) *Any natural transformation* $\mathcal{M}_{d,m} \longrightarrow N$ *to a* k*-scheme factorizes uniquely through a morphism* $M_{d,n} \longrightarrow N$ *over* k.

The second condition is essentially a consequence of the fact that $|\mathcal{O}(d)|_{\mathrm{sm}} \longrightarrow M_{d,n}$ is a categorical quotient. The fact that $|\mathcal{O}(d)|_{\mathrm{sm}} \subset |\mathcal{O}(d)|^{\mathrm{s}}$ and that $|\mathcal{O}(d)|^{\mathrm{s}} \longrightarrow |\mathcal{O}(d)|^{\mathrm{s}}//G$ is a geometric quotient implies the first one. For an outline of the details of the arguments see the discussion in [86, Sec. 5.2.].

Remark 1.14. As some geometric arguments make use of actual families, one often has to find substitutes for it. The following techniques are the most frequent ones:

(i) Instead of working with a universal family over $M_{d,n}$, which does not exist, one uses the universal family $\mathcal{X} \longrightarrow |\mathcal{O}(d)|_{\mathrm{sm}}$ and the fact that $|\mathcal{O}(d)|_{\mathrm{sm}} \longrightarrow M_{d,n}$ is a geometric quotient.
(ii) Assume $n > 0$, $d \geq 3$, and $(n, d) \neq (1, 3)$. Then, according to Theorem **1**.3.12, there exists an open and dense subset $V \subset |\mathcal{O}(d)|_{\mathrm{sm}}$ such that $\mathrm{Aut}(X, \mathcal{O}_X(1)) = \{\mathrm{id}\}$ for all $[X] \in V$. We may choose V to be invariant under G. Then there exists a universal family
$$\bar{\mathcal{X}} \longrightarrow \bar{V} \subset M_{d,n}$$
over the dense open subset $\bar{V} := V//G \subset M_{d,n}$. Explicitly, set $\bar{\mathcal{X}} := \mathcal{X}_V/G$. It would be useful to have control over the closed set $M_{d,n} \setminus \bar{V}$, e.g. to know its codimension, cf. Remark **1**.3.14.
(iii) Luna's étale slice theorem can be applied and yields the following: For any $x := [X] \in |\mathcal{O}(d)|_{\mathrm{sm}}$ there exists a G_x-invariant smooth locally closed subscheme $[X] \in S \subset |\mathcal{O}(d)|_{\mathrm{sm}}$, the *slice* through $[X]$, such that both natural morphisms
$$S \times^{G_x} G \longrightarrow |\mathcal{O}(d)|_{\mathrm{sm}} \text{ and } S/G_x \longrightarrow M_{d,n}$$
are étale. The morphism $S \longrightarrow S/G_x$ is finite and over S a 'universal' family exists, the pull-back of $\mathcal{X} \longrightarrow |\mathcal{O}(d)|_{\mathrm{sm}}$. In this sense, universal families exist étale locally over appropriate finite covers. See for example [99] for more on Luna's étale slice theorem.
(iv) Universal family may not even exist in formal neighbourhoods of points $[X] \in M_{d,n}$. Using the notation in Section **1**.3.3, for any X the restriction of the moduli functor $\mathcal{M}_{d,n}$ to $(\mathrm{A}rt/k) \hookrightarrow (\mathrm{S}ch/k)^o$, $A \longmapsto \mathrm{Spec}(A)$, is the union of all
$$F_X \simeq F_{X,\mathcal{O}_X(1)}$$
(under the numerical assumptions of Proposition **1**.3.9). This yields a finite morphism $\mathrm{Def}(X, \mathcal{O}_X(1)) \simeq \mathrm{Def}(X) \longrightarrow M_{d,n}$ onto the formal neighbourhood of $[X] \in$

$M_{d,n}$, which is in fact the quotient by Aut(X). Furthermore, over Def$(X, \mathcal{O}_X(1)) \simeq$ Def(X) there does exist a 'universal' family. This is a formal variant of (iii).

(v) Finally, using finite level structures, to be explained later, there exists a finite morphism $\tilde{M}_{d,n} \longrightarrow M_{d,n}$ with a 'universal' family $\tilde{\mathcal{X}} \longrightarrow \tilde{M}_{d,n}$.

Remark 1.15. The non-existence of a universal family or, equivalently, the possibility of non-trivial automorphisms, is also responsible for the difference between the field of moduli and the (or, rather, a) field of definition. This is expressed by saying that for non-closed fields k the map $\mathcal{M}_{d,n}(k) \longrightarrow M_{d,n}(k)$ is usually not bijective.

To make this precise, let $X \subset \mathbb{P}^{n+1}_{\bar{k}}$ be a hypersurface of degree d and $[X] \in M_{d,n}(\bar{k})$ the corresponding closed point in the moduli space. The moduli space $M_{d,n}$ is defined over the ground field k and so $[X] \in M_{d,n}$ has a residue field $k \subset k_{[X]} \subset \bar{k}$, the *field of moduli* of X, which is finite over k. However, X may not be defined over its field of moduli $k_{[X]}$, but only over some finite extension $k_{[X]} \subset k_X$ of it (which is not necessarily unique), i.e. there exists a variety X_o over k_X such that $X \simeq X_o \times_{k_X} \bar{k}$. Moreover, for all $\sigma \in \mathrm{Aut}(k_X/k_{[X]})$ there exists a polarized automorphism $\varphi_\sigma \colon X_o^\sigma \xrightarrow{\sim} X_o$ over k_X. In fact, $k_{[X]}$ is the fixed field of all $\sigma \in \mathrm{Aut}(\bar{k}/k)$ with $X^\sigma \simeq X$.

The isomorphisms φ_σ do not necessarily define a descent datum, as X_o may have non-trivial automorphisms. But if Aut(X) is trivial, then indeed $k_{[X]}$ is a field of definition (which in this case is unique), so $k_{[X]} = k_X$. As a consequence, one finds that (at least) over the open subset $\bar{V} \subset M_{d,n}$ of hypersurfaces without isomorphisms the field of definition and the field of moduli coincide for $[X] \in \bar{V}$, i.e. X is defined over the residue field $k_{[X]}$ of $[X] \in M_{d,n}$.

2 Stack

3 Period approach

3

Fano varieties of lines

With any (cubic) hypersurface $X \subset \mathbb{P}^{n+1}$ one associates its Fano variety of lines $F(X)$ or, more generally, of m-planes, contained in X. For a smooth cubic surface $S \subset \mathbb{P}^3$ the Fano variety $F(S)$ consists of 27 reduced points corresponding to the 27 lines contained in S. In higher dimensions, the Fano varieties $F(X)$ are even more interesting and have become a central topic of study in the theory of cubic hypersurfaces, especially in dimension three and four.

The classical references for Fano varieties of lines and planes are [4, 8]. For cubic hypersurfaces many arguments simplify dramatically and we will restrict to those whenever this is the case. For enumerative aspects we recommend [4, 59].

1 Construction and infinitesimal behaviour

We shall begin with an outline of the techniques that go into the construction of the Fano variety of linear subspaces $\mathbb{P}^m \subset \mathbb{P}^{n+1}$ contained in a given projective variety $X \subset \mathbb{P}^{n+1}$. The main tool is Grothendieck's Quot-scheme, which also allows one to gain information about the tangent space of the Fano variety at a point corresponding to $\mathbb{P}^m \subset X$.

1.1 Let k be an arbitrary field and $X \subset \mathbb{P} := \mathbb{P}^{n+1}_k$ be an arbitrary subvariety and $0 \leq m \leq n+1$. Then consider the *Fano functor*

$$\underline{F}(X,m)\colon (Sch/k)^o \longrightarrow (Sets)$$

that sends a k-scheme T (of finite type) to the set of all T-flat closed subschemes $L \subset T \times X$ such that all fibres $L_t \subset X_{k(t)} \subset \mathbb{P}_{k(t)}$ are linear subspaces of dimension m. We shall mostly be interested in the case of lines, i.e. $m = 1$, and will write $\underline{F}(X) := \underline{F}(X,1)$ in this case.

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Remark 1.1. Here are a few examples and easy observations.

(i) For $X = \mathbb{P}$, one obtains the the *Grassmann functor*

$$\underline{F}(\mathbb{P}, m) = \underline{\mathbb{G}}(m, \mathbb{P}).$$

(ii) For $m = 0$, the functor $\underline{F}(X, 0)$ is the functor of points h_X.

(iii) For closed subschemes $X \subset X' \subset \mathbb{P}$ there is a natural inclusion

$$\underline{F}(X, m) \subset \underline{F}(X', m) \subset \underline{F}(\mathbb{P}, m) = \underline{\mathbb{G}}(m, \mathbb{P}).$$

(iv) Let $P_m(\ell) \coloneqq \binom{m+\ell}{\ell}$ and $\underline{\mathrm{Hilb}}^{P_m}(X)$ be the Hilbert functor that sends a k-scheme T to the set of all T-flat closed subschemes $Z \subset T \times X$ with fibrewise Hilbert polynomial $\chi(Z_t, \mathcal{O}_{Z_t}(\ell)) = P_m(\ell)$. Then

$$\underline{F}(X, m) = \underline{\mathrm{Hilb}}^{P_m}(X).$$

Here, we leave it as an exercise to show that any closed subvariety $Z \subset \mathbb{P}$ with Hilbert polynomial P_m is indeed a linear subspace $\mathbb{P}^m \subset \mathbb{P}$.

Theorem 1.2. *The Fano functor $\underline{F}(X, m)$ is represented by a projective k-scheme $F(X, m)$, the* Fano variety of m-planes *in $X \subset \mathbb{P}$.*

There are various ways to argue, but in the end, the proof always comes down to the representability of the Grassmann functor.

• Use (iv) in the above remark and the representability of $\underline{\mathrm{Hilb}}^P(X)$ (for arbitrary projective X and P) by the Hilbert scheme $\mathrm{Hilb}^P(X)$. This in turn is a special case of the representability of the Grothendieck Quot-functor $\underline{\mathrm{Quot}}^P_{X/\mathcal{E}}$ of quotients $\mathcal{E} \twoheadrightarrow \mathcal{F}$ of Hilbert polynomial P. Indeed, $\underline{\mathrm{Hilb}}^P(X) \simeq \underline{\mathrm{Quot}}^P_{X/\mathcal{O}_X}$. Recall that the existence of the Quot-scheme is eventually reduced to the existence of the Grassmann variety, cf. [88, Ch. 2.2].

• The inclusion $\underline{F}(X, m) \subset \underline{\mathbb{G}}(m, \mathbb{P})$ in (iii) above describes a closed sub-functor. As the Grassmann functor $\underline{\mathbb{G}}(m, \mathbb{P})$ is representable by the Grassmann variety $\mathbb{G}(m, \mathbb{P})$, $\underline{F}(X, m)$ is represented by a closed subscheme $F(X, m) \subset \mathbb{G}(m, \mathbb{P})$.

Let us spell out the second approach a bit further. But first recall that $\mathbb{G}(m, \mathbb{P}^{n+1}) \simeq \mathrm{Gr}(m+1, n+2)$, where $\mathrm{Gr}(m+1, n+2)$ is the Grassmann variety of linear subspaces of $k^{\oplus n+2}$ of dimension $m+1$ or, in other words,

$$\mathbb{G}(m, \mathbb{P}^{n+1}) \simeq \mathrm{Gr}(m+1, n+2) \simeq \mathrm{Quot}^{P \equiv n+1-m}_{\mathrm{Spec}(k)/V},$$

where $V = k^{\oplus n+2}$ and so $\mathbb{P} = \mathbb{P}^{n+1} = \mathbb{P}(V)$. The isomorphism between the corresponding functors $\underline{\mathrm{Gr}}(m+1, n+2) \xrightarrow{\sim} \underline{\mathbb{G}}(m, \mathbb{P}^{n+1})$ is given by $[\mathcal{G} \subset V \otimes \mathcal{O}_T] \longmapsto \mathbb{P}(\mathcal{G})$.

Also recall that $\mathbb{G}(m, \mathbb{P})$ is an irreducible, smooth, projective variety of dimension

$$\dim(\mathbb{G}(m, \mathbb{P})) = (m+1) \cdot (n+1-m).$$

It is naturally embedded into $\mathbb{P}(\bigwedge^{n+1} V)$ via the Plücker embedding

$$\mathbb{G}(m,\mathbb{P}) \hookrightarrow \mathbb{P}\left(\bigwedge^{m+1} V\right),\ L = \mathbb{P}(W) \longmapsto [\det(W)].$$

Under this embedding, $\mathcal{O}(1)|_{\mathbb{G}} \simeq \bigwedge^{m+1}(\mathcal{S}^*)$. Here, $\mathcal{S}$ is the universal subbundle, which sits in the universal exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow V \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{1.1}$$

The universal family of m-planes over $\mathbb{G}(m,\mathbb{P})$ is the $\mathbb{P}^m$-bundle associated with $\mathcal{S}$:

$$p\colon \mathbb{L}_{\mathbb{G}} \coloneqq \mathbb{P}(\mathcal{S}) \longrightarrow \mathbb{G}(m,\mathbb{P}).$$

The inclusion $\mathcal{S} \subset V \otimes \mathcal{O}_{\mathbb{G}}$ corresponds to the natural embedding

$$\mathbb{L}_{\mathbb{G}} \subset \mathbb{G}(m,\mathbb{P}) \times \mathbb{P}^{n+1}.$$

The induced projection $\mathbb{L}_{\mathbb{G}} \longrightarrow \mathbb{P}^{n+1}$ satisfies $q^*\mathcal{O}_{\mathbb{P}}(1) \simeq \mathcal{O}_p(1)$.

Assume now that $X \subset \mathbb{P}$ is a hypersurface defined by $F \in H^0(\mathbb{P},\mathcal{O}(d)) \simeq S^d(V^*)$. Dualizing (1.1) and taking symmetric powers yields a natural surjection

$$S^d(V^*) \otimes \mathcal{O}_{\mathbb{G}} \twoheadrightarrow S^d(\mathcal{S}^*)$$

and hence a map $S^d(V^*) \longrightarrow H^0(\mathbb{G}, S^d(\mathcal{S}^*))$. Let $s_F \in H^0(\mathbb{G}, S^d(\mathcal{S}^*))$ denote the image of $F \in S^d(V^*) = k[x_0,\ldots,x_{n+1}]_d$ under this map. Then the Fano variety of m-planes on X is the closed subvariety of the Grassmann variety defined as the zero-locus of s_F:

$$F(X,m) = V(s_F) \subset \mathbb{G}(m,\mathbb{P}).$$

In particular, whenever $F(X,m)$ is non-empty, then

$$\begin{aligned}\dim(F(X,m)) &\geq \dim(\mathbb{G}(m,\mathbb{P})) - \operatorname{rk}(S^d(\mathcal{S}^*))\\ &= (m+1)\cdot(n+1-m) - \binom{m+d}{d}.\end{aligned} \tag{1.2}$$

Moreover, in case of equality the class of $F(X,m)$, in the Chow ring or just in cohomology, can be expressed as

$$[F(X,m)] = \mathrm{c}_r(S^d(\mathcal{S}^*)), \tag{1.3}$$

where $r = \operatorname{rk}(S^d(\mathcal{S}^*)) = \binom{m+d}{d}$. We will come back to this later, see Section 3.2. For more general subvarieties $X \subset \mathbb{P}$ the argument is similar: If $X = \bigcap V(F_i)$, then $F(X,m) = \bigcap V(s_{F_i})$, where $s_{F_i} \in H^0(\mathbb{G}, S^{d_i}(\mathcal{S}^*))$, $d_i = \deg(F_i)$.

We shall denote the universal family of m-planes over $F(X,m)$ by

$$\mathbb{L} \longrightarrow F(X,m),$$

which is nothing but the restriction $\mathbb{L} = \mathbb{L}_{\mathbb{G}}|_{F(X,m)} = \mathbb{P}(\mathcal{S}|_{F(X,m)})$ of $\mathbb{L}_{\mathbb{G}}$ to $F(X,m) \subset$

$\mathbb{G}(m,\mathbb{P})$. The composition of this natural inclusion with the Plücker embedding $\mathbb{G}(m,\mathbb{P})$ yields the Plücker embedding of the Fano variety of X

$$F(X,m) \hookrightarrow \mathbb{G}(m,\mathbb{P}) \hookrightarrow \mathbb{P}\left(\bigwedge^{m+1} V\right).$$

The restriction of the hyperplane line bundle $\mathcal{O}(1)|_{F(X,m)} \simeq \bigwedge^{m+1}(\mathcal{S}^*|_{F(X,m)})$ is called the *Plücker polarization* and its first Chern class will be denoted

$$g = \mathrm{c}_1(\mathcal{S}^*|_{F(X,m)}) \in \mathrm{CH}^1(F(X,m)), \tag{1.4}$$

often also considered as a class in $H^2(F(X,m),\mathbb{Z})(1)$.

The following universal variant will be very useful. Consider the universal hypersurface $\mathcal{X} \longrightarrow |\mathcal{O}(d)| = \mathbb{P}^N$, cf. Section **1**.2. Then consider

$$\underline{F}(\mathcal{X},m)\colon (Sch/|\mathcal{O}(d)|)^o \longrightarrow (Set)$$

that sends $T \longrightarrow |\mathcal{O}(d)|$ to the set of all T-flat closed subschemes $L \subset \mathcal{X}_T \subset T \times \mathbb{P}$ parametrizing m-planes $\mathbb{P}^m \subset \mathbb{P}$ in the fibres of $\mathcal{X} \longrightarrow |\mathcal{O}(d)|$. Using the relative version of the Quot-scheme or of the Grassmannian, one finds that $\underline{F}(\mathcal{X},m)$ is represented by a projective scheme

$$F(\mathcal{X},m) \longrightarrow |\mathcal{O}(d)|.$$

By functoriality the fibre over $[X] \in |\mathcal{O}(d)|$ is $F(X,m)$ and one should think of $F(\mathcal{X},m)$ as parametrizing pairs $(L \subset X)$ of m-planes contained in hypersurfaces of degree d. As in the absolute case, $F(\mathcal{X},m)$ can be realized as a closed subscheme of the relative Grassmanian

$$F(\mathcal{X},m) = V(s_G) \subset |\mathcal{O}(d)| \times \mathbb{G}(m,\mathbb{P}),$$

where $s_G \in H^0(|\mathcal{O}(d)| \times \mathbb{G}, \mathcal{O}(1) \boxtimes S^d(\mathcal{S}^*))$ is the image of the universal equation $G \in H^0(|\mathcal{O}(d)| \times \mathbb{P}, \mathcal{O}(1) \boxtimes \mathcal{O}_{\mathbb{P}}(d)) = H^0(|\mathcal{O}(d)|, \mathcal{O}(1)) \otimes S^d(V^*)$ under the map

$$\mathcal{O}(1) \boxtimes (S^d(V^*) \otimes \mathcal{O}_{\mathbb{G}}) \twoheadrightarrow \mathcal{O}(1) \boxtimes S^d(\mathcal{S}^*). \tag{1.5}$$

Let us now look at the other projection $\pi\colon F(\mathcal{X},m) \longrightarrow \mathbb{G}(m,\mathbb{P})$. From the description of $F(\mathcal{X},m)$ as $V(s_G) \subset |\mathcal{O}(d)| \times \mathbb{G}(m,\mathbb{P})$ one deduces the isomorphism

$$F(\mathcal{X},m) \simeq \mathbb{P}(\mathcal{K}) \longrightarrow \mathbb{G}(m,\mathbb{P}).$$

Here, $\mathcal{K} \coloneqq \mathrm{Ker}\left(S^d(V^*) \otimes \mathcal{O}_{\mathbb{G}} \longrightarrow S^d(\mathcal{S}^*)\right)$. This allows one to compute the dimension of the universal Fano scheme.

Proposition 1.3. *The relative Fano variety $F(\mathcal{X},m)$ of m-planes in hypersurfaces of degree d in $\mathbb{P}$ is an irreducible, smooth, projective variety of dimension*

$$\dim(F(\mathcal{X},m)) = (m+1)\cdot(n+1-m) + \binom{n+1+d}{d} - \binom{m+d}{d} - 1. \qquad \square$$

The first part of the following immediate consequence confirms (1.2).

Corollary 1.4. *If for an arbitrary hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d the Fano variety $F(X,m)$ is not empty, then*

$$\begin{aligned}\dim(F(X,m)) &\geq \dim(F(\mathcal{X},m)) - \dim|\mathcal{O}(d)| \\ &= (m+1)\cdot(n+1-m) - \binom{m+d}{d}.\end{aligned} \tag{1.6}$$

Moreover, equality holds in (1.6) for generic $X \in |\mathcal{O}(d)|$ unless $F(X)$ is empty. □

The case of interest to us is $d = 3$ and $m = 1$. In this case, (1.6) becomes

$$\dim(F(X)) \geq 2n - 4$$

for non-empty $F(X)$. Using deformation theory, we shall see that $F(X)$ really is non-empty of dimension $2n-4$ for all smooth cubic hypersurfaces of dimension at least two. This shall be explained next.

Remark 1.5. Also relevant for us is the case $d = 3$ and $m = 2$. Then $\dim(F(X,2)) \geq 3n - 13$ as soon as $F(X) \neq \emptyset$. The right hand side is non-negative for $n \geq 5$. For $n < 5$ one can conclude that $F(X,2) = \emptyset$ for generic $X \in |\mathcal{O}(3)|$. So, for example, the generic cubic threefold and the generic cubic fourfold do not contain planes.

1.2 Any further study of the Fano variety of m-planes needs at least some amount of deformation theory. Let us begin with a recollection of some classical facts and a reminder of the main arguments. Most of the following can be found in standard textbooks, e.g. [61, 79, 88, 97, 135]. As smoothness is preserved under base change, we may assume for simplicity that k is algebraically closed.

As the Fano variety of m-planes is a special case of the Hilbert scheme which in turn is a special case of the Quot-scheme, let us start with the latter.

Let $q \coloneqq [\mathcal{E} \twoheadrightarrow \mathcal{F}_0] \in \mathrm{Quot} = \mathrm{Quot}_{X/\mathcal{E}}$ be a closed point in the Quot-scheme of quotients of a given coherent sheaf $\mathcal{E}$ on X. We denote the kernel by $\mathcal{K}_0 \coloneqq \mathrm{Ker}(\mathcal{E} \twoheadrightarrow \mathcal{F}_0)$. Then there exists a natural isomorphism [73, Exp. 221, Sec. 5]

$$T_q\mathrm{Quot} \simeq \mathrm{Hom}(\mathcal{K}_0, \mathcal{F}_0).$$

Moreover, if $\mathrm{Ext}^1(\mathcal{K}_0, \mathcal{F}_0) \simeq 0$, then Quot is smooth at q.

Let us quickly recall the main arguments for both statements. See [88, Ch. 2.2] or [61, Ch. 6.4] for technical details. By the functorial property of the Quot-scheme, the tangent space $T_q\mathrm{Quot}$ parametrizes quotients $\mathcal{E}_{k[\varepsilon]} \twoheadrightarrow \mathcal{F}$ of $\mathcal{E}_{k[\varepsilon]} = \mathcal{E} \boxtimes k[\varepsilon]$ on $X_\varepsilon \coloneqq X \times \mathrm{Spec}(k[\varepsilon])$ which are flat over $k[\varepsilon]$ and the restriction of which to $X \subset X_{k[\varepsilon]}$ gives back q. It is convenient to study the following more general situation. Assume an extension $q_A = [\mathcal{E}_A \twoheadrightarrow \mathcal{F}]$ of $q = [\mathcal{E} \twoheadrightarrow \mathcal{F}_0]$ to $X_A = X \times \mathrm{Spec}(A)$ has been found already. Here, A is a local Artinian k-algebra with residue field k. Consider a small extension

$A' \twoheadrightarrow A = A'/I$, i.e. a local Artinian k-algebra A' with maximal ideal $\mathfrak{m}_{A'}$ such that $I \cdot \mathfrak{m}_{A'} = 0$. Any further extension of q_A to $q_{A'} = [\mathcal{E}_{A'} \twoheadrightarrow \mathcal{F}']$ leads to a commutative diagram of vertical and horizontal short exact sequences of the form

$$\begin{array}{ccccc}
\mathcal{K}_0 \otimes_k I & \hookrightarrow & \mathcal{E}_0 \otimes_k I & \twoheadrightarrow & \mathcal{F}_0 \otimes_k I \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}' & \hookrightarrow & \mathcal{E}_{A'} & \twoheadrightarrow & \mathcal{F}' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K} & \hookrightarrow & \mathcal{E}_A & \twoheadrightarrow & \mathcal{F}.
\end{array}$$

Here, one uses that $\mathcal{F}' \otimes_{A'} I \simeq \mathcal{F}_0 \otimes_k I$, etc. Next observe that

$$\mathcal{F}' \simeq \mathrm{Coker}\,(\psi\colon \mathcal{K} \longrightarrow \mathcal{E}_{A'}/(\mathcal{K}_0 \otimes_k I)),$$

where ψ is the obvious map . Furthermore, the composition of ψ with the projection $\varphi\colon \mathcal{E}_{A'}/(\mathcal{K}_0 \otimes_k I) \twoheadrightarrow \mathcal{E}_A$ is the given inclusion $\mathcal{K} \hookrightarrow \mathcal{E}_A$. Conversely, one can define an extension $\mathcal{F}'$ in this way if the short exact sequence of coherent sheaves on X_A

$$0 \longrightarrow \mathcal{F}_0 \otimes_k I \longrightarrow \varphi^{-1}(\mathcal{K}) \longrightarrow \mathcal{K} \longrightarrow 0 \tag{1.7}$$

is split. The class of (1.7) is an element

$$\mathfrak{o} \in \mathrm{Ext}^1_{X_A}(\mathcal{K}, \mathcal{F}_0 \otimes_k I) \simeq \mathrm{Ext}^1_X(\mathcal{K}_0, \mathcal{F}_0) \otimes_k I,$$

where we use flatness of $\mathcal{K}$ for the isomorphism. If this class is zero and a split has been chosen, then all other extensions differ by elements in

$$\mathrm{Hom}_{X_{A'}}(\mathcal{K}, \mathcal{F}_0 \otimes_k I) \simeq \mathrm{Hom}_X(\mathcal{K}_0, \mathcal{F}_0) \otimes I.$$

Hence, Quot is (formally) smooth at $q = [\mathcal{E} \twoheadrightarrow \mathcal{F}_0]$ if $\mathrm{Ext}^1(\mathcal{K}_0, \mathcal{F}_0) = 0$ and the possible extension of q to $X \times \mathrm{Spec}(k[\varepsilon])$ are parametrized by $\mathrm{Hom}(\mathcal{K}_0, \mathcal{F}_0)$.

Applied to $\mathrm{Hilb}(X) \simeq \mathrm{Quot}_{X/\mathcal{O}_X}$, one finds that the tangent space at $[Z] \in \mathrm{Hilb}(X)$ is given by

$$T_{[Z]}\mathrm{Hilb}(X) \simeq \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$$

and that $\mathrm{Hilb}(X)$ is smooth at $[Z]$ if $\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) = 0$. Now assume that $Z \subset X$ is a regular embedding with normal bundle $\mathcal{N}_{Z/X}$. Then $\mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \simeq H^0(Z, \mathcal{N}_{Z/X})$ and the local to global spectral sequence $E_2^{p,q} = H^p(Z, \mathcal{E}xt^q(\mathcal{I}_Z, \mathcal{O}_Z)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{I}_Z, \mathcal{O}_Z)$ provides an exact sequence

$$H^1(Z, \mathcal{N}_{Z/X}) \hookrightarrow \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) \longrightarrow H^0(Z, \mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}_Z)) \longrightarrow H^2(Z, \mathcal{N}_{Z/X}).$$

Furthermore, the local obstructions in $\mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}_Z)$ to deform a smooth subvariety are all trivial. Hence, the obstrucion space $\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z)$ is indeed isomorphic to $H^1(Z, \mathcal{N}_{Z/X})$.

Example 1.6. Let us test this in the case of $\mathbb{G}(m,\mathbb{P}) = \mathrm{Hilb}^{P_m}(\mathbb{P})$. We know that at $[L = \mathbb{P}(W)] \in \mathbb{G} = \mathbb{G}(m,\mathbb{P})$ the tangent space $T_{[L]}\mathbb{G}$ is isomorphic to $\mathrm{Hom}(W, V/W)$ or, more globally, that $\mathcal{T}_{\mathbb{G}} \simeq \mathcal{H}om(\mathcal{S},\mathcal{Q})$ with $\mathcal{S}$ and $\mathcal{Q}$ as in (1.1). On the other hand, $T_{[L]}\mathrm{Hilb}(\mathbb{P}) \simeq \mathrm{Hom}(\mathcal{I}_L,\mathcal{O}_L)$. And indeed, there is a natural isomorphism between the two descriptions obtained by applying $\mathrm{Hom}(\ ,\mathcal{O}_L)$ to the Koszul complex

$$\cdots \longrightarrow \bigwedge^2 (V/W)^* \otimes \mathcal{O}(-2) \longrightarrow (V/W)^* \otimes \mathcal{O}(-1) \longrightarrow \mathcal{I}_L \longrightarrow 0,$$

associated with the equations $(V/W)^* \hookrightarrow V^*$ for $L = \mathbb{P}(W)$, and by observing that $\mathrm{Hom}((V/W)^* \otimes \mathcal{O}(-1),\mathcal{O}_L) \simeq (V/W) \otimes H^0(L,\mathcal{O}_L(1)) \simeq \mathrm{Hom}(W, V/W)$.

Applied to the case $[L] \in F(X,m) = \mathrm{Hilb}^{P_m}(X)$ for an m-plane $L \subset X$ in a variety $X \subset \mathbb{P}$ which is smooth (along L), one obtains the following result.

Proposition 1.7. *Let $L \subset X$ be an m-plane contained in a variety $X \subset \mathbb{P}^{n+1}$ which is smooth along L. Then the tangent space $T_{[L]}F(X,m)$ of the Fano variety $F(X,m)$ at the point $[L] \in F(X,m)$ corresponding to L is naturally isomorphic to $H^0(L,\mathcal{N}_{L/X})$:*

$$T_{[L]}F(X,m) \simeq H^0(L,\mathcal{N}_{L/X}).$$

Moreover, if $H^1(L,\mathcal{N}_{L/X}) = 0$, then $F(X,m)$ is smooth at $[L]$. □

Remark 1.8. There exists a relative version of the above. Assume $\mathcal{X} \longrightarrow S$ is a projective morphism over a locally Noetherian base S and $\mathcal{E}$ a coherent sheaf on $\mathcal{X}$. Then the relative Quot-scheme $\pi\colon \mathrm{Quot}_{\mathcal{X}/S/\mathcal{E}} \longrightarrow S$ parametrizes T-flat quotients $\mathcal{E}_T \twoheadrightarrow \mathcal{F}_0$ on $\mathcal{X} \times_S T$ for all S-schemes T. It is a locally projective S-scheme with fibres $\pi^{-1}(s) = \mathrm{Quot}_{X/\mathcal{E}|_X}$, where $X = \mathcal{X}_s$. In particular, the relative tangent space at a closed point $q = [\mathcal{E}|_X \twoheadrightarrow \mathcal{F}] \in \mathrm{Quot}_{X/\mathcal{E}|_X} \subset \mathrm{Quot}_{\mathcal{X}/S/\mathcal{E}} \longrightarrow S$ is the tangent space of the fibre $\pi^{-1}(s) = \mathrm{Quot}_{X/\mathcal{E}|_X}$, i.e.

$$T_q\pi^{-1}(s) \simeq T_q\mathrm{Quot}_{X/\mathcal{E}|_X} \simeq \mathrm{Hom}_X(\mathcal{K}_0,\mathcal{F}_0),$$

where $\mathcal{K}_0 = \mathrm{Ker}(\mathcal{E}|_X \twoheadrightarrow \mathcal{F}_0)$. More interestingly, if locally (in X) no obstruction occur to deform $\mathcal{E}|_X \twoheadrightarrow \mathcal{F}_0$ and $H^1(X_s, \mathcal{H}om(\mathcal{K}_0,\mathcal{F}_0)) = 0$, then π is smooth at q.

This applies to our situation. Consider the universal family $\mathcal{X} \longrightarrow |\mathcal{O}(d)|$ of hypersurfaces of degree d and let $F(\mathcal{X},m) \longrightarrow |\mathcal{O}(d)|$ be the associated family of Fano varieties of m-planes in the fibres. Then the morphism $F(\mathcal{X},m) \longrightarrow |\mathcal{O}(d)|$ is smooth at a point $[L]$ corresponding to an m-plane $L \subset X$ in a smooth (along L) fibre $X = \mathcal{X}_s$ if $H^1(X,\mathcal{N}_{L/X}) = 0$.

To compute the normal bundle $\mathcal{N}_{L/X}$ of an m-plane $\mathbb{P}^m \simeq L \subset X$ we use the short exact sequence

$$0 \longrightarrow \mathcal{N}_{L/X} \longrightarrow \mathcal{N}_{L/\mathbb{P}}|_L \longrightarrow \mathcal{N}_{X/\mathbb{P}}|_L \longrightarrow 0 \tag{1.8}$$

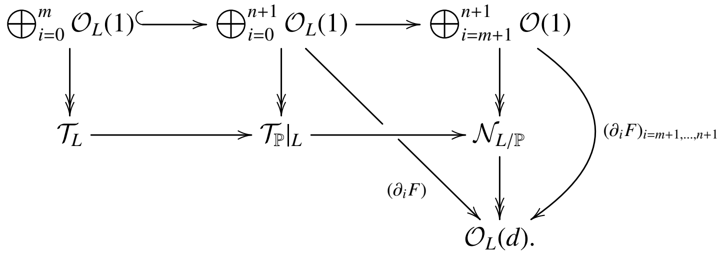
of locally free sheaves on $L \simeq \mathbb{P}^m$, where we again assume that X is smooth or at least smooth along L.

The normal bundle $\mathcal{N}_{L/\mathbb{P}}$ can be readily computed from a comparison of the Euler sequences for $L \simeq \mathbb{P}^m$ and $\mathbb{P} = \mathbb{P}^{n+1}$. One finds $\mathcal{N}_{L/\mathbb{P}} \simeq \mathcal{O}(1)^{\oplus n+1-m}$. More precisely, if $L = \mathbb{P}(W) \subset \mathbb{P} = \mathbb{P}(V)$, then $\mathcal{N}_{L/\mathbb{P}} \simeq \mathcal{O}_L(1) \otimes (V/W)$.

If now $X \subset \mathbb{P}$ is a smooth (at least along L) hypersurface of degree d, then the exact sequence (1.8) becomes

$$0 \longrightarrow \mathcal{N}_{L/X} \longrightarrow \mathcal{O}_L(1)^{\oplus n+1-m} \longrightarrow \mathcal{O}_L(d) \longrightarrow 0. \tag{1.9}$$

After the appropriate coordinate change, the surjection can be understood as given by $\partial_i F$, $i = m+1, \dots, n+1$. Indeed, assume that $L = V(x_{m+1}, \dots, x_{n+1})$. Then



Observe that (1.9) has the following numerical consequences

$$\det(\mathcal{N}_{L/X}) = \mathcal{O}_L((n+1-m)-d), \quad \mathrm{rk}(\mathcal{N}_{L/X}) = n-m, \text{ and}$$

$$\begin{aligned}\chi(\mathcal{N}_{L/X}) &= \chi(\mathcal{O}_L(1)) \cdot (n+1-m) - \chi(\mathcal{O}_L(d)) \\ &= (m+1)\cdot(n+1-m) - \binom{m+d}{d},\end{aligned}$$

which equals the right hand side of (1.6).

For the case $m = 1$ and $d = 3$ this allows one to classify all normal bundles.

Lemma 1.9. *Let $L \subset X$ be a line in a smooth (along L) cubic hypersuface $X \subset \mathbb{P}^{n+1}$. Then $\mathcal{N}_{L/X} \simeq \mathcal{O}_L(a_1) \oplus \cdots \oplus \mathcal{O}_L(a_{n-1})$, $a_1 \geq \cdots \geq a_{n-1}$, with*

$$(a_1, \dots, a_{n-1}) = \begin{cases} (1, \dots, 1, 0, 0) & \text{or} \\ (1, \dots, 1, 1, -1). \end{cases}$$

Proof As $L \simeq \mathbb{P}^1$, any locally free sheaf is isomorphic to a direct sum of invertible sheaves, so $\mathcal{N}_{L/X} \simeq \bigoplus \mathcal{O}_L(a_i)$ with $\sum a_i = n-3$. On the other hand, the inclusion $\mathcal{N}_{L/X} \subset \mathcal{O}_L(1)^{\oplus n}$ yields $a_i \leq 1$. This immediately proves the result. □

Note that for $n = 2$ there is in fact only one possibility, namely $\mathcal{N}_{L/X} \simeq \mathcal{O}_L(-1)$. For

$n > 2$ there are two cases and both can be geometrically realized, see [59, Prop. 6.30]. Obviously, $(1, \dots, 1, 0, 0)$ describes the generic case. Equivalently,

$$F_2 \coloneqq \{ [L] \mid h^0(L, \mathcal{N}_{L/X}(-1)) \geq n-2 \} \subset F(X)$$

is a closed subscheme and, moreover, $\dim(F_2) \leq n-2$ according to [34, Cor. 7,6]. Lines with $(a_i) = (1, \dots, 1, 0, 0)$ and $(a_i) = (1, \dots, 1, 1, -1)$ are called *lines of the first type* and *of the second type*, respectively.

Exercise 1.10. Consider the Gauss map for the cubic hypersurface $X = V(F) \subset \mathbb{P}^{n+1}$

$$\gamma_X \colon \mathbb{P}^{n+1} \twoheadrightarrow \mathbb{P}^{n+1},\ x \longmapsto [\partial_0 F(x) : \cdots : \partial_{n+1} F(x)].$$

Show that a line $L \subset X$ is of the first type if and only if $\gamma_X \colon L \xrightarrow{\sim} \gamma_X(L)$ is an isomorphism onto a smooth plane conic. It is of the second type if $\gamma_X \colon L \twoheadrightarrow \gamma_X(L)$ is a degree two covering of a line. See [34, Sec. 6].

The lemma also shows that for a line $\mathbb{P}^1 \simeq L \subset X$ contained in a smooth (along L) cubic hypersurface $X \subset \mathbb{P}^{n+1}$ the Fano variety $F(X)$ is smooth of dimension $2n-4$ at $[L] \in F(X)$. Indeed, smoothness follows from Proposition 1.7 and $H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0$ for $a = 0, 1$. For the dimension observe that $h^0(L, \bigoplus \mathcal{O}(a_i)) = 2n-4$ in the two cases $(a_i) = (1, \dots, 1, 0, 0)$ and $(a_i) = (1, \dots, 1, 1, -1)$.

Corollary 1.11. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface, $n \geq 2$. Then the Fano variety of lines $F(X)$ is smooth, projective, and of dimension $2n-4$.*

Proof The preceding discussion essentially proves the claim. It remains to show that $F(X)$ is non-empty. For this consider the Fermat cubic $X_0 = V(x_0^3 + \cdots + x_{n+1}^3)$ which is smooth for $\mathrm{char}(k) \neq 3$. Then clearly the line $L_0 \coloneqq V(x_0 + x_1, x_2 + x_3, x_4, \dots, x_{n+1})$ is contained in X_0 and hence $F(X_0) \neq \emptyset$. For $\mathrm{char}(k) = 3$ with $\xi = \sqrt{-1} \in k$ one may take $X_0 \coloneqq V(\sum_0^n x_i x_{i+1}^2 + x_0^3)$, see Section **1**.2.2, and the line $L_0 \coloneqq V(x_0 - \xi\, x_1, x_2, \dots, x_{n+1})$. Of course, for the assertion one may assume $k = \bar{k}$.

According to Remark 1.8, the vanishing $H^1(L_0, \mathcal{N}_{L_0/X}) = 0$ not only proves that the fibre $F(X_0)$ of $F(\mathcal{X}) \twoheadrightarrow |\mathcal{O}(3)|$ over $[X_0] \in |\mathcal{O}(d)|$ is smooth at $[L_0]$ but that in fact the morphism is smooth at $[L_0]$. In particular, the projective morphism $F(\mathcal{X}) \twoheadrightarrow |\mathcal{O}(d)|$ is surjective which proves $F(X) \neq \emptyset$ for all cubics. □

The existence of lines in cubic hypersurfaces has the following immediate consequence. The assumption on the field can be weakened. One only needs the existence of one line contained in X.

Corollary 1.12. *For any cubic hypersurface $X \subset \mathbb{P}^{n+1}$ of dimension $n > 1$ defined over an algebraically closed field, there exists a rational dominant map of degree two*

$$\mathbb{P}^n \dashrightarrow X.$$

In particular, cubic hypersurfaces are unirational, cf. Remark **1**.*5.3.*

Proof Pick a line $L \subset X$ and consider the projectivization of the restricted tangent bundle $\mathbb{P}(\mathcal{T}_X|_L) \twoheadrightarrow L$. A point in $\mathbb{P}(\mathcal{T}_X|_L)$ is represented by a tangent vector $0 \neq v \in T_xX$, which then defines a unique line $L_v \subset \mathbb{P}$ passing through x with $T_xL \subset T_xX$ spanned by v. Then, either the line L_v is contained in X or, and this is the generic case, it is not and then L_v intersects X in a unique point $y_v \in X$ with the property that the scheme theoretic intersection $L_v \cap X$ is $2x + y_v$. Note that $y_v = x$ can occur.

This defines a rational map

$$\mathbb{P}(\mathcal{T}_X|_L) \dashrightarrow X\ ,\ v \longmapsto y_v,$$

which is regular on a dense open subset intersecting each fibre $\mathbb{P}(T_xX)$, $x \in L$, unless X contains a hyperplane. For a generic point $y \in X$ in the image, there exist at most two lines $y \in L_i$, $i = 1, 2$, that intersect L and are tangent to X at the intersection point. Indeed, as $X \cap \overline{yL}$ is a cubic in $\overline{yL} \simeq \mathbb{P}^2$ and the L_i have to pass through the intersection of the residual quadric with L. Hence, the rational map is generically of degree two and dominant. □

In Section **5**.**??** we shall describe the indeterminacy locus explicitly for $n = 3$. See Remark **1**.5.3 for an alternative argument proving unirationality of cubic hypersurfaces containing a linear subspace $\mathbb{P}^{n-2}$ and, in fact, rationally for $n \geq 7$.

Remark 1.13. There is a general expectation (conjecture of Debarre–de Jong) that for $d \leq n+1$ and $\text{char}(k) = 0$ or at least $\text{char}(k) \geq d$ the Fano variety of lines $F(X)$ is smooth of the expected dimension $2n - d - 1$. For $d \leq 6$ this has been proved in characteristic zero in [17] for which we also refer for further references. See also [59, Prop. 6.40], where the claim is reduced to the case $d = n + 1$.

Remark 1.14. Note that for $m \geq 2$ and an m-plane $L \subset X$ contained in a smooth (along L) hypersurface X of degree d such that $\mathcal{N}_{L/X} \simeq \bigoplus \mathcal{O}(a_i)$ the Fano variety $F(X, m)$ is smooth at $[L]$ of dimension $\sum h^0(\mathbb{P}^m, \mathcal{O}(a_i))$. However, in contrast to locally free sheaves on $\mathbb{P}^1$, there is a priori no reason why $\mathcal{N}_{L/X}$ on $L \simeq \mathbb{P}^m$ should be a direct sum of invertible sheaves. Also the dimension of $F(X, m)$ is harder to control as other cohomology groups $H^i(L, \mathcal{N}_{L/X})$ enter the picture.

2 Global properties and a geometric Torelli theorem

TEXT

2.1 Recall that $\det(\mathcal{S}^*) \simeq \mathcal{O}(1)|_{\mathbb{G}}$ for the Plücker embedding $\mathbb{G} \hookrightarrow \mathbb{P}(\bigwedge^{m+1} V)$. The following result is [4, Prop. 1.8].

Lemma 2.1. *For a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ the canonical bundle ω_F of the Fano variety of lines $F = F(X) \subset \mathbb{G}(1,\mathbb{P}) \hookrightarrow \mathbb{P}^{\binom{n+2}{2}-1}$ is*

$$\omega_F \simeq \mathcal{O}(4-n)|_F.$$

Proof As the Fano variety is the zero set $F = V(s_F) \subset \mathbb{G}(1,\mathbb{P})$ of a regular section $s_F \in H^0(\mathbb{G}, S^3(\mathcal{S}^*))$, the normal bundle sequence for $F \subset \mathbb{G}$ takes the form

$$0 \longrightarrow \mathcal{T}_F \longrightarrow \mathcal{T}_{\mathbb{G}}|_F \longrightarrow S^3(\mathcal{S}^*)|_F \longrightarrow 0$$

and the adjunction formula yields

$$\omega_F = \det(\mathcal{T}_F^*) \simeq \left(\omega_{\mathbb{G}} \otimes \det(S^3(\mathcal{S}^*))\right)|_F.$$

As $\mathcal{T}_{\mathbb{G}} \simeq \mathcal{H}om(\mathcal{S},\mathcal{Q})$, one has $\omega_{\mathbb{G}} \simeq \det(\mathcal{S} \otimes \mathcal{Q}^*) \simeq \det(\mathcal{S})^{n+1-m} \otimes \det(\mathcal{Q}^*)^{m+1} \simeq \mathcal{O}(-n-2)|_{\mathbb{G}}$. Thus, it remains to prove that $\det(S^3(\mathcal{S})) \simeq \det(\mathcal{S})^6$ which one deduces from a computation involving the splitting principle: Write formally $\mathcal{S} = L_0 \oplus L_1$ and observe $S^3(L_0 \oplus L_1) \simeq L_0^3 \oplus (L_0^2 \otimes L_1) \oplus (L_0 \otimes L_1^2) \oplus L_1^3$. □

Thus, for smooth cubic threefolds $X \subset \mathbb{P}^4$ the Fano variety of lines $F(X)$ is a smooth projective surface with very ample canonical bundle, in particular $F(X)$ is of general type. For smooth cubic fourfolds $X \subset \mathbb{P}^5$ the Fano variety $F(X)$ has trivial canonical bundle $\omega_F \simeq \mathcal{O}_F$ and we will later see that it is a four-dimensional hyperkähler manifold. Eventually, for $n > 4$ the Fano variety becomes a Fano variety in the sense that its anti-canonical bundle ω_X^* is (very) ample. In short:

$$\omega_{F(X)} = \begin{cases} \text{ample} & \text{if } n = 3, \\ \text{trivial} & \text{if } n = 4, \\ \text{anti-ample} & \text{if } n > 4. \end{cases}$$

We will later determine the (rational) cohomology of $F(X)$ for any smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$, cf. Section **1**.3.4. At least in characteristic zero, the positivity property of the canonical bundle $\omega_{F(X)}$ already implies certain vanishings, e.g. $H^q(F(X),\mathcal{O}) = 0$ for $n > 4$ and $q > 0$. See Corollary 3.7 for an alternative approach. This allows one to prove the following result for $n > 4$.

Corollary 2.2. *Let $F = F(X)$ be the Fano variety of lines in a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ over $\mathbb{C}$. Then $H^1(F,\mathbb{Z}) = 0$ for $n \geq 4$ and, in particular,* $\mathrm{Pic}^0(F) = 0$*. For $n \geq 5$ one has* $\mathrm{Pic}(F) \simeq H^2(F,\mathbb{Z})(1)$. □

Apart from the Fano variety of lines on a cubic surface, all others are connected, see [4, Thm. 1.16] and [8, Thm. 6]. This leads to the following strengthening of Corollary 1.11.[1]

[1] One could think of applying the Fulton–Lazarsfeld connectivity [106, Thm. 7.2.1], to prove its connectivity whenever $\dim(F(X,m)) > 0$. However, this would need the ampleness of $S^d(\mathcal{S}^*)$ which is just wrong.

Proposition 2.3. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface of dimension $n > 2$. Then $F(X)$ is an irreducible, smooth, projective variety of dimension $2n-4$.*

Proof More generally, Barth and Van den Ven [8] prove that the Fano variety of lines $F(X)$ on any, not necessarily smooth, hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d is connected if $d < 2(n-1)$. They argue by bounding the dimension of the ramification locus (of the Stein factorization) of $\mathbb{L} \twoheadrightarrow X$. Altman and Kleiman [4] use instead the Koszul complex

$$\cdots \longrightarrow \bigwedge^2(S^3(\mathcal{S})) \longrightarrow S^3(\mathcal{S}) \longrightarrow \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{O}_{F(X)} \longrightarrow 0.$$

The induced spectral sequence

$$E_1^{p,q} = H^q(\mathbb{G}, \bigwedge^{-p}(S^3(\mathcal{S}))) \Rightarrow H^{p+q}(F(X), \mathcal{O}_{F(X)})$$

combined with generalized Bott vanishing results for Grassmann varieties:

$$H^p(\mathbb{G}, \bigwedge^p(S^3(\mathcal{S}))) = 0 \text{ for all } p \neq 0$$

yield $H^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}) \xrightarrow{\sim} H^0(F(X), \mathcal{O}_{F(X)})$. Hence, $F(X)$ is connected.

For alternative arguments see Example 3.11 and Exercise 2.6, the latter for the case $n \geq 4$. □

Exercise 2.4. Let $X \subset \mathbb{P}^{n+1}$ be cubic hypersurface and assume m is an integer with $m^2 + 11m \leq 6n$. Show that there exists a linear subspace $\mathbb{P}^m \subset X \subset \mathbb{P}^{n+1}$ of dimension m contained in X.

Remark 2.5. For $n > 2$, the universal line $\mathbb{L} = \mathbb{P}(\mathcal{S}|_{F(X)}) \twoheadrightarrow F(X)$ maps onto X or, equivalently, through every point $x \in X$ there exists at least one line $x \in L \subset X$, possibly defined only over a finite extension of the residue field of x.

To prove this claim we may assume that k is algebraically closed. For a fixed point $x \in X$, let $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ be a hyperplane not containing x. We may assume $x = [1 : 0 : \cdots : 0]$ and $\mathbb{P}^n = V(x_0)$. If $X = V(F)$, then the projective tangent space is the hyperplane $T_xX = V(\sum x_i\, \partial_i F(x))$ and any line $x \in L \subset T_xX$ has intersection multiplicity $\mathrm{mult}_x(X, L) \geq 2$. For dimension reasons, there exists a point

$$y \in \mathbb{P}^n \cap T_xX \cap X \cap V(\partial_0 F).$$

Then let $L = \overline{xy}$ be the line connecting the two points. We may choose coordinates such that $y = [0 : 1 : 0 : \cdots : 0]$ and then $F|_L$ is the polynomial $F(x_0, x_1, 0, \ldots, 0)$. By definition $\partial_0 F(y) = 0$ and by the Euler equation also $\partial_1 F(y) = 0$. Therefore, $\mathrm{mult}_y(X, L) \geq 2$. However, a line $L \subset \mathbb{P}^{n+1}$ intersecting a cubic hypersurface $X \subset \mathbb{P}^{n+1}$ in two distinct points with multiplicity at least two at both points is contained in X.

Alternatively, to prove surjectivity, one may first take hyperplane sections to reduce to the case of smooth cubic threefolds $Y \subset \mathbb{P}^4$. Then the result follows from [34, Cor.

8.2]. Another more direct argument can be found in [35, Lem. 2.1]. The above proof is taken from [70].

Note that the argument also shows that the fibre of $\mathbb{L} \longrightarrow X$ over x has the expected dimension $n-3$ if and only if the intersection $\mathbb{P}^n \cap T_xX \cap X \cap V(\partial_0 F)$ is of this expected dimension.

Exercise 2.6. Assume $n \geq 4$ and observe that then $\mathbb{P}^n \cap T_xX \cap X \cap V(\partial_0 F)$ in Remark 2.5 is connected by Bertini's theorem. Deduce from this that $F(X)$ is connected, thus proving Proposition 2.3 again for cubic hypersurfaces of dimension $n \geq 4$.

2.2 The following 'geometric global Torelli theorem' generalizes a well known result for $n = 3$ which we shall explain in Section **5**.2.3. It turns out, that the general proof is less geometric but in the end much easier. We follow [31].

Proposition 2.7. *Assume $X, X' \subset \mathbb{P}^{n+1}$ are smooth cubic hypersurfaces of dimension $n > 2$ and let $F(X)$ and $F(X')$ be their Fano varieties of lines endowed with the natural Plücker polarizations $\mathcal{O}_F(1)$ and $\mathcal{O}_{F'}(1)$, respectively.*

Then $X \simeq X'$ if and only if $(F(X), \mathcal{O}_F(1)) \simeq (F(X'), \mathcal{O}_{F'}(1))$ as polarized varieties. For $n \neq 4$ this is equivalent to $F(X) \simeq F(X')$ as unpolarized varieties.

Proof Any isomorphism $X \simeq X'$ is induced by an automorphism of the ambient projective space, cf. Corollary **1**.3.6. Therefore, it naturally induces an isomorphism between the Fano varieties of lines, which in addition is automatically polarized.

Conversely, a polarized automorphism $(F(X), \mathcal{O}_F(1)) \simeq (F(X'), \mathcal{O}_{F'}(1))$ is induced by an automorphism of the ambient Plücker space $\mathbb{P}(\bigwedge^2 V) \simeq \mathbb{P}(\bigwedge^2 V)$. We make use of the classical fact that the Grassmannian $\mathbb{G} = \mathbb{G}(1, \mathbb{P}^{n+1}) \subset \mathbb{P}(\bigwedge^2 V)$ is cut out by quadrics, see [66, Ex. 8.12]. Combining this with the injectivity of the restriction map $H^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(2)) \hookrightarrow H^0(F(X), \mathcal{O}_F(2))$, which is proved by a spectral sequence argument as in the proof of Proposition 2.3, cf. [4], one finds that $\mathbb{G}$ is cut out by all quadrics containing $F(X)$ or, alternatively, containing $F(X')$. In other words, $F(X) \simeq F(X')$ is induced by an automorphism of the ambient Grassmannian

$$\begin{array}{ccccc} F(X) & \hookrightarrow & \mathbb{G} & \hookrightarrow & \mathbb{P}(\bigwedge^2 V) \\ \Big| \wr & & \Big| \wr & & \Big| \wr \\ F(X') & \hookrightarrow & \mathbb{G} & \hookrightarrow & \mathbb{P}(\bigwedge^2 V). \end{array}$$

However, any automorphism of $\mathbb{G}$ compatible with its Plücker embedding is induced by an automorphism of $\mathbb{P}(V)$, see [33].

For $n \neq 4$, the canonical bundle $\omega_F \simeq \mathcal{O}_F(4-n)$ is a non-trivial, possibly negative, multiple of the Plücker polarization $\mathcal{O}_F(1)$. Clearly, any isomorphism $F(X) \simeq F(X')$ respects the canonical bundle. Using that $\mathrm{Pic}(F(X))$ is torsion free for $n > 4$, this implies that any isomorphism $F(X) \simeq F(X')$ is automatically polarized. To see that $\mathrm{Pic}(F(X))$ is torsion free at least in characteristic zero, observe that for a torsion line bundle L one has $\chi(F(X), L) = \chi(F(X), \mathcal{O}) = 1$, as $F(X)$ is a Fano variety for $n > 4$, with $H^i(F(X), L) = 0$ for $i > 0$ by Kodaira vanishing and, therefore, $L \simeq \mathcal{O}$.[2] For $n = 3$, the fact that $\omega_F \simeq \mathcal{O}(1)|_F$ suffices to conclude. Without the torsion freeness of the Picard group, the above arguments need to be adapted. □

The proof above shows more, namely that any (polarized) isomorphism $F(X) \simeq F(X')$ is induced by an isomorphism $X \simeq X'$. For $n = 3$ we will provide a different proof which relies on an isomorphism between the restriction of the tautological bundle $\mathcal{S}_F$ and the tangent sheaf $\mathcal{T}_F$, see Proposition **5**.2.9.

3 Cohomology and motives

According to Proposition 2.7 the Fano variety of lines $F(X)$ contained in a smooth cubic hypersurface X determines the hypersurface. Therefore, essentially all and, in particular, all cohomological and motivic information about X should be encoded by $F(X)$. In this section we shall study the cohomology of $F(X)$ and we shall do this by first looking at the motive of $F(X)$.

3.1 As the Fano variety $F(X)$ itself, its motive is an interesting object to study. Now, the motive of $F(X)$ may mean various things. Here, we are interested in its class in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$.

Recall that the Grothendieck ring $K_0(\mathrm{Var}_k)$ of varieties over a field k is by definition the abelian group generated by classes $[Y]$ of quasi-projective varieties modulo one relation $[Y] = [Z] + [U]$, the scissor relation, for any closed subset $Z \subset Y$ with open complement $U = Y \setminus Z$. Then $K_0(\mathrm{Var}_k)$ becomes a ring with multiplication given by the product $[Y] \cdot [Y'] = [Y \times Y']$. The *Lefschetz motive* is the class $\ell := [\mathbb{A}^1]$ of the affine line. An important consequence of the scissor relation is the fact that $[Y] = [F] \cdot [Z]$ for a Zariski locally trivial fibration $Y \twoheadrightarrow Z$ with fibre F. See for example [5, Ch. 13] for more details.

In [64], the class $[F(X)] \in K_0(\mathrm{Var}_k)$ is related to the class $[X^{[2]}] \in K_0(\mathrm{Var}_k)$ of the Hilbert scheme $X^{[2]}$ of subschemes of X of length two. The Hilbert scheme can be obtained as the blow-up of the symmetric product $X^{(2)} := (X \times X)/\mathfrak{S}_2$ along the diagonal $X \simeq \Delta \subset X^{(2)}$. Hence, in $K_0(\mathrm{Var}_k)$ one has

$$[X^{[2]}] - [\mathbb{P}^{n-1}] \cdot [X] = [X^{(2)}] - [X]. \tag{3.1}$$

[2] Thanks to S. Stark for the argument. Alternatively, one can use [46]. As an aside, $\mathrm{Pic}(F(X))$ is also torsion free for $n = 4$, as then $F(X)$ is a hyperkähler manifold, see Chapter **??**.**??** and, therefore, $\mathrm{Pic}(F(X) \simeq \mathrm{NS}(F(X))$, cf. Remark 3.9.

Proposition 3.1 (Galkin–Shinder). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Then in $K_0(\mathrm{Var}_k)$*

$$[X^{[2]}] = [\mathbb{P}^n] \cdot [X] + \ell^2 \cdot [F(X)]$$

and

$$[X^{(2)}] = (1 + \ell^n) \cdot [X] + \ell^2 \cdot [F(X)].$$

Proof We follow closely the argument in [64], where one also finds a version for singular cubics.

Consider the universal family $F(X) \longleftarrow \mathbb{L} \longrightarrow X$ of lines contained in X. As $\mathbb{L} \longrightarrow F(X)$ is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{S}|_{F(X)}) \longrightarrow F(X)$, its class in $K_0(\mathrm{Var}_k)$ is given by

$$[\mathbb{L}] = [\mathbb{P}^1] \cdot [F(X)]. \tag{3.2}$$

Similarly, let $\mathbb{G} = \mathbb{G}(1, \mathbb{P}) \longleftarrow \mathbb{L}_{\mathbb{G}} \longrightarrow \mathbb{P}$ be the universal family of lines in $\mathbb{P}$ and let $\mathbb{L}_G|_X$ be the pre-image of X under the second projection. Then $\mathbb{L}_{\mathbb{G}}|_X$ parametrizes pairs (x, L) consisting of a line $L \subset \mathbb{P}$ and a point $x \in X \cap L$. It can also be described as the $\mathbb{P}^n$-bundle $\mathbb{P}(\mathcal{T}_{\mathbb{P}}|_X) \longrightarrow X$, cf. the construction in the proof of Corollary 1.12. This shows that in $K_0(\mathrm{Var}_k)$ one has

$$[\mathbb{L}_G|_X] = [\mathbb{P}^n] \cdot [X]. \tag{3.3}$$

Next, consider the morphism

$$\mathbb{L}_{\mathbb{G}}|_X \setminus \mathbb{L} \longrightarrow X^{[2]}$$

that sends (x, L) to the residual intersection $[(L \cap X) \setminus \{x\}] \in X^{[2]}$. It is inverse to the morphism

$$X^{[2]} \setminus \mathbb{L}^{[2]} \longrightarrow \mathbb{L}_{\mathbb{G}}|_X$$

that sends $[Z] \in X^{[2]}$ to (x, L_Z), where $L_Z \subset \mathbb{P}$ is the unique line containing the length two-subscheme $Z \subset \mathbb{P}$ and x is the residual point of the inclusion $Z \subset X \cap L_Z$. Here, $\mathbb{L}^{[2]}$ is the relative symmetric product of the universal line $p \colon \mathbb{L} \longrightarrow F(X)$, which equivalently can be described as the relative Hilbert scheme of subschemes of length two in the fibres of p or, still equivalently, as $\mathbb{L}^{[2]} \simeq \mathbb{P}(S^2(\mathcal{S}^*|_{F(X)})) \longrightarrow F(X)$. In particular,

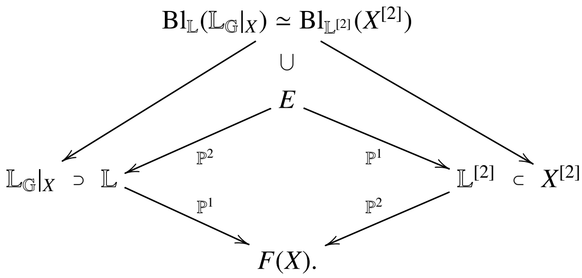
$$[\mathbb{L}^{[2]}] = [\mathbb{P}^2] \cdot [F(X)] \tag{3.4}$$

in $K_0(\mathrm{Var}_k)$. The isomorphism

$$\mathbb{L}_{\mathbb{G}}|_X \setminus \mathbb{L} \simeq X^{[2]} \setminus \mathbb{L}^{[2]}$$

together with (3.2), (3.3), and (3.4) yields the first assertion. The second follows from (3.1). □

Remark 3.2. A closer inspection reveals, see [151, Prop. 2.9] for further details, that the construction in the proof above leads to the following picture:



Here, E is the exceptional divisor of the two blow-up morphisms. In fact, $\mathrm{Bl}_{\mathbb{L}}(\mathbb{L}_{\mathbb{G}}|_X) \simeq \mathrm{Bl}_{\mathbb{L}^{[2]}}(X^{[2]})$ is an irreducible component of the incidence variety

$$\{\, (x, L, Z) \mid x \in L \cap X,\ Z \subset L \cap X \,\} \subset X \times \mathbb{L}_{\mathbb{G}}|_X \times X^{[2]}.$$

Standard formulae for cohomology or motives of smooth blow-ups and projective bundles can be applied. For example, for the category of rational Chow motives $\mathrm{Mot}(k)$ one knows that

$$\mathfrak{h}(\mathrm{Bl}_{\mathbb{L}}(\mathbb{L}_{\mathbb{G}}|_X)) \oplus \mathfrak{h}(\mathbb{L}) \simeq \mathfrak{h}(\mathbb{L}_G|_X) \oplus \mathfrak{h}(E)$$

and

$$\mathfrak{h}(\mathrm{Bl}_{\mathbb{L}^{[2]}}(X^{[2]})) \oplus \mathfrak{h}(\mathbb{L}^{[2]}) \simeq \mathfrak{h}(X^{[2]}) \oplus \mathfrak{h}(E),$$

see [5, 123]. This can be combined with $\mathfrak{h}(\mathbb{L}_G|_X) \simeq \bigoplus_{i=0}^n \mathfrak{h}(X)(-i)$ and

$$\begin{aligned} \mathfrak{h}(\mathbb{L}) &\simeq \mathfrak{h}(F(X)) \oplus \mathfrak{h}(F(X))(-1) \\ \mathfrak{h}(\mathbb{L}^{[2]}) &\simeq \mathfrak{h}(F(X)) \oplus \mathfrak{h}(F(X))(-1) \oplus \mathfrak{h}(F(X))(-2). \end{aligned}$$

Here, $\mathfrak{h}(X)(-i) \coloneqq \mathfrak{h}(X) \otimes (\mathbb{P}^1, [\mathbb{P}^1 \times x])^{\otimes i}$ is the twist with the i-th power of the Lefschetz motive. The isomorphism $\mathrm{Bl}_{\mathbb{L}}(\mathbb{L}_{\mathbb{G}}|_X) \simeq \mathrm{Bl}_{\mathbb{L}^{[2]}}(X^{[2]})$ then yields a formula which, if cancellation holds in $\mathrm{Mot}(k)$, would look like this:

$$\mathfrak{h}(F(X))(-2) \oplus \bigoplus_{i=0}^{n} \mathfrak{h}(X)(-i) \simeq \mathfrak{h}(X^{[2]}). \tag{3.5}$$

That cancellation really holds in this case was proved in [104], so (3.5) is true. The formula can be combined with the isomorphism

$$\mathfrak{h}(X^{[2]}) \simeq S^2\mathfrak{h}(X) \oplus \bigoplus_{i=1}^{n-1} \mathfrak{h}(X)(-i), \tag{3.6}$$

which shows that finite-dimensionality of $\mathfrak{h}(X)$ in the sense of Kimura implies finite-dimensionality of $\mathfrak{h}(F(X))$, cf. [104, Thm. 4].[3]

One also expects the following isomorphism to hold:

$$\mathfrak{h}(F(X))(-2) \oplus \mathfrak{h}(X) \oplus \mathfrak{h}(X)(-n) \simeq S^2\mathfrak{h}(X), \tag{3.7}$$

which would follow from the above if cancellation holds. Using the decomposition $\mathfrak{h}(X) \simeq \mathfrak{h}^n(X)_{\mathrm{pr}} \oplus \bigoplus_{i=0}^n \mathbb{Q}(-i)$, cf. Remark 1.7, and cancellation, this then becomes

$$\mathfrak{h}(F(X))(-2) \oplus \mathbb{Q}(-n) \simeq S^2\mathfrak{h}^n(X)_{\mathrm{pr}} \oplus \bigoplus_{i=1}^{n-1} \mathfrak{h}^n(X)_{\mathrm{pr}}(-i) \oplus \bigoplus_{0<i\leq j<n} \mathbb{Q}(-i-j). \tag{3.8}$$

Here, $\mathbb{Q}(1)$ is the Tate motive $(\mathrm{Spec}(k), \mathrm{id}, 1)$, the dual of the Lefschetz motive.

3.2 Combining the information obtained from the description of $F(X) \subset \mathbb{G}(1, \mathbb{P})$ as the zero set $V(s_F)$ and the description of its class $[F(X)] \in K_0(\mathrm{Var}_k)$, one can deduce the following numerical information, see [4, Prop. 1.6] and [64, Cor. 5.2]. The case $n = 3$ goes back to [22].

Proposition 3.3 (Altman–Kleiman, Galkin–Shinder). *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface and $F(X)$ its Fano variety of lines considered with its projective embedding $F(X) \hookrightarrow \mathbb{G}(1, \mathbb{P}) \hookrightarrow \mathbb{P}^{\binom{n+2}{2}-1}$. Then degree and Euler number of $F(X)$ are given by the following formulae*

$$\deg(F(X)) = 27 \cdot \frac{(2n-4)!}{n! \cdot (n-1)!} \cdot (3n^2 - 7n + 4) \tag{3.9}$$

and

$$e(F(X)) = \frac{e(X) \cdot (e(X) - 3)}{2} = \frac{2^{2n+4} + (-2)^{n+2} \cdot (6n+1) + 3n \cdot (3n+1) - 20}{18}. \tag{3.10}$$

Proof As a special case of (1.3) we know that $[F(X)] = \mathrm{c}_4(S^3(\mathcal{S}^*))$ in the cohomology or in the Chow ring of $\mathbb{G}$. Writing formally $\mathcal{S}^* = L_0 \oplus L_1$ and $S^3(\mathcal{S}^*) = L_0^3 \oplus (L_0^2 \otimes L_1) \oplus (L_0 \otimes L_1^2) \oplus L_1^3$ allows one to compute

$$\begin{aligned} \mathrm{c}_4(S^3(\mathcal{S}^*)) &= 9 \cdot (5\, \ell_0^2 \ell_1^2 + 2\, (\ell_0^3 \ell_1 + \ell_0 \ell_1^3)) \\ &= 9 \cdot (2\, \mathrm{c}_1(\mathcal{S}^*)^2 + \mathrm{c}_2(\mathcal{S}^*)) \cdot \mathrm{c}_2(\mathcal{S}^*), \end{aligned}$$

where $\ell_i = \mathrm{c}_1(L_i)$. Hence,

$$\deg(F(X)) = 9 \int_{\mathbb{G}} \mathrm{c}_1(\mathcal{S}^*)^{2n-4} \cdot (2\, \mathrm{c}_1(\mathcal{S}^*)^2 + \mathrm{c}_2(\mathcal{S}^*)) \cdot \mathrm{c}_2(\mathcal{S}^*).$$

Using standard Schubert calculus (Pieri's and Gambelli's formulae), this is turned in [4] into a rather complicated formula which then can be simplified to the above.

[3] It is not known whether smooth cubic hypersurfaces of dimension $5 \neq n > 3$ are Kimura finite-dimensional. For $n = 5$ finite-dimensionality is known (with thanks to R. Laterveer for the explanation).

In principle, the second assertion can also be deduced by Schubert calculus, as

$$e(F(X)) = \int_{F(X)} \mathrm{c}_{2n-4}(\mathcal{T}_{F(X)}) = \int_{\mathbb{G}} \left(\frac{c(\mathcal{T}_{\mathbb{G}})}{\mathrm{c}(S^3(\mathcal{S}^*))}\right)_{2n-4} \cdot \mathrm{c}_4(S^3(\mathcal{S}^*)).$$

However, there is a more illuminating way of doing this. Indeed, taking Euler numbers in the second equation in Proposition 3.1 yields

$$e(X^{(2)}) = 2 \cdot e(X) + e(F(X)),$$

where we use the additivity and the multiplicativity of the Euler number and $e(\ell^n) = 1$, cf. [20]. However, taking cohomology commutes with taking symmetric products, i.e. $H^*(X^{(n)}) = S^n H^*(X)$ (say with coefficients in a field of characteristic zero), cf. [72, Prop. 5.2.3] or [111]. Therefore, $e(X^{(2)}) = \binom{e(X)+1}{2}$.[4] This proves the first equality in (3.10) and the second follows from (1.6). □

Remark 3.4. Recall from Section **1**.1.4 that the Euler numbers $e(X_n)$ of all smooth cubic hypersurfaces $X_n \subset \mathbb{P}^{n+1}$ can also be described by the generating series

$$\sum_{n=0}^{\infty} e(X_n)\, z^{n+1} = \frac{3\,z}{(1-z)^2\,(2\,z+1)}.$$

Does there exist a similar description for the generating series of $e(F(X_n))$? Indeed, a formal computation using Mathematica[5] reveals

$$\sum_{n=0}^{\infty} e(F(X_n))\, z^{n+1} = \frac{27\,(1-2\,z)\,z^3}{(-1+z)^3\,(1+2\,z)^2\,(-1+4\,z)}.$$

However, a conceptual understanding in the sense of Theorem **1**.1.11 is not known.

n	$\omega_{F(X)}$	$\dim(F(X))$	$\deg(F(X))$	$e(F(X))$
2	$\mathcal{O}$	0	27	27
3	$\mathcal{O}(1)$	2	45	27
4	$\mathcal{O}$	4	108	324
5	$\mathcal{O}(-1)$	6	297	702

[4] The closed formula due to MacDonald says $\sum_{n=0}^{\infty} e(X^{(n)})\, z^n = (1-z)^{-e(X)} = \exp\left(e(X) \cdot \sum_{r=1}^{\infty} z^r/r\right)$.

[5] . . . with thanks to P. Magni.

3.3 The computation of the Euler number is only a shadow of the full cohomological information available. There are various ways to unpack the information encoded by the above considerations. We shall focus on the Hodge theoretic content and so assume from now on that $k = \mathbb{C}$.

To start, let $\mathrm{HS}_{\mathbb{Z},n}$ be the additive category of polarizable, pure Hodge structures of weight n.[6] Recall that the Tate twist defines an equivalence

$$\mathrm{HS}_{\mathbb{Z},n} \xrightarrow{\sim} \mathrm{HS}_{\mathbb{Z},n-2},\ H \longmapsto H(1) \coloneqq H \otimes \mathbb{Z}(1),$$

where $\mathbb{Z}(1) = (2\pi i)\,\mathbb{Z}$ is the pure Hodge structure of weight $(-1,-1)$ geometrically realized by the dual of $H^2(\mathbb{P}^1,\mathbb{Z})$. We let $\mathrm{HS}_{\mathbb{Z}} = \bigoplus \mathrm{HS}_{\mathbb{Z},n}$ be the additive category of graded pure, integral Hodge structures and denote its Grothendieck group by $K_0(\mathrm{HS}_{\mathbb{Z}})$. By definition, this is the group generated by isomorphism classes of integral polarizable odge structures with the condition that $[H] + [H'] = [H \oplus H']$. In particular, two Hodge structures H and H' define the same class $[H] = [H']$ in $K_0(\mathrm{HS}_{\mathbb{Z}})$ if and only if there exists a Hodge structure H_0 such that $H \oplus H_0 \simeq H' \oplus H_0$.

Recall that according to [20], $K_0(\mathrm{Var}_{\mathbb{C}})$ can also be described as the quotient of the free abelian group generated by isomorphism classes of smooth projective varieties by the relation

$$[\mathrm{Bl}_Z(Y)] + [Z] = [Y] + [E]$$

for every blow-up $\mathrm{Bl}_Z(Y) \longrightarrow Y$ of a smooth projective variety Y along a smooth projective subvariety $Z \subset Y$ with exceptional divisor E. Using that for each such smooth blow-up there exists an isomorphism of polarizable integral Hodge structures

$$H^*(\mathrm{Bl}_Z(Y),\mathbb{Z}) \oplus H^*(Z,\mathbb{Z}) \simeq H^*(Y,\mathbb{Z}) \oplus H^*(E,\mathbb{Z}),$$

one finds that there exists a ring homomorphism

$$K_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow K_0(\mathrm{HS}_{\mathbb{Z}}),\ [X] \longmapsto H^*(X,\mathbb{Z}). \tag{3.11}$$

Under this map, $\ell = [\mathbb{A}^1] = [\mathbb{P}^1] - [\mathrm{pt}]$ is sent to $\mathbb{Z}(-1)$.

Corollary 3.5. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Then as classes in $K_0(\mathrm{HS}_{\mathbb{Z}})$ one has*

$$[H^*(F(X),\mathbb{Z})] + [H^*(\mathbb{P}^n,\mathbb{Z})(2)] \cdot [H^*(X,\mathbb{Z})] = [H^*(X^{[2]},\mathbb{Z})(2)]. \tag{3.12}$$

Proof Apply (3.11) to Proposition 3.1 and twist by $\mathbb{Z}(2)$. □

Next consider the natural functor

$$\mathrm{HS}_{\mathbb{Z}} \longrightarrow \mathrm{HS}_{\mathbb{Q}},\ H \longmapsto H \otimes_{\mathbb{Z}} \mathbb{Q},$$

[6] The notation is borrowed from [58].

to the category of graded pure, polarizable, rational Hodge structures. It induces a linear map

$$K_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow K_0(\mathrm{HS}_{\mathbb{Z}}) \longrightarrow K_0(\mathrm{HS}_{\mathbb{Q}}). \tag{3.13}$$

As the category of graded pure, polarizable, rational Hodge structures $\mathrm{HS}_{\mathbb{Q}}$ is semi-simple (cf. [127, Cor. 2.12]), the natural map

$$\mathrm{HS}_{\mathbb{Q}} \hookrightarrow K_0(\mathrm{HS}_{\mathbb{Q}})$$

is injective, i.e. two rational Hodge structures H and H' are isomorphic if and only if $[H] = [H']$ in $K_0(\mathrm{HS}_{\mathbb{Q}})$.[7] Thus, (3.12) becomes an isomorphism of rational Hodge structures

$$H^*(F(X), \mathbb{Q}) \oplus (H^*(\mathbb{P}^n, \mathbb{Q}) \otimes H^*(X, \mathbb{Q}))\,(2) \simeq H^*(X^{[2]}, \mathbb{Q})(2).$$

There is a shortcut to arrive at this isomorphism by just applying cohomology to (3.5), i.e. using the commutativity of the diagram

$$\begin{array}{ccccc} (\mathrm{SmProj}(\mathbb{C})) & \longrightarrow & & \longrightarrow & \mathrm{Mot}(\mathbb{C}) \\ \downarrow & & & & \downarrow \\ K_0(\mathrm{Var}_{\mathbb{C}}) & \longrightarrow & K_0(\mathrm{HS}_{\mathbb{Z}}) & \longrightarrow & K_0(\mathrm{HS}_{\mathbb{Q}}). \end{array}$$

Similarly, either by applying cohomology to (3.6) or by using (3.13), one obtains an isomorphism of Hodge structures

$$H^*(X^{[2]}, \mathbb{Q}) \simeq S^2 H^*(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{n-1} H^*(X, \mathbb{Q})(-i).$$

Altogether this leads to the isomorphism of Hodge structures

$$H^*(F(X), \mathbb{Q}) \oplus H^*(X, \mathbb{Q})(2) \oplus H^*(X, \mathbb{Q})(2-n) \simeq S^2(H^*(X, \mathbb{Q}))(2) \tag{3.14}$$

and, after decomposing into primitive parts,

$$H^*(F(X), \mathbb{Q}) \oplus \mathbb{Q}(2-n) \tag{3.15}$$

$$\simeq \bigoplus_{i=1}^{n-1} H^n(X, \mathbb{Q})_{\mathrm{pr}}(2-i) \oplus \bigoplus_{0<i\leq j<n} \mathbb{Q}(2-i-j) \tag{3.16}$$

$$\oplus \begin{cases} S^2(H^n(X, \mathbb{Q})_{\mathrm{pr}})(2) & \text{for } n \equiv 0\,(2) \\ \bigwedge^2(H^n(X, \mathbb{Q})_{\mathrm{pr}})(2) & \text{for } n \equiv 1\,(2), \end{cases}$$

cf. [64]. The latter can also be obtained by taking cohomology of (3.8).

[7] Injectivity does not hold for integral Hodge structures. Indeed, there exist elliptic curves E, E', and E_0 such that E and E' are non-isomorphic but $E \times E_0 \simeq E' \times E_0$, see [141]. In this case $[H^1(E, \mathbb{Z})] = [H^1(E', \mathbb{Z})]$ in $K_0(\mathrm{HS}_{\mathbb{Z}})$, but $H^1(E, \mathbb{Z})$ and $H^1(E', \mathbb{Z})$ are non-isomorphic Hodge structures. Thanks to B. Moonen for the reference.

3.4 The last formula allows one to compute all Betti and Hodge numbers of $F(X)$. Combined with Hirzebruch's formula for the generating series $\sum_{n=0}^{\infty}\chi_y(X_n)z^{n+1}$ of all χ_y-genera of cubic hypersurfaces, see Theorem **1**.1.11, it yields the following.

Corollary 3.6. *The χ_y-genera of the Fano varieties of lines $F(X_n)$ of a smooth cubic hypersurface $X_n \subset \mathbb{P}^{n+1}$ are given by*

$$\sum_{n=0}^{\infty}\chi_y(F(X_n))\,z^{n+1} = TBC.$$

Proof □

Before making this explicit in low-dimensional cases, we shall draw a few further consequences of (3.15).

Corollary 3.7. *Let X be a smooth cubic hypersurface of dimension n.*

(i) *If n is even, then $H^*(X,\mathbb{Q}) = H^{\rm ev}(X,\mathbb{Q})$ and $H^*(F(X),\mathbb{Q}) = H^{\rm ev}(F(X),\mathbb{Q})$.*
(ii) *If n is odd, then $H^*(X,\mathbb{Q}) = H^{\rm ev}(X,\mathbb{Q}) \oplus H^n(X,\mathbb{Q})$ and*

$$H^*(F(X),\mathbb{Q}) = H^{\rm ev}(F(X),\mathbb{Q}) \oplus \bigoplus_{i=1}^{n-1} H^n(X,\mathbb{Q})_{\rm pr}(2-i).$$ □

The description of the first cohomology yields an alternative proof of Corollary 2.2.

Corollary 3.8. *Assume $n \geq 4$. Then the Picard variety* $\mathrm{Pic}^0(F(X))$ *of the Fano variety of lines $F(X)$ on a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ is trivial.*

Proof Indeed, as $H^{\rm odd}(F(X),\mathbb{Q})$ is trivial for even n and otherwise $H^{\rm odd}(F(X),\mathbb{Q}) \simeq \bigoplus_{i=1}^{n-1} H^n(X,\mathbb{Q})_{\rm pr}(2-i)$, one finds $H^1(F(X),\mathbb{Q}) = 0$ for $n \geq 4$. Hence, $H^{0,1}(F(X)) = 0$ which implies that $\mathrm{Pic}^0(F(X))$ is trivial. □

Remark 3.9. In fact, the $F(X)$ are known to be simply connected for $n \geq 4$.[8] For $n = 4$ it is due to [16] and for $n > 4$ the result follows from [142]. See [46, Prop. 1& Ex. 3.3], where it is also shown that $\mathrm{Pic}(F(X)) \simeq \mathbb{Z}\cdot\mathcal{O}_F(1)$ for $n > 4$.

Alternatively, one can use the fact that for $n > 4$ the Fano variety of lines $F(X)$ is a Fano variety, i.e. has negative canonical bundle. Quite generally, due to results of Campana and Kollár rationally connected varieties Z are simply connected, cf. [44, 43]. Roughly, at least in characteristic zero, this follows from the observation that any finite étale cover $\pi\colon \tilde{Z} \longrightarrow Z$ would again be Fano and, therefore, $1 = \chi(\tilde{Z},\mathcal{O}) = \chi(Z,\mathcal{O})\cdot \deg(\pi)$.

[8] I am indebted to R. Laterveer and S. Stark for pointing this out to me and providing the references.

The middle cohomology $H^n(X,\mathbb{Q})$ of the cubic hypersurface X carries the most information. As we will see again and again, for the Fano variety of lines it is the cohomology in degree $n-2$, which is below the middle for all n. And indeed, the next result says that the two are intimately related. See Section 4 for an alternative argument.

Corollary 3.10. *Let X be a smooth cubic hypersurface of dimension n.*

(i) *If n is even, then*

$$H^{n-2}(F(X),\mathbb{Q}) \simeq H^n(X,\mathbb{Q})_{\rm pr}(1) \oplus \bigoplus \mathbb{Q}(2-i-j),$$

where the direct sum is over all $0 < i \leq j < n$ *such that* $2(i+j) = n+2$.

(ii) *If n is odd, then* $H^{{\rm odd}<n-2}(F(X),\mathbb{Q}) = 0$ *and*

$$H^{n-2}(F(X),\mathbb{Q})_{\rm pr} = H^{n-2}(F(X),\mathbb{Q}) \simeq H^n(X,\mathbb{Q})(1) \simeq H^n(X,\mathbb{Q})_{\rm pr}(1). \qquad \square$$

Example 3.11. Let us start by computing $H^0(F(X),\mathbb{Q})$. For this, compare the proof of (3.15). We distinguish the two cases:

(i) For $n = 2$, we obtain the isomorphism of vector spaces

$$H^0(F(X),\mathbb{Q}) \oplus \mathbb{Q} \simeq S^2(H^2(X,\mathbb{Q})_{\rm pr}) \oplus H^2(X,\mathbb{Q})_{\rm pr} \oplus \mathbb{Q}.$$

Taking dimension while using $b_2(X)_{\rm pr} = 6$, yields

$$b_0(F(X)) + 1 = 21 + 6 + 1.$$

Hence, $b_0(F(X)) = 27$, i.e. $F(X)$ consists of 27 isolated points. We stress that using étale cohomology, the same conclusion can be drawn for smooth cubic surfaces over arbitrary algebraically closed fields.

(ii) For $n > 2$, one finds $H^0(F(X),\mathbb{Q}) \simeq \mathbb{Q}$ (where the right hand side comes from $\mathbb{Q}(2-1-1)$). This proves again that $F(X)$ is connected, cf. Proposition 2.3 and Exercise 2.6.

We shall exploit (3.15) to compute the Hodge diamond of $F(X)$ for smooth cubic hypersurfaces of dimensions $n \leq 5$, cf. [64]. For the computation of the Hodge diamonds for the corresponding cubic hypersurface see Section **1**.1.4.

***n* = 3**: Here, the formula leads to the following isomorphisms of Hodge structures

$$H^1(F(X),\mathbb{Q}) \simeq H^3(X,\mathbb{Q})_{\rm pr}(1)$$

and

$$H^2(F(X),\mathbb{Q}) \simeq \textstyle\bigwedge^2 H^3(X,\mathbb{Q})_{\rm pr}(2).$$

Note that a priori the formula involves a direct summand $\mathbb{Q}(1)$ on both sides, which then cancels out. For the Hodge diamond this yields:

$b_0(F(X)) = 1$			1		
$b_1(F(X)) = 10$		5		5	
$b_2(F(X)) = 45$	10		25		10

***n* = 4**: In this case, the cohomology is concentrated in even degree.

$$H^2(F(X), \mathbb{Q}) \simeq H^4(X, \mathbb{Q})_{\mathrm{pr}}(1) \oplus \mathbb{Q}(-1)$$

and

$$H^4(F(X), \mathbb{Q}) \simeq S^2(H^4(X, \mathbb{Q})_{\mathrm{pr}})(2) \oplus H^4(X, \mathbb{Q})_{\mathrm{pr}} \oplus \mathbb{Q}(-2).$$

There is an additional summand $\mathbb{Q}(-2)$ on both sides which we have suppressed. For the Betti and Hodge numbers this implies

$b_0(F(X)) = 1$			1		
$b_2(F(X)) = 23$		1	21	1	
$b_4(F(X)) = 276$	1	21	232	21	1

***n* = 5**: Here one finds the following isomorphisms of Hodge structures

$$H^1(F(X), \mathbb{Q}) = 0,\ H^2(F(X), \mathbb{Q}) \simeq \mathbb{Q}(-1),\ H^3(F(X), \mathbb{Q}) \simeq H^5(X, \mathbb{Q})_{\mathrm{pr}}(1),$$

$$H^4(F(X), \mathbb{Q}) \simeq \mathbb{Q}(-2)^{\oplus 2},\ H^5(F(X), \mathbb{Q}) \simeq H^5(X, \mathbb{Q})_{\mathrm{pr}},$$

and

$$H^6(F(X), \mathbb{Q}) \simeq \textstyle\bigwedge^2 (H^5(X, \mathbb{Q})_{\mathrm{pr}})(2) \oplus \mathbb{Q}(-3).$$

Thus, the non-trivial part of the Hodge diamond below the middle looks like this:

$b_0(F(X)) = 1$			1		
$b_1(F(X)) = 0$					
$b_2(F(X)) = 1$			1		
$b_3(F(X)) = 42$		21		21	
$b_4(F(X)) = 2$			2		
$b_5(F(X)) = 42$		21		21	
$b_6(F(X)) = 862$	210		442		210

Remark 3.12. Instead of considering Hodge structures of hypersurfaces over $\mathbb{C}$ and of their Fano varieties, one could look at hypersurfaces over finite fields. In [64] one finds the following formula

$$|F(X)(\mathbb{F}_q)| = \frac{|X(\mathbb{F}_q)|^2 - 2\,(1 + q^n)\,|X(\mathbb{F}_q)| + |X(\mathbb{F}_{q^2})|}{2\,q^2},$$

which is a direct consequence of Proposition 3.1 or its version (3.6) in Mot(k)and similar to the arguments at the end of the proof of Proposition **1**.3.3 to compute $e(F(X))$. One way to see this is by using that the Zeta function $Z(X,t) = \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r}\right)$ can also be written as $Z(X,t) = \sum t^{\deg(Z)}$ (with the sum running over all zero cycles) and so $|X^{(2)}(\mathbb{F}_q)| = (1/2)\,(|X(\mathbb{F}_q)|^2 + |X(\mathbb{F}_{q^2})|)$. In principle, this allows one to write $Z(F(X),t)$ in terms of $Z(X,t)$ (which according the Weil conjectures has a rather special form for cubic hypersurfaces, cf. Section **1**.1.6). This has been studied out in [45].

4 The Fano correspondence

The way we related $H^n(X,\mathbb{Q})$ and $H^{n-2}(F(X),\mathbb{Q})$ was rather abstract and we shall now explain a more direct and much more canonical way. This makes use of the *Fano correspondence*, i.e. the natural diagram:

$$\begin{array}{ccc} \mathbb{L} & \xrightarrow{\;q\;} & X \\ {\scriptstyle p}\big\downarrow & & \\ F(X). & & \end{array} \tag{4.1}$$

4.1 Viewed as a correspondence it yields homomorphisms of integral Hodge structures

$$\varphi := p_* \circ q^* \colon H^m(X,\mathbb{Z}) \longrightarrow H^{m-2}(F(X),\mathbb{Z})(-1) \tag{4.2}$$

for all k. Depending on the context, it may also be useful to consider the correspondence on the level of Chow groups

$$\varphi\colon \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^{*-1}(F(X)) \tag{4.3}$$

or to use other types of cohomology theories.

The key idea for the following computations is to use the decomposition of $H^*(\mathbb{L},\mathbb{Z})$ obtained from viewing $\mathbb{L} \longrightarrow F(X)$ as the projective bundle $\mathbb{P}(\mathcal{S}_F) \longrightarrow F := F(X)$:

$$H^*(\mathbb{L},\mathbb{Z}) \simeq p^*H^*(F(X),\mathbb{Z}) \oplus u \cdot p^*H^{*-2}(F(X),\mathbb{Z})(-1).$$

Here, $u := \mathrm{c}_1(\mathcal{O}_p(1))$ and the pull-back map $p^*\colon H^*(F(X),\mathbb{Z}) \longrightarrow H^*(\mathbb{L},\mathbb{Z})$ is injective. Moreover, $u^2 + u \cdot p^*\mathrm{c}_1(\mathcal{S}_F) + p^*\mathrm{c}_2(\mathcal{S}_F) = 0$ and $p_*(p^*\gamma + u \cdot p^*\gamma') = \gamma'$. Similar formulae hold for Chow groups.

Lemma 4.1. *The correspondence* $\varphi\colon H^4(X,\mathbb{Z}) \longrightarrow H^2(F(X),\mathbb{Z})(-1)$ *maps the square of the hyperplane class* h^2 *to the Plücker polarization* g*, cf. (1.4). Similarly,* $h^2 \in \mathrm{CH}^2(X)$ *is mapped to* $\mathrm{c}_1(\mathcal{O}_F(1)) \in \mathrm{CH}^1(F(X))$ *under (4.3).*

Proof Recall that $\mathbb{L} \simeq \mathbb{P}(\mathcal{S}_F) \subset \mathbb{P}(V \otimes \mathcal{O}_F) \simeq F \times \mathbb{P}(V)$ is induced by $\mathcal{S}_F \subset V \otimes \mathcal{O}_F$ and, thus, $\mathcal{O}_p(1) \simeq q^*\mathcal{O}(1)$. Hence, $p_*q^*h^2 = p_*(q^*\mathrm{c}_1(\mathcal{O}(1))^2) = p_*(u^2) = -\mathrm{c}_1(\mathcal{S}_F) = g$. □

The next proposition generalizes results from [16, 34] in the case of $n = 3, 4$.

Proposition 4.2. *The correspondence* $\varphi\colon H^n(X,\mathbb{Z}) \longrightarrow H^{n-2}(F(X),\mathbb{Z})(-1)$ *is injective on* $H^n(X,\mathbb{Z})_{\mathrm{pr}}$ *and satisfies*

$$(\alpha.\beta) = -\frac{1}{6}\int_{F(X)} \varphi(\alpha)\cdot\varphi(\beta)\cdot g^{n-2} \tag{4.4}$$

for all primitive classes $\alpha, \beta \in H^n(X,\mathbb{Z})_{\mathrm{pr}}$.

The pairing on the left hand side of (4.4) is the standard intersection pairing on the middle cohomology $H^n(X,\mathbb{Z})$. On the right hand side, the pairing is the Hodge–Riemann pairing associated with the Plücker polarization g.

Proof The injectivity of the map $\varphi\colon H^n(X,\mathbb{Z})_{\mathrm{pr}} \hookrightarrow H^{n-2}(F(X),\mathbb{Z})(-1)$ follows from (4.4) which is proved by the following computation. The pull-back of $\alpha \in H^n(X)$ can be written uniquely as

$$q^*\alpha = p^*\varphi'(\alpha) + u\cdot p^*\varphi(\alpha). \tag{4.5}$$

If α is primitive, then $h\cdot\alpha = 0$ and hence $u\cdot q^*\alpha = 0$. Using $u^2 = -p^*\mathrm{c}_2(\mathcal{S}_F) + u\cdot p^*g$, this becomes $-p^*(\varphi(\alpha)\cdot\mathrm{c}_2(\mathcal{S}_F)) + u\cdot p^*(\varphi'(\alpha) + g\cdot\varphi(\alpha)) = 0$, which implies (i) $\varphi'(\alpha) + g\cdot\varphi(\alpha) = 0$ and $\varphi(\alpha)\cdot\mathrm{c}_2(\mathcal{S}_F) = 0$. The latter then implies (ii) $u^2\cdot p^*\varphi(\alpha) = u\cdot p^*(g\cdot\varphi(\alpha))$.

Taking the product of (4.5) and the corresponding equation for another primitive class β yields

$$q^*(\alpha\cdot\beta) = p^*(\varphi'(\alpha)\cdot\varphi'(\beta)) + u\cdot p^*(\varphi(\alpha)\cdot\varphi'(\beta) + \varphi'(\alpha)\cdot\varphi(\beta)) + u^2\cdot p^*(\varphi(\alpha)\cdot\varphi(\beta)).$$

Under p_* the first summand on the right hand side becomes trivial, while by means of (i) the direct image p_* of the second can be written as $-2(g\cdot\varphi(\alpha)\cdot\varphi(\beta))$. Applying (ii) to the last summand shows that it equals $u\cdot p^*(g\cdot\varphi(\alpha)\cdot\varphi(\beta))$. Altogether, one obtains

$$p_*q^*(\alpha\cdot\beta) = -g\cdot\varphi(\alpha)\cdot\varphi(\beta).$$

The left hand side can also be written as $(\alpha.\beta)\cdot p_*q^*[\mathrm{pt}]$. Taking product with g^{n-3} and integrating proves

$$(\alpha.\beta)\cdot\deg(p(q^{-1}(z))) = -\int_{F(X)} \varphi(\alpha)\cdot\varphi(\beta)\cdot g^{n-2}$$

for generic $z \in X$. The claim then follows from Lemma 4.4. □

Remark 4.3. Note that for n odd, $H^{n-2}(F(X),\mathbb{Q}) = H^{n-2}(F(X),\mathbb{Q})_{\mathrm{pr}}$, cf. Corollary 3.10, and so φ maps $H^n(X,\mathbb{Q})_{\mathrm{pr}}$ to $H^{n-2}(F(X),\mathbb{Q})_{\mathrm{pr}}(-1)$. This also holds true for $n = 4$ but the argument is more involved: One may assume that X is general, in which case

$H^n(X,\mathbb{Q})_{\rm pr}$ is an irreducible Hodge structure. As $H^{3,1}(X)$ is sent to $H^{2,0}(F(X))$, the whole primitive cohomology $H^4(X,\mathbb{Q})_{\rm pr}$ is mapped into the minimal sub-Hodge structure of $H^2(F(X),\mathbb{Q})$ containing the one-dimensional $H^{2,0}(F(X))$, hence into $H^2(F(X),\mathbb{Q})_{\rm pr}$ and isomorphically onto it. In [92, Thm. 4] it is claimed in full generality that the composition of φ with the projection onto the primitive cohomology yields an isomorphisms $H^n(X,\mathbb{Z})_{\rm pr} \xrightarrow{\sim} H^{n-2}(F(X),\mathbb{Z})_{\rm pr}(-1)$. However, the projection does usually not map into integral cohomology and, therefore, one needs to at least invert some integers. For n odd or $n = 4$ one certainly obtains injections $H^n(X,\mathbb{Z})_{\rm pr} \hookrightarrow H^{n-2}(F(X),\mathbb{Z})_{\rm pr}(-1)$, which we shall see to be an isomorphism for $n = 3$ and $n = 4$.

Lemma 4.4. *For $n \geq 2$ the generic fibre of the morphism $q\colon \mathbb{L} \longrightarrow X$ is of dimension $n-3$ and degree six with respect to the Plücker polarization g, i.e.*

$$\int_{q^{-1}(z)} g^{n-3} = 6.$$

Proof Pick a hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ not containing z. Then the linear embedding

$$\mathbb{P}^n \hookrightarrow \mathbb{G}(1,\mathbb{P}) \hookrightarrow \mathbb{P}(\textstyle\bigwedge^2 V),\ y \longmapsto \overline{yz},$$

induces an isomorphism $\{y \in \mathbb{P}^n \mid \overline{yz} \subset X\} \simeq p(q^{-1}(z))$. As in Remark 2.5, it yields

$$\{y \in \mathbb{P}^n \mid \overline{yz} \subset X\} \simeq \mathbb{P}^n \cap T_zX \cap X \cap P_zX.$$

Here, $T_zX = V(\sum x_i\partial_i F(z))$ is the tangent space of $X = V(F)$ at $z \in X$ and $P_zX = V(\sum z_i\partial_i F)$ is its polar, cf. also Section **4**.2.4. For generic choices of $z \in X$ and $\mathbb{P}^n$, this is a transversal intersection of the cubic X, the quadric P_zX, and the two hyperplanes $\mathbb{P}^n$ and T_zX and, therefore, of degree six. □

Example 4.5. For $n = 3$, i.e. smooth cubic threefolds, the result says that there are exactly six lines passing through every point in a Zariski open subset of Y. We shall come back to this in Section **5**.1.

Remark 4.6. In [8] it is shown that for $n \geq 4$ the fibres of $q\colon \mathbb{L} \longrightarrow X$ are connected. This is done by proving that the codimension of the ramification locus is at least of codimension two, cf. the proof of Proposition 2.3.

4.2 Let us study a few more formal aspects of the correspondence (4.1). On the level of cohomology we are interested in the two maps:

$$\varphi \coloneqq p_* \circ q^*\colon H^n(X,\mathbb{Z}) \longrightarrow H^{n-2}(F(X),\mathbb{Z})(-1)$$

and

$$\psi \coloneqq q_* \circ p^*\colon H^{3n-6}(F(X),\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z})(3-n).$$

The degree shift for the map ψ is due to $q\colon \mathbb{L} \longrightarrow X$ having generic fibre of dimension $n-3$. Note that Poincaré duality for X and $F(X)$ yields natural isomorphisms

$$H^n(X,\mathbb{Z})^* \simeq H^n(X,\mathbb{Z}) \quad \text{and} \quad H^{n-2}(F(X),\mathbb{Z})^* \simeq H^{3n-6}(F(X),\mathbb{Z})$$

(up to torsion). The projection formula shows that φ and ψ are dual to each other, i.e.

$$(\varphi(\alpha).\gamma)_F = (\alpha.\psi(\gamma))_X$$

for all $\alpha \in H^n(X,\mathbb{Z})$ and $\gamma \in H^{3n-6}(F(X),\mathbb{Z})$. Here, $(\ .\ )_X$ and $(\ .\ )_F$ denote the intersection pairings on X and F. In [140] the correspondence ψ is considered as a map $H_{n-2}(F(X),\mathbb{Z}) \longrightarrow H_n(X,\mathbb{Z})$. It is shown to be surjective, which yields an alternative proof of Proposition 4.2, and to be an isomorphism up to torsion for n odd, which follows from a comparison of Betti numbers.

The same formalism works on the level of Chow groups, but one has to distinguish between n even and odd.

Assume $\mathbf{n} \equiv \mathbf{0}\,(\mathbf{2})$ and write $n = 2m$. Then (4.1) induces maps

$$\mathrm{CH}^{3m-3}(F(X)) \xrightarrow{\ \psi\ } \mathrm{CH}^m(X) \xrightarrow{\ \varphi\ } \mathrm{CH}^{m-1}(F(X)).$$

Using the compatibility with the cycle class maps, one obtains commutative diagrams

$$\begin{array}{ccccc} \mathrm{CH}^{3m-3}(F(X)) & \xrightarrow{\ \psi\ } & \mathrm{CH}^m(X) & \xrightarrow{\ \varphi\ } & \mathrm{CH}^{m-1}(F(X)) \\ \downarrow & & \downarrow & & \downarrow \\ H^{6m-6}(F(X),\mathbb{Z})(3m-3) & \xrightarrow{\ \psi\ } & H^{2m}(X,\mathbb{Z})(m) & \xrightarrow{\ \varphi\ } & H^{2m-2}(F(X),\mathbb{Z})(m-1). \end{array}$$

To avoid potential confusion, let us stress that the diagram is not supposed to suggest that the rows are exact or even that the compositions are zero.

For $n \equiv 1\,(2)$ write $n = 2m-1$ and consider as above

$$\mathrm{CH}^{3m-4}(F(X)) \xrightarrow{\ \psi\ } \mathrm{CH}^m(X) \xrightarrow{\ \varphi\ } \mathrm{CH}^{m-1}(F(X)).$$

However, in this case the cycle map does not relate this to the middle cohomology of X. Instead, one has to restrict to the homologically trivial parts and use the Abel–Jacobi maps to intermediate Jacobians, which for a smooth projective variety Z of dimension N are the complex tori

$$J^{2k-1}(Z) := \frac{H^{2k-1}(Z,\mathbb{C})}{F^kH^{2k-1}(Z) + H^{2k-1}(Z,\mathbb{Z})} \simeq \frac{F^{N-k+1}H^{2N-2k+1}(Z)^*}{H_{2N-2k+1}(Z,\mathbb{Z})}.$$

Both description are used in the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{CH}^{3m-4}(F(X))_{\mathrm{hom}} & \xrightarrow{\psi} & \mathrm{CH}^{m}(X)_{\mathrm{hom}} & \xrightarrow{\varphi} & \mathrm{CH}^{m-1}(F(X))_{\mathrm{hom}} \\
\downarrow & & \downarrow & & \downarrow \\
J^{3n-6}(F(X)) & \xrightarrow{\psi} & J^{n}(X) & \xrightarrow{\varphi} & J^{n-2}(F(X)) \\
\simeq \frac{F^{(n-1)/2}H^{n-2}(F(X))^*}{H^{n-2}(F(X),\mathbb{Z})} & & \simeq \frac{F^{(n+1)/2}H^{n}(X)^*}{H_{n}(X,\mathbb{Z})} & & \simeq \frac{H^{3n-6}(F(X),\mathbb{C})}{F^{3m-3}H^{n-2}(F(X))+H^{3n-6}(F(X),\mathbb{Z})}
\end{array}$$

Note that the intermediate Jacobian $J^n(X)$ is selfdual and the two maps in the bottom row are naturally dual to each other.

4

Cubic surfaces

The general theory presented in Chapter 1 applied to the case of smooth cubic surfaces $S \subset \mathbb{P}^3$ provides us with some crucial information.

On the purely numerical side, we have seen that the Hodge diamond is only non-trivial in bidegree (p, p), i.e. $H^1(S, \mathcal{O}_S) = H^0(S, \Omega_S) = 0$ and $H^2(S, \mathcal{O}_S) = H^0(S, \Omega_S^2) = 0$. Moreover, $H^{1,1}(S) = H^1(S, \Omega_S) \simeq k^7$, see Sections **1**.1.2 and **1**.1.3.

The linear system of all cubics $|\mathcal{O}(3)| \simeq \mathbb{P}^{19}$ comes with a natural PGL(4)-action and its GIT quotient, the moduli space of semi-stable cubic surface, is four-dimensional, see Sections **1**.2.1 and **1**.1.3

We have also seen that the Fano variety $F(S)$ of lines in S is non-empty, smooth, and zero-dimensional of degree 27, see Proposition **3**.3.3 and Example **3**.3.11. Hence, over an algebraically closed field k the Fano variety $F(S)$ consists of 27 isolated k-rational points or, in other words, any smooth cubic surface $S \subset \mathbb{P}^3$ defined over an algebraically closed field contains exactly 27 lines. In this chapter we denote them by $\ell_1, \ldots, \ell_{27}$ or, viewing S as a blow-up of $\mathbb{P}^2$, as $E_1, \ldots, E_6, L_1, \ldots, L_6, L_{12}, \ldots, L_{56}$, see below. There are more classical arguments to deduce this result and we will touch upon some of the techniques in this chapter. However, we will have to resist the temptation to dive into the classical theory too much and instead refer to the rich literature on the subject, see for example [13, 53, 78, 81, 98, 112].

1 Picard group

Let $S \subset \mathbb{P}^3$ be a smooth cubic surface over an arbitrary field k. Its Picard group $\text{Pic}(S)$ coincides with the Néron–Severi group $\text{NS}(S) = \text{Pic}(S)/\text{Pic}^0(S)$, as $H^1(S, \mathcal{O}_S) = 0$. It is endowed with the intersection product $(\mathcal{L}.\mathcal{L}')$, which satisfies the Hodge index theorem and in particular the inequality $(\mathcal{L}.\mathcal{L}')^2 \geq (\mathcal{L})^2 \cdot (\mathcal{L}')^2$ for all line bundles $\mathcal{L}, \mathcal{L}'$ such that $(\mathcal{L})^2 \geq 0$.

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1.1 The only line bundles that come for free on any smooth cubic surface are $\mathcal{O}_S(1) := \mathcal{O}(1)|_S$ and its powers $\mathcal{O}_S(a)$. For example, the canonical bundle is determined by the adjunction formula, see Lemma **1**.1.3,

$$\omega_S \simeq \mathcal{O}_S(-1),$$

with a very ample dual $\omega_S^* \simeq \mathcal{O}_S(1)$. For an arbitrary line bundle $\mathcal{L}$ on S the Hirzebruch–Riemann–Roch formula takes the form

$$\chi(S,\mathcal{L}) = \frac{(\mathcal{L})^2 + (\mathcal{L}.\mathcal{O}_S(1))}{2} + 1, \tag{1.1}$$

as $\chi(S,\mathcal{O}_S) = 1$.

Lemma 1.1. *Any numerically trivial line bundle $\mathcal{L}$ on a smooth cubic surface S is trivial. In particular,* $\mathrm{Pic}(S)$ *is torsion free of finite rank and, in particular,* $\mathrm{Pic}^0(S) = 0$*.*

Proof Indeed, if $\mathcal{L}$ is not trivial, then $(\mathcal{L}.\mathcal{O}_S(1)) = 0$ implies $H^0(S,\mathcal{L}) = 0$ and $H^2(S,\mathcal{L}) \simeq H^0(S,\mathcal{L}^* \otimes \omega_S)^* = 0$. Hence, $\chi(S,\mathcal{L}) \leq 0$, which contradicts (1.1) showing $\chi(S,\mathcal{L}) = 1$. □

Corollary 1.2. *For a smooth cubic surface $S \subset \mathbb{P}^3$ over an arbitrary field k one has*

$$\mathrm{Pic}(S) \simeq \mathrm{NS}(S) \simeq \mathrm{Num}(S) \simeq \mathbb{Z}^{\oplus \rho(S)}$$

with $1 \leq \rho(S) \leq 7$*. For a field extension $k \subset k'$ the base change map*

$$\mathrm{Pic}(S) \hookrightarrow \mathrm{Pic}(S_{k'}) \tag{1.2}$$

is injective. Moreover, if k is algebraically closed, then $\rho(S) = 7$ *and any further base change (1.2) is an isomorphism.*

Proof Use that an invertible sheaf $\mathcal{L}$ on S is trivial if and only if $H^0(S,\mathcal{L}) \neq 0$ and $H^0(S,\mathcal{L}^*) \neq 0$. As $H^0(S_{k'},\mathcal{L}_{k'}) \simeq H^0(S,\mathcal{L}) \otimes_k k'$, this shows the injectivity of (1.2).

Assume k is algebraically closed. The Kummer sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(\ )^n} \mathbb{G}_m \longrightarrow 0\,,$$

with $n = \ell^m$ prime to char(k), yields injections $\mathrm{Pic}(S) \otimes \mathbb{Z}/\ell^m\mathbb{Z} \hookrightarrow H^2_{\acute{e}t}(S,\mu_{\ell^m})$ with a cokernel contained in $H^2(S,\mathbb{G}_m)$. Taking limits, one obtains

$$\mathrm{Pic}(S) \otimes \mathbb{Z}_\ell \hookrightarrow H^2_{\acute{e}t}(S,\mathbb{Z}_\ell(1)) \simeq \mathbb{Z}_\ell(1)^{\oplus 7}, \tag{1.3}$$

as $b_2(S) = 7$, cf. Section **1**.1.6. Together with (1.2), this proves $\rho(S) \leq 7$ for arbitrary fields.

For k algebraically closed, the Brauer group is trivial, i.e. $\mathrm{Br}(S) = H^2(S,\mathbb{G}_m) = 0$. This is analogous to $H^2(S,\mathcal{O}_S^*) = H^2(S,\mathcal{O}_S)/H^2(S,\mathbb{Z}) = 0$ for $k = \mathbb{C}$. Hence, (1.3) is an isomorphism and, therefore, $\rho(S) = 7$. The last assertion follows from a standard spreading out argument and the fact that for $k = \bar{k}$ the Picard variety Pic_S consists of isolated, reduced, k-rational points, cf. [86, Lem. 17.2.2]. □

Example 1.3. Examples of smooth cubics with $\rho(S) < 7$ can be easily produced. For example, if $\mathcal{S} \longrightarrow |\mathcal{O}(3)|$ is the universal cubic surface, then the scheme-theoretic generic fibre S_η satisfies $\mathrm{Pic}(\mathcal{S}_\eta) \simeq \mathbb{Z} \cdot \mathcal{O}(1)|_{\mathcal{S}_\eta}$. Here, $\mathcal{S}_\eta$ is a smooth cubic surface over the (non algebraically closed) function field $k(\eta) \simeq k(t_1, \dots, t_{19})$.

Similarly, if $\mathcal{S}_{\mathbb{P}^1} \longrightarrow \mathbb{P}^1 \hookrightarrow |\mathcal{O}(3)|$ is a Lefschetz pencil, then the other projection $\tau\colon \mathcal{S}_{\mathbb{P}^1} \longrightarrow \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ in the smooth intersection $S_1 \cap S_2 \subset \mathbb{P}^3$ of two smooth cubics. Hence, $\mathrm{Pic}(\mathcal{S}_{\mathbb{P}^1}) \simeq \mathbb{Z} \cdot \mathcal{O}(1)|_{\mathcal{S}_{\mathbb{P}^1}} \oplus \mathbb{Z} \cdot \mathcal{O}(E)$ by the blow-up formula, where $E = \mathbb{P}(\mathcal{N}_{S_1 \cap S_2/\mathbb{P}^3})$ is the exceptional divisor of τ. Therefore, the fibre $\mathcal{S}_\eta$ over the generic point $\eta \in \mathbb{P}^1$, with residue field $k(\eta) \simeq k(t)$, satisfies $\mathrm{Pic}(\mathcal{S}_\eta) \simeq \mathbb{Z}$.

It should be possible to construct in a similar manner examples of smooth cubic surfaces with arbitrary prescribed Picard number $1 \leq \rho \leq 7$.

It is an entirely different matter to produce cubic surfaces with prescribed Picard number over special types of fields, like number fields or finite fields. See below for examples and comments.

The Picard group $\mathrm{Pic}(S) \simeq \mathrm{NS}(S) \simeq \mathbb{Z}^{\oplus \rho(S)}$ together with its non-degenerate intersection pairing defines a lattice of signature $(1, \rho(S) - 1)$. It is an odd lattice, because $(\mathcal{O}_S(1))^2 = 3$. The orthogonal complement $\mathcal{O}_S(1)^\perp \subset \mathrm{Pic}(S)$ is negative definite of rank ≤ 6.

For $k = \mathbb{C}$ the exponential sequence yields an isomorphism of lattices

$$\mathrm{Pic}(S) \simeq H^2(S, \mathbb{Z}).$$

As $H^2(S, \mathbb{Z})$ is unimodular and odd, it is isomorphic to $\mathrm{I}_{1,6}$ and $\mathcal{O}_S(1)^\perp \simeq H^2(S, \mathbb{Z})_{\mathrm{pr}} \simeq E_6(-1)$, cf. Corollary **1**.1.13 and Proposition **1**.1.14. The same conclusions hold over an arbitrary algebraically closed field, as we will show next.

Corollary 1.4. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface over an algebraically closed field k. Then*

$$\mathrm{Pic}(S) \simeq \mathrm{I}_{1,6} \text{ and } \mathcal{O}_S(1)^\perp \simeq E_6(-1).$$

For an explicit basis of both lattices in terms of lines see Section 3.4.

Proof Completely geometric arguments for this description exist. For example, one can use that S is a blow-up of $\mathbb{P}^2$ in six points, which we, however, will deduce from the description of the Picard group, or that S admits a conic fibration $S \longrightarrow \mathbb{P}^1$ with five singular fibres, see Section 2.5 and [138, IV.2.5]. We shall here derive the claim from the description of the intersection pairing $H^2(S, \mathbb{Z})$ of a smooth cubic surface over $\mathbb{C}$.

In characteristic zero, the assertion follows from the complex case. The general case can be reduced to it by means of the specialization map

$$\mathrm{Pic}(S_{\bar\eta}) \hookrightarrow \mathrm{Pic}(S_{\bar t}).$$

Here, $S \longrightarrow \mathrm{Spec}(R)$ is a smooth family of cubic surfaces over a DVR, t and η are the closed and generic points with residue fields $k(t)$ and $k(\eta)$ of positive and zero characteristic, respectively. Specialization is injective, because it is compatible with the intersection product. But $\mathrm{Pic}(S_{\bar\eta}) \simeq \mathrm{I}_{1,6}$ is a unimodular lattice and any isometric embedding of finite index of a unimodular lattice is an isomorphism. Once $\mathrm{Pic}(S)$ is determined, its primitive part is described as in the proof of Proposition **1**.1.14. □

Remark 1.5. The Galois group $\mathrm{Gal}(\bar k/k)$ naturally acts on $\mathrm{Pic}(S_{\bar k})$ and its sublattice $\mathcal{O}_S(1)^\perp \simeq E_6(-1)$. It therefore defines a subgroup $G \subset \mathrm{O}(E_6)$, which is in fact contained in the Weyl group $W(E_6) \subset \mathrm{O}(E_6)$, cf. Section **1**.2.5. Alternatively, the Galois group acts on the configuration of lines $\mathcal{L}(S)$, see Section 3.6, whose automorphism group is $W(E_6)$. Which subgroups can be realized in this way? It is a classical fact that for the scheme theoretic generic cubic surface, which lives over the function field of $|\mathcal{O}(3)|$, leads to $G = W(E_6)$, see Corollary 1.12.

1.2 We next aim at a purely numerical characterization of lines contained in smooth cubic surfaces.

Remark 1.6. (i) Observe that any $\mathbb{P}^1 \simeq L \subset S$ with $(L)^2 = -1$ is in fact a line, i.e. the degree of L as a subvariety of the ambient $\mathbb{P}^3$ is $\deg(L) = 1$ or, still equivalently, $(\mathcal{O}_S(1).\mathcal{O}(L)) = \deg(\mathcal{O}_S(1)|_L) = 1$. Indeed, by adjunction $\mathcal{O}(-2) \simeq \omega_L \simeq (\omega_S \otimes \mathcal{O}(L))|_L = \mathcal{O}_S(-1)|_L \otimes \mathcal{O}(-1)$.

(ii) For a geometrically integral curve $C \subset S$, one obtains from (1.1)

$$1 \geq 1 - h^1(C,\mathcal{O}_C) = \chi(C,\mathcal{O}_C) = \chi(S,\mathcal{O}_S) - \chi(S,\mathcal{O}_S(-C)) = -\frac{(C)^2 - \deg(C)}{2}$$

and, therefore,

$$(C)^2 \geq \deg(C) - 2 \geq -1.$$

If, in addition, $(C)^2 = -1$, then automatically $\deg(C) = 1$ and $h^1(\mathcal{O}_C) = 1$. Hence, again, $L \coloneqq C \simeq \mathbb{P}^1$ is a line. So, combining (i) and (ii), we find that a (-1)-curve, i.e. a geometrically integral curve with $(C)^2 = -1$, on a smooth cubic surface is the same thing as a line.

(iii) Similarly, if $\mathcal{L} \in \mathrm{Pic}(S)$ with $(\mathcal{L}.\mathcal{O}_S(1)) = 1$ and $(\mathcal{L})^2 = -1$, then $\chi(S,\mathcal{L}) = 1$ by (1.1) and, therefore, $H^0(S,\mathcal{L}) \neq 0$. Hence, $\mathcal{L} \simeq \mathcal{O}_S(L)$ for some curve $L \subset S$. But $\deg(L) = (\mathcal{L}.\mathcal{O}_S(1)) = 1$ then implies that L is geometrically integral and hence a line.

Note that these arguments only use the numerical properties of $(S,\mathcal{O}_S(1))$ and the fact that $\omega_S \simeq \mathcal{O}_S(-1)$, which will be useful later on, see for example the proof of Proposition 2.5.

Thus, if $\mathrm{Pic}(S) \simeq \mathrm{I}_{1,6}$ and $\alpha \in \mathrm{I}_{1,6}$ is a characteristic vector of square $(\alpha)^2 = 3$ (cf. proof of Proposition **1**.1.14), then there are natural bijections

$$\begin{aligned} & \{\, \mathbb{P}^1 \simeq L \subset S \mid \text{line} \,\} \\ \simeq \quad & \{\, C \subset S \mid \text{integral},\ (C)^2 = -1 \,\} \\ \simeq \quad & \{\, \beta \in \mathrm{I}_{1,6} \mid (\beta)^2 = -1,\ (\alpha.\beta) = 1 \,\}, \end{aligned}$$

see also the proof below.

We draw two immediate but crucial consequences from it. The first one is usually deduced from a concrete geometric reasoning, which is avoided in the present approach.

Corollary 1.7. *Assume $S \subset \mathbb{P}^3$ is a smooth cubic surface over an algebraically closed field. Then S contains six pairwise disjoint lines $\ell_1, \ldots, \ell_6 \subset S$.*

Proof By Corollary 1.4, the Picard lattice is $\mathrm{Pic}(S) \simeq \mathrm{I}_{1,6}$ and this is all that is needed in the following. As argued in the proof of Proposition **1**.1.14, the class $\alpha = (3, -1 \ldots, -1) \in \mathrm{I}_{1,6}$ (with the harmless but convenient sign change), written in the standard basis $v_0, \ldots, v_6$, and the hyperplane section h_S are both characteristic classes of the same square $(\alpha)^2 = (h_S)^2 = 3$. Hence, after applying an appropriate orthogonal transformation, they coincide. But then the classes v_i, $i = 1, \ldots, 6$ correspond to line bundles $\mathcal{L}_i$ with $(\mathcal{L}_i)^2 = -1$ and $(\mathcal{L}_i.\mathcal{O}_S(1)) = 1$. According to the above remark, $\mathcal{L}_i \simeq \mathcal{O}(\ell_i)$, where the curves $\ell_i \subset S$ are lines. As $(\mathcal{L}_i.\mathcal{L}_j) = (v_i.v_j) = 0$ for $i \neq j$, they are pairwise disjoint. □

Remark 1.8. It is curious to observe that one can reverse the flow of information and deduce from the geometry of a cubic surface information about the lattice $\mathrm{I}_{1,6}$ or E_6. For example, the fact that the Fano variety $F(S)$ of lines on a smooth cubic surface over an algebraically closed field consists of 27 isolated, smooth k-rational points translates into the fact that in the lattice $\mathrm{I}_{1,6}$ there exist exactly 27 classes ℓ with $(\ell.(3, 1, \ldots, 1)) = 1$ and $(\ell)^2 = -1$.

Corollary 1.9. *On a smooth cubic surface $S \subset \mathbb{P}^3$ over an arbitrary field k an invertible sheaf $\mathcal{L}$ is ample if and only if $(\mathcal{L})^2 > 0$ and $(\mathcal{L}.L) > 0$ for every line $L \subset S_{\bar{k}}$.*

Proof Only the 'if-direction' requires a proof. Recall the Nakai–Moishezon criterion for smooth projective surfaces over arbitrary fields, cf. [6]: An invertible sheaf $\mathcal{L}$ is ample if and only if $(\mathcal{L})^2 > 0$ and $(\mathcal{L}.C) > 0$ for every curve $C \subset S$. It is of course enough to test integral curves C, but we may not necessarily be able to reduce to geometrically integral ones. For this reason, one has to take all lines in the base change $S_{\bar{k}}$ into account. As $\mathcal{L}$ is ample if and only if its base change to $S_{\bar{k}}$ is ample, one can thus reduce to the case $k = \bar{k}$. Then any integral curve C is geometrically integral and either $(C)^2 = -1$, in which case $\mathbb{P}^1 \simeq C$ is a line, or $(C)^2 \geq 0$. To prove $(\mathcal{L}.C) > 0$ in the latter case we shall apply the Hodge index theorem. Over an algebraically closed field, S contains at least one line (and in fact 27 of them). Thus, $\mathcal{L}$ and C are contained in the same connected component of the positive cone and, hence, $(\mathcal{L}.C) \geq 0$.

The same remark as at the end of Remark 1.6 applies: Only the numerical properties of $(S, \mathcal{O}_S(1))$ and the fact that $\omega_S \simeq \mathcal{O}_S(-1)$ have been used in the proof. □

We summarize the situation by a description of the ample cone and the effective cone. By definition, the *effective cone* is the cone of all finite, non-negative real linear combinations of curves

$$\mathrm{NE}(S) := \left\{ \sum a_i[C_i] \mid a_i \in \mathbb{R}_{\geq 0} \right\},$$

where $C_i \subset S$ are arbitrary irreducible (or integral) curves. Its dual is the nef cone which can also be described as the closure of the (open) *ample cone*

$$\mathrm{Amp}(S) := \left\{ \sum a_i\mathcal{L}_i \mid a_i \in \mathbb{R}_{>0},\ \mathcal{L}_i \text{ ample} \right\}.$$

Proposition 1.10. *Let S be a smooth cubic surface over an algebraically closed field. Then the effective cone is*

$$\mathrm{NE}(S) = \left\{ \sum_{i=1}^{27} a_i[L_i] \mid a_i \in \mathbb{R}_{\geq 0} \right\},$$

the rational polyhedron spanned by the 27 *lines $\ell_1, \ldots, \ell_{27} \subset S$. The ample cone is the interior of its dual* $\mathrm{NE}(S)^*$ *(again rationally polyhedral):*

$$\mathrm{Amp}(S) = \mathrm{Int}\,(\mathrm{NE}(S)^*)\,.$$

□

This result in particular shows that the ample cones of smooth cubic surfaces over algebraically closed fields all look the same. This is in stark contrast to other types of surfaces, for example K3 surfaces, cf. [86, Ch. 8].

For non algebraically closed fields these cones can be described via the inclusion $\mathrm{Pic}(S) \hookrightarrow \mathrm{Pic}(S_{\bar{k}})$ as $\mathrm{Amp}(S) = \mathrm{Amp}(S_{\bar{k}}) \cap \mathrm{Pic}(S)$ and, dually, $\mathrm{NE}(S) = \mathrm{NE}(S_{\bar{k}}) \cap \mathrm{Pic}(S)$. Hence, rephrasing Corollary 1.9, $\mathcal{L}$ is ample if and only if $(\mathcal{L}.C) > 0$ for all curves C which after base change to the algebraic closure are unions (with multiplicities) of lines.

Remark 1.11. It is not difficult to prove that an ample invertible sheaf on a cubic surface is automatically very ample, see [78, V. Thm. 4.11].

1.3 Consider the family of all smooth cubic surfaces $\mathcal{S} \longrightarrow U := |\mathcal{O}(3)|_{\rm sm}$. In Section **1**.2.5 we discussed the monodromy group of this family, i.e. the image of the natural representation

$$\pi_1(U) \longrightarrow \mathrm{O}(H^2(S, \mathbb{Z})),$$

where $S = \mathcal{S}_0$ is a distinguished smooth fibre. According to Theorem **1**.2.9, this is the group $\tilde{\mathrm{O}}^+(H^2(S, \mathbb{Z}))$ of all orthogonal transformations of the lattice $H^2(S, \mathbb{Z})$ with trivial spinor norm that respect the hyperplane class. In fact, what has been argued in the discussion there is that the monodromy group, as a subgroup of the orthogonal group of the lattice $H^2(S, \mathbb{Z})_{\rm pr} \simeq E_6(-1)$, is the Weyl group $W(E_6)$. Recall that its order is

$$|W(E_6)| = 51.840 = 2^7 \cdot 3^4 \cdot 5$$

and that it is a subgroup of index two of $\mathrm{O}(E_6)$, only the orthogonal transformation given by a global sign change is missing, cf. [40, Sec. 15]. We shall rephrase this in terms of the family of 27 lines. Recall from Section **1**.2.5 that the relative Fano variety of lines of $\mathcal{S} \longrightarrow U$ is an étale morphism

$$F := F(\mathcal{S}/U) \longrightarrow U$$

of degree 27. Any choice of a line $Ll \subset S = \mathcal{S}_0$, induces a natural map $\rho \colon \pi_1(U) \longrightarrow \mathfrak{S}_{27}$, the image of which is also called the monodromy group of the family. The image is isomorphic to the Galois group of the covering, cf. [76, Sec. 1].

Corollary 1.12. *The Galois group or, equivalently, the monodromy group* $\mathrm{Im}(\rho) \subset \mathfrak{S}_{27}$ *of the universal family* $F \longrightarrow U$ *of the* 27 *lines in smooth cubic surfaces* $S \subset \mathbb{P}^3$ *is isomorphic to the Weyl group* $W(E_6)$*.* □

2 Representing cubic surfaces

Cubic surfaces can be viewed from different angles and can be described geometrically in various ways. Each representation highlights particular features. We will briefly describe the most common ones.

2.1 To start, let us try to realize cubic surfaces as blow-ups of simpler surfaces.

Let $S \subset \mathbb{P}^3$ be a smooth cubic over an arbitrary field k and let $\mathbb{P}^1 \simeq E \subset S$ be a smooth, integral, rational curve. Assume E is a (-1)-curve, i.e. $(E)^2 = -1$ or, equivalently, that E is a line. Then S is the blow-up

$$\tau \colon S \longrightarrow \bar{S}$$

of a smooth projective surface $\bar{S}$ in a point $x \in \bar{S}$ with exceptional line E. This is a special case of Castelnuovo's theorem [6, 13, 78]. Alternatively, one could study the linear system $|\mathcal{O}_S(1)\otimes\mathcal{O}(E)|$, which can be checked to be base point free.

More generally, one proves the following.

Lemma 2.1. *Assume $E_1,\ldots,E_m\subset S$ are m pairwise disjoint (-1)-curves. Then S is isomorphic to the blow-up $\tau\colon S\simeq \mathrm{Bl}_{\{x_i\}}(\bar S)\twoheadrightarrow \bar S$ of a smooth, projective surface $\bar S$ with $E_i=\tau^{-1}(x_i)$ as exceptional lines. Furthermore, one has*

(i) *The Picard number satisfies $m\leq\rho(S)-1\leq 6$.*
(ii) *If $m=6$, then $\bar S\simeq\mathbb{P}^2$.*
(iii) *If $m=5$, then $\bar S\simeq \mathrm{Bl}_x(\mathbb{P}^2)$ or $\bar S\simeq\mathbb{P}^1\times\mathbb{P}^1$.*

Proof Indeed, the blow-up of a smooth surface in one point increases the Picard number by one. As $\bar S$ is projective, $\rho(\bar S)\geq 1$. This proves (i).

If $m=6$, then $\bar S$ is minimal and its canonical bundle $\omega_{\bar S}$ satisfies $\omega_S\simeq\tau^*\omega_{\bar S}\otimes\mathcal{O}(\sum E_i)$, where E_i, $i=1,\ldots,6$, are the exceptional lines. Thus, $(\omega_{\bar S})^2=9$. Hence, by classification theory of minimal surfaces of Kodaira dimension $-\infty$, one concludes $\bar S\simeq\mathbb{P}^2$. Note that ruled surfaces over curves of positive genus can be excluded by using $H^1(S,\mathcal{O}_S)=0$.

If $m=5$, then, similarly, $(\omega_{\bar S})^2=8$. Now, if $\bar S$ is not minimal, then it can be blown down once more and the resulting surfaces will then have to be $\mathbb{P}^2$. If $\bar S$ is minimal, then by classification theory S is a Hirzebruch surface, i.e. $\bar S\simeq\mathbb{F}_n:=\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(n))$ over $\mathbb{P}^1$ with $0\leq n\neq 1$. We need to exclude all the cases $0<n$. However, $C_n=\mathbb{P}(\mathcal{O}(n))\subset\mathbb{F}_n$ is a smooth rational curve with $(C_n)^2=-n$. Its strict transform in S is thus a smooth rational curve $\tilde C_n$ with self-intersection $(\tilde C_n)^2\leq -n$. Hence, according to Remark 1.6, $n=0$ or $n=1$. □

Remark 2.2. Consider the two cases $\tau\colon S\twoheadrightarrow\mathbb{P}^2$ and $\tau\colon S\twoheadrightarrow\mathbb{P}^1\times\mathbb{P}^1$ as above, which are given by the linear systems $\mathcal{O}_S(1)\otimes\mathcal{O}(\sum_{i=1}^m E_i)$ with $m=6$ and $m=5$, respectively. Hence, for degree reasons,

$$\mathcal{O}_S(1)\simeq\tau^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}\left(-\sum_{i=1}^6 E_i\right)\ \text{and}\ \mathcal{O}_S(1)\simeq\tau^*\mathcal{O}(2,2)\otimes\mathcal{O}\left(-\sum_{i=1}^5 E_i\right),$$

respectively. Here, $\mathcal{O}(2,2):=\mathcal{O}_{\mathbb{P}^1}(2)\boxtimes\mathcal{O}_{\mathbb{P}^1}(2)$ on $\mathbb{P}^1\times\mathbb{P}^1$. Numerically, in the second case $\mathcal{O}_S(1)$ could a priori also be, for example, $\tau^*\mathcal{O}(4,1)\otimes\mathcal{O}(-\sum_{i=1}^5 E_i)$. However, in this case the first ruling would lead to a family of lines on S, which we know not to exist.

2.2 Assume a smooth cubic $S\subset\mathbb{P}^3$ contains six pairwise disjoint lines $E_1,\ldots,E_6\subset S$. The induced classes $[E_i]\in\mathrm{Pic}(S_{\bar k})\simeq \mathrm{I}_{1,6}$ generate a sublattice $\mathrm{I}_{0,6}\subset\mathrm{I}_{1,6}$. In fact, together with $\mathcal{O}_S(1)\otimes\mathcal{O}(\sum E_i)\in\mathrm{Pic}(S)$ they form a standard basis of $\mathrm{I}_{1,6}$ and so in particular $\mathrm{Pic}(S)\xrightarrow{\sim}\mathrm{Pic}(S_{\bar k})=\mathrm{I}_{1,6}$.

Thus, as a consequence of Corollary 1.7, we obtain the following classical description of cubic surfaces as blow-ups of $\mathbb{P}^2$. The assumption of k being algebraically closed can be weakened to $\mathrm{Pic}(S)$ being a unimodular lattice of rank seven or, equivalently, $\mathrm{Pic}(S) \simeq \mathrm{I}_{1,6}$.

Proposition 2.3. *Let $S \subset \mathbb{P}^3$ be a smooth cubic surface over an algebraically closed field k. Then S is isomorphic to the blow-up $\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2)$ of $\mathbb{P}^2$ in six distinct points $x_i \in \mathbb{P}^2$, $i = 1, \ldots, 6$.* □

Remark 2.4. Assume S can be presented as $\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2)$ as in the proposition. Then there are three sets of curves readily visible that will turn out to be lines:

(i) The exceptional lines $E_1, \ldots, E_6$.
(ii) The strict transform L_{ij}, $i \neq j$, of the line $\bar{L}_{ij} \subset \mathbb{P}^2$ passing through $x_i \neq x_j \in \mathbb{P}^2$.
(iii) The strict transform L_i of a smooth conic $\bar{L}_i \subset \mathbb{P}^2$ passing through the five points $x_{j \neq i} \in \mathbb{P}^2$.

TBC: Picture of all lines.

Let us count them. There are six of type (i). There are 15 of type (ii) under the assumption that no three points are collinear, i.e. that no $x_k \in \bar{L}_{ij}$ for any k distinct from i and j. To count the curves of type (iii), observe that $|\mathcal{O}_{\mathbb{P}^2}(2)|$ is of dimension five. Hence, for arbitrary five points, there exists a conic C containing them all. As the conic C is either smooth or the union of two distinct lines or a double line, under the assumption that no three of the points $x_1, \ldots, x_6$ are collinear, there exists indeed an L_i for every i. This yields another six curves exactly when the six points are not all contained in one conic.

As it turns out, these conditions are automatically satisfied, see Remark 2.6. Moreover, under these conditions the L_{ij} and the L_i are indeed lines, i.e. $(L_{ij})^2 = (L_i)^2 = -1$. Hence, starting with a smooth cubic surface S, the lines of type (i), (ii), and (iii) account for 27, and hence all, lines.

Let us now address the converse and consider the blow-up $\tau\colon S := \mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2) \twoheadrightarrow \mathbb{P}^2$

in six distinct points $x_1, \ldots, x_6 \in \mathbb{P}^2$. Is S then automatically a cubic surface? It turns out that the same conditions on the points $\{x_i\}$ as above have to be imposed.

Proposition 2.5. *Assume $x_1, \ldots, x_6 \in \mathbb{P}^2$ are general in the sense that no three of them are collinear and that they are not all contained in one conic. Then the blow-up $\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2)$ is isomorphic to a cubic surface $S \subset \mathbb{P}^3$.*

Proof More precisely, one shows that the invertible sheaf

$$\mathcal{L} := \tau^*\mathcal{O}(3) \otimes \mathcal{O}\left(-\sum E_i\right)$$

is very ample and that the image of the induced closed embedding $\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2) \xrightarrow{\sim} S \subset \mathbb{P}^3$ is a cubic surface. Here, $E_1, \ldots, E_6$ denote the exceptional divisors.

Classically the assertion is proved by showing that $\mathcal{L}$ separates points and tangent directions, cf. [13, 78]. We shall instead give an argument that uses the general Nakai–Moishezon criterion and some of our earlier considerations.

First note that numerically $(\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2), \mathcal{L})$ indeed behaves like a cubic surface. By the blow-up formula, its Néron–Severi lattice is isomorphic to $\mathrm{I}_{1,6}$ with $\mathcal{L}$ corresponding to the characteristic vector $(3, -1, \ldots, -1)$ and, in particular, $(\mathcal{L})^2 = 3$. Hence, Corollary 1.9 is valid, see the comment at the end of its proof. In fact, only $(\mathcal{L})^2 = 3$ and $\omega_S \simeq \mathcal{L}^*$ are needed. Therefore, $\mathcal{L}$ is ample if and only if $(\mathcal{L}.L) > 0$ for every $\mathbb{P}^1 \simeq L \subset \mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2)$ with $(L)^2 = -1$. If L is one of the exceptional lines, then clearly, $(\mathcal{L}.L) = -(\mathcal{L}.E_i) = 1$. If not, let $D := \tau(L)$ be its image. Then $D \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ for some d. Denote by $m_i := \mathrm{mult}_{x_i}(D)$ the multiplicity of D at the point x_i. Thus, $m_i = 0$ if $x_i \notin D$ and $m_i = 1$ if x_i is a smooth point of D. Moreover, L is the strict transform of D and $\tau^*D = L + \sum m_i E_i$. The latter shows $d^2 = (D)^2 = (\tau^*D)^2 = -1 + 2\sum m_i - \sum m_i^2 = 5 - \sum(m_i - 1)^2$ from which we deduce that $d = 1$ or $d = 2$, i.e. D is a line or a conic. Now, $(\mathcal{L}.L) \leq 0$ is equivalent to $3d \leq \sum m_i$, which for $d = 1, 2$ reads $3 \leq \sum m_i$ and $6 \leq \sum m_i$, respectively. Hence, for $d = 1$, the line D passes through at least three of the points $x_1, \ldots, x_6$. If $d = 2$ and D is a smooth conic, then D contains all $x_1, \ldots, x_6$. If $d = 2$ and D is singular, i.e. consists of two lines, then one of the two contains at least three of the points. However, for general points $x_1, \ldots, x_6$ these two situations are excluded. Hence, $\mathcal{L}$ is indeed ample.

In order to prove that $\mathcal{L}$ is very ample, one first shows that there exists a smooth curve $C \in |\mathcal{L}|$, which then is an elliptic curve. As $H^0(\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2), \mathcal{L}) \simeq H^0(\mathbb{P}^2, \mathcal{I}_{\{x_i\}} \otimes \mathcal{O}_{\mathbb{P}^2}(3))$ the existence of C follows from Bertini's theorem with base points, cf. [78, III. Rem. 10.9.2]. In other words, there exists a smooth elliptic curve in $\mathbb{P}^2$ passing through $x_1, \ldots, x_6$. Its strict transform is C, still a smooth elliptic curve. Next observe that the restriction map

$$H^0(\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2), \mathcal{L}) \twoheadrightarrow H^0(C, \mathcal{L}|_C)$$

is surjective due to $H^1(\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2), \mathcal{O}) = H^1(\mathbb{P}^2, \mathcal{O}) = 0$. As $\deg(\mathcal{L}|_C) = 3$ and any line bundle of degree three on a smooth elliptic curve is very ample, one deduces that $\mathcal{L}$ is base point free. As $(\mathcal{L})^2 = 3$, the induced morphism $\varphi_{\mathcal{L}}\colon \mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2) \longrightarrow \mathbb{P}^3$ is either of degree one or three. However, the latter would imply that $S := \mathrm{Im}(\varphi_{\mathcal{L}})$ is a plane which contradicts $h^0(\mathcal{L}) = 4$. Hence, $\varphi_{\mathcal{L}}$ is generically injective. It does not contract any curve, as $\mathcal{L}$ is ample, and is therefore the normalization of its image S, a possibly singular cubic surface. However, the natural injection $H^0(S, \mathcal{O}_S(m)) \hookrightarrow H^0((\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2), \mathcal{L}^m)$ is a bijection, as both spaces are of the same dimension. Using that $\mathcal{L}^m$, $m \gg 0$, is very ample, this suffices to conclude that $\mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2) \xrightarrow{\sim} S$. □

Remark 2.6. The proof also reveals that whenever a smooth cubic surface S is viewed as a blow-up $S = \mathrm{Bl}_{\{x_i\}}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$, then the points $x_1, \ldots, x_6 \in \mathbb{P}^2$ have to be in general position.

Dimension check: The choice of six generic points in $\mathbb{P}^2$ modulo the action of PGL(3) is indeed $\dim |\mathcal{O}_{\mathbb{P}^3}(3)| - \dim \mathrm{PGL}(4) = 4$, the dimension of the moduli space of smooth cubic surfaces, cf. Section **1**.2.1.

2.3 A similar analysis can be done for blow-ups $\tau\colon S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ in five points. If the five points $x_1, \ldots, x_5 \in \mathbb{P}^1 \times \mathbb{P}^1$ are completely arbitrary, then $\mathcal{L} := \tau^*\mathcal{O}(2,2) \otimes \mathcal{O}(-\sum E_i)$ may not be ample. For example, if two points x_1, x_2 are contained in the same fibre $\bar{F}$ of one of the two projections, then $(\mathcal{L}.F) = 0$ for the strict transform F of that fibre. Similarly, not four of them can lie on the diagonal.

Example 2.7. Work out the exact conditions for the five points in $\mathbb{P}^1 \times \mathbb{P}^1$ that ensure that the blow-up is a cubic surface.

Dimension check: The choice of five general points on $\mathbb{P}^1 \times \mathbb{P}^1$ modulo the action of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is again four-dimensional, the dimension of the moduli space of cubic surfaces.

2.4 Let now $u \in S$ be a point not contained in any line. Consider the projection of S from the point u to a generic plane $\mathbb{P}^2 \subset \mathbb{P}^3$. This yields morphisms

$$S \xleftarrow{\ \sigma\ } \tilde{S} := \mathrm{Bl}_u(S) \xrightarrow{\ \varphi\ } \mathbb{P}^2,$$

determined by the linear system $|\mathcal{I}_u \otimes \mathcal{O}_S(1)|$ on S or, alternatively, by the complete base point free linear system $|\sigma^*\mathcal{O}_S(1) \otimes \mathcal{O}(-E)|$ on $\tilde{S}$, where $E := \sigma^{-1}(u)$. The fibre $\varphi^{-1}(y)$ over $y \in \mathbb{P}^2$ consists of the intersection of the line $\overline{uy}$ through u and y with S or, more precisely, the residual intersection $\overline{uy} \setminus u$.

Thus, as we may assume that u is not contained in any line, $\varphi\colon \tilde{S} \longrightarrow \mathbb{P}^2$ is a finite morphism of degree two ramified along the intersection of S with the polar quadric $P_uS := V(\sum u_i\, \partial_i F)$. To see the last assertion, choose coordinates such that $u = [1 : 0 : 0 : 0]$, $\mathbb{P}^2 = V(x_0)$, and $y = [0 : 1 : 0 : 0]$. Then $P_uS = V(\partial_0 F)$. The intersection of S

with the line $V(x_2, x_3)$ through u and y is singular at $z = [z_0 : z_1 : 0 : 0] \neq u$, i.e. z is a branch point of the projection, if and only if $\partial_0 F(z) = 0$.

Note that $C \coloneqq S \cap P_u S$ is singular at u. Indeed, the tangent plane of $P_u S$ at $u \in P_u S$ is given by $\sum_i x_i \sum_j u_j \, (\partial_i \partial_j F)(u) = \sum_i x_i \sum_j u_j \, (\partial_j \partial_i F)(u) = 2 \sum_i x_i \, (\partial_i F)(u)$, which is also the equation for the tangent plane of S at u. Similarly, one checks that $S \cap P_u S$ is smooth in every other point. The strict transform $\tilde{C} \subset \tilde{S}$ of C is the branch curve of φ. Observe that this implies that $\tilde{C}$ is contained in the linear system of $\varphi^* \omega^*_{\mathbb{P}^2} \otimes \omega_{\tilde{S}} \simeq \varphi^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes \sigma^* \mathcal{O}_S(-1) \otimes \mathcal{O}(E) \simeq \varphi^* \mathcal{O}_{\mathbb{P}^2}(2) \simeq \sigma^* \mathcal{O}_S(2) \otimes \mathcal{O}(-2E)$. This confirms $C \in |\mathcal{O}_S(2)|$. Also note that the smoothness of S implies that $\tilde{C}$ is smooth, i.e. C is smooth away from u and has multiplicity two at u. Moreover, $D \coloneqq \varphi(\tilde{C}) \subset \mathbb{P}^2$ is a smooth quartic.

Summarizing, the blow-up of a smooth cubic surface in a point not contained in any line is a double cover of $\mathbb{P}^2$ ramified over a smooth quartic curve. The converse of the construction holds as well:

Proposition 2.8. *Assume $k = \bar{k}$ and let $\varphi\colon \tilde{S} \longrightarrow \mathbb{P}^2$ be a double cover ramified along a smooth quartic curve $D \subset \mathbb{P}^2$. Then there exists a (-1)-curve in $\tilde{S}$ the contraction $\tilde{S} \longrightarrow S$ of which is isomorphic to a smooth cubic surface S.*

TBC: Picture of the double cover, preimage of the bitangent.

Proof First note that $\omega_{\tilde{S}} \simeq \varphi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \varphi^* \mathcal{O}_{\mathbb{P}^2}(-1)$. Next, let $E \subset \tilde{S}$ be an irreducible component of one of the 28 bitangents ℓ of D, cf. Section 3.7. Then E is a (-1)-curve and we show that its contraction yields a cubic surface.

Compute the normal bundle $\mathcal{N}_{E/\tilde{S}}$ as the kernel of $\varphi^* \mathcal{N}_{\ell/\mathbb{P}^2} \longrightarrow \mathcal{O}_{(D \cap \ell)_{\mathrm{red}}}$ to see that indeed $(E)^2 = -1$. Let $\sigma\colon \tilde{S} \longrightarrow S$ be the contraction of E. If we let $\mathcal{O}_S(1)$ be the dual of ω_S, then $\sigma^* \mathcal{O}_S(1) \otimes \mathcal{O}(-E) \simeq \varphi^* \mathcal{O}_{\mathbb{P}^2}(1)$. To conclude, one argues as in the proof of Proposition 2.5. First, twisting the structure sequence for $E \subset \tilde{S}$ with $\varphi^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}(E)$ shows that $h^0(S, \mathcal{O}_S(1)) = 4$. Therefore, the associated linear system defines a map to $S \longrightarrow \mathbb{P}^3$, which is readily seen to be regular. Using ampleness of $\varphi^* \mathcal{O}_{\mathbb{P}^2}(1)$, one shows that it is in fact an embedding. Eventually, observe that $(\mathcal{O}_S(1))^2 = (\varphi^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}(E))^2 = 2\,(\mathcal{O}_{\mathbb{P}^2}(1))^2 + 2\,(\varphi^* \mathcal{O}_{\mathbb{P}^2}(1).E) + (E)^2 = 3$. □

Dimension check: The moduli space $\mathcal{M}_g$ of smooth curves of genus $g = 3$ is of dimension $3g - 3 = 6$. The canonical embedding of the non-hyperelliptic ones yield smooth plane curves $D \subset \mathbb{P}^2$ of degree four. Cubic surfaces together with the choice of the additional point $u \in S$ needed for the passage to plane quartic curves also make up for a six-dimensional family.

2.5 We apply the general construction of Section **1**.5.1. So, pick a line $L \subset S$ in a smooth cubic surface and consider the linear projection

$$\varphi\colon S \twoheadrightarrow \mathbb{P}^1$$

from L to a generic line $\mathbb{P}^1 \subset \mathbb{P}^3$. Usually, the linear projection is only rational, but as L is of codimension it is regular in this case or, equivalently, $\mathrm{Bl}_L(S) \longrightarrow S$ is an isomorphism. The fibres $\varphi^{-1}(y)$ are the residual conics of the intersection $L \subset S \cap \overline{yL}$. In particular, $(\varphi^{-1}(y).L) = 2$ and $\varphi\colon L \longrightarrow \mathbb{P}^1$ is of degree two.

According to Lemma **1**.5.1 there are exactly five singular fibres. For degree reasons, none of the fibres is multiple. Hence, the singular fibres $\varphi^{-1}(y_i)$, $i = 1, \ldots, 5$ consist of two lines intersecting each other and both intersecting L.

TBC: Picture of fibration with the six singular fibres, all meeting the line.

2.6

3 Lines on cubic surfaces

Once the 27 lines have been found and described geometrically, one can study the configurations from various angles. We collect a few observations concerning their positions, which have been studied classically, just by looking at the three types of lines (i)-(iii) as introduced in Remark 2.4. In the following, we let S be a smooth cubic surface with fixed six pairwise disjoint lines $E_1, \ldots, E_6$ viewed as the exceptional lines of a contraction $S \longrightarrow \mathbb{P}^2$.

The 27 lines on a cubic surface are among the most studied geometric objects in classical mathematics. That there are only finitely many was proved by Cayley in 1849 and then Salamon immediately observed that there are exactly 27 of them. Both papers, with the same title, appeared in the same volume of the Cambridge and Dublin Math. Journal [30, 134]. We recommend the introduction to [81] and the essay [41] for more on the history and a discussion of the various notations.

3.1 Any line $L \subset S$ can be realized as an exceptional line E_1' of some blow-down $S \twoheadrightarrow \mathbb{P}^2$. In other words, for any line $L \subset S$ there exist five lines $E_2', \ldots, E_6'$ such that $E_1' \coloneqq L, E_2', \ldots, E_6'$ are pairwise disjoint lines.

This is clear if L is of type (i), i.e. if L is already one of the exceptional lines E_i. For those of type (ii) and (iii) just observe that $L_{12}, L_{13}, L_{14}, L_{15}, E_6, L_6$ is a collection of pairwise disjoint lines involving at least one line of each type. That the first five are pairwise disjoint is easy and also that E_6 and L_6 are disjoint. To see that $L_6 \cap L_{1j} = \emptyset$ for $j \neq 6$, observe that the intersections of their images, a conic and a line in$\mathbb{P}^2$, satisfies $\bar{L}_6 \cap \bar{L}_{1j} = \{x_1, x_j\}$. Therefore, the intersection is transversal at both points and, hence, the intersection of the strict transforms is empty.

Remark 3.1. Note that $L_i \cap L_{ij} \neq \emptyset \neq L_j \cap L_{ij}$. Indeed, either $\bar{L}_i \cap \bar{L}_{ij}$ consists of x_j and another point $x \notin \{x_k\}$ or of x_j with multiplicity two. In the first case, L_i and L_{ij} intersect in x (or rather in the unique point lying above x), while in the second case they meet in the point in E_j corresponding to the common tangent direction of $\bar{L}_i$ and $\bar{L}_{ij}$ at x_j.

3.2 Any two disjoint lines are alike, i.e. any two disjoint lines L, L' can be complemented to $E_1' \coloneqq L, E_2' \coloneqq L', E_3', \ldots, E_6'$ of six pairwise disjoint lines, which then can be viewed as the exceptional lines of a blow-down of S to $\mathbb{P}^2$.

Indeed, we only have to consider the following three cases: (i) $L = E_1$ and $L' = E_2$, (ii) $L = E_1$ and $L' = L_{23}$, and (iii) $L = E_1$ and $L' = L_1$. Of course, (i) can be completed by $E_3, \ldots, E_6$ and for (ii) and (iii) use a configuration of the type considered above already: $E_1, L_1, L_{23}, L_{24}, L_{25}, L_{26}$.

3.3 For every line $L \subset S$ there exist exactly 10 lines intersecting L. Moreover, these 10 lines come in pairs $\{\ell_1, \ell_1', \ldots, \ell_5, \ell_5'\}$ such that every two pairs say, $\{\ell_1, \ell_1'\}$ and $\{\ell_2, \ell_2'\}$, are disjoint, i.e. $(\ell_1 \cup \ell_1') \cap (\ell_2 \cup \ell_2') = \emptyset$. Furthermore, all triangles L, ℓ_i, ℓ_i' are coplanar, i.e. there exists a plane $\mathbb{P}^2 \subset \mathbb{P}^3$ with $S \cap \mathbb{P}^2 = L \cup \ell_i \cup \ell_i'$.

According to Section 3.1, we may assume $L = E_6$. Going through the list, one finds that indeed E_6 intersects only $L_{16}, L_{26}, L_{36}, L_{46}, L_{56}$ and L_1, L_2, L_3, L_4, L_5. We let $\ell_i \coloneqq L_{i6}$ and $\ell_i' \coloneqq L_i$, $i = 1, \ldots, 5$.

Then check that for $i \neq j \in \{1, \ldots, 5\}$, for example $i = 1$, $j = 2$, one has $L_{i6} \cap L_{j6} = \emptyset$, $L_{i6} \cap L_j = \emptyset$ (see the arguments in Section 3.1), and $L_i \cap L_j = \emptyset$. For the last one use that, for example $\bar{L}_1 \cap \bar{L}_2$ consists of the four points $x_3, \dots, x_6$. Hence, the intersection is transversal and, therefore, the intersection $L_1 \cap L_2$ of their strict transforms is empty. It remains to verify that L, ℓ_i, ℓ'_i are coplanar. For this assume $i = 1$ and observe that

$$\begin{aligned} & \mathcal{O}(E_6) \otimes \mathcal{O}(L_{16}) \otimes \mathcal{O}(L_1) \\ \simeq\ & \mathcal{O}(E_6) \otimes (\tau^*\mathcal{O}(1) \otimes \mathcal{O}(-E_1 - E_6)) \otimes (\tau^*\mathcal{O}(2) \otimes \mathcal{O}(-\textstyle\sum_{i>1} E_i)) \\ \simeq\ & \tau^*\mathcal{O}(3) \otimes \mathcal{O}(-\textstyle\sum E_i) \simeq \mathcal{O}_{\mathbb{P}^3}(1)|_S. \end{aligned}$$

Note that the plane containing L, ℓ_i, ℓ'_i is tangent at each point of the three lines.

Remark 3.2. Each of the coplanar unions $L \cup \ell_i \cup \ell'_i$ is either a triangle, i.e. it has three singular points, or consists of three lines all going through one point. This corresponds to the two possibilities that the line $\bar{L}_{i6}$ and the conic $\bar{L}_i$ intersect transversally in x_6 or with multiplicity two, so $\bar{L}_{i6}$ tangent to $\bar{L}_i$ at x_6.

From the above count, one deduces that every smooth cubic surface admits exactly 45 *tritangent planes*, i.e. planes that intersect the cubic in the union of three pairwise distinct planes.

It is possible to show that there are nine of the tritangent plane which cut out all the 27 lines contained in S. In this sense the union of all lines on a cubic surface is described as the intersection with a (highly degenerate) surface of degree nine.

To prove the existence of the five pairs of lines intersecting $L \subset S$ one could alternatively use the linear projection $\varphi\colon S \dashrightarrow \mathbb{P}^1$ from L, see Section 2.5. They occur as the five singular fibres $\varphi^{-1}(y_i) = \ell_i \cup \ell'_i$ of φ.

3.4 The above discussion of the geometry of lines is useful when it comes to writing down explicit bases of $\mathrm{Pic}(S) \simeq \mathrm{I}_{1,6}$ and $\mathcal{O}_S(1)^\perp \simeq \mathrm{E}_6(-1)$ in terms of lines.

If $f_0, \dots, f_6$ denotes the standard basis of $\mathrm{I}_{1,6}$, set $f_i = E_i$, $i = 1, \dots, 6$. Then consider a tritangent plane, for example $E_6 + L_1 + L_{16}$, which corresponds to $3f_0 + f_1 + \cdots + f_6$. So, $E_1, \dots, E_6$ together with $E_6 + L_1 + L_{16}$ already generate a sublattice of $\mathrm{I}_{1,6}$ of index three. To generate all of $\mathrm{I}_{1,6}$ by lines, observe that $(L_1.E_i) = 1$, $i = 2, \dots, 6$, $(L_1.E_1) = 0$ and, therefore $L_1 = 2f_0 + 0f_1 + f_2 + \cdots + f_6$.

Spelling out the comments in the proof of Proposition **1**.1.14, a basis of $\mathcal{O}_S(1)^\perp \simeq E_6(-1)$ is then given by $e_1 = E_1 - E_2, e_2 = E_2 - E_3, e_3 = E_3 - E_4, e_5 = E_4 - E_5, e_6 = E_5 - E_6$, an $e_4 = -E_1 + E_4 + E_5 + 2E_6 + L_{16}$.

3.5 For any pair of disjoint lines L, L' there exist exactly five lines $\ell_1, \dots, \ell_5$ meeting both. Moreover, those five lines are pairwise disjoint. According to Section 3.2, we may assume $L = E_1$ and $L' = E_2$. The lines meeting E_1 are

$$L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_2, L_3, L_4, L_5, L_6$$

and those meeting E_2 are

$$L_{12}, L_{23}, L_{24}, L_{25}, L_{26}, L_1, L_3, L_4, L_5, L_6.$$

Hence, the ones meeting both lines, E_1 and E_2, are $L_{12}, L_3, L_4, L_5, L_6$, which we have seen to be pairwise disjoint already.

This collection of five pairwise disjoint lines is special and not at all like, for example, the lines $E_1, \ldots, E_5$. Namely, there is no further line disjoint to all of the lines $L_{12}, L_3, L_4, L_5, L_6$. Indeed, the lines E_i all intersect at least one of them. The lines L_{1j}, $j = 3, \ldots, 6$, intersect L_j (cf. Remark 3.1), the lines L_{ij}, $2 < i < j$, intersect L_{12}, and L_1, L_2 also both intersect L_{12}, see Remark 3.1.

As a consequence of Lemma 2.1, one finds

Corollary 3.3. *Any pair of disjoint lines $L, L' \subset S$ gives rise to a blow-down*

$$S \longrightarrow L \times L' \simeq \mathbb{P}^1 \times \mathbb{P}^1$$

contracting exactly the five lines intersecting both lines L, L'. □

There is a very geometric way of describing this blow-down, cf. [13]. Namely, for any point $x \in S \setminus (L \cup L')$ the plane $\overline{xL'}$ spanned by L' and x intersects L in exactly one point u_x. Similarly, $\overline{xL}$ intersects L' in u'_x. This defines

$$S \setminus (L \cup L') \longrightarrow L \times L',\ x \longmapsto (u_x, u'_x),$$

which can be extended to all of S by replacing $\overline{xL'}$ for $x \in L'$ by the tangent plane T_xS (which contains L'). Also, in this description one sees that exactly the lines $\ell_1, \ldots, \ell_5$ are contracted. Their images are the points (u_i, u'_i), where $\ell_i \cap L = \{u_i\}$ and $\ell_i \cap L' = \{u'_i\}$.

3.6 Consider the *configuration*

$$\mathcal{L} := \mathcal{L}(S) := \{ \ell_1, \ldots, \ell_{27} \}$$

of all lines contained in a cubic surface S. By definition, it not only encodes the set of all lines, but also their intersection numbers (but not, for examples, the intersection points and, in particular, not whether there are triple intersection points), and is independent of the actual surface S. Alternatively, view $\mathcal{L}$ as the graph with vertices ℓ_i and with two vertices ℓ_i, ℓ_j connected if the two lines intersect. Its complement, i.e. the graph with the same set of vertices but with vertices connected by an edge if and only if they are not connected in $\mathcal{L}$, is the so-called *Schläfli graph*.

Note that naming the first six lines as $\ell_1 = E_1, \ldots, \ell_6 = E_6$ as the exceptional lines of a blow-up $S \longrightarrow \mathbb{P}^2$ determines uniquely the remaining 21 lines. For example, L_{12} is the unique line that intersects ℓ_1 and ℓ_2 but no $\ell_3, \ldots, \ell_6$ and L_1 is the unique line that intersects $\ell_2, \ldots, \ell_6$ but not ℓ_1. Moreover, according to Lemma 2.1, any subset $\{\ell_{i_1}, \ldots, \ell_{i_6}\}$ of six pairwise disjoint lines can be realized as the exceptional lines of a blow-up $S \longrightarrow \mathbb{P}^2$.

In other words, for any two choices $\ell_1, \ldots, \ell_6$ and $\ell'_1, \ldots, \ell'_6$ of six pairwise disjoint lines, there exists an automorphism $g\colon \mathcal{L} \xrightarrow{\sim} \mathcal{L}$ of the configuration with $g(\ell_i) = \ell'_i$. Thus, choosing six pairwise disjoint lines $\ell_1, \ldots, \ell_6$ is equivalent to giving an element in $\mathrm{Aut}(\mathcal{L})$. This allows one to compute the order

$$|\mathrm{Aut}(\mathcal{L})| = 27 \cdot 16 \cdot 10 \cdot 6 \cdot 2 = 2^7 \cdot 3^4 \cdot 5 = 51.849.$$

Indeed, there are 27 choices for E_1, then 16 choices for E_2, etc. Of course, it is no coincidence that the order is the order of the Weyl group $W(E_6)$.

The configuration of lines presented by the entries of the matrix

$$\begin{pmatrix} E_1 & E_2 & E_3 & E_4 & E_5 & E_6 \\ L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \end{pmatrix}$$

is what is called a *Schläfli double six*. It has the property that each line intersects exactly those lines in the matrix that are neither contained in the same row nor in the same column. As L_i, L_j, $i \neq j$, are disjoint, there are exactly 30 intersection points.

Example 3.4. Show that there exist 36 double sixes in each smooth cubic surface.

It is a classical fact that any double six of actual lines in $\mathbb{P}^3$ contained in a cubic surface S determines the surface uniquely. In fact, given five skew lines in $\mathbb{P}^3$ and one that intersects them all (think of $E_1, \ldots, E_5$ and L_6), there exists a unique cubic surface containing the six lines as part of a double six.

In Section **1**.2.5 we have seen that the monodromy group of the family of all smooth cubics is the Weyl group $W(E_6)$. As the discriminant divisor has degree 2^6, see Theorem **1**.2.2, the Weyl group is generated by 2^6 reflections. In [39] it has been shown that $W(E_6)$ can also be generated by six reflections and one transformation that is given by interchanging the two rows of a double six.

3.7 Let us now make use of the description of a cubic surface S as a double cover of $\mathbb{P}^2$, cf. Section 2.4. So, fix a point $u \in S$ not contained in any of the lines and let $\varphi\colon \tilde{S} = \mathrm{Bl}_u(S) \twoheadrightarrow \mathbb{P}^2$ the projection onto a generic plane. We denote the exceptional line of the blow-up by E and the ramification curve by $D := \varphi(\tilde{C}) \subset \mathbb{P}^2$, a smooth quartic curve. Then there exists a natural bijection between the 28 bitangent lines of D and the 27 lines in S together with E:

$$\{\, \ell \subset \mathbb{P}^2 \mid \text{ bitangents to } D \,\} \longleftrightarrow \{\, \ell_1, \ldots, \ell_{27} \subset S \mid \text{ lines } \} \cup \{E\}.$$

First, observe that each line $\ell_i \subset S$, simultaneously considered as a curve in $\tilde{S}$, satisfies $1 = (\ell_i.\mathcal{O}_S(1)) = (\ell_i.\varphi^*\mathcal{O}_{\mathbb{P}^2}(1))$. Hence, $\varphi(\ell_i) \subset \mathbb{P}^2$ is a line and $\ell_i \xrightarrow{\sim} \bar{\ell}_i := \varphi(\ell_i)$ is an isomorphism. Similarly, $(E.\varphi^*\mathcal{O}_{\mathbb{P}^2}(1)) = 1$ and, therefore, $E \xrightarrow{\sim} \bar{E} := \varphi(E) \subset \mathbb{P}^2$

is also a line. However, lines in $\mathbb{P}^2$ whose pre-image under φ split off a copy of the line cannot intersect D transversally at any point. Hence, it has to be a bitangent of D.[1]

Lemma 3.5. *Let $D \subset \mathbb{P}^2$ be a smooth quartic curve over an algebraically closed field k with* $\mathrm{char}(k) \neq 2$. *Then D admits exactly* 28 *bitangent lines.*

Proof As a first step one observes that $\omega_D \simeq \mathcal{O}_{\mathbb{P}^2}(1)|_D$. Therefore, a bitangent through $x, y \in D$ (or a hyperflex through $x = y \in D$) corresponds to an invertible sheaf $N \in \mathrm{Pic}^2(D)$ with $H^0(D, N) \neq 0$ and $N^2 \simeq \omega_D$. The number of square roots of ω_D, called theta-characteristics, is of course $2^{2g(D)} = 64$. However, only 28 of them are effective. For this one has to compute the degree of the map $S^2(D) \twoheadrightarrow \mathrm{Pic}^2(D) \twoheadrightarrow \mathrm{Pic}^4(D)$.

Alternatively, one could use the Plücker formula for smooth curves $C \subset \mathbb{P}^2$ with only bitangents and simple flexes. It turns out that there exist 24 flexes and 28 bitangents, see [76, Sec. II]. □

As $\varphi^{-1}(\bar{\ell}_i) \twoheadrightarrow \bar{\ell}_i$ is of degree two, $\varphi^{-1}(\bar{\ell}_i) = \ell_i \cup \ell'_i$ with $\ell'_i \xrightarrow{\sim} \varphi(\ell'_i) = \bar{\ell}_i$. The two curves ℓ_i and ℓ'_i intersect in the pre-image of the points of contact $\bar{\ell}_i \cap D$. Note that ℓ'_i does not correspond to a line in S, as two lines in S intersect at most in one point and there transversally. Instead, $(\sigma(\ell'_i).\mathcal{O}_S(1)) = 2$ and $(\ell'_i.E) = 1$, i.e. u is a smooth point of the curve $\sigma(\ell'_i)$. Indeed, $\ell_i \cup \ell'_i = \varphi^{-1}(\bar{\ell}_i)$ is a curve in the linear system of $\varphi^*\mathcal{O}_{\mathbb{P}^2}(1) \simeq \sigma^*\mathcal{O}_S(1) \otimes \mathcal{O}(-E)$. Hence, $2 = (\ell_i \cup \ell'_i.\varphi^*\mathcal{O}_{\mathbb{P}^2}(1)) = 1 + (\ell'_i.\sigma^*\mathcal{O}_S(1) \otimes \mathcal{O}(-E))$. As ℓ'_i is not a line and, hence, $(\ell'_i.\mathcal{O}_S(1)) > 1$, one has $(\ell'_i.E) \geq 1$ and in fact $(\ell'_i.E) = 1$, for the two lines $\varphi(\ell'_i) = \bar{\ell}_i$ and $\varphi(E)$ intersect in one point only.

For example, for the line $\bar{L}_{12}$ there exists a unique conic Q through x_3, x_4, x_5, x_6 and u intersecting $\bar{L}_{12}$ in two points distinct from x_1, x_2. The strict transform $\tilde{Q} \subset \tilde{S}$ of Q is contained in the linear system of $\sigma^*(\tau^*\mathcal{O}(2) \otimes \mathcal{O}(-\sum_{i \neq 1,2} E_i)) \otimes \mathcal{O}(-E)$. This line bundle is indeed isomorphic to $\varphi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}(-L_{12})$, and, hence, $L_{12} \cup \tilde{Q} = \varphi^{-1}(\varphi(L_{12}))$. Note that $\tilde{Q}$ is a (-1)-curve in $\tilde{S}$, but not its image in S.

Remark 3.6. For the universal family $\mathcal{D} \twoheadrightarrow U \coloneqq |\mathcal{O}_{\mathbb{P}^2}(4)|_{\mathrm{sm}}$ of smooth quartic curves in $\mathbb{P}^2$, the relative family of bitangents $B(\mathcal{D}/U) \twoheadrightarrow U$ is an étale map of degree 28. Its Galois group or, equivalently, its monodromy group $\mathrm{Im}(\pi_1(U) \twoheadrightarrow \mathfrak{S}_{28})$ is isomorphic to $\mathrm{Sp}_6(\mathbb{Z}/2\mathbb{Z})$, which is of order $2^5 \cdot 3^2$. See [76, Sec. II:4] and compare this to Corollary 1.12 and the discussion Section **1**.2.5.

3.8 A point $x \in S$ in a smooth cubic surface S is called an *Eckardt point* if the tangent plane at $x \in S$ intersects S in three lines through x. How many Eckardt points can a smooth cubic surface have?

[1] By definition, a bitangent of D is a line in $\mathbb{P}^2$ that intersects D in two points x, y with multiplicity (at least) two. The case $x = y$ is allowed, in which the bitangent has multiplicity four at this point. This is sometimes also called a hyperflex. The locus of smooth quartic curves with a hyperflex is a divisor in the moduli space of curves of genus three, cf. [42, 84].

In Remark 3.2 we have seen examples of this, namely three lines consisting of an exceptional line E_6, the strict transform L_i, $i \neq 6$, of the conic $\bar{L}_i$ (which contains x_6), and the strict transform L_{i6} of the line $\bar{L}_{i6}$ tangent to $\bar{L}_i$ (at x_6). As for each i there exist only two lines through x_i tangent to $\bar{L}_i$ at some point, each conic $\bar{L}_i$ will give rise to at most two Eckardt points. So altogether, there exist at most 12 Eckardt points of this type.

However, Eckardt points may arise in a different way namely as the triple intersection $\bar{L}_{i_1 i_2} \cap \bar{L}_{i_3 i_4} \cap \bar{L}_{i_5 i_6}$ with $\{i_1, \dots, i_6\} = \{1, \dots, 6\}$. Generically, this triple intersection would consist of three points, but star shaped configuration are of course possible.[2]

For generic choices of points $x_1, \dots, x_6 \in \mathbb{P}^2$ we do not expect any of these two possibilities to occur. Namely, neither will any of the conics $\bar{L}_i$ be tangent to any line $\bar{L}_{ij}$ nor will the line configuration show stars.

Proposition 3.7. *The number of Eckardt points on a cubic surface S is one of the following numbers:* $0, 1, 2, 3, 4, 6, 9, 10$*, or* 18.

Remark 3.8. The article [147] (see also the author's thesis) studies the loci $H_k \subset |\mathcal{O}_{\mathbb{P}^3}(3)|_{\mathrm{sm}}$ of smooth cubic surfaces with at least k Eckardt points. They are invariant under the action of PGL(4) and, thus, determine closed subschemes

$$\bar{H}_k := H_k/\mathrm{PGL}(4) \subset M_3 = M_{3,3}$$

of the four-dimensional moduli space of smooth cubic surfaces. For example, it turns out that $\bar{H}_1 \subset M_3$ is an irreducible divisor, i.e. of dimension three, and that $\bar{H}_k$ is zero-dimensional for $k \geq 10$.

This is of course compatible with the above proposition. As soon as the surface S contains more than 10 Eckardt points, it contains 18 Eckardt points.

Moreover, $\bar{H}_{10}$ consists of exactly two points, corresponding to the Clebsch surface and the Fermat cubic. The latter admits 18 Eckardt points and is the only point in $\bar{H}_{11} = \cdots = \bar{H}_{18}$. We refer to [53] for more details.

4 Moduli space

4.1

[2] I would guess that some combinatorial argument shows that at most six points can occur in this way. However, the next result may be valid without it. A priori it could happen that whenever there are more stars in the line configuration associated with the six points, then fewer conics $\bar{L}_i$ are tangent to those lines.

5

Cubic threefolds

This chapter is devoted to cubic hypersurfaces $Y \subset \mathbb{P}(V) \simeq \mathbb{P}^4$ of dimension three. We will be mostly interested in smooth ones, but (mildly) singular ones will also occur. Cubic threefolds and their Fano surfaces of lines have a long and distinguished history in algebraic geometry, going back to the Italian school, and Gino Fano [60] in particular, and the landmark article of Clemens and Griffiths [34], proving irrationality of all smooth cubic threefolds and introducing the intermediate Jacobian as a key tool. Cubic threefolds have also served as a testing ground for the Weil conjectures already in [22] and their geometry has been investigated further in the series of papers of Tyurin [145, 146, 144], Murre [120, 121], Beauville [9, 10], and many others

Before getting started, let us collect the basic facts on cubic threefolds that follow from the general theory as presented in Chapter **1**. For simplicity we usually work over $\mathbb{C}$, see Section **??** for cubic threefolds over other fields.

0.1 The canonical bundle of a smooth cubic threefold $Y \subset \mathbb{P}^4$ is $\omega_Y \simeq \mathcal{O}_Y(-2)$, which is the square of the dual of the ample generator of $\mathrm{Pic}(Y) \simeq \mathbb{Z} \cdot \mathcal{O}_Y(1)$. The non-trivial Betti numbers of Y are as follows:

$$b_0(Y) = b_2(Y) = b_4(Y) = b_6(Y) = 1, \text{ and } b_3(Y) = 10$$

and, therefore, its Euler number $e(Y) = -6$, see Section **1**.1.

0.2 As for cubic surfaces, the geometry of lines on cubic threefolds is particularly rich and interesting. In dimension three however, every point is contained in a line and a generic point is contained in exactly six lines, cf. Example **3**.4.5. As for a smooth cubic threefold $\mathrm{Pic}(Y) \simeq \mathbb{Z} \cdot \mathcal{O}_Y(1)$, there are no planes contained in Y.

The general theory of Fano varieties of lines as outlined in Section **3**.1 provides us with very useful information:

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(i) The Fano variety of lines $F := F(Y)$ of a smooth cubic threefold Y is a smooth, irreducible, projective surface, the *Fano surface*.
(ii) The canonical bundle ω_F of F is ample and $\omega_F \simeq \mathcal{O}_F(1)$. Here, $\mathcal{O}_F(1)$ is the Plücker polarization induced by $F \hookrightarrow \mathbb{G}(1,\mathbb{P}^4) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$, see Lemma **3**.2.1.
(iii) Its degree with respect to the Plücker polarization $g = c_1(\mathcal{O}_F(1))$ is

$$\deg(F) = \int_F g^2 = 45.$$

(iv) The Euler number of the Fano surface is $e(F) = 27$, see Proposition **3**.3.3 and Section 2.1.
(v) The Hodge diamond (half of it) of F is, cf. Section **3**.3.4:

$$\begin{array}{lccccc} b_0(F(Y)) = 1 & & & 1 & & \\ b_1(F(Y)) = 10 & & 5 & & 5 & \\ b_2(F(Y)) = 45 & 10 & & 25 & & 10. \end{array}$$

(vi) For the universal family of lines on Y

$$F(Y) \xleftarrow{\ p\ } \mathbb{L} \xrightarrow{\ q\ } Y$$

the morphism $q\colon \mathbb{L} \twoheadrightarrow Y$ is generically finite of degree six cf. Lemma **3**.4.4.
(vii) The Fano correspondence $\varphi = p_* \circ q^*\colon H^3(Y,\mathbb{Q}) \xrightarrow{\sim} H^1(F,\mathbb{Q})(-1)$ is an isomorphism of rational Hodge structures which according to Proposition **1**.4.2 satisfies

$$(\alpha.\beta) = -\frac{1}{6}\int_F \varphi(\alpha)\cdot\varphi(\beta)\cdot g,$$

(viii) There exist isomorphisms of Hodge structures, cf. Section **3**.3.4,

$$H^3(Y,\mathbb{Q})(1) \simeq H^1(F(Y),\mathbb{Q}) \text{ and } \bigwedge^2 H^3(Y,\mathbb{Q})(2) \simeq \bigwedge^2 H^1(F(Y),\mathbb{Q}) \simeq H^2(F(Y),\mathbb{Q}),$$

where the first one is given by the Fano correspondence and the second one by taking exterior product, see below. As Deligne's invariant cycle theorem [149, V. Thm. 16.24] shows that for very general Y the only rational Hodge class in $\bigwedge^2 H^3(Y,\mathbb{Q})$, up to scaling, is the one given by the intersection product on Y, the sheer existence of the second isomorphism already shows that for the very general cubic threefold $Y \subset \mathbb{P}^4$ the Picard number is

$$\rho(F(Y)) = \operatorname{rk}\operatorname{NS}(F(Y)) = 1.$$

1 Lines on the threefold and curves on the Fano

We consider natural curves in the Fano surface $F(Y)$ of a smooth cubic threefold $Y \subset \mathbb{P}^4$ over an arbitrary algebraically closed field. Firstly, there is the curve of lines of the second type $R \subset F(Y)$, cf. Section **3**.1.2. Its pre-image to $\mathbb{L}$ turns out to be the ramification divisor of $q\colon \mathbb{L} \longrightarrow Y$. Secondly, for each line $L \subset Y$ one considers the closure $C_L \subset F(Y)$ of the curve of all lines $L \neq L' \subset Y$ intersecting L. It comes with a natural fixed point free involution, the quotient of which is the discriminant curve of the linear projection of Y from L.

1.1 To understand the geometry of $F = F(Y)$ we need to study the surjective morphism $q\colon \mathbb{L} \twoheadrightarrow Y$. Note that both varieties are smooth projective and of dimension three. Therefore, the ramification locus $R(q) \subset \mathbb{L}$ of q, i.e. the closed set of points in which q fails to be smooth, is a surface.

Proposition 1.1. *The ramification divisor $R(q) \subset \mathbb{L}$ of the morphism $q\colon \mathbb{L} \twoheadrightarrow Y$ is contained in the linear system $|p^*\mathcal{O}_F(2)|$. It is the pull-back of a curve $R \subset F(Y)$ in the linear system $|\mathcal{O}_F(2)|$.*

Proof We consider the differential of q as a morphism of sheaves $dq\colon \mathcal{T}_{\mathbb{L}} \longrightarrow q^*\mathcal{T}_Y$. Then by definition $R(q)$ is the zero locus of $\det(dq)\colon \det(\mathcal{T}_{\mathbb{L}}) \longrightarrow q^*\det(\mathcal{T}_Y)$, which we consider as a section of $\omega_{\mathbb{L}} \otimes q^*\omega_Y^*$. Now, $q^*\omega_Y^* \simeq q^*\mathcal{O}_Y(2) \simeq \mathcal{O}_p(2)$, see Lemma **3**.4.1.

Furthermore, the relative Euler sequence $0 \longrightarrow \mathcal{O}_{\mathbb{L}} \longrightarrow p^*\mathcal{S}_F \otimes \mathcal{O}_p(1) \longrightarrow \mathcal{T}_p \longrightarrow 0$ for the projective bundle $p\colon \mathbb{L} \simeq \mathbb{P}(\mathcal{S}_F) \longrightarrow F$ shows that $\omega_{\mathbb{L}}$ is isomorphic to $\omega_p \otimes p^*\omega_F \simeq p^*\det(\mathcal{S}_F^*) \otimes \mathcal{O}_p(-2) \otimes p^*\mathcal{O}_F(1)$. Hence, $\det(dq) \in H^0(\mathbb{L}, p^*\mathcal{O}_F(2)) \simeq H^0(F, \mathcal{O}_F(2))$. □

Remark 1.2. The proposition goes back to Fano. In [34, Sec. 10] the argument uses the observation that for the generic hyperplane section $S \coloneqq Y \cap \mathbb{P}^3$ the pre-image $q^{-1}(S)$ is the blow-up $p\colon q^{-1}(S) \longrightarrow F(Y)$ in the 27 points $[\ell_i] \in F(Y)$ corresponding to the 27 lines $\ell_i \subset S$ contained in the cubic surface S.

It seems that in [120, Cor. 1.9] a local computation is used to show that R is actually smooth. However, for special smooth cubics the morphism $q\colon \mathbb{L} \longrightarrow Y$ may contract curves $E_x \subset \mathbb{L}$ to so-called Eckardt points $x \in Y$, i.e. points lying on infinitely many lines. A cubic $Y \subset \mathbb{P}^4$ can admit at most 30 Eckardt points, cf. [131]. In these cases, R will contain the finitely many curves $p(E_x)$ (which are smooth elliptic) as irreducible components. We will see that R is in fact ample and, therefore, connected. Thus, whenever R is reducible, it is in fact singular.

The kernel of the tangent map $T_{(L,x)}\mathbb{L} \longrightarrow T_xY$ at a point $(L, x) \in \mathbb{L} \subset F \times Y$ is the space of first order deformations of $L \subset Y$ through $x \in L$. This space is naturally isomorphic to the subspace $H^0(L, \mathcal{N}_{L/Y} \otimes \mathcal{I}_x) \subset H^0(L, \mathcal{N}_{L/Y})$. As the ideal sheaf $\mathcal{I}_x$ of $x \in L \simeq \mathbb{P}^1$ is isomorphic to $\mathcal{O}_L(-1)$, this space is non-zero if and only if $L \subset Y$ is a line of the second type, i.e. $\mathcal{N}_{L/Y} \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$, cf. Lemma **3**.1.9.

Corollary 1.3. *The morphism $q\colon \mathbb{L} \longrightarrow Y$ is smooth at $(L, x) \in \mathbb{L}$ if and only if $L \subset Y$ is a line of the first type.* □

Note that the non-vanishing of $H^0(L, \mathcal{N}_{L/Y} \otimes \mathcal{I}_x)$ only depends on $L \subset Y$ and not on the point $x \in L$. In particular, $q\colon \mathbb{L} \longrightarrow Y$ is ramified along all of $L = p^{-1}([L]) \subset \mathbb{L}$ or not at all. This confirms that $R(f) = p^{-1}R$ for a curve $R \subset F(Y)$.

Corollary 1.4. *The ramification divisor $R(q) \subset \mathbb{L}$ is the pull-back of a curve $R \subset F(Y)$ that parametrizes all lines $L \subset Y$ of the second type.* □

The set R of all lines of the second type $[L] \in F$ is viewed with this scheme structure, i.e. as a curve in the linear system $|\mathcal{O}_F(2)|$. Note that $R(q)$, and hence R, cannot be empty. Indeed, otherwise $\mathbb{L} \longrightarrow Y$ would be an étale covering of degree six of the simply connected threefold Y. As $\mathbb{L}$ is connected, this is absurd. It is maybe also worth pointing out that a priori it is not even clear that lines of the second type do not define isolated points in $F(Y)$ or that not every line is of the second type. A stronger version of the latter is proved in Corollary 1.6.

Clearly, two distinct lines $L_1, L_2 \subset \mathbb{P}^4$ (contained in the cubic Y or not) do not intersect at all or in exactly one point (and there transversally). In the second case, they are contained in a unique plane. Similarly, for infinitesimal deformations one distinguishes between these two cases:

(i) Let $L \subset Y$ be a line of the first type. Then the lines $L_t \subset Y$ corresponding to $t \in F(Y)$ close to $[L] \in F(Y)$ are disjoint to L. Indeed, a first order deformation keeping the intersection non-empty, would correspond to a global section of $\mathcal{N}_{L/Y} \otimes \mathcal{I}_x \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ for some point $x \in L$.

(ii) On the other hand, if $L \subset Y$ is of the second type, then for each point $x \in L$, the subspace $H^0(L, \mathcal{N}_{L/Y} \otimes \mathcal{I}_x) \subset H^0(L, \mathcal{N}_{L/Y}) \simeq T_{[L]}F(Y)$ is one-dimensional and corresponds to a deformation $\operatorname{Spec} k[\varepsilon] \times \{x\} \subset L_\varepsilon \subset \operatorname{Spec} k[\varepsilon] \times Y$. Its image under the second projection is contained in the intersection of a plane $\mathbb{P}^2 \simeq P_L \subset \mathbb{P}^4$ with Y, i.e. $P_L \cap Y$ is the double line $2L$ in the plane P_L. Note that the subspace $H^0(\mathcal{N}_{L/Y} \otimes \mathcal{I}_x) \subset H^0(\mathcal{N}_{L/Y})$ is independent of x and so is P_L. So, we have proved:

Lemma 1.5. *Let $L \subset Y$ be a line in a smooth cubic threefold. Then L is of the second type if and only if there exists a (unique) plane $\mathbb{P}^2 \simeq P_L \subset \mathbb{P}^4$, that is tangent to Y at every point of L, i.e. $2L \subset P_L \cap Y$.* □

Corollary 1.6. *The generic line $L \subset Y$ is of the first type. In fact, for a dense open subset of lines $[L] \in F(Y)$ any plane $\mathbb{P}^2 \subset \mathbb{P}^4$ containing L intersects Y in two further distinct lines, i.e. such that $\mathbb{P}^2 \cap Y$ is reduced.*

Proof Lines of the second type are parametrized by the curve R. So, any line $[L] \in F(Y) \setminus R$ is of the first type. Furthermore, for any line of the second type $L' \subset Y$ there exists a unique plane $\mathbb{P}^2 \simeq P_{L'} \subset \mathbb{P}^4$ which intersects Y along L' with multiplicity two. The residual curves $L \subset Y$ of $L' \subset P_{L'} \cap Y$, $[L'] \in R$, sweep out a curve $R' \subset F(Y)$. Then any line $[L] \in Y \setminus (R \cup R')$ has the required property. □

1.2 We move to the next class of curves. For a fixed line $[L] \in F$ we define the curve

$$C_L \subset F(Y)$$

as the closure of the curve of all lines $L' \subset Y$ different from L but with non-empty intersection $L \cap L'$. To make this rigorous, one may use the formalism of Section **3**.4.2 and consider

$$\varphi = p_* \circ q^* \colon \mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^1(F(Y)) \simeq \mathrm{Pic}(F(Y)).$$

By definition, p_* is trivial on components of $q^{-1}(L)$ with positive fibre dimension over $F(Y)$, e.g. the class of the component $p^{-1}([L]) \subset q^{-1}(L)$ is mapped to zero under p_*. The image of the class of the line $L \subset Y$ under φ is a line bundle $\mathcal{O}(C_L)$ that comes with a natural section (up to scaling) vanishing along $p_*(q^{-1}(L))$.

As explained before, L is of the second type if and only if $[L] \in C_L$. Assume now that L is of the first type, i.e. $[L] \in F(Y) \setminus C_L$. In this case, $q^{-1}(L)$ is the disjoint union of $p^{-1}([L]) \simeq L$ and a curve mapping isomorphically onto C_L:

$$q^{-1}(L) = p^{-1}([L]) \sqcup C_L.$$

Indeed, for $[L'] \in C_L$ the line $L' = q(p^{-1}([L']))$ intersects L transversally in exactly one point. Hence, $p^{-1}([L'])$ and $q^{-1}(L)$ intersect with multiplicity one.

Remark 1.7. Any curve in $F(Y)$ intersects the ample curve $R \subset F(Y)$ of all lines of the second type. Applied to C_L, this shows that any line $L \subset Y$ intersects some line $L' \subset Y$ of the second type.

Lemma 1.8. *For any two lines $L_1, L_2 \subset Y$ the curves $C_{L_1}, C_{L_2} \subset F(Y)$ are algebraically equivalent. Moreover, $\mathcal{O}(C_L)^{\otimes 3}$ is algebraically equivalent to $\mathcal{O}_F(1) \simeq \omega_F$.*

Proof As $[C_L]$ is the image of $[L] \in \mathrm{CH}^2(Y)$ under $\varphi\colon \mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^1(F(Y))$ and all lines, which are parametrized by the connected Fano surface $F(Y)$, are algebraically equivalent to each other, the algebraic equivalence class of C_L is independent of L.

For the second statement use Lemma **3**.4.1. As $h^2 \in \mathrm{CH}^2(Y)$ is represented by any intersection with a plane $\mathbb{P}^2 \subset \mathbb{P}^4$, it can be written as a sum of three lines $h^2 = [L_1] + [L_2] + [L_3]$. Hence, $\mathrm{c}_1(\mathcal{O}_F(1)) = \varphi(h^2) = [C_{L_1}] + [C_{L_2}] + [C_{L_3}]$, which is algebraically equivalent to $3[C_L]$. □

Exercise 1.9. To make the above more explicit, consider a plane $\mathbb{P}^2 \subset \mathbb{P}^4$ as $V(s_1) \cap V(s_2)$ for $s_1, s_2 \in V^* = H^0(\mathbb{P}^4, \mathcal{O}(1)) \simeq H^0(F, \mathcal{S}_F^*)$. Then the the zero set of the image of $s_1 \wedge s_2$ under $\bigwedge^2 V^* \longrightarrow H^0(F(Y), \bigwedge^2 \mathcal{S}_F^*) \simeq H^0(F(Y), \mathcal{O}_F(1))$ is the set of all lines $L = \mathbb{P}(W)$ with $(s_1 \wedge s_2)|_W = 0$. The latter is equivalent to $L \cap \mathbb{P}^2 \neq \emptyset$ or, equivalently, to $L \cap (L_1 \cup L_2 \cup L_3) \neq \emptyset$, i.e. $[L] \in C_{L_1} \cup C_{L_2} \cup C_{L_3}$. Here, $\mathbb{P}^2 \cap Y = L_1 \cup L_2 \cup L_3$.

Exercise 1.10. The self-intersection and the (arithmetic) genus of C_L are given by

$$(C_L)^2 = 5 \text{ and } g(C_L) = 11.$$

Exercise 1.11. Let $L \subset Y$ be a line of the first type. Show that the map that sends $[L'] \in C_L \subset F$ to the point of intersection of L and L' defines a morphism

$$C_L \longrightarrow L$$

of degree five. Use that $q\colon \mathbb{L} \longrightarrow Y$ is of degree six.

Corollary 1.12. *The Plücker class $g = \mathrm{c}_1(\mathcal{O}_F(1)) \in H^2(F(Y), \mathbb{Z})$ is divisible by three and so is the Hodge–Riemann pairing $\int_F \gamma_1 \cdot \gamma_2 \cdot g$ on $H^1(F(Y), \mathbb{Z})$, cf. Proposition **1**.4.2.* □

1.3 We study the linear projection from a line $L \subset Y$ as a special case of the construction in Section **1**.5.1.

Let $L = \mathbb{P}(W) \subset \mathbb{P}^4 = \mathbb{P}(V)$ be a line contained in the smooth cubic hypersurface $Y \subset \mathbb{P}^4$. Assume $\mathbb{P}^2 \subset \mathbb{P}^4$ is a plane disjoint to L, of which we think as $\mathbb{P}(V/W)$. The linear projection $Y \setminus L \dashrightarrow \mathbb{P}^2$ from L onto this plane is the rational map associated with the linear system $|\mathcal{O}_Y(1) \otimes \mathcal{I}_L| \subset |\mathcal{O}_Y(1)|$. It is resolved by a simple blow-up $\tau\colon \mathrm{Bl}_L(Y) \longrightarrow Y$. The induced morphism $\varphi\colon \mathrm{Bl}_L(Y) \longrightarrow \mathbb{P}^2$ is then associated with the complete linear system $|\tau^*\mathcal{O}_Y(1) \otimes \mathcal{O}(-E)|$.

The fibre over a point $y \in \mathbb{P}^2$ is the residual conic of $L \subset Y \cap \overline{y\,L} \subset \overline{y\,L} \simeq \mathbb{P}^2$. The conic is smooth or a union of two lines L_1, L_2, possibly non-reduced or with $L_i = L$. Note that when $L_i = L$, the plane $\overline{y\,L} \simeq \mathbb{P}^2$ intersects Y with higher multiplicity along L and hence L would be of the second type. Therefore, if L was chosen to be of the first type, then the fibres of $\varphi\colon \mathrm{Bl}_L(Y) \longrightarrow \mathbb{P}^2$ are either smooth conics or possibly non-reduced unions of two lines different from L.

Corollary 1.13. *Let $L \subset Y$ be a line in a smooth cubic hypersurface $Y \subset \mathbb{P}^4$, generic in the sense of Corollary 1.6, i.e. $[L] \in F(Y) \setminus (R \cup R')$. Then the linear projection from L defines a morphism*

$$\mathrm{Bl}_L(Y) \twoheadrightarrow \mathbb{P}^2,$$

with a smooth discriminant curve $D_L \subset \mathbb{P}^2$ of degree five and genus six. Moreover, the fibre over a point $y \in D_L$ is the reduced union of two lines $\varphi^{-1}(y) = L_1 \cup L_2$.

Proof This is mostly a consequence of Proposition **1**.5.1. The fibre over $y \in D_L$ cannot be a double line L', as then $\mathbb{P}^2 \cap Y = L \cup L'$ with L' of the second type, which is excluded for L generic. The smoothness of D_L is a local computation, see [9, 22]. □

Remark 1.14. The abstract approach matches nicely with the intuitive picture. Here are two comments in this direction.

(i) That D_L is of degree five, i.e. $D_L \in |\mathcal{O}(5)|$ can be related to the fact that a line in a smooth cubic surface $S \subset \mathbb{P}^3$ is intersected by five mutually disjoint pairs of lines, see Section **4**.3.3. Indeed, if $L \subset Y \subset \mathbb{P}^4$ is intersected with a generic hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ containing L, then $D_L \subset \mathbb{P}^2$ is intersected with a generic line $\mathbb{P}^1 \subset \mathbb{P}^2$. The fibres over the intersection points $y \in D_L \cap \mathbb{P}^1$ are the pairs of lines in Y contained in the cubic surface $S \coloneqq Y \cap \mathbb{P}^3$ intersecting L, of which there are exactly five.

(ii) For a line of the first kind the exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}(\mathcal{N}_{L/Y}) \subset \mathrm{Bl}_L(Y)$ has normal bundle $\mathcal{O}(0,-1)$. Hence, the restriction of $\varphi^*\mathcal{O}(1) \simeq \tau^*\mathcal{O}(1) \otimes \mathcal{O}(-E)$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is $\mathcal{O}(1,1)$. In particular, the composition $\mathbb{P}^1 \times \mathbb{P}^1 \simeq \mathbb{P}(\mathcal{N}_{L/Y}) \subset \mathrm{Bl}_L(Y) \twoheadrightarrow \mathbb{P}^2$ is a morphism of degree two, which confirms the geometric description: $\varphi^{-1}(y) \cap L$ is the intersection of the residual conic of $L \subset \overline{yL}$ with L.

Consider the restriction $\varphi^{-1}(D_L) \twoheadrightarrow D_L$ of the linear projection to the discriminant curve. We assume that L is generic in the sense of Corollary 1.6, so that all fibres are reduced singular conics, i.e. unions of two distinct lines. The curve parametrizing lines in the fibres is thus an étale cover

$$\widetilde{D}_L \twoheadrightarrow D_L$$

of degree two. The morphism is indeed unramified which can be shown by a direct computation or an argument involving abstract deformation theory. As $\widetilde{D}_L$ parametrizes lines in Y, it comes with a classifying morphism $\widetilde{D}_L \longrightarrow F(Y)$ which is easily seen to be a closed immersion. Alternatively, $\widetilde{D}_L$ can be obtained as the Stein factorization of the composition of the normalization of $\varphi^{-1}(D_L)$ with φ. The morphism to $F(Y)$ can then be viewed as follows: The natural rational map $\varphi^{-1}(D_L) \dashrightarrow \mathbb{L}$ is regular on the complement of the section of $\varphi^{-1}(D_L) \twoheadrightarrow D_L$ given by the intersection point of the two lines in each fibre. Composing with $p\colon \mathbb{L} \twoheadrightarrow F(Y)$ yields $\widetilde{D}_L \hookrightarrow F(Y)$.

Lemma 1.15. *For a generic line $L \subset Y$ the curves $\widetilde{D}_L, C_L \subset F(Y)$ coincide. Furthermore, $\widetilde{D}_L = C_L$ is a smooth curve of genus* 11.

Proof Indeed, $\widetilde{D}_L$ and C_L both parametrize all lines $L \neq L' \subset Y$ intersecting L. We know that the (arithmetic) genus of C_L is $g(C_L) = 11$ and the same is true for $\widetilde{D}_L$ by Hurwitz's formula. This is enough to conclude equality. Smoothness of $\widetilde{D}_L$ follows from the smoothness of D_L. □

Corollary 1.16. *For a general line $L \subset Y$, i.e. $[L] \in F(Y) \setminus (R \cup R')$, the curve $\widetilde{D}_L = C_L$ is connected.*

Proof For a general cubic Y, one has $\rho(F(Y)) = 1$ and hence $\widetilde{D}_L = C_L$ would be numerically equivalent to a multiple of each of its irreducible components. However, then the existence of more than one irreducible curve would contradict $(C_L)^2 = 5$. As under deformations of $L \subset Y$ with $L \in F(Y) \setminus (R \cup R')$ the topology of the situation does not change, this proves the assertion in general. □

See also [67] for an alternative proof that does not reduce to the case $\rho(F(Y)) = 1$. The idea there is that sending a line $[L'] \in C_L$ to its intersection with L defines a map $C_L \twoheadrightarrow L$ of degree five, cf. Exercise 1.11. If C_L is not irreducible, then one of the irreducible components is rational, which would contradict the injectivity of the Albanese map in Corollary 2.6, or hyperelliptic. However, $F(Y)$ is not covered by hyperelliptic curves, cf. [67, Sec. 3].

2 Albanese of the Fano surface

Fix a point $t_0 = [L_0] \in F = F(Y)$ and consider the Albanese morphism

$$a\colon F \longrightarrow A := \mathrm{Alb}(F) = H^{1,0}(F)^*/H_1(F,\mathbb{Z}),\ t \longmapsto \left(\alpha \longmapsto \int_{t_0}^{t} \alpha\right).$$

According to the numerical results, A is an abelian variety of dimension five.

The goal of this section is to compare the following two pictures

$$\begin{array}{ccc} \mathbb{L} = \mathbb{P}(\mathcal{S}_F) & \xrightarrow{\ q\ } & Y \subset \mathbb{P}(V) \simeq \mathbb{P}^4 \\ {\scriptstyle p}\downarrow & & \\ F & & \end{array} \qquad \begin{array}{ccccc} \mathbb{P}(\mathcal{T}_F) & \dashrightarrow & \mathbb{P}(\mathcal{T}_A) & \longrightarrow & \mathbb{P}(T_0A) \simeq \mathbb{P}^4 \\ \downarrow & & \downarrow & & \\ F & \xrightarrow{\ a\ } & A & & \end{array} \quad (\tilde{q}\colon \mathbb{P}(\mathcal{T}_F) \to \mathbb{P}(T_0A)) \tag{2.1}$$

We will show that they describe the same geometric situation.

2.1 Let us begin with some preliminary comments:

(i) The natural inclusion $\mathcal{S}_F \subset V \otimes \mathcal{O}_F$ yields an embedding $\mathbb{L} \simeq \mathbb{P}(\mathcal{S}_F) \subset F \times \mathbb{P}(V)$, which is in fact nothing but the composition of the two inclusions $\mathbb{L} \subset F \times Y$ and $F \times Y \subset F \times \mathbb{P}(V)$. Thus, the relative tautological line bundle is described by $\mathcal{O}_p(1) \simeq q^*\mathcal{O}(1)$, cf. Lemma **3**.4.1, and the pull-back yields a homomorphism

$$H^0(\mathbb{P}^4, \mathcal{O}(1)) \xrightarrow{\ \sim\ } H^0(Y, \mathcal{O}_Y(1)) \hookrightarrow H^0(\mathbb{L}, \mathcal{O}_p(1)) \simeq H^0(F, \mathcal{S}_F^*). \tag{2.2}$$

It is injective, because $Y = q(\mathbb{L}) \subset \mathbb{P}(V)$ is not contained in any hyperplane. However, at this point it is not clear that the map is also surjective or, equivalently, that the morphism $q\colon \mathbb{L} \longrightarrow \mathbb{P}(V)$ is the morphism associated with the complete linear system $|\mathcal{O}_p(1)|$.

(ii) The differential of the Albanese morphism $a\colon F \longrightarrow A = \mathrm{Alb}(F)$ induces a homomorphism between the tangent sheaves $da\colon \mathcal{T}_F \longrightarrow a^*\mathcal{T}_A$, but a priori this does not automatically induce a morphism $\mathbb{P}(\mathcal{T}_F) \longrightarrow \mathbb{P}(a^*\mathcal{T}_A) \longrightarrow \mathbb{P}(\mathcal{T}_A)$. For this we will have to argue that $da\colon T_tF \longrightarrow T_{a(t)}A$ is injective for all $t \in F$. Note that the tangent bundle $\mathcal{T}_A$ is trivial, which yields a natural projection $\mathbb{P}(\mathcal{T}_A) \simeq A \times \mathbb{P}(T_0A) \longrightarrow \mathbb{P}(T_0A)$.

(iii) Finally note that there is indeed an isomorphism $V \simeq T_0A$. Namely, compose $T_0A \simeq H^{1,0}(F)^* \simeq H^{2,1}(Y)^*$ with the dual of $H^{2,1}(Y) \simeq R_1 \simeq V^*$ provided by Theorem **1**.4.18. Here, $R = \bigoplus R_i \simeq \mathbb{C}[V^*]/(\partial_i F)$ is the Jacobian ring of $Y = V(F)$. However, in the discussion below, the isomorphism between the two spaces will be obtained in a different manner.

The first step is to show that $\mathcal{T}_F$ and $\mathcal{S}_F$ are naturally isomorphic. Evidence for the existence of such an isomorphisms comes from

$$\det(\mathcal{T}_F) \simeq \omega_F^* \simeq \mathcal{O}_F(-1) \simeq \det(\mathcal{S}_F) \ \text{ and } \ \mathrm{c}_2(\mathcal{T}_F) = e(F) = 27 = \mathrm{c}_2(\mathcal{S}_F).$$

The latter is shown by the following argument, which *a posteriori* explains geometrically the curious observation that $e(F)$ is the number of lines on a cubic surface. Consider a generic hyperplane section $S \coloneqq Y \cap V(s)$, $s \in H^0(\mathbb{P}^4, \mathcal{O}(1))$, which is a cubic surface $S \subset V(s) \simeq \mathbb{P}^3$. Let s_F be the image of s under $H^0(\mathbb{P}(V), \mathcal{O}(1)) \longrightarrow H^0(F, \mathcal{S}_F^*)$. Its zero set $V(s_F) \subset F$ is the set of lines $[L] \in F$ with $s|_L = 0$, i.e. the set of all lines contained in the cubic surface S.

Proposition 2.1. *Let $Y \subset \mathbb{P}^4$ be a smooth cubic threefold and $F = F(Y)$ its Fano variety of lines. Then there exists a natural isomorphism*

$$\mathcal{T}_F \simeq \mathcal{S}_F$$

between the tangent bundle $\mathcal{T}_F$ of F and the restriction $\mathcal{S}_F$ of the universal subbundle $\mathcal{S} \subset V \otimes \mathcal{O}_{\mathbb{G}}$ under the natural embedding $F \subset \mathbb{G}(1, \mathbb{P}^4)$.

Proof The result was originally proved in [34, 145] by very clever geometric arguments. We follow the more algebraic approach in [4].

Recall that $F \subset \mathbb{G} = \mathbb{G}(1, \mathbb{P}^4)$ is the zero set of the regular regular section $s_Y \in H^0(\mathbb{G}, S^3(\mathcal{S}^*))$. The latter is the image of the equation in $S^3(V^*)$ defining Y under the natural surjection $S^3(V^*) \twoheadrightarrow S^3(\mathcal{S}^*)$, see Section **3**.1.1. Hence, the normal bundle sequence has the form

$$0 \longrightarrow \mathcal{T}_F \longrightarrow \mathcal{T}_{\mathbb{G}}|_F \longrightarrow S^3(\mathcal{S}_F^*) \longrightarrow 0. \tag{2.3}$$

Deformation theory, see Section **3**.1.2, provides us with descriptions of the fibres of the two tangent bundles:

$$T_{[L]}F \simeq H^0(L, \mathcal{N}_{L/Y}) \text{ and } T_{[L]}\mathbb{G} \simeq H^0(L, \mathcal{N}_{L/\mathbb{P}}).$$

Moreover, fibrewise (2.3) is described as the cohomology sequence of the exact sequence $0 \longrightarrow \mathcal{N}_{L/Y} \longrightarrow \mathcal{N}_{L/\mathbb{P}} \longrightarrow \mathcal{O}_L(3) \longrightarrow 0$ of normal bundles for the nested inclusion $L \subset Y \subset \mathbb{P}^4$. The global version of the latter is the exact sequence of normal bundles of the nested inclusion $\mathbb{L} \subset F \times Y \subset F \times \mathbb{P}$:

$$0 \longrightarrow \mathcal{N}_{\mathbb{L}/F\times Y} \longrightarrow \underset{\simeq p^*\mathcal{Q}_F \otimes \mathcal{O}_p(1)}{\mathcal{N}_{\mathbb{L}/F\times\mathbb{P}}} \longrightarrow \underset{\simeq q^*\mathcal{O}(3) \simeq \mathcal{O}_p(3)}{\mathcal{N}_{F\times Y/F\times\mathbb{P}}|_{\mathbb{L}}} \longrightarrow 0. \qquad (2.4)$$

We use $\mathcal{N}_{\mathbb{L}_{\mathbb{G}}/\mathbb{G}\times\mathbb{P}} \simeq p^*\mathcal{Q} \otimes \mathcal{O}_p(1)$, which is the global version of the natural isomorphism $\mathcal{N}_{L/\mathbb{P}} \simeq V/W \otimes \mathcal{O}(1)$ for a line $L = \mathbb{P}(W) \subset \mathbb{P}(V)$, cf. the discussion in Section **1**.1.2. Restriction to F yields $\mathcal{N}_{\mathbb{L}/F\times\mathbb{P}} \simeq p^*\mathcal{Q}_F \otimes \mathcal{O}_p(1)$. Taking the direct image of (2.4) under $p\colon \mathbb{L} \longrightarrow F$ gives back (2.3):

$$0 \longrightarrow \underset{\simeq \mathcal{T}_F}{p_*\mathcal{N}_{\mathbb{L}/F\times Y}} \longrightarrow \underset{\simeq \mathcal{T}_{\mathbb{G}}|_F}{p_*\mathcal{N}_{\mathbb{L}/F\times\mathbb{P}}} \longrightarrow \underset{\simeq S^3(\mathcal{S}_F^*)}{p_*\mathcal{O}_p(3)} \longrightarrow 0. \qquad (2.5)$$

The key to describing $\mathcal{T}_F$ is to view it as the direct image of $\mathcal{N} := \mathcal{N}_{\mathbb{L}/F\times Y}$. Taking determinants of (2.4) shows

$$\textstyle\bigwedge^2 \mathcal{N} \simeq \det(\mathcal{N}) \simeq \det\left(p^*\mathcal{Q}_F \otimes \mathcal{O}_p(1)\right) \otimes \mathcal{O}_p(-3) \simeq p^*\det(\mathcal{Q}_F).$$

Applying $\bigwedge^2$ and $\otimes\, \mathcal{O}_p(-3)$ to (2.4), this yields the exact sequence

$$0 \longrightarrow \textstyle\bigwedge^2\mathcal{N} \otimes \mathcal{O}_p(-3) \longrightarrow p^*\bigwedge^2\mathcal{Q}_F \otimes \mathcal{O}_p(-1) \longrightarrow \mathcal{N} \longrightarrow 0.$$

As $p_*\mathcal{O}_p(-1) \simeq 0 \simeq R^1p_*\mathcal{O}_p(-1)$, taking direct images gives

$$\mathcal{T}_F \simeq p_*\mathcal{N} \simeq R^1p_*(\textstyle\bigwedge^2\mathcal{N} \otimes \mathcal{O}_p(-3)) \simeq \det(\mathcal{Q}_F) \otimes R^1p_*\mathcal{O}_p(-3).$$

By relative Serre duality, cf. [78, III. Ex. 8.4], $R^1p_*\mathcal{O}_p(-3) \simeq p_*(\mathcal{O}_p(1))^* \otimes \det(\mathcal{S}_F)$ and, therefore, $\mathcal{T}_F \simeq p_*\mathcal{N} \simeq \det(\mathcal{Q}_F) \otimes \mathcal{S}_F \otimes \det(\mathcal{S}_F) \simeq \mathcal{S}_F$. □

Note that $\mathcal{S}_F$ is naturally viewed as a subbundle $\mathcal{S}_F \subset V \otimes \mathcal{O}_F$ and, as we will see, $\mathcal{T}_F$ as a subbundle $\mathcal{T}_F \subset a^*\mathcal{T}_A \simeq T_0A \otimes \mathcal{O}_F$. However, the above result does not yet show that there exists an isomorphism $\mathcal{S}_F \simeq \mathcal{T}_F$ that would be compatible with those inclusions after an appropriate isomorphism $V \simeq T_0A$.

Corollary 2.2. *The natural map in (2.2) is an isomorphism*

$$V^* \xrightarrow{\sim} H^0(F, \mathcal{S}_F^*).$$

Hence, $q\colon \mathbb{L} \twoheadrightarrow Y \subset \mathbb{P}(V)$ *is the morphism associated with the complete linear system* $|\mathcal{O}_p(1)|$.

Proof Use $\dim H^0(F, \mathcal{S}_F^*) = \dim H^0(F, \mathcal{T}_F^*) = h^1(F, \mathcal{O}_F) = 5$. □

2.2 So far, we have shown that there exists an isomorphism $\mathbb{L} \simeq \mathbb{P}(\mathcal{S}_F) \simeq \mathbb{P}(\mathcal{T}_F)$, but not that the two morphisms in (2.1) are related. In fact, we have not yet even properly defined the morphism $\mathbb{P}(\mathcal{T}_F) \longrightarrow \mathbb{P}(T_0 A)$. This will be done next.

The first step towards it will be to show that exterior product yields an isomorphism $\bigwedge^2 H^1(F, \mathbb{Q}) \xrightarrow{\sim} H^2(F, \mathbb{Q})$ of Hodge structures. Due to Section **3**.3.3 we in fact know already that the two Hodge structures are isomorphic

$$\bigwedge^2 H^1(F, \mathbb{Q}) \simeq \bigwedge^2 H^3(Y, \mathbb{Q})(2) \simeq H^2(F, \mathbb{Q}),$$

but not that this isomorphism is given by exterior product.

Lemma 2.3. *The exterior product defines isomorphisms*

$$\bigwedge^2 H^{1,0}(F) \xrightarrow{\sim} H^{2,0}(F) \ \ \textit{and} \ \ \bigwedge^2 H^1(F, \mathbb{Q}) \xrightarrow{\sim} H^2(F, \mathbb{Q}).$$

Proof Using the isomorphism $\mathcal{S}_F \simeq \mathcal{T}_F$, the Hodge theoretic assertion is turned into the geometric assertion that the natural $\bigwedge^2 H^0(F, \mathcal{S}_F^*) \longrightarrow H^0(F, \bigwedge^2 \mathcal{S}_F^*)$ is an isomorphism. Using the commutative diagram

$$\begin{array}{ccc} \bigwedge^2 V^* & \xrightarrow{\sim} & \bigwedge^2 H^0(F, \mathcal{S}_F^*) \\ \wr\downarrow & & \downarrow \\ H^0(\mathbb{P}(\bigwedge^2 V), \mathcal{O}(1)) & \longrightarrow & H^0(F, \bigwedge^2 \mathcal{S}_F^*) \simeq H^0(F, \mathcal{O}_F(1)) \end{array}$$

and the fact that all spaces are of the same dimension ten, it suffices to show that the Plücker embedding $F \subset \mathbb{P}(\bigwedge^2 V)$ is not contained in any hyperplane. This can either be argued geometrically or using the Koszul complex as in Proposition **3**.2.3.

As the map $\bigwedge^2 H^1(F, \mathbb{Q}) \longrightarrow H^2(F, \mathbb{Q})$ is topologically defined, its injectivity is independent of the smooth cubic threefold. It is thus enough to check it for one $F = F(Y)$. However, for the very general cubic $\bigwedge^2 H^3(Y, \mathbb{Q})(2) \simeq \bigwedge^2 H^1(F, \mathbb{Q})$ is the direct sum $\mathbb{Q}(-1) \oplus H$ of two irreducible Hodge structures of weight two. This can be seen, for example, as a consequence of the monodromy description Theorem **1**.2.9. The first summand is pure and spanned by the intersection product on $H^3(Y, \mathbb{Q})$. It is mapped onto the line spanned by the Plücker polarization $\mathbb{Q} \cdot g$. Due to the irreducibility of the Hodge structure H, the map $H \longrightarrow H^2(F, \mathbb{Q})$ is injective if and only if $\bigwedge^2 H^{1,0}(F) \longrightarrow H^{2,0}(F)$ is non-trivial, which we have shown already. Moreover, $H^{2,0}(F)$ is contained in $H^2(F, \mathbb{C})_{\rm pr}$ and, therefore, $H \hookrightarrow H^2(F, \mathbb{Q})_{\rm pr}$. Altogether, this proves the injectivity of $\bigwedge^2 H^1(F, \mathbb{Q}) \longrightarrow H^2(F, \mathbb{Q})$ and, due to dimension reasons, its bijectivity. □

Geometrically, the first injectivity is equivalent to saying that the image of the Albanese morphism $a\colon F \longrightarrow A$ is a surface.

Corollary 2.4. *The Albanese morphism $a\colon F \longrightarrow A$ is unramified, i.e. for all $t \in F$ the tangent map $da_t\colon T_tF \longrightarrow T_{a(t)}A$ is injective. In particular, the Albanese map defines the morphism $\tilde{q}$ in (2.1)*

$$\tilde{q}\colon \mathbb{P}(\mathcal{T}_F) \longrightarrow \mathbb{P}(\mathcal{T}_A) \longrightarrow \mathbb{P}(T_0A).$$

Proof Assume da_t is not injective for some $t \in F$. Then the induced map

$$\bigwedge^2 T_tF \longrightarrow \bigwedge^2 T_{a(t)}A$$

is trivial. However, this map is dual to the map

$$\bigwedge^2 T^*_{a(t)}A \simeq \bigwedge^2 H^{1,0}(A) \simeq \bigwedge^2 H^{1,0}(F) \xrightarrow{\sim} H^{2,0}(F) \simeq H^0(F, \omega_F) \longrightarrow \omega_F \otimes k(t),$$

which then is also trivial. As ω_F is very ample and, in particular, globally generated, this is absurd. □

Lemma 2.5. *The morphism $\tilde{q}\colon \mathbb{P}(\mathcal{T}_F) \longrightarrow \mathbb{P}(T_0A)$ is the morphism associated with the complete linear system $|\mathcal{O}_p(1)|$*

Proof First, $\tilde{q}^*\mathcal{O}(1) \simeq \mathcal{O}_p(1)$, as $\mathbb{P}(\mathcal{T}_F) \subset \mathbb{P}(a^*\mathcal{T}_0) \simeq \mathbb{P}(T_0A) \times A$ is induced by the inclusion $\mathcal{T}_F \hookrightarrow \mathcal{T}_A \simeq T_0A \otimes \mathcal{O}_F$. It remains to show that the linear system is complete, i.e. that the pull-back map $H^0(\mathbb{P}(T_0A), \mathcal{O}(1)) \longrightarrow H^0(\mathbb{P}(\mathcal{T}_F), \mathcal{O}_p(1))$ is a bijection. Both sides are of dimension five, so it suffices to prove the injectivity. If the map were not injective, then all tangent spaces $T_tF \hookrightarrow T_0A$ would be contained in a hyperplane. But this would contradict the bijectivity of the dual map $H^0(A, \Omega_A) \longrightarrow H^0(F, \Omega_F)$, which is the pull-back of one-forms under the Albanese map $a\colon F \longrightarrow A$. □

This proves the main result of this section:

Proposition 2.6. *There exists isomorphisms $\mathcal{S}_F \simeq \mathcal{T}_F$ and $V \simeq T_0A$ inducing a commutative diagram*

$$\begin{array}{ccc}
\mathbb{P}(\mathcal{S}_F) & \xrightarrow{q} & \mathbb{P}(V) \\
\wr\downarrow & & \downarrow\wr \\
\mathbb{P}(\mathcal{T}_F) & \xrightarrow[\tilde{q}]{} & \mathbb{P}(T_0A)
\end{array}$$

□

Corollary 2.7. *The Albanese morphism $a\colon F \longrightarrow A$ is unramified and generically injective.*

Proof The first assertion is Corollary 2.4. To prove the injectivity generically, we choose the above isomorphism $V \simeq T_0A$ such that the two inclusions

$$\mathcal{T}_F \hookrightarrow a^*\mathcal{T}_A \simeq T_0A \otimes \mathcal{O}_F \quad\text{and}\quad \mathcal{S}_F \hookrightarrow V \otimes \mathcal{O}_F$$

coincide. Hence, the morphism $F \longrightarrow \mathbb{G}(1, \mathbb{P}(T_0A))$, $t \longmapsto [T_tF \subset T_0(A)]$ is identified with the Plücker embedding $F \hookrightarrow \mathbb{G}(1, \mathbb{P}(V))$. However, if for all points $t \in a(F)$ and distinct points $t_1 \neq t_2 \in a^{-1}(t)$ the tangent spaces $T_{t_1}F \subset T_0A$ and $T_{t_2}F \subset T_0A$ are different, then the generic fibre consists of just one point. □

Remark 2.8. In fact, it has been shown by Beauville that $a := F \hookrightarrow A$ is injective and hence a closed immersion [11, Thm. 4]. We will see that this assertion is equivalent to saying that the invertible sheaves $\mathcal{O}(C_{L_1})$ and $\mathcal{O}(C_{L_2})$ associated with two distinct lines $L_1 \neq L_2 \subset Y$ are never isomorphic.

2.3 The following is a special case of the 'geometric global Torelli theorem', see Proposition **3**.2.7. The result in dimension three [34] predates the general result and the proof is different.

Proposition 2.9. *Two smooth cubic threefolds $Y, Y' \subset \mathbb{P}^4$ are isomorphic if and only if their Fano surfaces $F(Y)$ and $F(Y')$ are isomorphic:*

$$Y \simeq Y' \quad \Leftrightarrow \quad F(Y) \simeq F(Y').$$

Proof For any smooth cubic threefold $Y \subset \mathbb{P}^4$, the Picard group $\mathrm{Pic}(Y)$ is generated by $\mathcal{O}_Y(1)$. Hence, any isomorphism $Y \simeq Y'$ is induced by an automorphism of the ambient $\mathbb{P}^4$ and, therefore, induces an isomorphism $F(Y) \simeq F(Y')$ between their Fano surfaces. Conversely, any isomorphism $F(Y) \simeq F(Y')$ induces an isomorphism $Y \simeq Y'$ between the images of the natural morphisms $\mathbb{P}(\mathcal{T}_{F(Y)}) \twoheadrightarrow \mathbb{P}(T_0\mathrm{Alb}(F(Y)))$ and $\mathbb{P}(\mathcal{T}_{F(Y')}) \twoheadrightarrow \mathbb{P}(T_0\mathrm{Alb}(F(Y')))$, the differentials of the Albanese maps. □

Exercise 2.10. The same techniques show that for any smooth cubic threefold $Y \subset \mathbb{P}^4$ there exists a natural isomorphism of finite groups (cf. Corollary **1**.3.6)

$$\mathrm{Aut}(Y) \simeq \mathrm{Aut}(F(Y)).$$

Quotients of $F(Y)$ by subgroups of $\mathrm{Aut}(F(Y))$ have been studied in [132].

We complement this result by the following infinitesimal statement which moreover shows that the Fano surface $F(Y)$ stays a Fano surface after deformation, which is false for example for the Fano variety of lines on a smooth cubic fourfold, see Section **??**.**??**.

Proposition 2.11. *Let $Y \subset \mathbb{P}^4$ be a smooth cubic threefold and let $F = F(Y)$ be its Fano surface. Then the natural map*

$$H^1(Y, \mathcal{T}_Y) \twoheadrightarrow H^1(F, \mathcal{T}_F)$$

is an isomorphism.

Proof □

3 Albanese, Picard, and Prym

The general theory set up in Section **3**.4.2 provides us with a commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}^2(F)_{\mathrm{alg}} & \longrightarrow & \mathrm{CH}^2(Y)_{\mathrm{alg}} & \longrightarrow & \mathrm{CH}^1(F)_{\mathrm{alg}} \\ \downarrow & & \downarrow & & \downarrow \wr \\ A(F) & \twoheadrightarrow & J(Y) & \twoheadrightarrow & \mathrm{Pic}^0(F). \end{array}$$

Note that $\mathrm{CH}^2(F)_{\mathrm{alg}}$ is known to be big (over $\mathbb{C}$), while $\mathrm{CH}^1(F) \simeq \mathrm{Pic}(F)$.

The intermediate Jacobian $J(Y) := J^3(Y)$ of Y is self-dual and the two maps on the bottom are dual to each other. Indeed, they are induced by $\varphi\colon H^3(Y,\mathbb{Z}) \longrightarrow H^1(F,\mathbb{Z})(-1)$ and its dual $\psi\colon H^3(F,\mathbb{Z}) \longrightarrow H^3(Y,\mathbb{Z})$ (up to torsion, see Remark 3.8). Moreover, as the two maps induce isomorphisms between the cohomology groups with rational coefficients, they are isogenies of abelian varieties of dimension five.

The aim of this section is to show that all three abelian varieties are isomorphic and, moreover, can be identified with the Prym variety of $C_L \longrightarrow D_L$.

3.1 In order to understand the composition $A = A(F) \twoheadrightarrow J(Y) \twoheadrightarrow \mathrm{Pic}^0(F)$, we compose it with the Albanese map $a\colon F \longrightarrow A$, which depends on the additional choice of a point $t_0 = [L_0] \in F$ and factorizes via $F \longrightarrow \mathrm{CH}^2(F)_{\mathrm{alg}}$, $t \longmapsto [t] - [t_0]$ and the Abel–Jacobi map $\mathrm{CH}^2(F)_{\mathrm{alg}} \subset \mathrm{CH}^2(F)_{\mathrm{hom}} \longrightarrow A(F)$. Note that eventually Corollary 3.9 will imply that

$$\mathrm{CH}^2(Y)_{\mathrm{alg}} = \mathrm{CH}^2(Y)_{\mathrm{hom}} \text{ and } \mathrm{Pic}^0(F) \simeq \mathrm{CH}^1(F)_{\mathrm{alg}} = \mathrm{CH}^1(F)_{\mathrm{hom}}.$$

The latter is equivalent to $\mathrm{Pic}(F)/\mathrm{Pic}^0(F)$ or, still equivalently, $H^2(F,\mathbb{Z})$ being torsion free.

Lemma 3.1. *The composition* $F \longrightarrow A(F) \twoheadrightarrow J(Y) \twoheadrightarrow \mathrm{Pic}^0(F)$ *sends* $[L] \in F$ *to the invertible sheaf* $\mathcal{O}(C_L - C_{L_0})$.

Proof Clearly, the class of the point $t = [L] \in F$ under $\varphi\colon \mathrm{CH}^2(F) \longrightarrow \mathrm{CH}^2(Y)$ is mapped to the class $[L] \in \mathrm{CH}^2(Y)$ of the line $L \subset Y$. The image of the latter under the correspondence $\psi\colon \mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^1(F)$ is by construction $\mathcal{O}(C_L)$. □

The result can be extended to yield for every (smooth) curve $C \subset F$ a description for the composition

$$C \longrightarrow A(C) \longrightarrow A(F) \twoheadrightarrow J(Y) \twoheadrightarrow \mathrm{Pic}^0(F) \longrightarrow \mathrm{Pic}^0(C),$$

as $[L] \longmapsto \mathcal{O}(C_L - C_{L_0})|_C$, which is particularly useful when C is ample, e.g. $C = C_L$. In this case,

$$A(C) \twoheadrightarrow A(F) \text{ and } \mathrm{Pic}^0(F) \hookrightarrow \mathrm{Pic}^0(C),$$

as by Kodaira vanishing theorem $H^1(F,\mathcal{O}_F)\longrightarrow H^1(C,\mathcal{O}_C)$ is injective.

Let $[L]\in F$ be generic and consider C_L as the étale cover

$$\pi\colon C_L\simeq \widetilde{D}_L\twoheadrightarrow D_L$$

of degree two, see Section 1.3.

Corollary 3.2. *The composition*

$$\mathrm{Pic}^0(D_L)\xrightarrow{\pi^*}\mathrm{Pic}^0(C_L)\simeq A(C_L)\twoheadrightarrow A(F)\twoheadrightarrow J(Y)\twoheadrightarrow \mathrm{Pic}^0(F)$$

is trivial. Dually, the image of the restriction map $H^1(F,\mathbb{Z})\longrightarrow H^1(C_L,\mathbb{Z})$ is contained in the kernel of $\pi_*\colon H^1(C_L,\mathbb{Z})\longrightarrow H^1(D_L,\mathbb{Z})$.

Proof Observe that under the natural map

$$D_L\longrightarrow \mathrm{Pic}(D_L)\longrightarrow \mathrm{Pic}(C_L)\longrightarrow \mathrm{CH}^2(F)\longrightarrow \mathrm{CH}^2(Y)$$

a point $y\in D_L$ is mapped to $[L_1]+[L_2]\in \mathrm{CH}^2(Y)$, where L_1 and L_2 correspond to the two points of the fibre $\pi^{-1}(y)$. Clearly, $[L_1]+[L_2]=[L_1]+[L_2]+[L]-[L]=[\mathbb{P}^2]|_Y-[L]$, where $\mathbb{P}^2=\overline{y\,L}$. As $[\overline{y\,L}]\in \mathrm{CH}^2(\mathbb{P}^4)\simeq\mathbb{Z}$ is independent of $y\in D_L$, the class $[L_1]+[L_2]\in \mathrm{CH}^2(Y)$ is independent of y and, therefore, $D_L\longrightarrow \mathrm{CH}^2(Y)\longrightarrow \mathrm{CH}^1(F)$ is constant. □

By a purely topological description of étale coverings of degree two, one computes that the intersection pairing on $H^1(C_L,\mathbb{Z})$ restricted to

$$H^1(C_L,\mathbb{Z})^-\coloneqq \mathrm{Ker}\left(\pi_*\colon H^1(C_L,\mathbb{Z})\longrightarrow H^1(D_L,\mathbb{Z})\right)$$

is divisible by two, cf. Section 3.2. Hence, the Hodge–Riemann pairing $(\,.\,)_F$ on $H^1(F,\mathbb{Z})$ with respect to the Plücker polarization is of the form

$$(\gamma.\gamma')_F=\int_F\gamma\cdot\gamma'\cdot g=3\int_{C_L}\gamma|_{C_L}\cdot\gamma'|_{C_L}\in 6\,\mathbb{Z},$$

where we use Lemma 1.8 showing that $3\,[C_L]=g\in H^2(F,\mathbb{Z})$. Then, according to Proposition **3**.4.2, the Fano correspondence yields an injection of integral(!) symplectic lattices

$$\varphi\colon (H^3(Y,\mathbb{Z}),(\,.\,)_Y)\hookrightarrow (H^1(F,\mathbb{Z}),(-1/6)(\,.\,)_F)$$

of finite index. As the left hand side is unimodular, it has to be an isomorphism. We thus have proved the following result.

Corollary 3.3. *The Fano correspondence induces an isomorphism of Hodge structures*

$$\varphi\colon H^3(Y,\mathbb{Z})\xrightarrow{\sim} H^1(F,\mathbb{Z})(-1)$$

and the induced morphisms between the associated abelian varieties are isomorphisms

$$A(F)\xrightarrow{\sim} J(Y)\xrightarrow{\sim}\mathrm{Pic}^0(F).$$

□

3.2 The next aim is to relate the intermediate Jacobian $J(Y)$ to the Prym variety of the étale cover $C_L \twoheadrightarrow D_L$ for any generic line $L \subset Y$. Let us begin with a reminder on Prym varieties. See [9, 103, 117] for a detailed discussion and [62] for a historical account.

We consider an étale cover $\pi\colon C \twoheadrightarrow D$ of degree two between smooth projective curves. Its corresponding two-torsion line bundle $\mathcal{L}_\pi \simeq \pi_*\mathcal{O}_C/\mathcal{O}_D \in \mathrm{Pic}^0(D)$ satisfies $\pi^*\mathcal{L}_\pi \simeq \mathcal{O}_C$. We shall denote the covering involution by ι and its natural action on the Picard variety by $\iota^*\colon \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(C)$.

Lemma 3.4. *The pull-back* $\pi^*\colon \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(C)$ *yields an isomorphism*

$$\mathrm{Pic}(D)/\langle \mathcal{L}_\pi \rangle \simeq \mathrm{Ker}\,(\,\mathrm{Pic}(C) \xrightarrow{1-\iota^*} \mathrm{Pic}(C)\,).$$

Proof The morphism $1-\iota^*$ maps a line bundle $\mathcal{L}$ to $\mathcal{L} \otimes \iota^*\mathcal{L}^*$. Clearly, if $\mathcal{L} = \pi^*\mathcal{M}$, then $(1-\iota^*)(\mathcal{L}) \simeq \mathcal{O}_C$. For the other inclusion use that any ι^*-invariant invertible sheaf descends to an invertible sheaf on D.[1] Following [9], the descent can be shown explicitly as follows: Suppose $\mathcal{L} = \mathcal{O}(E)$ satisfies $\iota^*\mathcal{L} \simeq \mathcal{L}$. Write $\mathcal{L} \otimes \iota^*\mathcal{L}^*$ as the principal divisor (f) for some $f \in K(C)$ and observe that then $f \cdot \iota^* f$ has neither zeros nor poles, so we may assume $f \cdot \iota^* f = 1$ (we need k to admit square roots for this). Pick an element $g \in K(C)$ with $\iota^* g = -g$ and set $f_0 \coloneqq g \cdot (f-1)$. Then, $f = f_0 \cdot (\iota^* f_0)^{-1}$ and, therefore, $E_0 \coloneqq E - (f_0)$ is the pull-back of a divisor on D. We leave it to the reader to verify that $\mathcal{O}_D$ and $\mathcal{L}_\pi$ are the only line bundles with trivial pull-back to C. □

Definition 3.5. The *Prym variety* of an étale cover $\pi\colon C \twoheadrightarrow D$ of degree two is defined as

$$\mathrm{Prym}\,(C/D) \coloneqq \mathrm{Im}\,(\,\mathrm{Pic}^0(C) \xrightarrow{1-\iota^*} \mathrm{Pic}(C)\,).$$

Hence, there exists a natural exact sequence of abelian varieties

$$0 \longrightarrow \langle \mathcal{L}_\pi \rangle \longrightarrow \mathrm{Pic}^0(D) \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Prym}(C/D) \longrightarrow 0. \tag{3.1}$$

Alternatively, the Prym variety can be viewed as a connected component of the kernel of the norm map. Recall that the norm map $N\colon \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(D)$ is the push-forward map $\pi_*\colon \mathrm{CH}^1(C) \longrightarrow \mathrm{CH}^1(D)$ or, using Pic $\simeq$ Alb for curves, the natural map $\mathrm{Alb}(C) \longrightarrow \mathrm{Alb}(D)$. It can also be described as $N(\mathcal{L}) \simeq \det \pi_*\mathcal{L} \otimes (\det \pi_*\mathcal{O}_C)^*$. Clearly, N defined as π_* is a group homomorphism, which is not quite so apparent in the latter description. Then

$$\mathrm{Prym}\,(C/D) = \mathrm{Im}\,(1-\iota^*) \simeq \mathrm{Ker}\,(N)^o$$

is the connected component of the kernel containing $\mathcal{O}_C$. In fact, $\mathrm{Ker}(N)$ has exactly two connected components $\mathrm{Prym} \sqcup \mathrm{Prym}'$, non-canonically isomorphic to each other. For one

[1] One could think that the absence of fixed points is important here. It is not, although for the descent of invariant invertible sheaves on surfaces, fixed points do cause problems.

inclusion use $\pi_*\iota^* = \pi_*$ and compute $N(\mathcal{L} \otimes \iota^*\mathcal{L}^*) \simeq \det \pi_*(\mathcal{L}) \otimes (\det \pi_*\iota^*\mathcal{L})^* \simeq \mathcal{O}_D$. For the other, observe that $N\colon \mathrm{Pic}^0(C) \twoheadrightarrow \mathrm{Pic}^0(D)$ is surjective and hence $\mathrm{Prym}\,(C/D) \subset \mathrm{Ker}\,(N)^o$ is an inclusion of abelian varieties of the same dimension.

To summarize, in addition to the exact sequence (3.1) there is an exact sequence

$$0 \longrightarrow \mathrm{Prym} \sqcup \mathrm{Prym}' \longrightarrow \underset{\simeq A(C)}{\mathrm{Pic}^0(C)} \xrightarrow{N} \underset{\simeq A(D)}{\mathrm{Pic}^0(D)} \longrightarrow 0. \tag{3.2}$$

It may be helpful to describe both points of view in terms of integral Hodge structures. Recall that

$$\mathrm{Pic}^0(C) \simeq \frac{H^1(C, \mathcal{O}_C)}{H^1(C, \mathbb{Z})} \simeq \frac{H^0(C, \omega_C)^*}{H_1(C, \mathbb{Z})}$$

and similarly for $\mathrm{Pic}^0(D)$. As explained in [103, Ch. 12.4], $H^1(C,\mathbb{Z})$ admits a symplectic basis of the form $\tilde\lambda_0, \tilde\mu_0, \lambda_i^\pm, \mu_i^\pm$, $i = 1, \dots, g(D) - 1$ with $\tilde\lambda_0, \tilde\mu_0$ fixed by the action of ι^* on $H^1(C,\mathbb{Z})$ and $\iota^*(\lambda_i^\pm) = \lambda_i^\mp$, $\iota^*(\mu_i^\pm) = \mu_i^\mp$. This allows one to describe the eigenspaces $H^1(C,\mathbb{Z})^\pm \subset H^1(C,\mathbb{Z})$ of the involution ι^*:

$$H^1(C,\mathbb{Z})^+ = \langle \tilde\lambda_0, \tilde\mu_0, \lambda_i^+ + \lambda_i^-, \mu_i^+ + \mu_i^- \rangle \ \text{ and } \ H^1(C,\mathbb{Z})^- = \langle \lambda_i^+ - \lambda_i^-, \mu_i^+ - \mu_i^- \rangle.$$

The latter implies the fact alluded to before that the intersection pairing on $H^1(C,\mathbb{Z})^-$ is divisible by 2, which was used in the proof of Corollary 3.3. Moreover, the pull-back describes

$$\pi^*\colon H^1(D,\mathbb{Z}) = \langle \lambda_i, \mu_i \rangle_{i=0,\dots,g(D)-1} \hookrightarrow H^1(C,\mathbb{Z})^+,$$

given by $\lambda_0 \longmapsto \tilde\lambda_0$, $\mu_0 \longmapsto 2\,\tilde\mu_0$, $\lambda_i \longmapsto \lambda_i^+ + \lambda_i^-$, and $\mu_i \longmapsto \mu_i^+ + \mu_i^-$ for $i = 1, \dots, g(D) - 1$, as sublattice of index two. With this notation, topologically the étale covering $C \longrightarrow D$ can be constructed by cutting D along the standard loop representing μ_0 and glueing two copies of D along μ_0. This explains why in particular the pull-back of μ_0 yields $2\,\tilde\mu_0$.

Also observe that the image of

$$H^1(C,\mathbb{Z}) \longrightarrow H^1(C,\mathbb{Z})^+, \ \alpha \longmapsto \alpha + \iota^*\alpha$$

is contained in $\pi^*H^1(D,\mathbb{Z}) \subset H^1(C,\mathbb{Z})^+$ with index two. On the other hand,

$$H^1(C,\mathbb{Z}) \twoheadrightarrow H^1(C,\mathbb{Z})^-, \ \alpha \longmapsto \alpha - \iota^*\alpha$$

is surjective. The sequence (3.1) is induced by the exact sequence

$$0 \longrightarrow H^1(C,\mathbb{Z})^+ \longrightarrow H^1(C,\mathbb{Z}) \xrightarrow{1-\iota^*} H^1(C,\mathbb{Z})^- \longrightarrow 0, \tag{3.3}$$

which allows one to describe the Prym variety as

$$\mathrm{Prym}(C/D) \simeq \frac{H^1(C, \mathcal{O}_C)^-}{H^1(C,\mathbb{Z})^-} \simeq \frac{H^0(C, \omega_C)^{-*}}{H_1(C,\mathbb{Z})^-}.$$

Remark 3.6. The Prym variety is commonly viewed as a principally polarized abelian variety: Indeed, the last isomorphism together with the description of $H^1(C,\mathbb{Z})^-$ as $\langle\lambda_i^+ - \lambda_i^-, \mu_i^+ - \mu_i^-\rangle$ allows one to define a principal polarization on $\mathrm{Prym}(C/D)$ explicitly as given by the intersection pairing on $H^1(C,\mathbb{Z})^-$ up to the factor $(1/2)$.

The kernel of $\mathrm{Pic}^0(C) \twoheadrightarrow \mathrm{Prym}(C/D)$ can be written as the degree two quotient

$$\mathrm{Pic}^0(D) \simeq \frac{H^1(C,\mathcal{O}_C)^+}{H^1(D,\mathbb{Z})} \twoheadrightarrow \frac{H^1(C,\mathcal{O}_C)^+}{H^1(C,\mathbb{Z})^+}.$$

On the other hand, the sequence (3.2) corresponds to

$$0 \longrightarrow H^1(C,\mathbb{Z})^- \longrightarrow H^1(C,\mathbb{Z}) \xrightarrow{1+\iota^*} (1+\iota^*)H^1(C,\mathbb{Z}) \longrightarrow 0,$$

using the degree two quotient

$$\frac{H^1(C,\mathcal{O}_C)^+}{(1+\iota^*)H^1(C,\mathbb{Z})} \twoheadrightarrow \frac{H^1(C,\mathcal{O}_C)^+}{H^1(D,\mathbb{Z})} \simeq \mathrm{Pic}^0(D).$$

Let us now apply this to the cover $C_L \twoheadrightarrow D_L$ associated with any generic line $L \subset Y$.

Proposition 3.7 (Mumford)**.** *For a generic line $L \subset Y$, the inclusion $i\colon C_L \hookrightarrow F(Y)$ induces an isomorphism*

$$\mathrm{Prym}(C_L/D_L) \xrightarrow{\sim} A(F) \simeq J(Y) \simeq \mathrm{Pic}^0(F).$$

Proof The assertion follows from a comparison of (3.3) with the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\xi) \longrightarrow H^1(C_L,\mathbb{Z}) \xrightarrow{\xi} H^3(Y,\mathbb{Z}) \longrightarrow 0.$$

Here, ξ is the composition of the push-forward $i_*\colon H^1(C_L,\mathbb{Z}) \longrightarrow H^3(F(Y),\mathbb{Z})$ of the inclusion $i\colon C_L \hookrightarrow F(Y)$ and $\psi\colon H^3(F(Y),\mathbb{Z}) \twoheadrightarrow H^3(Y,\mathbb{Z})$. The surjectivity of i_* is a consequence of the ampleness of C_L and the Lefschetz hyperplane theorem $H^1(C_L,\mathbb{Z}) \simeq H_1(C_L,\mathbb{Z}) \twoheadrightarrow H_1(F(Y),\mathbb{Z}) \simeq H^3(F(Y),\mathbb{Z})$. As ψ is (up to torsion) dual to $\varphi\colon H^3(Y,\mathbb{Z}) \xrightarrow{\sim} H^1(F(Y),\mathbb{Z})$, which is an isomorphism according to Corollary 3.3, ψ it is surjective, too.

Due to Corollary 3.2, we know that $H^1(D_L,\mathbb{Z}) \longrightarrow H^3(F(Y),\mathbb{Z})$ is trivial and, hence, $H^1(D_L,\mathbb{Z}) \subset \mathrm{Ker}(\xi)$. Both are free $\mathbb{Z}$-modules of the same rank and $\mathrm{Ker}(\xi)$ is saturated, as its cokernel is the torsion free $H^3(Y,\mathbb{Z})$. On the other hand, $H^1(D_L,\mathbb{Z})$ is also contained with finite index in $H^1(C,\mathbb{Z})^+$, which is a saturated submodule of $H^1(C,\mathbb{Z})$ (its cokernel is the torsion free $H^1(C,\mathbb{Z})^-$). Hence, $\mathrm{Ker}(\xi)$ and $H^1(C,\mathbb{Z})^+$ both realize the saturation of $H^1(D_L,\mathbb{Z}) \subset H^1(C_L,\mathbb{Z})$ and, therefore coincide. But then $H^3(Y,\mathbb{Z}) \simeq H^1(C_L,\mathbb{Z})^-$ and $\mathrm{Prym}(C_L/D_L) \simeq J(Y)$. □

Remark 3.8. There is a subtlety that has been avoided in the above discussion: Is $\psi\colon H^3(F(Y),\mathbb{Z}) \twoheadrightarrow H^3(Y,\mathbb{Z})$ an isomorphism? This is true up to torsion in $H^3(F(Y),\mathbb{Z})$, which in any case does not effect the isomorphism $J(Y) \simeq A(F)$. Nevertheless, it would be interesting to decide whether there is non-trivial torsion in $H^3(F(Y),\mathbb{Z})$ or, similarly, in $H^2(F(Y),\mathbb{Z})$. The torsion freeness of the latter would in particular prove that numerical equals algebraic equivalence for line bundles on $F(Y)$, see below.

As a consequence, one obtains a description of the algebraically trivial part of the Chow group of curves on Y.

Corollary 3.9. *For a generic line $L\subset Y$ (in the sense of Corollary 1.6) the Abel–Jacobi map yields an isomorphism of groups*

$$\mathrm{CH}^2(Y)_{\mathrm{alg}} \xrightarrow{\sim} J(Y) \simeq \mathrm{Prym}(C_L/D_L).$$

Proof The result can be seen as an application of a result of Bloch and Srinivas [21, Thm. 1]: If Y is a smooth complex projective variety with $\mathrm{CH}_0(Y)\simeq\mathbb{Z}$, then the Abel–Jacobi map induces isomorphisms of groups

$$\mathrm{CH}^2(Y)_{\mathrm{alg}} \simeq \mathrm{CH}^2(Y)_{\mathrm{hom}} \xrightarrow{\sim} J(Y). \tag{3.4}$$

Clearly, as on a cubic threefold any two points can be connected by a chain of lines, cubic threefolds satisfy the assumption.

However, in our case of a smooth cubic threefold more direct arguments for the isomorphism $\mathrm{CH}^2(Y)_{\mathrm{alg}}\simeq J(Y)$ exist, see [120, 121, 122] or [9, Thm. 3.1]. □

Remark 3.10. The arguments heavily relied on the ground field being $\mathbb{C}$. In fact, already the definition of the intermediate Jacobian $J(Y)$ over other fields is problematic. However, $\mathrm{CH}^2(Y)$ and $\mathrm{Prym}(C_L/D_L)$ make perfect sense over arbitrary fields. And, indeed, Murre in [120] describes an algebraic approach that yields an isomorphism $\mathrm{CH}^2(Y)_{\mathrm{alg}}\simeq\mathrm{Prym}(C_L/D_L)$ over arbitrary algebraically closed fields. In fact, the isomorphism was originally stated up to elements of order two, but the divisibility of $\mathrm{CH}^2(Y)_{\mathrm{alg}}$ (pointed out by Bloch, see the review of [120]), yields the full statement.

Remark 3.11. In [34, App. A] one finds a geometric argument that shows a weaker version of the first isomorphism in (3.4), namely that the difference between $\mathrm{CH}^2(Y)_{\mathrm{hom}}$ and $\mathrm{CH}^2(Y)_{\mathrm{alg}}$ is annihilated by 6. More precisely, it is shown that $6\,\mathrm{CH}^2(Y)_{\mathrm{hom}}=\mathrm{CH}^2(Y)_{\mathrm{alg}}$. The idea is the following: Let $C\subset Y$ be any curve. Then a surface $C\subset S\subset Y$ is constructed such that $6\,C$ on S is rationally equivalent to the sum of $nH|_S$ and a sum of lines $\sum a_iL_i$. As $\mathrm{Pic}(Y)\simeq\mathbb{Z}$, this proves the assertion. The surface S is obtained as $q(\tilde S)$ with $\tilde S=p^{-1}(p(q^{-1}(C)))$ which parametrizes (L,x) with L a line containing x and intersecting C. Clearly, $\tilde S$ is a $\mathbb{P}^1$-bundle over $q^{-1}(C)$ and comes with a natural section $q^{-1}(C)\subset\tilde S$, which via q_* yields $6\,C$.

4 Global Torelli theorem and irrationality

6

Cubic fourfolds

1 Lattice and Hodge theory for cubic fourfolds and K3 surfaces

In the first section, we collect all facts from Hodge and lattice theory relevant for the study of cubic fourfolds. The curious relation between the lattice theory of cubic fourfolds and K3 surfaces has been systematically studied first by Hassett [80]. Earlier results in this direction are due to Beauville and Donagi [16]. [1]

1.1 As we have seen in Section **??.??**, as abstract lattices the middle cohomology and the primitive cohomology of a smooth cubic fourfold $X \subset \mathbb{P}^5$ are described by

$$H^4(X,\mathbb{Z}) \simeq \mathrm{I}_{21,2} \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{3,0},$$
$$H^4(X,\mathbb{Z})_{\mathrm{pr}} \simeq E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2,$$

where $h^2 = (1,1,1) \in \mathrm{I}_{3,0}$. It will be convenient to change the sign and introduce the *cubic lattice* and the *primitive cubic lattice* as

$$\bar{\Gamma} \coloneqq \mathrm{I}_{2,21} \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{0,3} \simeq H^4(X,\mathbb{Z})(-1),$$
$$\Gamma \coloneqq E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2(-1) \simeq H^4(X,\mathbb{Z})_{\mathrm{pr}}(-1).$$

In particular, from now on $(h^2)^2 = -3$. The twist should not be confused with the Tate twist of the Hodge structure. It turns out that $E_8(-1)^{\oplus 2}$, certainly the most interesting part of these lattices, will hardly play any role in our discussion. We shall henceforth abbreviate it by

$$E \coloneqq E_8(-1)^{\oplus 2}$$

and consequently write

$$\bar{\Gamma} \simeq E \oplus U^{\oplus 2} \oplus I_{0,3} \text{ and } \Gamma \simeq E \oplus U^{\oplus 2} \oplus A_2(-1).$$

Version Oct 14, 2018. The first two sections are still in very rough form. Feedback will be most welcome.

[1] This part is based on lectures given at the school 'Birational Geometry of Hypersurfaces' in Gargnano in March 2018.

In the discussion below, the intersection form of a K3 surface will be central, although there is a priori no geometric reason why K3 surfaces should enter the picture at all. In a first step, we shall deal with this purely on the level of abstract lattice theory and later add Hodge structures.

Recall that for a complex K3 surface S, its middle cohomology with the intersection form is the lattice

$$H^2(S,\mathbb{Z}) \simeq E \oplus U^{\oplus 3} \simeq E \oplus U_1 \oplus U_2 \oplus U_3 =: \Lambda,$$

see [86, Ch. 14]. The summands U_i, $i = 1,2,3$, are copies of the hyperbolic plane U. Indexing them will make the discussion more explicit and will help us to avoid ambiguities later on.

The full cohomology $H^*(S,\mathbb{Z})$ is also endowed with a unimodular intersection form. It is customary to introduce a sign in the pairing on $(H^0 \oplus H^4)(S,\mathbb{Z})$, which, however, does not change the abstract isomorphism type, for $U \simeq U(-1)$. The resulting lattice is the *Mukai lattice*

$$\begin{aligned}\widetilde{H}(S,\mathbb{Z}) &\coloneqq H^2(S,\mathbb{Z}) \oplus (H^0 \oplus H^4)(S,\mathbb{Z}) \simeq E \oplus U^{\oplus 3} \oplus U_4\\ &\simeq E \oplus U_1 \oplus U_2 \oplus U_3 \oplus U_4 =: \widetilde{\Lambda}.\end{aligned}$$

The standard basis of U consists of isotropic vectors e, f with $(e.f) = 1$. We shall denote the standard bases in the first three copies of U as $e_i, f_i \in U_i$, $i = 1,2,3$. However, in order to take into account the sign change in the Mukai pairing, we shall use the convention that $(e_4.f_4) = -1$ and that $e_4 = [S] \in H^0(S,\mathbb{Z})$ and $f_4 = [x] \in H^4(S,\mathbb{Z})$.

Next, we introduce an explicit embedding $A_2 \hookrightarrow \widetilde{\Lambda}$. Here, $A_2 = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ is the lattice of rank two given by the intersection form $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and we define

$$A_2 \hookrightarrow U_3 \oplus U_4 \subset \widetilde{\Lambda} \tag{1.1}$$

by $\lambda_1 \longmapsto e_4 - f_4$ and $\lambda_2 \longmapsto e_3 + f_3 + f_4$. The orthogonal complement $\langle \lambda_1, \lambda_2 \rangle^\perp = A_2^\perp \subset \widetilde{\Lambda}$ is the lattice

$$A_2^\perp = E \oplus U_1 \oplus U_2 \oplus A_2(-1),$$

where $A_2(-1) \subset U_3 \oplus U_4$ is spanned by $\mu_1 \coloneqq e_3 - f_3$ and $\mu_2 \coloneqq -e_3 - e_4 - f_4$ satisfying $(\mu_i)^2 = -2$ and $(\mu_1.\mu_2) = 1$.

Remark 1.1. We observe that $\lambda_1^\perp = E \oplus U_1 \oplus U_2 \oplus U_3 \oplus \mathbb{Z}(-2)$, where the last direct summand is generated by $e_4 + f_4$. Hence, $\lambda_1^\perp \simeq \Lambda \oplus \mathbb{Z}(-2)$. As $H^2(S,\mathbb{Z}) \simeq \Lambda$ and $H^2(S^{[2]},\mathbb{Z}) \simeq H^2(S,\mathbb{Z}) \oplus \mathbb{Z}(-2)$ for the Hilbert scheme $S^{[2]}$ of any K3 surface S, this can be read as a lattice isomorphism $\lambda_1^\perp \simeq H^2(S^{[2]},\mathbb{Z})$.

The discussion so far leads to the fundamental observation that there exists an isomorphism

$$\bar{\Gamma} \supset \Gamma \simeq A_2^{\perp} \subset \widetilde{\Lambda}$$

between the primitive cubic lattice Γ and the lattice $A_2^{\perp}$ inside the Mukai lattice $\widetilde{\Lambda}$.

For later use, we record that (1.1) induces inclusions of index three:

$$A_2 \oplus A_2(-1) \subset U_3 \oplus U_4 \text{ and } A_2 \oplus A_2^{\perp} \subset \widetilde{\Lambda},$$

where, for example, the quotient of the latter is generated by the image of the class $(1/3)(\mu_1 - \mu_2 - \lambda_1 + \lambda_2) = e_3 + f_4$.

Another technical result that will be crucial at some point later, is the following elementary statement which is surprisingly difficult to prove, cf. [3, Prop. 3.2].

Lemma 1.2. *Consider $A_2 \subset \widetilde{\Lambda}$ as before, let $U \hookrightarrow \widetilde{\Lambda}$ be an isometric embedding of a copy of the hyperbolic plane, and denote by $\overline{A_2 + U}$ the saturation of $A_2 + U \subset \widetilde{\Lambda}$. Then there exists an isometric embedding of a copy of the hyperbolic plane $U' \hookrightarrow \overline{A_2 + U}$ such that* $\mathrm{rk}(A_2 + U') = 3$.

Proof Clearly, if $\mathrm{rk}(\overline{A_2 + U}) = 3$, there is nothing to prove, so we can restrict to the case that the rank is four. □

Remark 1.3. To motivate the notion of Noether–Lefschetz (or Heegner) divisors for cubic fourfolds, let us recall the corresponding concept for K3 surfaces: For a primitive class $\ell \in \Lambda$ with $(\ell)^2 = d$, we write

$$\Lambda_d \coloneqq \ell^{\perp} \subset \Lambda.$$

As ℓ is in the same $\mathrm{O}(\Lambda)$-orbit as the class $e_2 + (d/2) f_2$, cf. [86, Cor. 14.1.10], it can abstractly be described as

$$\Lambda_d \simeq E \oplus U^{\oplus 2} \oplus \mathbb{Z}(-d).$$

It is important to note that the lattices Λ_d are in general not contained in $A_2^{\perp} \subset \widetilde{\Lambda}$.

We shall call any primitive vector $v \in \Gamma \simeq A_2^{\perp}$ with $(v)^2 < 0$ a *Noether–Lefschetz* vector. With such Noether–Lefschetz vector one naturally associates two lattices. On the cubic side, one defines

$$\mathbb{Z} h^2 \oplus \mathbb{Z} v \subset K_v \subset \bar{\Gamma}$$

as the saturation of $\mathbb{Z} h^2 \oplus \mathbb{Z} v \subset \bar{\Gamma}$. On the K3 side, we introduce the saturation

$$A_2 \oplus \mathbb{Z} v \subset L_v \subset \widetilde{\Lambda}.$$

Note that L_v is of rank three and signature $(2, 1)$, while K_v is of rank two and signature $(0, 2)$. Clearly, their respective orthogonal complements are isomorphic:

$$\bar{\Gamma} \supset K_v^{\perp} \simeq L_v^{\perp} \subset \widetilde{\Lambda},$$

as they are both described as $v^\perp \subset \Gamma \simeq A_2^\perp$. In particular, for the discriminants[2] we have

$$d := \mathrm{disc}(L_v) = \mathrm{disc}(K_v).$$

The situation has been studied in depth in [80], where one also finds the next result.

Lemma 1.4 (Hassett)**.** *Only the following two cases can occur:*

(i) *Either* $\mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v = K_v$*,* $A_2 \oplus \mathbb{Z}\,v = L_v$*, and*

$$d = \mathrm{disc}(K_v) = \mathrm{disc}(L_v) = -3\,(v)^2 \equiv 0\,(6)$$

(ii) *or* $\mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v \subset K_v$*,* $A_2 \oplus \mathbb{Z}\,v \subset L_v$ *are both of index three, and*

$$d = \mathrm{disc}(K_v) = \mathrm{disc}(L_v) = -\frac{1}{3}(v)^2 \equiv 2\,(6).$$

Proof The main ingredient is the standard formula, see e.g. [86, Sec. 14.0.2],

$$\mathrm{disc}(K_v) \cdot [K_v : \mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v]^2 = \mathrm{disc}(\mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v) = -3\,(v)^2.$$

Any $y \in K_v$ is of the form $y = s\,h^2 + t\,v$, with $s, t \in \mathbb{Q}$. From $(h.y) \in \mathbb{Z}$ one concludes $s \in (1/3)\,\mathbb{Z}$ and hence also $t \in (1/3)\,\mathbb{Z}$. This shows that $[K_v : \mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v] = 1, 3$, or $= 9$, but the last possibility is excluded as $(1/3)\,h^2 \notin \bar{\Gamma}$.

In the first case, i.e. $K_v = \mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v$, one finds $d = \mathrm{disc}(K_v) = -3\,(v)^2 \equiv 0\,(6)$. In the second case, so when the index is three, then $3\,d = -(v)^2 \equiv 0, 2, 4\,(6)$. On the other hand, K_v admits a basis consisting of h^2 and another class x. Indeed, pick any class $x \in K_v$ whose image generates the quotient $K_v/(\mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v) \simeq \mathbb{Z}/3\mathbb{Z}$. We may assume $3\,x = s\,h^2 + t\,v$ with $s, t \in \{\pm 1\}$ and, therefore, $K_v = \mathbb{Z}\,h^2 \oplus \mathbb{Z}\,x$. Hence, its discriminant satisfies $d = -3\,(x)^2 - (x.h^2)^2 \equiv 0, 2, 3, 5\,(6)$. Altogether this shows that $d \equiv 0, 2\,(6)$.

We claim that $d \equiv 0\,(6)$ holds if and only if $K_v = \mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v$. The 'if'-direction' was proved already. For the 'only if'-direction, assume that $d \equiv 0\,(6)$ but $[K_v : \mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v] = 3$. Pick $x \in K_v$ as above. Then, write $v = s\,h^2 + t\,x$, $s, t \in \mathbb{Z}$, and use $(v.h^2) = 0$ and the primitivity of v to show $v = r\,((x.h^2)\,h^2 + 3\,x)$ with $r = \pm 1, \pm(1/3)$ as v is primitive. However, $(x.h^2) \equiv 0\,(3)$ under the assumption that $d \equiv 0\,(6)$. Hence, $\pm v = m\,h^2 + x$, $m \in \mathbb{Z}$, and, therefore, $x \in \mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v$. This yields a contradiction and thus proves the assertion.

The assertions for the lattice L_v follows directly from the ones for K_v. □

Remark 1.5. Depending on the perspective, it may be useful to study the various cases from the point of view of d or, alternatively, of $(v)^2$. To have the results handy for later use, we restate the above discussion as

$$d \equiv 0\,(6) \quad \Rightarrow \quad (v)^2 = -d/3 \equiv 0\,(6) \text{ or } \equiv \pm 2\,(6),$$
$$d \equiv 2\,(6) \quad \Rightarrow \quad (v)^2 = -3\,d \equiv 0\,(6)$$

[2] The sign of the discriminant will be of no importance in our discussion, we tacitly work with its absolute value.

and

$$(v)^2 \equiv \pm 2\,(6) \quad \Rightarrow \quad d = -3\,(v)^2 \equiv 0\,(6),$$
$$(v)^2 \equiv 0\,(6) \quad \Rightarrow \quad d = -3\,(v)^2 \equiv 0\,(6) \text{ or } d = -(1/3)\,(v)^2 \equiv 2\,(6).$$

In particular, d determines $(v)^2$ uniquely, but not vice versa unless $(v)^2 \equiv 2, 4\,(6)$.

Proposition 1.6 (Hassett). *Let $v, v' \in \Gamma$ be two primitive vectors and assume that* $\mathrm{disc}(L_v) = \mathrm{disc}(L_{v'})$ *or, equivalently,* $\mathrm{disc}(K_v) = \mathrm{disc}(K_{v'})$. *Then there exist an orthogonal transformations $g \in \tilde{\mathrm{O}}(\Gamma)$ such that $g(v) = \pm v'$ and, in particular,*

$$L_{v'} \simeq L_{g(v)} \text{ and } K_{v'} \simeq K_{g(v)}.$$

See Section **1**.2.4 for the definition of $\tilde{\mathrm{O}}(\Gamma)$, which we also recall below.

Proof One applies Eichler's criterion, cf. [69, Prop. 3.3]: If an even lattice N is of the form $N \simeq N' \oplus U^{\oplus 2}$, then a primitive vector $v \in N$ with prescribed $(v)^2 \in \mathbb{Z}$ and $(1/n)\,\bar{v} \in A_N$, with n determined by $(v.N) = n\,\mathbb{Z}$, is unique up to the action of $\tilde{\mathrm{O}}(N)$. Apply this to $v \in \Gamma \simeq A_2^{\perp} \simeq E \oplus U^{\oplus 2} \oplus A_2(-1)$ and use that for any primitive $v \in \Gamma$, either $(v.\Gamma) = \mathbb{Z}$ or $= 3\,\mathbb{Z}$. This follows from $[\bar{\Gamma} : \Gamma \oplus \mathbb{Z}\,h^2] = 3$ and the unimodularity of $\bar{\Gamma}$.

(i) If $(v)^2 \equiv 0\,(6)$, there are two cases: Assume first that $d \equiv 2\,(6)$ or, equivalently, that $\mathbb{Z}\,v \oplus \mathbb{Z}\,h^2$ is not saturated. Then, one finds an element of the form $\alpha := (1/3)\,v + t\,h^2 \in \bar{\Gamma}$. As $(\alpha.w) \in \mathbb{Z}$ for all $w \in \Gamma$, this shows $(v.\Gamma) \subset 3\,\mathbb{Z}$. Hence, $n = 3$ and $(1/3)\,\bar{v} = \pm 1 \in A_\Gamma \simeq \mathbb{Z}/3\,\mathbb{Z}$.

Assume now that $d \equiv 0\,(6)$ and write $v = n_1 v_1 + n_2 v_2$ with $v_1 \in E \oplus U_1 \oplus U_2$ and $v_2 \in A_2(-1)$, both primitive, and $n_1, n_2 \in \mathbb{Z}$. If $n_1 \not\equiv 0\,(3)$, then there exists a class w in the unimodular lattice $E \oplus U_1 \oplus U_2 \subset \Gamma$ with $(v.w) \notin 3\,\mathbb{Z}$ and hence $(v.\Gamma) = \mathbb{Z}$. If $n_1 \equiv 0\,(3)$, then $n_2 \not\equiv 0\,(3)$, as v is primitive. However, in this case $(1/3)\,(v \pm h^2) = (n_1/3)v_1 + (1/3)\,(n_2 v_2 \pm h^2) \in \bar{\Gamma}$ and so $\mathbb{Z}\,v \oplus \mathbb{Z}\,h^2$ is not saturated, contradicting $d \equiv 0\,(6)$.

(ii) If $(v)^2 \equiv 2, 4\,(6)$ and hence $(v)^2 \not\equiv 0\,(3)$, then $(v.\Gamma) = \mathbb{Z}$, $n = 1$, and $\bar{v} \in A_\Gamma$ is trivial.

Hence, in case (i) and (ii), if indeed d and not only $(v)^2$ is fixed, then $(v)^2 = (v')^2$ and $(1/n)\,\bar{v} = (1/n)\,\bar{v}' \in A_\Gamma$ (up to sign). □

Remark 1.7. Due to the uniqueness, no information is lost when explicit classes $v \in \Gamma \simeq A_2^{\perp}$ are chosen for any given d. In the sequel, we will work with the following ones.

(i) For $d \equiv 0\,(6)$, one may choose $v_d := e_1 - (d/6)\,f_1 \in U_1 \subset \Gamma$. Observe that indeed, as explained in the general context above, $(v_d)^2 = -d/3$ and that the lattice $A_2 \oplus \mathbb{Z}\,v_d$ is saturated (use $A_2 \subset U_2 \oplus U_3$ and $v_d \in U_1$), i.e.

$$L_d := L_{v_d} = A_2 \oplus \mathbb{Z}\,v_d.$$

Similarly,

$$K_d \coloneqq K_{v_d} = \mathbb{Z}\, h^2 \oplus \mathbb{Z}\, v_d,$$

which again shows $(v_d)^2 = -d/3$. Their orthogonal complement is

$$\Gamma_d \coloneqq L_d^\perp \simeq K_d^\perp \simeq E \oplus U_2 \oplus A_2(-1) \oplus \mathbb{Z}\,(e_1 + (d/6)\, f_1)$$

and their discriminant group

$$A_{K_d^\perp} \simeq A_{K_d} \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/(d/3)\mathbb{Z}$$

is cyclic if and only if $9 \nmid d$.

(ii) For $d \equiv 2\,(6)$, one sets $v_d \coloneqq 3\,(e_1 - ((d-2)/6)\, f_1) + \mu_1 - \mu_2 \in U_1 \oplus A_2(-1)$. Then both inclusions

$$A_2 \oplus \mathbb{Z}\, v_d \subset L_d \coloneqq L_{v_d} \text{ and } \mathbb{Z}\, h^2 \oplus \mathbb{Z}\, v_d \subset K_d \coloneqq K_{v_d}$$

are of index three, for example $v_d - \lambda_1 + \lambda_2$ and $v_d - h^2$ are divisible by 3. Use $\lambda_1 = e_4 - f_4$, $\lambda_2 = e_3 + f_3 + f_4$, $\mu_1 = e_3 - f_3$, and $\mu_2 = -e_3 - e_4 - f_4$. The latter corresponding to $(1,-1,0), (0,1,-1) \in \mathbb{Z}^{\oplus 3}$. In this case, see [2, 80, 143],

$$\Gamma_d \coloneqq L_d^\perp \simeq K_d^\perp \simeq E \oplus U_2 \oplus (\mathbb{Z}^{\oplus 3}, (\ .\ )_A) \text{ with } A \coloneqq \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & (d-2)/3 \end{pmatrix}$$

and L_d and K_d are given by the matrices $-A$ and

$$\begin{pmatrix} -3 & 1 \\ 1 & -(d+1)/3 \end{pmatrix}.$$

The discriminant groups for $d \equiv 2\,(6)$ are cyclic $A_{K_d^\perp} \simeq A_{K_d} \simeq \mathbb{Z}/d\mathbb{Z}$.

In addition to the orthogonal group

$$\tilde{\mathrm{O}}(\Gamma) \coloneqq \{\, g \in \mathrm{O}(\bar{\Gamma}) \mid g(h^2) = h^2 \,\}$$

(see Section **1**.2.4), which we will also think of as $\tilde{\mathrm{O}}(\Gamma) = \{\, g \in \mathrm{O}(\Gamma) \mid \bar{g} \equiv \mathrm{id} \text{ on } A_\Gamma \,\}$, we need to consider

$$\begin{array}{c} \tilde{\mathrm{O}}(\Gamma, K_d) \coloneqq \{\, g \in \tilde{\mathrm{O}}(\Gamma) \mid g(K_d) = K_d, \text{i.e. } g(v_d) = \pm v_d \,\} \\ \bigcup \\ \tilde{\mathrm{O}}(\Gamma, v_d) \coloneqq \{\, g \in \tilde{\mathrm{O}}(\Gamma) \mid g|_{K_d} = \mathrm{id}, \text{i.e. } g(v_d) = v_d \,\}. \end{array}$$

Observe that $\tilde{\mathrm{O}}(\Gamma, v_d)$ can be identified with the subgroup of all $g \in \mathrm{O}(\Gamma_d)$ with trivial action on the discriminant $A_{\Gamma_d} \simeq A_{K_d}$. Also, by definition, $\tilde{\mathrm{O}}(\Gamma, v_d) \subset \tilde{\mathrm{O}}(\Gamma, K_d)$ is a subgroup of index one or two. Note that the natural homomorphism $\tilde{\mathrm{O}}(\Gamma, K_d) \longrightarrow \mathrm{O}(K_d)$ is neither surjective (let alone injective) nor is its image contained in $\tilde{\mathrm{O}}(K_d)$.

Lemma 1.8 (Hassett)**.** *The subgroup* $\tilde{\mathrm{O}}(\Gamma, v_d) \subset \tilde{\mathrm{O}}(\Gamma, K_d)$ *is of index at most two. More precisely, one distinguishes the following cases:*

(i) *If* $d \equiv 0\,(6)$*, then*

$$\tilde{\mathrm{O}}(\Gamma, v_d) \subset \tilde{\mathrm{O}}(\Gamma, K_d)$$

has index two.

(ii) *If* $d \equiv 2\,(6)$*, then*

$$\tilde{\mathrm{O}}(\Gamma, v_d) = \tilde{\mathrm{O}}(\Gamma, K_d).$$

Proof (i) According to Lemma 1.4, $d \equiv 0\,(6)$ if and only if $\mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v_d = K_d$, which is contained in $\mathrm{I}_{0,3} \oplus U_1$. Let $g \in \tilde{\mathrm{O}}(\Gamma)$ be the orthogonal transformation defined by $g = \mathrm{id}$ on $E \oplus U_2 \oplus \mathrm{I}_{0,3}$ and by $g = -\mathrm{id}$ on U_1. Then g is an element in $\tilde{\mathrm{O}}(\Gamma, K_d) \setminus \tilde{\mathrm{O}}(\Gamma, v_d)$.

(ii) Now, $d \equiv 2\,(6)$ if and only if $\mathbb{Z}\,h^2 \oplus \mathbb{Z}\,v_d \subset K_d$ has index three and then $v_d = 3\,(e_1 - ((d-2)/6)\,f_1) + \mu_1 - \mu_2$ with $\mu_1 = (1,-1,0), \mu_2 = (0,1,-1) \in A_2(-1) \subset \mathrm{I}_{0,3}$ and $h^2 = (1,1,1)$. Now observe that $(1/3)\,(v_d - h^2) \in K_d$, but $(1/3)\,(-v_d - h^2) \notin K_d$. □

1.2 It turns out that certain geometric properties of cubic fourfolds are encoded by lattice-theoretic properties of Noether–Lefschetz vectors $v \in \Gamma$. The following ones are relevant for our purposes. It is a matter of choice, whether they are read as conditions on d or on the primitive $v \in \Gamma$. For $d \in \mathbb{Z}$ one considers the following conditions:

$(*)$	$\Leftrightarrow$	There exists an L_d.
$(**')$	$\Leftrightarrow$	There exists an L_d and an embedding $U(n) \hookrightarrow L_d$ for some $n \neq 0$.
$(**)$	$\Leftrightarrow$	There exists an L_d and a primitive embedding $U \hookrightarrow L_d$.
$(***)$	$\Leftrightarrow$	There exists an L_d and a primitive embedding $U \hookrightarrow L_d$ with $\lambda_1 \in U$.

Remark 1.9. (i) The following implications trivially hold

$$(***) \Rightarrow (**) \Rightarrow (**') \Rightarrow (*).$$

(ii) Each of the conditions in fact splits in two, distinguishing between $d \equiv 0\,(6)$ and $d \equiv 2\,(6)$. We shall write accordingly $(*)_0$, $(*)_2$, $(**')_0$, $(**')_2$, etc.

Lemma 1.10. *Condition* $(**)$ *holds if and only if there exists an isomorphism of lattices*

$$\varepsilon\colon \Gamma_d \xrightarrow{\sim} \Lambda_d.$$

In this case, one also has an isomorphism (independent of the choice of ε*) of groups*

$$\tilde{\mathrm{O}}(\Gamma, v_d) \simeq \tilde{\mathrm{O}}(\Lambda_d).$$

Proof Assume that there exists a (primitive) hyperbolic plane $U \hookrightarrow L_d$. As the composition with the inclusion $L_d \subset \widetilde{\Lambda}$ can be identified with $U_4 \hookrightarrow \widetilde{\Lambda}$ up to the action of $\mathrm{O}(\widetilde{\Lambda})$, see [86, Thm. 14.1.12], one has $U^\perp \simeq \Lambda$. Hence, $\Gamma_d \simeq L_d^\perp \subset U^\perp \simeq \Lambda$ is a primitive sublattice of corank one, signature $(2, 19)$, discriminant d, and is, therefore, isomorphic to Λ_d. Conversely, if $L_d^\perp \simeq \Gamma_d \simeq \Lambda_d \subset \Lambda \subset \widetilde{\Lambda}$, then $U_4 \subset L_d$. Here, one again uses that up to $\mathrm{O}(\widetilde{\Lambda})$, there exists only one primitive embedding $L_d \hookrightarrow \widetilde{\Lambda}$.

For the isomorphism between the two orthogonal groups, just recall that they are both described as the subgroup of all orthogonal transformations of $\Gamma_d \simeq \Lambda_d$ acting trivially on the discriminant $A_{\Gamma_d} \simeq A_{\Lambda_d} \simeq \mathbb{Z}/d\mathbb{Z}$. □

Remark 1.11. As any isometric embedding $U \hookrightarrow L_d$ splits, see [86, Example 14.0.3], one concludes that for d satisfying $(**')_0$ and $(**')_2$, respectively, that

$$(**')_0: \quad A_2 \oplus \mathbb{Z}\, v_d \simeq L_d \simeq U \oplus \mathbb{Z}(d) \text{ and } (v_d)^2 = -(1/3)\, d$$

$$(**')_2: \quad A_2 \oplus \mathbb{Z}\, v_d \hookrightarrow L_d \simeq U \oplus \mathbb{Z}(d) \text{ index three and } (v_d)^2 = -3\, d.$$

Remark 1.12. For a numerical description of these conditions one needs the following classical facts determining which numbers are represented by A_2, see [37, 96].

(i) For a given even, positive integer d there exists a vector $w \in A_2$ with $(w) = d$ if and only if the prime factorization of $d/2$ satisfies

$$\frac{d}{2} = \prod p^{n_p} \text{ with } n_p \equiv 0\,(2) \text{ for all } p \equiv 2\,(3). \tag{1.2}$$

(ii) For a given even, positive integer d there exists a primitive vector $w \in A_2$ with $(w)^2 = d$ if and only if

$$\frac{d}{2} = \prod p^{n_p} \text{ with } n_p = 0 \text{ for all } p \equiv 2\,(3) \text{ and } n_3 \leq 1. \tag{1.3}$$

Proposition 1.13. *Numerically,* $(*)$, $(**')$, $(**)$*, and* $(***)$ *are described by:*

(i) $(*) \quad \Leftrightarrow \quad d \equiv 0, 2\,(6)$.

(ii) $(**') \quad \Leftrightarrow \quad \exists\, w \in A_2 : (w)^2 = d \quad \Leftrightarrow \quad (1.2)$.

(iii) $(**) \quad \Leftrightarrow \quad \exists\, w \in A_2$ primitive: $(w)^2 = d \quad \Leftrightarrow \quad (1.3) \quad \Leftrightarrow \quad \exists\, a, n \in \mathbb{Z} \colon d = \frac{2n^2+2n+2}{a}$.

(iv) $(***) \quad \Leftrightarrow \quad \exists\, a, n \in \mathbb{Z} \colon d = \frac{2n^2+2n+2}{a^2}$.

Proof The first assertion follows from Lemma 1.4.

To prove (ii), one has to distinguish between the two cases $d \equiv 0\,(6)$ and $d \equiv 2\,(6)$. Assume first that $(**')_0$ holds. Then $L_d = A_2 \oplus \mathbb{Z}\, v_d$, which contains the isotropic vector $e \in U(n) \subset L_d$. Written as $e = w_0 + a\, v_d$ for some $w_0 \in A_2$ and $a \in \mathbb{Z}$, one has $(w_0)^2 = a^2 d/3$. Hence, $a^2 d/6$ satisfies (1.2) and, therefore, $d/2$ does. The latter then yields the existence of some $w \in A_2$ with $(w)^2 = d$. Assume now we are in case $(**')_2$, then the standard basis vector $e \in U \subset L_d$ itself might not be contained in $A_2 \oplus \mathbb{Z}\, v_d$, but $3\, e$ is and replacing e by $3\, e$ and $(1/3)$ by 3, one can argue as before.

Conversely, if $d/2$ satisfies (1.2), then we can pick $w \in A_2$ with $(w)^2 = d/3$ for $d \equiv 0\,(6)$ and with $(w)^2 = 3\,d$ for $d \equiv 2\,(6)$. Then $e := w + v_d$ is isotropic. Furthermore, there exists $w' \in A_2$ with $m := (e.w') = (w.w') \neq 0$. Then $f := m\,w' - ((w')^2/2)\,e$ satisfies $(f)^2 = 0$ and $(e.f) = (e.w')^2 =: n$, which yields an embedding $U(n) \hookrightarrow A_2 \oplus \mathbb{Z}\,v_d \subset L_d$ proving $(**')$.

Turning to (iii) and using the notation in (ii), observe that in case $(**)_0$, which implies $(**')_0$, the class w_0 has to be primitive. Indeed, if $w_0 = p\,w_1$ for some prime p, then $p \mid a$ or $p \mid d/3$. On the other hand, writing $f = w_0' + a'\,v_d$ yields the contradiction $1 = (e.f) = p\,(w_1.w_0') + aa'd/3 \equiv 0\,(p)$. Hence, $a^2d/6$ satisfies (1.3) and, therefore, $d/2$ does, i.e. there exists a primitive $w \in A_2$ with $(w)^2 = d/2$. The argument for $(**)_2$ is similar: If $e = w_0 + a\,v_d$, one argues as before. If not, then $3\,e = w_0 + a\,v_d$ and if $w_0 = p\,w_1$, then $p \neq 3$. All other primes are excluded as before.

For the converse in this situation, we use the arguments above and pick a primitive $w \in A_2$ with $(w)^2 = d/3$ or $= 3d$, respectively. As $A_{A_2} \simeq \mathbb{Z}/3\mathbb{Z}$, either $(w.A_2) = \mathbb{Z}$ or $= 3\,\mathbb{Z}$. If $(w)^2 = d/3$, then the former holds (because $3^2 \nmid d$) and, therefore, w' above can be chosen such that $m = 1$. Hence, there exists $U \hookrightarrow L_d$. If $(w.A_2) = 3\,\mathbb{Z}$, so in particular $(w)^2 = 3\,d$ and $d \equiv 2\,(6)$, then the class $e := w \pm v_d$ is of the form $e = 3\,e'$ with $e' \in L_d$. Therefore, the two classes e' and $f' := w' - ((w')^2/2)\,e'$, where $w' \in A_2$ is chosen such that $(w.w') = 3$, define an embedding $U \hookrightarrow L_d$.

As we will not use the presentation of d as $(2n^2 + 2n + 2)/a$ and $(2n^2 + 2n + 2)/a^2$, respectively, we leave the proof of the other equivalences to the reader, see [80, Prop. 6.1.3] and [2, Sec. 3]. □

$(***)$			14				26				38	42
$(**)$			14				26				38	42
$(**')$	8		14	18		24	26		32		38	42
$(*)$	8	12	14	18	20	24	26	30	32	36	38	42

$(***)$							62					
$(**)$							62				74	78
$(**')$			50				62		68		74	78
$(*)$	44	48	50	54	56	60	62	66	68	72	74	78

1.3 In the theory of K3 surfaces, there are good reasons to pass from the K3 lattice $\Lambda \simeq H^2(S,\mathbb{Z})$ to the Mukai lattice $\widetilde{\Lambda} \simeq \widetilde{H}(S,\mathbb{Z}) \simeq H^2(S,\mathbb{Z}) \oplus U_4$, see [86, Ch. 16] for a survey and references. A similar extension of lattices, though slightly more technical due to the non-triviality of the canonical bundle, turns out to be useful for cubics and their comparison with K3 surfaces.

We have already constructed and fixed an isomorphism $\Gamma \simeq E \oplus U_1 \oplus U_2 \oplus A_2(-1) \simeq$

$A_2^\perp$, where $A_2(-1) \oplus A_2 \hookrightarrow U_3 \oplus U_4$. On the cubic side, one also finds a natural sublattice isomorphic to $U_3 \oplus U_4$, namely $H^{*\neq 4}(X,\mathbb{Z})$. However, the distinguished $A_2(-1) \subset \Gamma$ sits in $H^4(X,\mathbb{Z})$, so this has to be modified. Moreover, we will embed A_2 into rational cohomology $H^*(X,\mathbb{Q})$ and the intersection product on $H^*(X,\mathbb{Q})$ is modified by more than a mere sign

Definition 1.14. The *Mukai pairing* on $H^*(X,\mathbb{Q})$ is defined as

$$(\alpha.\alpha') := -\int e^{\frac{c_1(X)}{2}} \cdot \alpha^* \cdot \alpha'. \tag{1.4}$$

Here, $(\alpha_0 + \alpha_2 + \alpha_4 + \alpha_6 + \alpha_8)^* := \alpha_0 - \alpha_2 + \alpha_4 - \alpha_6 + \alpha_8$ and

$$e^{\frac{c_1(X)}{2}} = e^{\frac{3h}{2}} = 1 + \frac{3}{2}h + \frac{9}{8}h^2 + \frac{27}{48}h^3 + \frac{81}{384}h^4.$$

Warning: Unlike the Mukai pairing for K3 surfaces, the pairing (1.4) is not symmetric.

Definition 1.15. The *Mukai vector* of a coherent sheaf $E \in \mathrm{Coh}(X)$, or a complex $E \in \mathrm{D^b}(X)$, or simply a class $E \in K_{\mathrm{top}}(X)$ is defined as

$$v(E) := \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)}.$$

One easily computes

$$\sqrt{\mathrm{td}(X)} = 1 + \frac{3}{4}h + \frac{11}{32}h^2 + \frac{15}{128}h^3 + \frac{121}{6144}h^4.$$

Using the general fact $\sqrt{\mathrm{td}}^* = e^{-\frac{c_1(X)}{2}} \cdot \sqrt{\mathrm{td}}$ and the Grothendieck–Riemann–Roch formula, one expresses the Euler–Poincaré pairing of two coherent sheaves as

$$\chi(E,E') = -(v(E).v(E')). \tag{1.5}$$

Note that the left hand side is not symmetric, as ω_X is not trivial. This confirms the observation that (1.4) is not symmetric.

Example 1.16. For our purposes the following classes are of importance:

$$w_0 := v(\mathcal{O}_X) = \sqrt{\mathrm{td}(X)},\ \ w_1 := v(\mathcal{O}_X(1)) = e^h \cdot \sqrt{\mathrm{td}(X)},$$

$$\text{and } w_2 := v(\mathcal{O}_X(2)) = e^{2h} \cdot \sqrt{\mathrm{td}(X)}.$$

In a sense to be made more precise, these classes are responsible for (.) not being symmetric. Explicitly, they are

$$w_0 = 1 + \frac{3}{4}h + \frac{11}{32}h^2 + \frac{15}{128}h^3 + \frac{121}{6144}h^4,\ \ w_1 = 1 + \frac{7}{4}h + \frac{51}{32}h^2 + \frac{385}{384}h^3 + \frac{2921}{6144}h^4,$$

$$\text{and }\ w_2 = 1 + \frac{11}{4}h + \frac{132}{32}h^2 + \frac{1397}{384}h^3 + \frac{16025}{6144}h^4.$$

In addition to the classes w_0, w_1, w_2, one also needs the following ones

$$v(\lambda_1) \coloneqq 3 + \frac{5}{4}h - \frac{7}{32}h^2 - \frac{77}{384}h^3 + \frac{41}{2048}h^4.$$

$$v(\lambda_2) \coloneqq -3 - \frac{1}{4}h + \frac{15}{32}h^2 + \frac{1}{384}h^3 - \frac{153}{2048}h^4.$$

Remark 1.17. The notation suggests that the $v(\lambda_i)$, $i = 1,2$, are Mukai vectors of some natural (complexes of) sheaves. This is almost true, as we explain next. Consider an arbitrary line $L \subset X$ and the two natural sheaves $\mathcal{O}_L(i)$, $i = 1,2$, in X. Their Mukai vectors are

$$u_i \coloneqq v(\mathcal{O}_L(i)) = \begin{cases} \frac{1}{3}h^3 + \frac{5}{12}h^4 & \text{if } i = 2 \\ \frac{1}{3}h^3 + \frac{9}{12}h^4 & \text{if } i = 2. \end{cases}$$

Under the right orthogonal projection $H^*(X,\mathbb{Q}) \longrightarrow \{w_0, w_1, w_2\}^\perp$ they are mapped to λ_i. Explicitly,

$$v(\lambda_1) = u_1 - w_1 + 4w_0 \text{ and } v(\lambda_2) = u_2 - w_2 + 4w_1 - 6w_0. \tag{1.6}$$

Here, one uses

$$\begin{array}{rclcl} (w_i.w_j) & = & \chi(\mathcal{O}_X(i), \mathcal{O}_X(j)) & = & \chi(X, \mathcal{O}_X(j-i)), \\ (w_i.u_j) & = & \chi(\mathcal{O}_X(i), \mathcal{O}_L(j)) & = & \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j-i)), \\ (u_i, w_j) & = & \chi(\mathcal{O}_L(i), \mathcal{O}_X(j)) & = & \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i-j-3)), \\ (u_i, u_j) & = & \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j-i)). & & \end{array}$$

Lemma 1.18. *If $H^*(X,\mathbb{Q})$ is considered with the negative Mukai pairing, then*

$$A_2 \hookrightarrow H^*(X,\mathbb{Q}),\ \lambda_i \longmapsto v(\lambda_i)$$

defines an isometric embedding. Furthermore,

(i) $v(\lambda_1), v(\lambda_2) \in \{w_0, w_1, w_2\}^\perp$.
(ii) $w_0, w_1, w_2, v(\lambda_1), v(\lambda_2) \in \mathbb{Q}[h]$ *are linearly independent.*
(iii) $\{w_0, w_1, w_2, v(\lambda_1), v(\lambda_2)\}^\perp = H^4(X,\mathbb{Q})_{\mathrm{pr}} = {}^\perp\{w_0, w_1, w_2, v(\lambda_1), v(\lambda_2)\}$*, on which the Mukai pairing coincides with the intersection product (up to sign).*
(iv) *The Mukai pairing (.) is symmetric on the right orthogonal complement*

$$\{w_0, w_1, w_2\}^\perp \subset H^*(X,\mathbb{Q}).$$

Proof The first assertion can be proved by a computation or using (1.6). Similarly, (i) follows from the observation that $v(\lambda_i)$ is the orthogonal projection of u_i and (ii) is again proven by a computation. Finally, (ii) implies (iii) and (iv) can be deduced from (iii). □

Corollary 1.19. *The lattices* $A_2^\perp \simeq \Gamma \simeq H^4(X,\mathbb{Z})_{\rm pr} \subset H^*(X,\mathbb{Q})$ *and* $A_2 \simeq \mathbb{Z}\,v(\lambda_1) \oplus \mathbb{Z}\,v(\lambda_2) \subset H^*(X,\mathbb{Q})$ *are orthogonal with respect to the Mukai pairing (1.4). The induced embedding of their direct sum* $A_2^\perp \oplus A_2$ *extends to*

$$A_2^\perp \oplus A_2 \subset \widetilde{\Lambda} \hookrightarrow H^*(X,\mathbb{Q}). \tag{1.7}$$

A more conceptual understanding of these calculations is provided by [3]. In particular, cohomology with rational coefficients $H^*(X,\mathbb{Q})$ is replaced by integral topological K-theory. Denote by $K_{\rm top}(X)$ the topological K-theory of all complex vector bundles. Traditionally, the Chern character is used to identify $K_{\rm top}(X) \otimes \mathbb{Q}$ with $H^*(X,\mathbb{Q}) = H^{2*}(X,\mathbb{Q})$. For our purposes the Mukai vector is better suited

$$v\colon K_{\rm top}(X) \hookrightarrow K_{\rm top}(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X,\mathbb{Q}).$$

Note that the torsion freeness $K_{\rm top}(X)$ follows from the torsion freeness of $H^*(X,\mathbb{Z})$ and the Atiyah–Hirzebruch spectral sequence. Then $K_{\rm top}(X)$ is equipped with a non-degenerate but non-symmetric linear form with values in $\mathbb{Q}$. Due to (1.5), it takes values in $\mathbb{Z}$ on the image of the highly non-injective map $K(X) \longrightarrow K_{\rm top}(X)$. Clearly, the classes $[\mathcal{O}_X(i)]$, $i = 0,1,2$, and $[\mathcal{O}_L(i)]$, $i = 1,2$, are all contained in the image. We shall be interested in the right orthogonal complement of the former three classes and introduce the notation:

$$K'_{\rm top}(X) \coloneqq \{\,[\mathcal{O}_X],[\mathcal{O}_X(1)],[\mathcal{O}_X(2)]\,\}^\perp \subset K_{\rm top}(X).$$

Proposition 1.20 (Addington–Thomas)**.** *The restriction of the Mukai pairing* $(\ .\ ) = -\chi(\ ,\ )$ *to* $K'_{\rm top}(X)$ *is symmetric. Moreover, as abstract lattices*

$$\widetilde{\Lambda} \simeq K'_{\rm top}(X).$$

Proof Note that $v\colon K'_{\rm top}(X) \otimes \mathbb{Q} \xrightarrow{\sim} \{w_0,w_1,w_2\}^\perp$. Hence, Lemma 1.18 implies the first assertion. The original proof [3] of the second assertion uses derived categories. Here is a sketch of a more direct, purely topological argument. Consider the right orthogonal projection $p\colon K_{\rm top}(X) \twoheadrightarrow K'_{\rm top}(X)$. It really is defined over $\mathbb{Z}$, as $(w_i)^2 = 1$. Analogously to (1.6), one has $p[\mathcal{O}_L(1)] = [\mathcal{O}_L(1)] - [\mathcal{O}_X(1)] + 4[\mathcal{O}_X]$ and $p[\mathcal{O}_L(2)] = [\mathcal{O}_L(2)] - [\mathcal{O}_X(2)] + 4[\mathcal{O}_X(1)] - 6[\mathcal{O}_X]$. Hence, $\lambda_i \longmapsto p[\mathcal{O}_L(i)]$ defines an isometric embedding $A_2 \hookrightarrow K'_{\rm top}(X)$.

First, $H^4(X,\mathbb{Z})_{\rm pr} \subset H^*(X,\mathbb{Q})$ is contained in $v(K'_{\rm top}(X))$. Indeed, $H^4(X,\mathbb{Z})_{\rm pr}$ is spanned by classes of all vanishing spheres and those lift to $K_{\rm top}(X)$. After fixing an isometry $E \oplus U^{\oplus 2} \oplus A_2(-1) \simeq A_2^\perp \simeq \Gamma \simeq H^4(X,\mathbb{Z})_{\rm pr} \subset K'_{\rm top}(X)$, this yields an isometric embedding $\Gamma \oplus A_2 \hookrightarrow K'_{\rm top}(X)$ and allows one to view $\mu_1,\mu_2 \in A_2(-1)$ as classes in $K'_{\rm top}(X)$.
Second, one needs to show that the class $(1/3)(\mu_1 - \mu_2 - \lambda_1 + \lambda_2) \in (A_2(-1) \oplus A_2) \otimes \mathbb{Q} \subset K_{\rm top}(X) \otimes \mathbb{Q}$ is integral, i.e. that it is contained in $K_{\rm top}(X)$. This presumably can be achieved algebraically on some specific cubic fourfold. Hence, the embedding in step

one extends to an isometric embedding $\widetilde{\Lambda} \hookrightarrow K'_{\text{top}}(X)$ of finite index. The unimodularity of $\widetilde{\Lambda}$ then implies the second assertion. □

1.4 We now endow the various lattices considered above with natural Hodge structures. Let us first briefly recall the well known theory for K3 surfaces, see [86, Ch. 16] for further details and references.

For any complex K3 surface S its second cohomology $H^2(S,\mathbb{Z})$, which as a lattice is isomorphic to Λ, comes with a natural Hodge structure of weight two given by the $(2,0)$-part $H^{2,0}(S)$. The full Hodge structure is then determined by additionally requiring $H^{1,1}(S) \perp H^{2,0}(S)$ with respect to the intersection pairing.

The global Torelli theorem for complex K3 surfaces asserts that two K3 surfaces S and S' are isomorphic if and only if there exists a Hodge isometry $H^2(S,\mathbb{Z}) \simeq H^2(S',\mathbb{Z})$, i.e. an isomorphism of integral Hodge structures that is compatible with the intersection pairing:

$$S \simeq S' \Leftrightarrow \exists\, H^2(S,\mathbb{Z}) \simeq H^2(S',\mathbb{Z}) \text{ Hodge isometry.}$$

Let (S,L) be a polarized K3 surface. Then the primitive cohomology $H^2(S,\mathbb{Z})_{L\text{-pr}} \subset H^2(S,\mathbb{Z})$ is endowed with the induced structure. Its $(2,0)$-part is again $H^{2,0}(S)$ and its $(1,1)$-part is the primitive part of $H^{1,1}(S)$, i.e. the kernel of $(L.\ ) \colon H^{1,1}(S) \longrightarrow \mathbb{C}$. The polarized version of the global Torelli theorem is the statement that two polarized K3 surfaces (S,L) and (S',L') are isomorphic if and only if there exists a Hodge isometry $H^2(S,\mathbb{Z}) \simeq H^2(S',\mathbb{Z})$ inducing $H^2(S,\mathbb{Z})_{L\text{-pr}} \simeq H^2(S',\mathbb{Z})_{L\text{-pr}}$:

$$(S,L) \simeq (S',L) \Leftrightarrow \exists\, H^2(S;\mathbb{Z}) \simeq H^2(S',\mathbb{Z}),\ L \longmapsto L',\ \text{Hodge isometry}$$

The result will be stated again in moduli theoretic terms in Theorem 2.1.

Warning: A Hodge isometry $H^2(S,\mathbb{Z})_{L\text{-pr}} \simeq H^2(S',\mathbb{Z})_{L\text{-pr}}$ does not necessarily extend to a Hodge isometry between the full cohomology. Hence, in general, the existence of a Hodge isometry between the primitive Hodge structures of two polarized K3 surfaces does not imply that (S,L) and (S',L') are isomorphic. In fact, even the unpolarized K3 surfaces S and S' may be non-isomorphic.

Next comes the Mukai Hodge structure $\widetilde{H}(S,\mathbb{Z})$. The underlying lattice is $H^*(S,\mathbb{Z})$ with the sign change in $U_4 = (H^0 \oplus H^4)(S,\mathbb{Z})$. The Hodge structure of weight two is again given by the $(2,0)$-part being $\widetilde{H}^{2,0}(S) \coloneqq H^{2,0}(S)$ and the condition that $\widetilde{H}^{1,1}(S) \perp H^{2,0}(S)$ with respect to the Mukai pairing. In particular, $U \simeq U_4 = (H^0 \oplus H^4)(S,\mathbb{Z})$ is contained in $\widetilde{H}^{1,1}(S,\mathbb{Z})$. The derived global Torelli theorem is the statement that for two projective K3 surfaces S and S' there exists an exact, $\mathbb{C}$-linear equivalence $\mathrm{D}^{\mathrm{b}}(S) \simeq \mathrm{D}^{\mathrm{b}}(S')$ between their bounded derived categories of coherent sheaves if and only of there exists a Hodge isometry $\widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{H}(S',\mathbb{Z})$:

$$\mathrm{D}^{\mathrm{b}}(S) \simeq \mathrm{D}^{\mathrm{b}}(S') \Leftrightarrow \exists\, \widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{H}(S',\mathbb{Z}) \text{ Hodge isometry.}$$

A twisted K3 surface (S,α) consists of a K3 surface S together with a Brauer class $\alpha \in \mathrm{Br}(S) \simeq H^2(S,\mathcal{O}_S^*)$ (we work in the analytic topology). Choosing a lift $B \in H^2(S,\mathbb{Q})$ of α under the natural morphism $H^2(S,\mathbb{Q}) \twoheadrightarrow \mathrm{Br}(S)$ induced by the exponential sequence allows one to introduce a natural Hodge structure $\widetilde{H}(S,\alpha,\mathbb{Z})$ of weight two associated with (S,α). As a lattice, this is just $\widetilde{H}(S,\mathbb{Z})$, but the $(2,0)$-part is now given by $\widetilde{H}^{2,0}(S,\alpha) \coloneqq \mathbb{C}\,(\sigma + \sigma \wedge B)$, where $0 \neq \sigma \in H^{2,0}(S)$. This defines a Hodge structure by requiring, as before, that $\widetilde{H}^{1,1}(S,\alpha) \perp \widetilde{H}^{2,0}(S,\alpha)$ with respect to the Mukai pairing. Although the definition depends on the choice of B, the Hodge structures induced by two different lifts B and B' of the same Brauer class α are Hodge isometric albeit not canonically, see [90].

The twisted version of the derived global Torelli theorem is the statement that the bounded derived categories of twisted coherent sheaves on (S,α) and (S',α') are equivalent if and only if there exists a Hodge isometry $\widetilde{H}(S,\alpha,\mathbb{Z}) \simeq \widetilde{H}(S',\alpha',\mathbb{Z})$ preserving the natural orientation of the four positive directions, cf. [86, Ch. 16.4] and [130]:

$$\mathrm{D^b}(S,\alpha) \simeq \mathrm{D^b}(S',\alpha') \Leftrightarrow \exists\, \widetilde{H}(S,\alpha,\mathbb{Z}) \simeq \widetilde{H}(S',\alpha',\mathbb{Z}) \text{ oriented Hodge isometry.}$$

Next consider $H^4(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})_{\mathrm{pr}}$ of a smooth cubic fourfold X. Its natural Hodge structure is of weight four. It is determined by the one-dimensional $H^{3,1}(X)$ and the condition that $H^{3,1}(X) \perp H^{2,2}(X)$ with respect to the intersection product.

The global Torelli theorem for smooth cubic fourfolds, which we will state again as Theorem 2.12 in moduli theoretic terms, is the statement that two smooth cubic fourfolds X and X' are isomorphic (as varieties over $\mathbb{C}$, without embedding) if and only if there exists a Hodge isometry $H^4(X,\mathbb{Z})_{\mathrm{pr}} \simeq H^4(X',\mathbb{Z})_{\mathrm{pr}}$:

$$X \simeq X' \Leftrightarrow \exists\, H^4(X,\mathbb{Z})_{\mathrm{pr}} \simeq H^4(X',\mathbb{Z})_{\mathrm{pr}} \text{ Hodge isometry.}$$

Note that any such Hodge isometry can be extended to a Hodge isometry $H^4(X,\mathbb{Z}) \simeq H^4(X',\mathbb{Z})$ that maps h_X^2 to $\pm h_{X'}^2$. The situation here is easier compared to the case of polarized K3 surfaces as the discriminant of $H^4(X,\mathbb{Z})_{\mathrm{pr}}$ is just $\mathbb{Z}/3\mathbb{Z}$.

To relate $H^4(X,\mathbb{Z})$ of a cubic fourfolds to K3 surfaces one has to change the sign of the intersection product, so that as abstract lattices $H^4(X,\mathbb{Z}) \simeq \bar{\Gamma}$ and $H^4(X,\mathbb{Z})_{\mathrm{pr}} \simeq \Gamma$ (this is not reflected by the notation), and Tate shift the Hodge structure to obtain $H^4(X,\mathbb{Z})(1)$ and $H^4(X,\mathbb{Z})_{\mathrm{pr}}(1)$, which are now Hodge structures of weight two.

Definition 1.21. The integral Hodge structure $\widetilde{H}(X,\mathbb{Z})$ of K3 type associated with a smooth cubic fourfold X is the lattice

$$\widetilde{H}(X,\mathbb{Z}) \coloneqq K'_{\mathrm{top}}(X)$$

with the Hodge structure of weight two given by $\widetilde{H}^{2,0}(X) \coloneqq v^{-1}(H^{3,1}(X))$ and the requirement that $\widetilde{H}^{1,1}(X)$ and $\widetilde{H}^{2,0}(X)$ are orthogonal with respect to the Mukai pairing on $K_{\mathrm{top}}(X)$.

The Mukai vector $K_{\mathrm{top}}(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X, \mathbb{Q})$ induces an isometry

$$\widetilde{H}(X, \mathbb{Z}) = K'_{\mathrm{top}}(X) \simeq \widetilde{\Lambda} \subset H^*(X, \mathbb{Q})$$

with $\widetilde{\Lambda} \subset H^*(X, \mathbb{Q})$ provided by (1.7). Observe that there is a natural isometric inclusion of Hodge structures

$$H^4(X, \mathbb{Z})_{\mathrm{pr}}(1) \subset \widetilde{H}(X, \mathbb{Z}).$$

Moreover, the sublattice A_2 is algebraic, i.e. $A_2 \subset \widetilde{H}^{1,1}(X, \mathbb{Z})$, and its orthogonal Hodge structure is $A_2^\perp \simeq H^4(X, \mathbb{Z})_{\mathrm{pr}}(1)$. Also note that according to Remark 1.1 $\lambda_1^\perp \subset \widetilde{H}(X, \mathbb{Z})$ is a sub Hodge structure with underlying lattice isomorphic to $\Lambda \oplus \mathbb{Z}(-2)$.

Remark 1.22. Once, the Kuznetsov category $\mathcal{A}_X \subset \mathrm{D^b}(X)$ has been introduced, one also writes $\widetilde{H}(\mathcal{A}_X, \mathbb{Z}) = \widetilde{H}(X, \mathbb{Z})$. The notation $\widetilde{H}(X, \mathbb{Z})$ is analogous to the notation $\widetilde{H}(S, \mathbb{Z})$ for K3 surfaces and the Hodge structure plays a similar role. In fact, as a consequence of the above discussion we know that as lattices $\widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(S, \mathbb{Z})$ and the analogy goes further: For a K3 surface, the algebraic part naturally contains a hyperbolic plane:

$$U \simeq (H^0 \oplus H^4)(S, \mathbb{Z}) \hookrightarrow \widetilde{H}^{1,1}(S, \mathbb{Z}).$$

Similarly, for a smooth cubic fourfold the algebraic part naturally contains a copy of A_2:

$$v\colon A_2 \simeq \mathbb{Z}\, p[\mathcal{O}_L(1)] \oplus \mathbb{Z}\, p[\mathcal{O}_L(2)] \hookrightarrow \widetilde{H}^{1,1}(X, \mathbb{Z}).$$

Their respective orthogonal complements are

$$H^2(S, \mathbb{Z}) = U^\perp \hookrightarrow \widetilde{H}(S, \mathbb{Z}) \ \text{ and } \ H^4(X, \mathbb{Z})_{\mathrm{pr}}(1) = A_2^\perp \hookrightarrow \widetilde{H}(X, \mathbb{Z}),$$

in terms of which the global Torelli theorem is formulated in both instances. Also, $e_4 - f_4 = (1, 0, -1) \in \widetilde{H}^{1,1}(S, \mathbb{Z})$ and $v(\lambda_1) \in \widetilde{H}^{1,1}(X, \mathbb{Z})$ are both algebraic classes satisfying $(e_4 - f_4)^2 = 2 = (v(\lambda_1))^2$. Their orthogonal complements are isometric.

Definition 1.23. Let (S, L) be a polarized K3 surface and X a smooth cubic fourfold.

(i) We say (S, L) and X are *associated*, $(S, L) \sim X$, if there exists an isometric embedding of Hodge structures

$$H^2(S, \mathbb{Z})_{L\text{-pr}} \hookrightarrow H^4(X, \mathbb{Z})_{\mathrm{pr}}(1). \tag{1.8}$$

(ii) We say S and X are *associated*, $S \sim X$, if there exists a Hodge isometry

$$\widetilde{H}(S, \mathbb{Z}) \simeq \widetilde{H}(X, \mathbb{Z}).$$

(iii) For $\alpha \in \mathrm{Br}(S)$ we say that the twisted K3 surface (S, α) and X are *associated*, $(S, \alpha) \sim X$, if there exists a Hodge isometry

$$\widetilde{H}(S, \alpha, \mathbb{Z}) \simeq \widetilde{H}(X, \mathbb{Z}).$$

First observe the immediate implication:

$$(S, L) \sim X \Rightarrow S \sim X.$$

Indeed, any isometric embedding (1.8) can be extended to an isometry $\widetilde{H}(S, \mathbb{Z}) \simeq \widetilde{H}(X, \mathbb{Z})$. This follows from the existence of the hyperbolic plane $U \subset H^2(S, \mathbb{Z})^{\perp}_{L\text{-pr}}$, cf. [86, Rem. 14.1.13].

As an aside, observe that a K3 surface S that is associated with a cubic fourfold in any sense is necessarily projective. Indeed, if for example $S \sim X$, then $\widetilde{H}^{1,1}(S, \mathbb{Z}) \simeq \widetilde{H}^{1,1}(X, \mathbb{Z})$ contains the positive plane A_2 and, therefore, $H^{1,1}(S, \mathbb{Z})$ contains at least one class of positive square.

The key to link $S \sim X$, (S, L), and $(S, \alpha) \sim X$ to the properties $(**)$ and $(**')$ is the following result in [3] generalized to the twisted case in [87].

Proposition 1.24 (Addington–Thomas, Huybrechts)**.** *Assume X is a smooth cubic fourfold.*

(i) *There exists a K3 surface S with $S \sim X$ if and only if there exists a (primitive) embedding $U \hookrightarrow \widetilde{H}^{1,1}(X, \mathbb{Z})$.*
(ii) *There exists a twisted K3 surface (S, α) with $(S, \alpha) \sim X$ if and only if there exists an embedding $U(n) \hookrightarrow \widetilde{H}^{1,1}(X, \mathbb{Z})$ for some $n \neq 0$.*

Proof Any Hodge isometry $\widetilde{H}(S, \mathbb{Z}) \simeq \widetilde{H}(X, \mathbb{Z})$ yields a hyperbolic plane $U \simeq (H^0 \oplus H^4)(S, \mathbb{Z}) \subset \widetilde{H}^{1,1}(S, \mathbb{Z}) \simeq \widetilde{H}^{1,1}(X, \mathbb{Z})$. Conversely, if $U \subset \widetilde{H}^{1,1}(X, \mathbb{Z}) \subset \widetilde{H}(X, \mathbb{Z})$, then as a lattice $U^{\perp} \simeq \Lambda$. Moreover, the Hodge structure of $\widetilde{H}(X, \mathbb{Z})$ induces a Hodge structure on $U^{\perp} \simeq \Lambda$ which due to the surjectivity of the period map [86, Thm. 7.4.1] is Hodge isometric to $H^2(S, \mathbb{Z})$ for some K3 surface S. However, as before, $U^{\perp} \simeq H^2(S, \mathbb{Z})$ extends to $\widetilde{H}(X, \mathbb{Z}) \simeq \widetilde{H}(S, \mathbb{Z})$. This proves (i).

For (ii), again one direction is easy, as $\widetilde{H}^{1,1}(S, \mathbb{Z})$ contains the B-field shift of $(H^0 \oplus H^4)(S, \mathbb{Z})$. More precisely, $\widetilde{H}^{1,1}(S, \alpha, \mathbb{Z}) = (\exp(B)\, \widetilde{H}^{1,1}(S, \mathbb{Q})) \cap \widetilde{H}(S, \mathbb{Z})$, which contains the lattice $(\langle 1, B, B^2/2 \rangle \cap \widetilde{H}(S, \mathbb{Z})) \oplus H^4(S, \mathbb{Z}) \simeq U(n)$, where n is minimal with $n\,(1, B, B^2) \in \widetilde{H}(S, \mathbb{Z})$. The other direction needs a surjectivity statement for twisted K3 surfaces which is an easy consequence of the surjectivity of the untwisted period map. □

Proposition 1.25. *Assume a smooth cubic fourfold X is associated with some K3 surface S, so $S \sim X$. Then there exists a polarized K3 surface $(S', L') \sim X$.*

Proof Assume $S \sim X$. Then there exists a Hodge isometry $\widetilde{H}(S, \mathbb{Z}) \simeq \widetilde{H}(X, \mathbb{Z})$. On the left hand side, one finds $U \simeq (H^0 \oplus H^4)(S, \mathbb{Z}) \subset \widetilde{H}^{1,1}(S, \mathbb{Z})$ and, on the right hand side, $A_2 \subset \widetilde{H}^{1,1}(X, \mathbb{Z})$. Consider the saturation of the sum of both as a lattice $\overline{U + A_2} \subset \widetilde{H}^{1,1}(S, \mathbb{Z})$. According to Lemma 1.2, there exists another hyperbolic plane $U' \subset \overline{U + A_2}$

with $\mathrm{rk}(U' + A_2) = 3$. Using the surjectivity of the period map, one finds another K3 surface S' and a Hodge isometry

$$\widetilde{H}(S',\mathbb{Z}) \simeq \widetilde{H}(S,\mathbb{Z}) \simeq \widetilde{H}(X,\mathbb{Z}) \tag{1.9}$$

inducing $H^2(S',\mathbb{Z}) \simeq U'^{\perp}$. But then $H^2(S',\mathbb{Z}) \cap A_2^{\perp} \subset H^2(S',\mathbb{Z})$ is of corank one and we can assume it to be of the form $H^2(S',\mathbb{Z})_{L'\text{-pr}}$. However, being contained in $A_2^{\perp}$ implies that under (1.9) $H^2(S',\mathbb{Z})_{L'\text{-pr}}$ embeds into $H^4(X,\mathbb{Z})_{\mathrm{pr}}(1)$, which ensures $(S',L') \sim X$. □

Corollary 1.26. *A smooth cubic fourfold X is associated with some polarized K3 surface, $(S,L) \sim X$, if and only if there exists an isometric embedding $U \hookrightarrow \widetilde{H}^{1,1}(X,\mathbb{Z})$.* □

2 Period domains and moduli spaces

The comparison of the Hodge theory of K3 surfaces and cubic fourfolds is now considered in families. Via period maps, this leads to an algebraic correspondence between the moduli space of polarized K3 surfaces of certain degrees and the moduli space of cubic fourfolds. The approach has been initiated by Hassett [80] and has turned out to be a beneficial point of view.

2.1 Here is a very brief reminder on some results, mostly due to Borel and Baily–Borel, on arithmetic quotients of orthogonal type. Let $(N, (\ .\ ))$ be a lattice of signature $(2, n_-)$ and set $V := N \otimes \mathbb{R}$. Then the period domain D_N associated with N is the Grassmannian of positive, oriented planes $W \subset V$, which alternatively can be described as

$$\begin{aligned} D_N &\simeq \{\, x \mid (x)^2 = 0,\ (x.\bar{x}) > 0 \,\} \subset \mathbb{P}(N \otimes \mathbb{C}) \\ &\simeq \mathrm{O}(2, n_-)/(\mathrm{O}(2) \times \mathrm{O}(n_-)). \end{aligned}$$

By definition, the period domain D_N associated with N has the structure of a complex manifold. This is turned into an algebraic statement by the following fundamental result [7]. It uses the fact that under the assumption on the signature of N the orthogonal group $\mathrm{O}(N)$ acts properly discontinuously on D_N.

Theorem 2.1 (Baily–Borel)**.** *Assume $G \subset \mathrm{O}(N)$ is a torsion free subgroup of finite index. Then the quotient*

$$G \setminus D_N$$

has the structure of a smooth, quasi-projective complex variety.

As G acts properly discontinuously as well, the stabilizers are finite and hence trivial.

This already proves the smoothness of the quotient $G \setminus D_N$. The difficult part of the theorem is to find a Zariski open embedding into a complex projective variety.

Finite index subgroups $G \subset \mathrm{O}(N)$ with torsion are relevant, too. In this situation, one uses Minkowski's theorem stating that the map $\pi_p \colon \mathrm{GL}(n,\mathbb{Z}) \longrightarrow \mathrm{GL}(n,\mathbb{F}_p)$, $p \geq 3$, is injective on finite subgroups or, equivalently, that its kernel is torsion free. Hence, for every finite index subgroup $G \subset \mathrm{O}(N)$ there exists a normal and torsion free subgroup $G_0 \coloneqq G \cap \mathrm{Ker}(\pi_p) \subset G$ of finite index.

Corollary 2.2. *Assume $G \subset \mathrm{O}(N)$ is a subgroup of finite index. Then the quotient*

$$G \setminus D_N$$

has the structure of a normal, quasi-projective complex variety with finite quotient singularities. □

Not only are these arithmetic quotients algebraic, also holomorphic maps into them are algebraic. This is the following remarkable GAGA style result, see [23].

Theorem 2.3 (Borel)**.** *Assume $G \subset \mathrm{O}(N)$ is a torsion free subgroup of finite index. Then any holomorphic map $\varphi \colon Z \longrightarrow G \setminus D_N$ from a complex variety Z is regular.*

Remark 2.4. Often, the result is applied to holomorphic maps to singular quotients $G \setminus D_N$, i.e. in situations when G is not necessarily torsion free. This is covered by the above only when $Z \longrightarrow G \setminus D_N$ is induced by a holomorphic map $Z' \longrightarrow G_0 \setminus D_N$, where $Z' \longrightarrow Z$ is a finite quotient and $G_0 \subset G$ is a normal, torsion free subgroup of finite index.

2.2 We shall be interested in (at least) three different types of period domains: For polarized K3 surfaces and for (special) smooth cubic fourfolds. These are the period domains associated with the lattices Γ, Γ_d, and Λ_d:

$$D \subset \mathbb{P}(\Gamma \otimes \mathbb{C}),\ D_d \subset \mathbb{P}(\Gamma_d \otimes \mathbb{C}),\ \text{and } Q_d \subset \mathbb{P}(\Lambda_d \otimes \mathbb{C}).$$

These period domains are endowed with the natural action of the corresponding orthogonal groups $\mathrm{O}(\Gamma)$, $\mathrm{O}(\Gamma_d)$, and $\mathrm{O}(\Lambda_d)$ and we will be interested in the following quotients by distinguished finite index subgroups of those:

$$\mathcal{C} \coloneqq \tilde{\mathrm{O}}(\Gamma) \setminus D = \mathrm{O}(\Gamma) \setminus D,\ \ \tilde{\mathcal{C}}_d \coloneqq \tilde{\mathrm{O}}(\Gamma, K_d) \setminus D_d,\ \ \tilde{\tilde{\mathcal{C}}}_d \coloneqq \tilde{\mathrm{O}}(\Gamma, v_d) \setminus D_d,\ \text{and}$$

$$\mathcal{M}_d \coloneqq \tilde{\mathrm{O}}(\Lambda_d) \setminus Q_d.$$

For the first equality note that $\tilde{\mathrm{O}}(\Gamma) \subset \mathrm{O}(\Gamma)$ is of index two, but $-\mathrm{id} \in \mathrm{O}(\Gamma) \setminus \tilde{\mathrm{O}}(\Gamma)$ acts trivially on D.

Due to Theorem 2.1 and 2.3, see also Remark 2.4, the induced maps $\tilde{\tilde{\mathcal{C}}}_d \longrightarrow\!\!\!\!\to \tilde{\mathcal{C}}_d \longrightarrow \mathcal{C}$

are regular morphisms between normal quasi-projective varieties. The image in $\mathcal{C}$ shall be denoted by $\mathcal{C}_d$, so that

$$\tilde{\tilde{\mathcal{C}}}_d \twoheadrightarrow \tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}_d \subset \mathcal{C}.$$

The condition $(*)$ will in the sequel be interpreted as the condition that $\mathcal{C}_d \neq \emptyset$.

Corollary 2.5 (Hassett)**.** *The naturally induced maps*

$$\tilde{\tilde{\mathcal{C}}}_d \twoheadrightarrow \tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}_d$$

are surjective, finite, and algebraic.

Furthermore, $\tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}_d$ is the normalization of $\mathcal{C}_d$ and $\tilde{\tilde{\mathcal{C}}}_d \twoheadrightarrow \tilde{\mathcal{C}}_d$ is a finite morphism between normal varieties, which is an isomorphism if d satisfies $(*)_0$ and of degree two if d satisfies $(*)_2$.

Proof Clearly, if d satisfies $(*)_2$, then $\tilde{\mathrm{O}}(\Gamma, K_d) = \tilde{\mathrm{O}}(\Gamma, v_d)$ by Lemma 1.8 and, therefore, $\tilde{\tilde{\mathcal{C}}}_d \simeq \tilde{\mathcal{C}}_d$. Otherwise, $\tilde{\tilde{\mathcal{C}}}_d \twoheadrightarrow \tilde{\mathcal{C}}_d$ is the quotient by the involution $g \in \tilde{\mathrm{O}}(\Gamma)$ defined by $g = \mathrm{id}$ on $E \oplus U_2 \oplus \mathrm{I}_{0,3}$ and $g = -\mathrm{id}$ on U_1, which indeed acts non-trivially on $\tilde{\tilde{\mathcal{C}}}_d$.

To prove that $\tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}_d$ is quasi-finite, use that $\tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}$ is algebraic with discrete and hence finite fibres. For a very general $x \in D_d$ such that there does not exist any proper primitive sublattice $N \subset \Gamma_d$ with $x \in N \otimes \mathbb{C}$, any $g \in \tilde{\mathrm{O}}(\Gamma)$ with $g(x) = x$ also satisfies $g(\Gamma_d) = \Gamma_d$ and, therefore, $g(K_d) = K_d$, i.e. $g \in \tilde{\mathrm{O}}(\Gamma, K_d)$. This proves that $\tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}$ is generically injective. Thus, once $\tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}$ is shown to be finite, and not only quasi-finite, it is the normalization of its image $\mathcal{C}_d$. We refer to [25, 80] for more details on this point. □

Remark 2.6. Note that while the fibre of $\tilde{\tilde{\mathcal{C}}}_d \twoheadrightarrow \tilde{\mathcal{C}}_d$ consists of at most two points, the fibres of $\tilde{\mathcal{C}}_d \twoheadrightarrow \mathcal{C}_d$ may contain more points, depending on the singularity type of the points in $\mathcal{C}_d$. For fixed d, the cardinality of the fibres is bounded, but not when d is allowed to grow.

Lemma 1.10 immediately yields the following result which eventually leads to the mysterious relation between K3 surfaces and cubic fourfolds.

Corollary 2.7. *Assume d satisfies $(**)$. We choose an isomorphism $\varepsilon\colon \Gamma_d \xrightarrow{\sim} \Lambda_d$.*

(i) *If d satisfies $(*)_0$, then ε naturally induces an isomorphism $\mathcal{M}_d \simeq \tilde{\tilde{\mathcal{C}}}_d$. Therefore, $\mathcal{M}_d$ comes with a finite morphism onto $\mathcal{C}_d$ generically of degree two:*

$$\Phi_\varepsilon\colon\ \mathcal{M}_d \simeq \tilde{\tilde{\mathcal{C}}}_d \xrightarrow{2:1} \tilde{\mathcal{C}}_d \xrightarrow{\text{norm}} \mathcal{C}_d \subset \mathcal{C}.$$

(ii) *If d satisfies $(*)_2$, then ε naturally induces an isomorphism $\mathcal{M}_d \simeq \tilde{\tilde{\mathcal{C}}}_d \simeq \tilde{\mathcal{C}}_d$. Therefore, $\mathcal{M}_d$ can be seen as the normalization of $\mathcal{C}_d \subset \mathcal{C}$:*

$$\Phi_\varepsilon\colon \mathcal{M}_d \simeq \tilde{\tilde{\mathcal{C}}} \simeq \tilde{\mathcal{C}}_d \xrightarrow{\text{norm}} \mathcal{C}_d \subset \mathcal{C}.$$ □

Remark 2.8. As indicated by the notation, the morphism $\Phi_\varepsilon\colon \mathcal{M}_d \twoheadrightarrow \mathcal{C}_d \subset \mathcal{C}$, which will be seen to link polarized K3 surfaces (S, L) of degree d with special cubic fourfolds X, depends on the choice of $\varepsilon\colon \Gamma_d \xrightarrow{\sim} \Lambda_d$. There is no distinguished choice for ε and, therefore, one should not expect to find a distinguished morphism $\mathcal{M}_d \longrightarrow \mathcal{C}_d$ that can be described by a geometric procedure associating a cubic fourfold X to a polarized K3 surface (S, L).[3]

To avoid any dependance on ε, one could think of defining a morphism from the finite quotient

$$\pi_d\colon \mathcal{M}_d = \tilde{\mathrm{O}}(\Lambda_d) \setminus Q_d \twoheadrightarrow \bar{\mathcal{M}}_d := \mathrm{O}(\Lambda_d) \setminus Q_d$$

to some meaningful quotient of $\mathcal{C}$. But, as the degree of π_d grows with d, there definitely is no reasonable quotient of $\mathcal{C}$ that would receive all of them. However, it seems plausible that a quotient $\mathcal{C}_d \longrightarrow \bar{\mathcal{C}}_d$ can be constructed that allows for a morphism $\bar{\mathcal{M}}_d \twoheadrightarrow \bar{\mathcal{C}}_d$. The derived point of view to be explained later will shed more light on this.

2.3 We start by recalling the central theorem in the theory of K3 surfaces: the global Torelli theorem. In the situation at hand, it is due to Pjateckiĭ-Šapiro and Šafarevič, see [86] for details, generalizations, and references.

Consider the coarse moduli space M_d of polarized K3 surface (S, L) with $(L)^2 = d$, which can be constructed as a quasi-projective variety either by (not quite) standard GIT methods, by using the theorem below, or as a Deligne–Mumford stack.

The period map associates with any $[(S, L)] \in M_d$ a point in $\mathcal{M}_d$. For this, choose an isometry $H^2(S, \mathbb{Z}) \simeq \Lambda$, called a marking, that maps $\mathrm{c}_1(L)$ to $\ell = e_2 + (d/2)f_2$ and, therefore, induces an isometry $H^2(S, \mathbb{Z})_{L\text{-pr}} \simeq \Lambda_d$. Then the $(2, 0)$-part $H^{2,0}(S) \subset H^2(S, \mathbb{C}) \simeq \Lambda \otimes \mathbb{C}$ defines a point in the period domain Q_d. The image point in the quotient $\tilde{\mathrm{O}}(\Lambda_d) \setminus Q_d$ is then independent of the choice of any marking. This defines the period map $\mathcal{P}\colon M_d \longrightarrow \mathcal{M}_d$ which Hodge theory reveals to be holomorphic. Note that both spaces, M_d and $\mathcal{M}_d$, are quasi-projective varieties with quotient singularities.

Theorem 2.9 (Pjateckiĭ-Šapiro and Šafarevič)**.** *The period map is an algebraic, open embedding*

$$\mathcal{P}\colon M_d \hookrightarrow \mathcal{M}_d = \tilde{\mathrm{O}}(\Lambda_d) \setminus Q_d. \tag{2.1}$$

Remark 2.10. Coming back to Remark 2.8, one might wonder how the image of M_d under the finite quotient $\pi_d\colon \mathcal{M}_d \longrightarrow \bar{\mathcal{M}}_d$, can be interpreted geometrically in terms of the polarized K3 surfaces (S, L) parametrized by M_d. There is no completely satisfactory answer to this, i.e. the image $\pi_d(M_d)$ is not known (and should probably not expected) to be the coarse moduli space of a nice geometric moduli functor. The best one can say is that for $(S, L) \in M_d$ with $\rho(S) = 1$, the fibre $\pi_d^{-1}(\pi_d(S, L))$ can be viewed as the set of all Fourier–Mukai partners of S, which come with a unique polarization, cf. [**? ?**].

[3] I wish to thank E. Brakkee and P. Magni for discussions concerning this point.

To understand the complement of the open embedding (2.1), note first that any $x \in Q_d$ is the period of some K3 surface S. This surface then comes with a natural line bundle L (up to the action of the Weyl group) corresponding to $\ell = e_2 + (d/2)f_2 \in \Lambda$. Furthermore, L is ample (again, possibly after applying the Weyl group action) if and only if there exists no $\delta \in \Lambda_d$ with $(\delta)^2 = -2$ orthogonal to x, i.e. $x \in Q_d \setminus \bigcup \delta^\perp$ with $\delta \in \Delta_d \coloneqq \Delta(\Lambda_d)$, the set of all (-2)-classes in Λ_d. Hence, the complement of $M_d \subset \mathcal{M}_d$ can be described as the quotient

$$\tilde{\mathrm{O}}(\Lambda_d) \setminus \bigcup \delta^\perp \subset \mathcal{M}_d. \tag{2.2}$$

s Note that $\tilde{\mathrm{O}}(\Lambda_d)$ acts on Δ_d and that the quotient (2.2) really is a finite union. In fact, it consists of at most two components due to the following result.[4]

Proposition 2.11. *The complement $\mathcal{M}_d \setminus M_d$ consists of either one or two irreducible Noether–Lefschetz divisors depending on d:*

(i) *If $d/2 \not\equiv 1\,(4)$, then the complement (2.2) of $M_d \subset \mathcal{M}_d$ is irreducible.*
(ii) *If $d/2 \equiv 1\,(4)$, then the complement (2.2) of $M_d \subset \mathcal{M}_d$ has of two irreducible components.*

Proof This is again an application of Eichler's criterion, see the proof of Proposition 1.6. For $\delta \in \Lambda_d$ with $(\delta)^2 = -2$, one has $(\delta.\Lambda_d) = n\mathbb{Z}$ with $n = 1$ or $n = 2$. In the first case, the residue class $(1/n)\bar{\delta} \in A_{\Lambda_d} \simeq \mathbb{Z}/d\mathbb{Z}$ is trivial. In the second case, $(1/2)\bar{\delta} \equiv 0$ or $\equiv d/2\,(d)$ in $\mathbb{Z}/d\mathbb{Z}$. However, the second case is only possible if $d/2 \equiv 1\,(4)$. Indeed, write $\delta = \delta' + \delta'' \in U_2^\perp \oplus U_2$ with $\delta'' \in \ell^\perp \cap U_2 = \mathbb{Z}(e_2 - (d/2)f_2)$. Then $(1/2)\delta' + (1/2)\delta'' + (m/2)\ell \in \Lambda$ for some $m \in \mathbb{Z}$. Hence, $(1/2)\delta' \in \Lambda$ and, therefore, $-2 = (\delta)^2 \equiv (\delta'')^2\,(8)$. Combine this with $(1/2)\delta'' + (m/2)\ell \in U_2$, which implies $(\delta'')^2 \equiv m^2 d\,(8)$. □

To be more explicit, one can write

$$M_d = \begin{cases} \mathcal{M}_d \setminus \delta_0^\perp & \text{if } \frac{d}{2} \not\equiv 1\,(4) \\ \mathcal{M}_d \setminus (\delta_0^\perp \cup \delta_1^\perp) & \text{if } \frac{d}{2} \equiv 1\,(4), \end{cases}$$

where δ_0, δ_1 are chosen explicitly as $\delta_0 = e_1 - f_1$ and $\delta_1 = 2e_1 + \frac{d/2-1}{2} f_1 + e_2 - (d/2) f_2$.

2.4 We now switch to the cubic side. The moduli space M of smooth cubic fourfolds can be constructed by means of standard GIT methods as the quotient

$$M = |\mathcal{O}_{\mathbb{P}^5}(3)|_{\mathrm{sm}} /\!/ \mathrm{PGL}(6).$$

As in the case of K3 surfaces, mapping a smooth cubic fourfold X to its period $H^{3,1}(X) \subset H^4(X,\mathbb{C})_{\mathrm{pr}} \simeq \Gamma \otimes \mathbb{C}$, which is a point in the period domain $D \subset \mathbb{P}(\Gamma \otimes \mathbb{C})$, defines a

[4] Thanks to O. Debarre for pointing this out to me.

holomorphic map $\mathcal{P}\colon M \longrightarrow \mathcal{C}$. In analogy to the situation for K3 surfaces, the following global Torelli theorem has been proven [148, 150, 110, 31, 89].

Theorem 2.12 (Voisin, Looijenga,...,Charles, Huybrechts–Rennemo,...)**.** *The period map is an algebraic, open embedding*

$$\mathcal{P}\colon M \hookrightarrow \mathcal{C} = \mathrm{O}(\Gamma) \setminus D.$$

This central result is complemented by a result of Laza and Looijenga, which can be seen as an analogue of Proposition 2.11, see [105, 110]. First note that for $d = 2$ and $d = 6$ the lattice K_d is given by the matrices $\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$, respectively, see Remark 1.7. Hence, if a smooth cubic fourfold X defined a point in $\mathcal{C}_6$, then $H^{2,2}(X, \mathbb{Z})_{\mathrm{pr}}$ would contain a class δ with $(\delta)^2 = 2$ contradicting [148, §4, Prop. 1]. In [80] one finds an argument using limiting mixed Hodge structures to also exclude the case $[X] \in \mathcal{C}_2$. So, $M \subset \mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6)$.

Theorem 2.13 (Laza, Looijenga)**.** *The period map identifies the moduli space M of smooth cubic fourfolds with the complement of $\mathcal{C}_2 \cup \mathcal{C}_6$:*

$$M = \mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6).$$

To complete the picture, we state the following result. We refrain from giving a proof, but refer to similar results in the theory of K3 surfaces [86, Prop. 6.2.9].

Proposition 2.14. *The union $\bigcup \mathcal{C}_d \subset \mathcal{C}$ of all $\mathcal{C}_d$ with d satisfying $(***)$ is analytically dense in $\mathcal{C}$. Consequently, the union of all $\mathcal{C}_d$ for satisfying $(**')$ (or $(**)$ or $(*)$) is analytically dense.*

Remark 2.15. On the level of moduli spaces, the theory of K3 surfaces is linked with the theory of cubic fourfolds in terms of the morphism

$$\Phi_\varepsilon\colon M_d \subset \mathcal{M}_d \longrightarrow \mathcal{C}_d \subset \mathcal{C},$$

cf. Corollary 2.7. Note that the image of a point $[(S, L)] \in M_d$ corresponding to a polarized K3 surface (S, L) can a priori be contained in the boundary $\mathcal{C} \setminus M = \mathcal{C}_2 \cup \mathcal{C}_6$. However, unless $d = 2$ or $d = 6$, generically this is not the case and the map defines a rational map

$$\Phi_\varepsilon\colon M_d \dashrightarrow M\ ,$$

which is generically of degree one or two.

2.5 In Section 1.4 we have linked Hodge theory of K3 surfaces and Hodge theory of cubic fourfolds. We will now cast this in the framework of period maps and moduli spaces, i.e. in terms of the maps Φ_ε.

Proposition 2.16. *A smooth cubic fourfold X and a polarized K3 surface* (S, L) *are associated,* $(S, L) \sim X$*, in the sense of Definition 1.23 if and only if* $\Phi_\varepsilon[(S, L)] = [X]$ *for some choice of* $\varepsilon\colon \Gamma_d \xrightarrow{\sim} \Lambda_d$*:*

$$(S, L) \sim X \;\Leftrightarrow\; \exists\, \varepsilon\colon \Phi_\varepsilon[(S, L)] = [X].$$

Proof Assume $\Phi_\varepsilon[(S, L)] = [X]$. Pick an arbitrary marking $H^2(S, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ with $\ell \longmapsto L$. Composing the induced isometry $H^2(S, \mathbb{Z})_{L\text{-pr}} \xrightarrow{\sim} \Lambda_d$ with $\varepsilon^{-1}\colon \Lambda_d \xrightarrow{\sim} \Gamma_d \subset \Gamma$ yields a point in $D_d \subset D$. Then there exists a marking $\Gamma \xrightarrow{\sim} H^4(X, \mathbb{Z})_{\text{pr}}$ such that X yields the same period point in D, which thus yields a Hodge isometric embedding $H^2(S, \mathbb{Z})_{L\text{-pr}} \hookrightarrow H^4(X, \mathbb{Z})_{\text{pr}}(1)$. Conversely, any such Hodge isometric embedding defines a sublattice of $\Gamma \simeq H^4(X, \mathbb{Z})_{\text{pr}}$ isomorphic to some $v^\perp$ which after applying some element in $\text{O}(\Gamma)$ becomes Γ_d, see Proposition 1.6. Composing with a marking of (S, L) yields the appropriate ε. □

Corollary 2.17. *Let X be a smooth cubic fourfold. Then* $X \sim (S, L)$ *for some polarized K3 surface* (S, L) *of degree d if and only if* $X \in \mathcal{C}_d$ *with d satisfying* $(**)$*.*

Proof Interpret $\mathcal{M}_d$ as the moduli space of quasi-polarized K3 surfaces (S, L), i.e. with L only big and nef but not necessarily ample. One then has to show that whenever there exists a Hodge isometric embedding $H^2(S, \mathbb{Z})_{L\text{-pr}} \hookrightarrow H^4(X, \mathbb{Z})_{\text{pr}}(1)$, then L is not orthogonal to any algebraic class $\delta_S \in H^2(S, \mathbb{Z})$ with $(\delta_S)^2 = -2$. Indeed, in this case L would be automatically ample. However, such a class δ_S would correspond to a class $\delta \in H^{2,2}(X, \mathbb{Z})_{\text{pr}}$ with $(\delta)^2 = 2$, which contradicts $[X] \in M = \mathcal{C} \setminus (\mathcal{C}_2 \cup \mathcal{C}_6)$. Of course, the argument is purely Hodge theoretic and one can easily avoid talking about quasi-polarized K3 surfaces. □

Remark 2.18. Note that a given cubic fourfold X can be associated with more than one polarized K3 surface (S, L) and, in fact, sometimes even with infinitely many (S, L). To start, there are the finitely many choices of $\varepsilon \in \text{O}(\Lambda_d)/\tilde{\text{O}}(\Lambda_d)$. Then, Φ_ε is only generically injective for d satisfying $(**)_2$ and even of degree two for $(**)_0$. And finally, X could be contained in more than $\mathcal{C}_d$. In fact, it can happen that $X \in \mathcal{C}_d$ for infinitely many d satisfying $(**)$. To be more precise, depending on the degree d, there may exist non-isomorphic K3 surfaces S and S' endowed with polarizations L and L', respectively, such there nevertheless exists a Hodge isometry $H^2(S, \mathbb{Z})_{L\text{-pr}} \simeq H^2(S', \mathbb{Z})_{L'\text{-pr}}$. Indeed, the latter may not extend to a Hodge isometry $H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z})$, see Section 1.4.

The situation is not quite as bad as it sounds. Although there may be infinitely many polarized K3 surfaces (S, L) associated with one X, only finitely many isomorphism types of unpolarized K3 surfaces S will be involved.

Remark 2.19. In [25] a geometric interpretation for the generic fibre of the rational map $\Phi_\varepsilon\colon M_d \dashrightarrow \mathcal{C}_d$ in the case $d \equiv 0\,(6)$ is described. It turns out that $\Phi_\varepsilon[(S, L)] = [(S', L')]$

implies that S' is isomorphic to $M(3, L, d/6)$, the moduli space of stable bundles on S with the indicated Mukai vector.

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