Higher Degree Symmetric Products of Curves

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1 Introduction

Torelli's theorem states, that a curve is uniquely determined by its polarized Jacobian variety. There are many variants and generalizations of this theorem. Based on [3] the goal of this thesis will be to establish the following generalization:

Theorem 1.1. *Let C and D be two smooth, projective, irreducible curves of genus $g > 2$ over an algebraically closed field k. If $C^{(d)} \cong D^{(d)}$ for some $d \geq 1$, then $C \cong D$.*

Here $C^{(d)}$ denotes the d-th symmetric power of the curve C, i.e. the quotient of the d-fold product C^d by the action of the symmetric group $\mathfrak{S}_d$. One can think of the points of $C^{(d)}$ as the degree d effective divisors on C. The statement of the theorem is then, that this data is enough to uniquely determine the underlying curve itself. This is an extension of Torelli's Theorem because the image of $C^{(g-1)}$ in $J(C)$ determines the canonical polarization. So Torelli's theorem would be the case $d = g - 1$ in Theorem 1.1.

In section 2 we will start with recalling some definitions and establishing some general constructions, the most important being, that we may reconstruct the Jacobian variety from the symmetric product. Section 3 will cover the case $d < 2g - 2$. The idea is to use the above mentioned interpretation of points of $C^{(d)}$ as effective divisors to deduce combinatorial statements.

Before we can continue with the proof for $d \geq 2g-2$ we have to do some preparations. In section 4 we will introduce Picard sheaves, which can be seen as a special case of Fourier–Mukai transformations. We will establish general properties of Picard sheaves with the two goals being the calculation of their Chern classes and to investigate their connection to the symmetric product. These statements will be used in section 5 to explicitly calculate the connection between the Chern classes associated to the symmetric products, which allows us to finish the proof with a criterion by Matsusaka.

Finally in section 6 we will investigate how Theorem 1.1 extends to smaller genus and how it does not. The main result will be, that there are non-isomorphic genus two curves with isomorphic second symmetric power, hence the theorem does not extend to the case $d = g = 2$.

Deutsche Zusammenfassung

Das Hauptziel dieser Arbeit wird sein, die folgende Verallgemeinerung von Torellis Theorem zu beweisen

Theorem. *Seien C und D zwei glatte, irreduzible, projective Kurven über einen allgebraisch abgeschlossenen Körper. Wenn für ein $d \geq 1$ gilt, dass $C^{(d)} \cong D^{(d)}$, dann gilt schon $C \cong D$.*

Hierbei bezeichnet $C^{(d)}$ das d-te symmetrische Produkt einer Kurve, also den Quotienten von C^d nach der Aktion der symmetrischen Gruppe durch Vertauschung. Dies ist eine Verallgemeinerung von Torellis Theorem, da die Polarisation der Jakobischen Varietät durch des Bild von $C^{(g-1)}$ in der Jakobischen induziert ist. Dies wird vom Fall $d = g - 1$ in unserem Theorem abgedeckt.

Der Beweis verläuft für $d < 2g-2$ über die Interpretation der Punkte von $C^{(d)}$ als effektive Grad d Divisoren auf C. Damit kann man den Fall durch kombinatorische Überlegungen abhandeln und braucht nur an einer Stelle Schnitttheorie, um einen Automorphismus von $J(C)$ zu konstruieren.

Für $d \geq 2g - 2$ läuft der Beweis sehr verschieden. Hier bildet $C^{(d)}$ ein projektives Bündel über $J(C)$ (im Fall $d = 2g - 2$ müssen wir vorher einen Punkt entfernen). Um die Chern Klassen dieses Bündels auszurechenen werden wir Picard Garben einführen, für die diese Berrechnung einfacher ist und ihre Verbindung zu den symmetrischen Produkt untersuchen. Sobald wir die Chern Klassen berechnet haben, können wir ein Kriterium von Matsusaka nutzen um den Beweis abzuschließen.

Im letzten Abschnitt werden wir untersuchen, wie sich das Theorem auf Genus kleiner drei erweitern lässt. Es wird sich herausstellen, dass es nicht isomorphe Kurven in Genus zwei gibt, mit isomorphen zweiten symmetrischen Produkt. Damit kann das Theorem insbesondere nicht auf den Fall $d = g = 2$ erweitert werden.

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Contents

2 Definitions and preparation

In this thesis a curve will be irreducible, smooth and projective, unless specified differently.

We first recall some important definitions.

Definiton 2.1. Let X be a variety. An Albanese variety for X with a given point $x \in X$ is an abelian variety $\mathrm{Alb}(X)$ with a morphism $a_x \colon X \to \mathrm{Alb}(X)$ with the following universal property. Any morphism $f \colon X \to A$ into an abelian variety A satisfying $f(x) = 0_A$ factors uniquely over $\mathrm{Alb}(X)$ by a morphism of abelian varieties.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "a_x"'] & & A \\ & \mathrm{Alb}(X) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Indeed any variety has an Albanese, see for example [11, II §3] and the Albanese is uniquely determined by its universal property. For a curve C we have an explicit model of $\mathrm{Alb}(C)$. We can consider $\mathrm{Pic}^0(C)$, the set of degree zero divisor modulo linear equivalence or equivalently degree zero line-bundles modulo isomorphism. One can equip it with the structure of an abelian variety to get the Jacobian $J(C)$. Given a point $c \in C$ the albanese morphism into $J(C) = \mathrm{Alb}(C)$ is then given by $c' \mapsto [c' - c]$ when working with divisors respectively $c' \mapsto \mathcal{O}(c' - c)$ if one works with line-bundles. Given a positive integer n one can further consider the n-fold product C^n of C. It comes with a natural $\mathfrak{S}_d$-action by permuting the factors.

Definiton 2.2. We define the n-th symmetric product $C^{(n)}$ of C to be the quotient of C^n by the $\mathfrak{S}_n$ action: $C^{(n)} = C^n/\mathfrak{S}_n$.

The points of $C^{(n)}$ correspond to effective degree n divisors on C. This gives rise to a natural morphism $\varphi_n \colon C^{(d)} \to J(C), (c_1 + \cdots + c_n) \mapsto [c_1 + \cdots + c_n - nc]$ given a base point c on C. For $0 < n < g$ denote the image of $C^{(n)}$ in $J(C)$ by W^n. For the second curve D the image of $D^{(n)}$ will be denoted by V^n. We will identify C with its image W^1. This is possible for $g(C) > 0$ as then a_c is injective for any $c \in C$.

Next we want to investigate, how we may reconstruct $J(C)$ from $C^{(n)}$.

Lemma 2.3. *For a smooth projective curve C there is an isomorphism $\mathrm{Alb}(C^{(n)}) \cong J(C)$ for any $n \geq 1$.*

In the setting of Theorem 1.1 the condition $C^{(d)} \cong D^{(d)}$ thus establishes $J(C) \cong J(D)$. As $\dim(J(C)) = g(C)$ this also shows, that we may drop the condition, that C and D have the same genus.

We will prove Lemma 2.3 in a more general setting. First we need a theorem which allows us to compare the maps from a product into an abelian variety with maps defined on the factors.

Theorem ([11, II. Thm. 3])**.** *Let V, W be varieties over k and let A be an abelian variety. Let $f \colon V \times W \to A$ be a rational map. Then there exist two rational maps $f_1 \colon V \dashrightarrow A, f_2 \colon W \dashrightarrow A$ such that for any point (P, Q) on $V \times W$ where both are defined one has $f(P, Q) = f_1(P) + f_2(Q)$ and f_1, f_2 are uniquely determined up to an additive constant.*

We can extend this from rational maps to morphisms via the following statement:

Theorem ([16, Thm. 3.1]). *Any rational map $f\colon V \dashrightarrow A$ from a non-singular variety into an abelian variety is defined on the whole of V.*

Proposition 2.4. *Let X and Y be non-singular varieties over k. Then $\mathrm{Alb}(X) \times \mathrm{Alb}(Y)$ together with the map $X \times Y \to \mathrm{Alb}(X) \times \mathrm{Alb}(Y)$ induced by $X \times Y \to X \to \mathrm{Alb}(X)$ and $X \times Y \to Y \to \mathrm{Alb}(Y)$ is an Abelian variety for $X \times Y$.*

Proof. We show directly, that $\mathrm{Alb}(X) \times \mathrm{Alb}(Y)$ satisfies the universal property of the Albanese variety of $X \times Y$ with base point (x_0, y_0). Here x_0 respectively y_0 are the base points chosen for $\mathrm{Alb}(X)$ and $\mathrm{Alb}(Y)$. Take any morphism $f\colon X \times Y \to A$ into an abelian variety A. By the above theorems we get $f_1\colon X \to A$ and $f_2\colon Y \to A$ which are unique up to a constant with $f(x, y) = f_1(x) + f_2(y)$. Chose the constant such that $f_1(x_0) = 0$, then $f(x_0, y_0) = 0$ implies $f_2(y_0) = 0$. By the universal property of $\mathrm{Alb}(X)$ and $\mathrm{Alb}(Y)$ we get morphisms $g_1\colon \mathrm{Alb}(X) \to A$ and $g_2\colon \mathrm{Alb}(Y) \to A$ such that

$$\begin{tikzcd} X \arrow[rr, "f_1"] \arrow[dr, "a_X"'] & & A \\ & \mathrm{Alb}(X) \arrow[ur, dashed, "g_1"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "f_2"] \arrow[dr, "a_Y"'] & & A \\ & \mathrm{Alb}(Y). \arrow[ur, dashed, "g_2"'] & \end{tikzcd}$$

So we get a map $g\colon \mathrm{Alb}(X) \times \mathrm{Alb}(Y) \to A, (a, b) \mapsto g_1(a) + g_2(b)$. Obviously g is a morphism of abelian varieties when equipping $\mathrm{Alb}(X) \times \mathrm{Alb}(Y)$ with pointwise addition. We then have $g(a_{X \times Y}(x, y)) = g_1(a_X(x)) + g_2(a_Y(y)) = f_1(x) + f_2(y) = f(x, y)$ for all $(x, y) \in X \times Y$. This morphism is unique, as any map $\mathrm{Alb}(X) \times \mathrm{Alb}(Y) \to A$ factors as a sum of morphisms $\mathrm{Alb}(X) \to A, \mathrm{Alb}(Y) \to A$, unique up to a constant. So consider

$$\begin{tikzcd} X \times Y \arrow[rr, "f"] \arrow[dr] & & A \\ & \mathrm{Alb}(X) \times \mathrm{Alb}(Y). \arrow[ur, shift left, "g'"] \arrow[ur, shift right, "g"'] & \end{tikzcd}$$

By factorizing both g and g' with fitting constants we get

$$\begin{tikzcd} X \arrow[rr, "f_1"] \arrow[dr] & & A \\ & \mathrm{Alb}(X) \arrow[ur, shift left, "g_1'"] \arrow[ur, shift right, "g_1"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "f_2"] \arrow[dr] & & A \\ & \mathrm{Alb}(Y). \arrow[ur, shift left, "g_2"] \arrow[ur, shift right, "g_2'"'] & \end{tikzcd}$$

Now we may use the universal property of $\mathrm{Alb}(X)$ and $\mathrm{Alb}(Y)$ to get uniqueness. $\square$

With this we are equipped to prove the promised generalization of Lemma 2.3:

Proposition 2.5. *Let X be a non-singular variety over k, then $\mathrm{Alb}(X^{(n)}) \cong \mathrm{Alb}(X)$ for any $n \in \mathbb{N}$.*

Proof. First by Proposition 2.4 we know $\mathrm{Alb}(X^n) = \mathrm{Alb}(X)^n$. Now for an abelian variety A and a morphism $f\colon X^{(n)} \to A$ we can consider the diagram:

$$\begin{tikzcd} X^n \arrow[r, "p"] \arrow[dr] & X^{(n)} \arrow[r, "f"] & A \\ & \mathrm{Alb}(X)^n. \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

The vertical composition is $\mathfrak{S}_n$-invariant, as p is. So the constructed morphism φ will be $\mathfrak{S}_n$-invariant as well. The following lemma concludes the proof. $\square$

Lemma 2.6. *Given an abelian variety B and a symmetric morphism of abelian varieties $\varphi\colon B^n \to A$ there is a unique factorization*

$$\begin{tikzcd} B^n \arrow[rr, "\varphi"] \arrow[dr, "s"'] & & A \\ & B \arrow[ur, dashed] & \end{tikzcd}$$

where $s(b_1, \dots, b_n) = b_1 + \dots + b_n$.

Proof. Uniqueness is the easy part, since s is surjective. In order to show existence we have to show, that φ is constant on the fibres of s. For this fix $b \in B$. Then we have $s^{-1}(b) = \{(b_1, \dots, b_n) \mid b_1 + \dots + b_n = b\}$, which is isomorphic to B^{n-1} via

$$(b_1, \dots, b_{n-1}) \mapsto (b_1, \dots, b_{n-1}, b - (b_1 + \dots + b_{n-1})).$$

On this fibre φ has the form

$$\begin{aligned} &\varphi(b_1, \dots, b_{n-1}, b - b_1 - \dots - b_{n-1}) \\ &= \varphi(b_1, \dots, b_{n-1}, b) - \varphi(0, \dots, 0, b_1) - \dots - \varphi(0, \dots, 0, b_{n-1}) \\ &= \varphi(b_1, \dots, b_{n-1}, b) - \varphi(b_1, 0, \dots, 0) - \dots - \varphi(0, \dots, 0, b_{n-1}, 0) \\ &= \varphi(b_1 - b_1, \dots, b_{n-1} - b_{n-1}, b) = \varphi(0, \dots, 0, b). \end{aligned}$$

Altogether we see, that φ is indeed constant on fibres. $\square$

This enables us to do the following construction, allowing us to assume that $C^{(d)} \cong D^{(d)}$ over a common Jacobian J: Use Lemma 2.3 to fix an abelian variety $J \cong J(C) \cong J(D)$. Then we are given a diagram

$$\begin{tikzcd} C^{(d)} \arrow[r, "\sim"] \arrow[d, "\varphi_1"'] & D^{(d)} \arrow[d, "\varphi_2"] \\ J & J. \end{tikzcd}$$

We can use the universal property of J being the Albanese of $C^{(d)}$ and $D^{(d)}$ to get morphisms $f, f'\colon J \to J$ fitting into a diagram

$$\begin{tikzcd} C^{(d)} \arrow[r, "\sim"] \arrow[d, "\varphi_1"'] & D^{(d)} \arrow[d, "\varphi_2"] \\ J \arrow[r, dashed, bend left, "f"] & J.. \arrow[l, dashed, bend left, "f'"] \end{tikzcd}$$

Now f, f' give isomorphisms on J. To see this apply the uniqueness part of the universal property of the Albanese variety, implying $f \circ f' = id_J = f' \circ f$. We may replace $D \cong V^1$ by $f^{-1}(V^1)$ to assume that f is id_J. Hence we are in the following situation

$$\begin{tikzcd} C^{(d)} \arrow[rr, "\sim"] \arrow[dr, "\varphi_1"'] & & D^{(d)} \arrow[dl, "\varphi_2"] \\ & J. & \end{tikzcd}$$

3 The case $d \leq 2g - 3$

We first want to prove the case $d \leq g - 1$, following Martens approach in [12]. So assume $C^{(d)} \cong D^{(d)}$, which by our preliminary construction implies $W^d = V^d$. Recall that W^d and V^d denote the images of $C^{(d)}$ respectively $D^{(d)}$ in J under φ_d. This will be enough to conclude $C \cong D$, we may even allow that W^d and V^d differ by a translation. First we will collect the statements we will need later in the proof.

3.1 Combinatorial preliminaries

We begin with a bit of notation. For a subvariety Z of J we denote by $Z_a = \{a+z \mid z \in Z\}$ its translate by any $a \in J$ and by Z^- its image under the reflection map $u \mapsto -u$ of J. Let K denote a canonical divisor on C. For subsets A and B of $J(C)$ we set

$$A \ominus B \coloneqq \bigcap_{b \in B} A_{-b}.$$

Here one can observe that $u \in A \ominus B$ exactly when $u + b \in A$ for all $b \in B$. The latter is the case precisely if $B_u \subset A$. For the sake of readability we will loosen our convention a bit and drop the index n from the maps $\varphi_n \colon C^{(n)} \to J$, so for an effective divisor A we have $\varphi(A) = [A - \deg(A)c]$.

Remark 3.1. In the following we will frequently use, that we may interpret the fibres of φ_r as complete linear systems. We can interpret $A, B \in C^{(n)}$ as effective divisors and $\varphi(A) = \varphi(B)$ implies $A - nc \sim B - nc$ so $A \sim B$ and $\varphi^{-1}(x)$ is the complete linear system of $x \in J$. More generally if A is of degree n and B is of degree m then $\varphi(A) = \varphi(B)$ if and only if $A - nP \sim B - mP$ so $A + (n - m)c \sim B$.

We will need a few combinatorial preliminaries.

Lemma 3.2. *Let $0 \leq r \leq t \leq g - 1$ and $a, b \in J$. Then*

(i) $W_a^r \subset W_b^t$ *exactly when* $a \in W_b^{t-r}$

(ii) $W_b^t \ominus W_a^r = W_{b-a}^{t-r}$

(iii) $B \subset A \ominus (A \ominus B)$ *for all subsets* $A, B \subset J$

Proof. (i) If $a \in W_b^{t-r}$ then $a = b + \varphi(A)$ for an effective degree $t - r$ divisor A on C. Hence for any $c = a + \varphi(B) \in W_a^r$ we have $c = b + \varphi(A) + \varphi(B) = b + \varphi(A + B) \in W_b^t$, using $A + B$ has degree $(t - r) + r$. This establishes $W_a^r \subset W_b^t$.

Conversely assume $W_a^r \subset W_b^t$. Since $a \in W_a^r \subset W_b^t$ we get an effective degree t divisor A with $a = \varphi(A) + b$. For any degree r effective divisor B we find by assumption an effective degree t divisor $\overline{B}$ with $a + \varphi(B) = b + \varphi(\overline{B})$. Then $\varphi(B) + \varphi(A) = \varphi(\overline{B})$ and thus $B + A \sim \overline{B} + rc$. By Riemann–Roch we have

$$h^0(K - A) = h^0(A) - \deg(A) + g - 1 = h^0(A) - t + g - 1 \geq h^0(A) \geq 1.$$

For $\overline{B}$ we get

$$h^0(K - \overline{B}) = h^0(\overline{B}) - \deg(\overline{B}) + g - 1 = h^0(\overline{B}) - t + g - 1 \geq g - t.$$

So we find an effective divisors A' and a linear system of dimension $g - t - 1$ of effective divisors $\overline{B'}$ with $A + A' \sim K \sim \overline{B} + \overline{B'}$. Note that this linear system is independent of the choice B. Combining this with $\varphi(B) + \varphi(A) = \varphi(\overline{B})$ implies $B + \overline{B'} \sim A' + rc$. Now B was an arbitrary effective degree r divisor so we get $h^0(A' + rc) \geq r + (g - t - 1) + 1 = g - t + r$ if we vary B and $\overline{B}$. So by Riemann Roch $h^0(K - A' - rc) \geq 1$ and we in particular find an effective divisor $\overline{A}$ of degree $t - r$ such that $\overline{A} + A' + rc \sim K$ giving us $\overline{A} + rc \sim K - A' \sim A$. Thus $\varphi(\overline{A}) = \varphi(A')$ and $a = b + \varphi(\overline{A}) \in W_b^{t-r}$, what was to be shown.

(ii) We have $u \in W_b^t \ominus W_a^r$ exactly if $W_{a+u}^r \subset W_b^t$. The later is by part (i) equivalent to $a + u \in W_b^{t-r}$ and thus to $u \in W_{b-a}^{t-r}$.

(iii) We have for all $b \in B$ that $(A \ominus B)_b \subset (A_{-b})_b = A$, hence $b \in A \ominus (A \ominus B)$.

□

The next lemma might seem a bit technical, we will need it in the proof of Lemma 3.4 and later in the actual proof.

Lemma 3.3. *Let $0 < r + 1 \leq t \leq g - 1$, $x \in W^1, y \in W^{t-r}$. Then $W_a^{r+1} \subset W_{a+x-y}^t$ or*

$$W_a^{r+1} \cap W_{a+x-y}^t = W_{a+x}^r \cup (W_a^{r+1} \cap (W_{a-y}^t \ominus (W^1)^-)). \tag{1}$$

Proof. We can write $x = \varphi(p)$ for a point p on C and $y = \varphi(A)$ for A an effective degree $t - r$ divisor. If p is a point of A then $a = y - x + x + a - y = \varphi(A - p) + a + x - y$ and thus $a \in W_{a+x-y}^{t-r-1}$. By Lemma 3.2.(i) we get $W_a^{r+1} \subset W_{a+x-y}^t$. So we assume p is not a point of A.

Consider some $u \in W_a^{r+1} \cup W_{a+x-y}^t$, so there are effective divisors E and E' on C of degree $r + 1$ respectively t with $u = \varphi(E) + a = \varphi(E') + a + x - y$. This gives us $\varphi(E) + \varphi(A) = \varphi(E') + \varphi(p)$ and therefore $E + A \sim E' + p$.

If $E + A = E' + p$, then p is a point of E, as we assumed it is no point of A. Then

$$\begin{aligned} u &= \varphi(E') + a + x - y = \varphi(E') + a + x - \varphi(A) \\ &= \varphi(E - p + A) + a + x - \varphi(A) = \varphi(E - p) + a + x \end{aligned}$$

with $E - p$ being an effective divisor of degree r. Thus $u \in W_{a+x}^r$ meaning it lies in the right hand side of (1).

For the final case we assume $E + A \neq E' + p$. Then $h^0(E + A) \geq 2$ as its linear system contains two effective divisors. Take any point $Q \in C$. Then by Riemann–Roch

$$\begin{aligned} h^0(E + A - Q) &= h^0(K - E - A + Q) + \deg(E + A - Q) - g + 1 \\ &\geq h^0(K - E - A) + \deg(E + A) - g + 1 - 1 = h^0(E + A) - 1 \geq 1. \end{aligned}$$

This especially gives us an effective degree t divisor Q' with $Q + Q' \sim E + A$. Then we get $u = \varphi(E) + a = \varphi(Q') + \varphi(Q) - \varphi(A) + a$, thus $u \in W^t_{a-y+\varphi(Q)}$. As Q runs over C, $\varphi(Q)$ runs over W^1 so

$$u \in \bigcap_{q \in W^1} W^t_{a-y+q} = W^t_{a-y} \ominus (W^1)^-.$$

Moreover $u = \varphi(E) + a$ and $\deg(E) = r + 1$ implies $u \in W^{r+1}_a$. In total we have shown the $\subset$ inclusion of (1).

Now consider the opposite inclusion. Assume first $u \in W^r_{a+x}$, so we have a degree r effective divisor E such that $u = \varphi(E) + a + x = \varphi(E + c) + a = \varphi(E + A) + a + x - y$, with $\deg(E + c) = r + 1$ and $\deg(E + A) = t$. This implies $u \in W^{r+1}_a \cap W^t_{a+x-y}$. Observe that $(W^t_{a-y} \ominus (W^1)^-)) \subseteq W^t_{a+x-y}$ as the left hand side is an intersection and the right hand side is a set of those we intersect. So in total

$$W^{r+1}_a \cap (W^t_{a-y} \ominus (W^1)^-) \subseteq W^{r+1}_a \cap W^t_{a+x-y}.$$

□

We come to the final combinatorial Lemma:

Lemma 3.4. *Let $1 \le t \le g - 1$ and assume $W^t = V^t_c$ for some $c \in J$. Suppose for some $1 \le r \le t$ and some $b \in J$ the intersection $V^1 \cap W^r_b$ contains two different points u and v, then V^1 is contained in a translate of W^r or $(W^r)^-$.*

Proof. Since u, v are in W^r_b and since $\ominus$ is defined as an intersection, we have

$$W^{t-r}_{-b} \overset{3.2.(ii)}{=} W^t \ominus W^r_b \subset W^t_{-u} \cap W^t_{-v} = V^t_{c-u} \cap V^t_{c-v}.$$

Now we may apply Lemma 3.3, with W replaced by V, $r = t - 1$, $a = c - u, x = u, y = v$, to the intersection on the right, to get

$$W^{t-r}_b \subset V^t_{c-u} \cap V^t_{c-v} = V^{t-1}_c \cup (V^t_{c-u} \cap (V^t_{c-u-v} \ominus (V^1)^-)).$$

Observe that W^{t-r}_b is irreducible so W^{t-r}_b is contained in one of the sets of the right hand side union. In the first case we get $W^{t-r}_b \subset V^{t-1}_c$ and hence

$$V^1 \overset{3.2.(ii)}{=} V^t_c \ominus V^{t-1}_c \subset W^t \ominus W^{t-r}_{-b} = W^t_{-b}.$$

Otherwise we have $W^{t-r}_b \subset V^t_{c-u-v} \ominus (V^1)^-$. Then

$$(V^1)^- \overset{3.2.(iii)}{\subset} V^t_{c-u-v} \ominus (V^t_{c-u-v} \ominus (V^1)^-) \subset W^t_{-u-v} \ominus W^{t-r}_{-b} = W^r_{b-u-v}.$$

In the first case V^1 is contained in a translate of W^r, in the second one it is contained in a translate of $(W^r)^-$, establishing the claim. □

As a last ingredient we will need to introduce some endomorphisms of J:

Definiton 3.5. Let Y be a divisor on J. Then we define an endomorphism $\alpha(W^1, Y)$ of J by

$$\alpha(W^1, Y)(u) = S(W^1 \cdot (Y_u - Y))$$

for $u \in J$. This means we intersect W^1 with a translate of Y minus Y and then add up the resulting points with multiplicity with respect to the addition on J. Analogous we define $\alpha(V^1, Y)$.

In order to analyse the morphisms we will need the following proposition, for a proof see [11].

Proposition 3.6 ([11, VI, Thm. 3])**.** *For a divisor X on J there is a unique $x \in J$ with*

$$X \sim W^{g-1}_x - W^{g-1}.$$

Moreover x can be expressed as $x = S(W^1 \cdot X)$.

Corollary 3.7. *In the above setting we have*

$$Y_u - Y \sim W^{g-1}_{\alpha(W^1,Y)(u)} - W^{g-1}.$$

Corollary 3.8. *We have $\alpha(W^1, W^{g-1}) = id_J$.*

Proof. We trivially have $W^{g-1}_u - W^{g-1} \sim W^{g-1}_u - W^{g-1}$ so the uniqueness part of Proposition 3.6 gives us the desired. □

Altogether we deduce

$$\begin{aligned}\alpha(V^1, Y)(u) = S(V^1 \cdot (Y_u - Y)) &= S(V^1 \cdot (W^{g-1}_{\alpha(W^1,Y)(u)} - W^{g-1})) \\ &= \alpha(V^1, W^{g-1}) \circ \alpha(W^1, Y)(u)\end{aligned}$$

and the analogous statement with V and W exchanged.

In particular

$$id_J = \alpha(V^1, V^{g-1}) = \alpha(V^1, W^{g-1}) \circ \alpha(W^1, V^{g-1})$$

$$id_J = \alpha(W^1, W^{g-1}) = \alpha(W^1, V^{g-1}) \circ \alpha(V^1, W^{g-1})$$

so $\alpha(V^1, W^{g-1})$ and $\alpha(W^1, V^{g-1})$ are inverse automorphisms of J.

We have collected all results we will need for the proof.

3.2 The proof

Recall we are assuming $W^d = V^d_b$ for some $b \in J$. Let r be the smallest integer for which an inclusion of the form $V^1 \subset W^{r+1}_a$ or $(V^1)^- \subset W^{r+1}_a$ holds for some $a \in J$. Assume $V^1 \subset W^{r+1}_a$, in the other case the argument is the same with an extra involution at the end. As $V^1 \subset V^d$ we surely have $r < d$. We consider intersections of the form $V^1 \cap W^{g-1}_{a+x-y}$ where $x \in W^1$ and $y \in W^{g-1-r}$. If for some x and all $y \in W^{g-1-r}$ we had $V^1 \subset W^{g-1}_{a+x-y}$, then

$$V^1_{-a-x} \subset \bigcap_{y \in W^{g-1-r}} W^{g-1}_{-y} = W^{g-1} \ominus W^{g-1-r} \overset{3.2.(ii)}{=} W^r.$$

This would imply $V^1 \subset W^r_{a+x}$, contradicting our minimality assumption for r. So we find an y such that for at least one $x \in W^1$ we have $V^1 \not\subseteq W^{g-1}_{a+x-y}$. Fix this y. We want to investigate, for which $x \in W^1$ we have $V^1 \not\subseteq W^{g-1}_{a+x-y}$. For a fixed $v \in V^1$ we have $\{z \in J \mid v - z \notin W^{g-1}_{a-y}\} = J \setminus (W^{g-1}_{a-y-v})^-$ is open, as $W^{g-1} \subset J$ is closed. This means

$$\{z \in J \mid V^1 \not\subseteq W^{g-1}_{a+z-y}\} = \bigcup_{v \in V^1} \{z \in J \mid v - z \notin W^{g-1}_{a+y}\}$$

is open as well and so is its intersection with W^1. So the x for which $V^1 \not\subseteq W^{g-1}_{a+x-y}$ form a non-empty, open and hence dense subset of W^1. Now recall $V^1 \subset W^{r+1}_a$, so $V^1 \not\subseteq W^{g-1}_{a+x-y}$ implies $W^{r+1}_a \not\subseteq W^{g-1}_{a+x-y}$. Then by Lemma 3.3 applied for $t = g-1$ we have

$$W^{r+1}_a \cap W^{g-1}_{a+x-y} = W^r_{a+x} \cup (W^{r+1}_a \cap (W^{g-1}_{a-y} \ominus (W^1)^-)).$$

Intersecting it with $V^1 \subset W^{r+1}_a$ yields

$$V^1 \cap W^{g-1}_{a+x-y} = (V^1 \cap W^r_{a+x}) \cup (V^1 \cap A)$$

with $A \subset J$ independent of x. If we take the sum over the intersection points, the left hand side is just $\alpha(V^1, W^{g-1})(a + x - y)$. On the right hand side, $V^1 \cap A$ gives some fixed element $z \in J$ and $V^1 \cap W^r_{a+x}$ contains by Lemma 3.4 exactly one point of V^1 with some multiplicity k. So we see that an open subset of the translate of W^1 by $a - y$ gets mapped by α into a translate of ${}^kV^1 := \{kv \mid v \in V^1\}$ by z. In total this yields an isomorphism ψ of J, which maps an open subset of W^1 into ${}^kV^1$. As W^1 is irreducible and ${}^kV^1$ is irreducible and closed, ψ has to map all of W^1 isomorphically onto ${}^kV^1$.

So in total we get a morphism

$$V^1 \xrightarrow{v \mapsto kv} {}^kV^1 \xrightarrow{\psi^{-1}} W^1$$

To this composition we want to apply Hurwitz's theorem:

Theorem (Hurwitz, [5, IV.2.4])**.** *Let $f\colon X \to Y$ be a finite separable morphism of curves. Let $n = \deg(f)$. Then*

$$2g(X) - 2 = n(2g(Y) - 2) + \deg(R)$$

where R denotes the ramification divisor.

As $g(V^1) = g(D) = g(C) = g(W^1) > 1$ we get, that the above composition is unramified and has degree one, so it is an isomorphism. This completes the proof in the case $d \leq g - 1$

Remark 3.9. With further arguments it is possible to prove $k = \pm 1$, which gives rise to a slightly more general theorem

Theorem. *Let C and D be complete non-singular curves and assume $J(C) = J(D) = J$. If for some t, $1 \leq t \leq g-1$ some translate of V^t coincides with W^t, then there exists an automorphism λ of J such that $\lambda(W^1)$ is a translate of V^1.*

For details see [12].

3.3 Extension to $g \leq d \leq 2g-3$

With a small trick we may apply Theorem 3.9 to the case $g \leq d \leq 2g-3$ as well. Let K denote a canonical divisor on C. Then by Riemann–Roch for every $E \in C^{(d)}$ we have

$$h^0(E) = h^0(K - E) + \deg(E) - g + 1.$$

So $h^0(E) > d - g + 1$ exactly when $h^0(K - E) > 0$. We can interpret the fibre of φ_d at $[B] \in J$ as complete linear system of dimension $h^0(B + dP_0) - 1$. So the divisors where the fibre dimension is bigger than $d - g$ form a subvariety isomorphic to W^{2g-2-d} under the isomorphism induced by $E \mapsto K - E$. The analogous result holds for D. By our preliminary construction $C^{(d)} \cong D^{(d)}$ respects fibres, so we get $W^{2g-2-d} = V^{2g-2-d}$. As $0 < 2g - 2 - d < g$ we may use Theorem 3.9 and get $C \cong D$.

4 Picard Sheaves and Chern class of the symmetric product

Before we are able to prove Theorem 1.1 for higher d, we first need to collect some prerequisites. Following [19] we will introduce Picard sheaves associated to a curve, which we use to calculate the Chern classes corresponding to $C^{(d)}$. For another approach to the calculation of the Chern classes see also [15]. From now on we will consider Pic^0 as line-bundles rather than as divisors.

4.1 Introduction of Picard sheaves and first properties

Let C be a complete, non-singular curve of genus g defined over an algebraically closed field k and $c \in C$ a fixed basepoint. We can consider the invertible sheaf $\xi_n = \mathcal{O}_C(nc)$ corresponding to the divisor nc. This gives an exact sequence of sheaves:

$$0 \longrightarrow \xi_{n-1} \longrightarrow \xi_n \longrightarrow \xi_n/\xi_{n-1} \longrightarrow 0 \tag{2}$$

The last sheaf has support c and restriction $\mathcal{O}_c$ there. This comes from the exact sequence

$$0 \to \mathcal{O}_C(-c) \to \mathcal{O}_C \to \mathcal{O}_c \to 0$$

by tensoring with ξ_n.

Given an abelian variety J, a curve C, $x \in J$ and $y \in C$ we have four canonical maps we will need in the definitions below, $\pi_J \colon J \times C \to J, \pi_C \colon J \times C \to C, i_y \colon J \to J \times C$ and $j_x \colon C \to J \times C$. These are given by $\pi_J(x, y) = x, \pi_C(x, y) = y, i_y(x) = (x, y)$ and $j_x(y) = (x, y)$.

Definiton 4.1. A pair $(J, \mathscr{P})$ of an abelian variety J and an invertible sheaf $\mathscr{P}$ on $J \times C$ is called a *Picard variety* for C if $\varphi(x) = j_x^*\mathscr{P}$ defines an isomorphism $\varphi \colon J \to \mathrm{Pic}^0(C)$. To ensure that $(J, \mathscr{P})$ is uniquely defined, we demand $i_c^*\mathscr{P} = \mathcal{O}_J$ for the base point $c \in C$. The sheaf $\mathscr{P}$ is called a Poincaré sheaf for C.

For any curve C the Jacobian can be equipped with a line bundle to form a Picard variety of C, see [11].

Definiton 4.2. The sheaves

$$\mathcal{E}_n = \pi_{J*}(\mathscr{P} \otimes \pi_C^* \xi_n) \text{ and } \mathcal{F}_n = R^1\pi_{J*}(\mathscr{P} \otimes \pi_C^* \xi_n)$$

are called Picard sheaves on J. Here $R^1\pi_{J*}$ denotes the first derived functor of π_{J*} or in other words the first higher directed image.

Remark 4.3. The motivation, why we consider $\mathscr{P} \otimes \pi_C^* \xi_n$ is its pull-back along the morphism $\tilde{\varphi_n}\colon C^{(n)} \times C \to J \times C$ induced by φ_n. For $A = c_1 + \cdots + c_n \in C^{(n)}$ we can consider $\tilde{\varphi_n}^* \mathscr{P}$ on the fibre of A. There it is just $(\tilde{\varphi_n} \circ j_A)^* \mathscr{P} = j_{\varphi_n(A)}^* \mathscr{P} = \mathcal{O}(c_1 + \cdots + c_n - nc)$. So ξ_n is chosen such that the fibre over $c_1 + \cdots + c_n$ is $\mathcal{O}_C(c_1 + \cdots + c_n)$. We will investigate this more thoroughly in section 4.4.

Proposition 4.4. *For each integer n there is an exact sequence*

$$0 \to \mathcal{E}_{n-1} \to \mathcal{E}_n \to \mathcal{O}_J \to \mathcal{F}_{n-1} \to \mathcal{F}_n \to 0. \tag{3}$$

Proof. We take the exact sequence (2) and apply the exact functor $\mathscr{P} \otimes \pi_C^*(-)$ to get a short exact sequence

$$0 \to \mathscr{P} \otimes \pi_C^* \xi_{n-1} \to \mathscr{P} \otimes \pi_C^* \xi_n \to \mathscr{P} \otimes \pi_C^*(\xi_n/\xi_{n-1}) \to 0.$$

Denote $\mathscr{P} \otimes \pi_C^*(\xi_n/\xi_{n-1})$ by $\mathcal{M}$. Now we can consider the long exact sequence we get from the right derived functor of π_{J*}:

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \pi_{J*}(\mathcal{M}) \to \mathcal{F}_{n-1} \to \mathcal{F}_n \to R^1\pi_{J*}(\mathcal{M})$$

The sheaf $\mathcal{M}$ has support $J \times c$ and restricts there to $\mathcal{O}_J$. So first we get $\pi_{J*}(\mathcal{M}) = \mathcal{O}_J$. Next we use, that we have an explicit description for $R^1\pi_{J*}(\mathcal{M})$, namely it is the sheaf associated to $U \mapsto H^1(\pi_J^{-1}(U), \mathcal{M}) \cong H^1(U, \mathcal{O}_J(U))$. So it is the same sheaf as $R^1(id_{j_*})(\mathcal{O}_J)$ which is 0 since id_{J*} is exact. □

To investigate the structure of Picard sheaves further we need a proposition from EGA. For this let $f\colon Y \to X$ be a proper morphism, $\mathscr{F}$ an f-flat $\mathcal{O}_Y$-module and x a point of X. We denote by $\mathscr{F}_x = \mathscr{F} \otimes_{\mathcal{O}_X} k(x)$ the fibres of $\mathscr{F}$ along f. Then:

Proposition 4.5 ([4, III, Prop. 4.6.1])**.** *If we have $H^n(f^{-1}(x), \mathscr{F}_x) = 0$ for some $n \geq 0$, then $R^n f_*(\mathscr{F}) = 0$ in some neighbourhood of x. Furthermore the canonical morphism*

$$R^{n-1} f_*(\mathscr{F})_x \to H^{n-1}(f^{-1}(x), \mathscr{F}_x)$$

is surjective.

The proposition is a bit stronger in EGA, there we have surjectivity for $\mathcal{F}_x$ replaced by $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_x/\mathfrak{m}_x^{p+1})$ but we will only need the case $p = 0$. We will apply this proposition for $\pi_J\colon J \times C \to J$ and $\mathscr{F}$ locally free, so $\mathscr{F}$ will surely be π_J-flat. The fibre of $\mathscr{F}$ at x is $j_x^* \mathscr{F}$ and the fibres of π_J can be identified with C.

Corollary 4.6. (i) *For all $n > 1$ we have $R^n \pi_{J*}(\mathscr{F}) = 0$*

(ii) *The natural morphism $R^1 \pi_{J*}(\mathscr{F})_x \to H^1(\pi_J^{-1}(x), \mathscr{F}_x)$ is surjective for all $x \in J$.*

(iii) *If* $H^1(\pi_J^{-1}(x), \mathscr{F}_x) = 0$*, there is a neighbourhood of* x *in which* $R^1\pi_{J*}(\mathscr{F})$ *is zero and* $R^0\pi_{J*}(\mathscr{F})$ *is locally free.*

(iv) *If* $H^0(\pi_J^{-1}(x), \mathscr{F}_x) = 0$*, there is a neighbourhood of* x *in which* $R^0\pi_{J*}(\mathscr{F})$ *is zero and* $R^1\pi_{J*}(\mathscr{F})$ *is locally free.*

Proof. The first two statements follow from Proposition 4.5 by observing dim $C = 1$ implies $H^n(f^{-1}(x), \mathscr{F}_x) = 0$ for $n > 1$.

To prove (iii) we apply Proposition 4.5 to get a neighbourhood U of x, where $R^1\pi_{J*}(\mathscr{F})$ is zero. Furthermore as the Euler characteristic of the fibres is locally constant we may assume the fibres $\mathscr{F}_{x'}$ for $x' \in U$ all have the same Euler characteristic. Combining this with the vanishing of $h^1(\pi_J^{-1}(x'), \mathscr{F}_{x'})$ implies $h^0(\pi_J^{-1}(x'), \mathscr{F}_{x'})$ is constant on U. Choose a basis for $H^0(\pi_J^{-1}(x), \mathscr{F}_x)$. Since the homomorphism $\pi_{J*}(\mathscr{F})_x \to H^0(\pi_J^{-1}(x), \mathscr{F}_x)$ is surjective, again using Proposition 4.5, we may extend this basis to sections in a neighbourhood $U' \subset U$. These sections remain linearly independent so $R^0\pi_J(\mathscr{F})$ is free in U'.

The proof for (iv) works similar but we have to use (ii) to get the surjectivity of $R^1\pi_{J*}(\mathscr{F})_x \to H^1(\pi_J^{-1}(x), \mathscr{F}_x)$. $\square$

The first statement will not be used later, but it motivates why we only consider $R^0\pi_{J*}$ and $R^1\pi_{J*}$, because those are the only non vanishing higher direct images. By applying the corollary to the situation of Picard sheaves we get:

Proposition 4.7 ([19, Proposition 2/3])**.** *We have that* $\mathcal{E}_n$ *is torsion-free for all* n*. If* $n < 0$ *then* $\mathcal{F}_n$ *is locally free of rank* $g - n + 1$*, if* $n < g$ *then* $\mathcal{E}_n$ *is zero and conversely if* $n > 2g - 2$ *then* $\mathcal{E}_n$ *is locally free of rank* $n - g + 1$ *and* $\mathcal{F}_n$ *is zero.*

Proof. Since the pushforward of a torsion free sheaf along a dominant morphism is torsion free and observing that $\mathscr{P} \otimes \pi_C^*\xi_n$ is torsionbfree as it is a line bundle, we get that $\mathcal{E}_n = \pi_{J*}(\mathscr{P} \otimes \pi_C^*\xi_n)$ is torsion free as well.

For the other statements we notice, that $H^0(\pi_J^{-1}(x), j_x^*\mathscr{P} \otimes \xi_n) = 0$ for $n < 0$ as $j_x^*\mathscr{P} \in \mathrm{Pic}^0(C)$ so $j_x^*\mathscr{P} \otimes \xi_n$ has degree $n < 0$ and hence no global sections. Then Corollary 4.6 gives that $\mathcal{F}_n$ is locally free for $n < 0$. To get the rank we apply Riemann–Roch to the fibre to get $h^1(\pi_J^{-1}(x), j_x^*\mathscr{P} \otimes \xi_n) = -\chi(j_x^*\mathscr{P} \otimes \xi_n) = -(n - g + 1)$. For $n < g$ the morphism $\varphi_n \colon C^{(n)} \to J(C)$ is not surjective. Take any point $x \in J$ not in the image, then the linear system of x is empty, hence $H^0(\pi_J^{-1}(x), j_x^*\mathscr{P} \otimes \xi_n) = 0$. Altogether we get that there are fibres where $\mathcal{E}_n$ vanishes. Since $\mathcal{E}_n$ is torsion free this already implies $\mathcal{E}_n = 0$.

If $n \geq 2g - 1$ we have $H^1(\pi_J^{-1}(x), j_x^*\mathscr{P} \otimes \xi_n) = H^0(C, \omega_\otimes(j_x^*\mathscr{P} \otimes \xi_n)^*) = 0$, where ω_C denotes the canonical line bundle on C. Here we used Serre duality and that there are no global section because of $\deg(\omega_C \otimes (j_x^*\mathscr{P} \otimes \xi_n)^*) = 2g - 2 - n < 0$. Then apply the Corollary 4.6 again. $\square$

Remark 4.8. For $0 \leq n \leq g - 1$ we still have that $\mathcal{F}_n$ is locally free on the set of $x \in J$ with $H^0(\pi_J^{-1}(x), j_x^*\mathscr{P} \otimes \xi_n) = 0$. This is the complement of W^n, as the fibres of φ_n are exactly the linear systems. Furthermore for $g \leq n \leq 2g - 2$ we know that $\mathcal{F}_n$ vanishes at all $x \in J$ for which $H^1(\pi_J^{-1}(x), j_x^*\mathscr{P} \otimes \xi_n) = 0$, which by Serre duality is the complement of $\theta(W^{2g-2-n})$ for $\theta \colon J \to J, \mathcal{L} \mapsto \omega_C \otimes \mathcal{L}^* \otimes \mathcal{O}_C(-(2g-2)c)$.

As special cases we see that $\mathcal{F}_0$ is locally free in the complement of one point which we will use quite frequently. In the other boarder case $\mathcal{F}_{2g-2}$ vanishes in the complement of one point κ. By using (3) one sees, that it has at most rank 1 there. On the other hand there is an epimorphism $\mathcal{F}_{2g-2,\kappa} \to H^1(C,\omega)$, so it has exactly rank 1 at κ. This will be useful for Proposition 4.21 as the corresponding projective fibred variety will consist of one point.

We need to investigate the connection between $\mathcal{E}_r$ and $\mathcal{F}_s$ for high r and low s.

Proposition 4.9. *For $r > 2g-2$ and $s < g$ there is an exact sequence*

$$0 \to \mathcal{E}_r \to \mathcal{M} \to \mathcal{F}_s \to 0$$

where $\mathcal{M}$ is a successive extension of $(r-s)$ copies of $\mathcal{O}_J$. Especially we get for the total Chern classes that $c(\mathcal{E}_r)c(\mathcal{F}_s) = c(\mathcal{M}) = 1$.

Proof. We want to use the exact sequence (2) to get a sequence

$$0 \to \xi_s \to \xi_r \to \mathcal{T} \to 0 \tag{4}$$

where $\mathcal{T}$ is an extension of $(r-s)$ copies of $\mathcal{O}_c$. Then we could apply $\mathscr{P} \otimes \pi_C^*(-)$ and the long exact sequence of $R\pi_*$ to get

$$\mathcal{E}_s \to \mathcal{E}_r \to \mathcal{M} \to \mathcal{F}_s \to \mathcal{F}_r$$

At last we observe that r and s are chosen such that $\mathcal{E}_s = 0 = \mathcal{F}_r$.

So we are left to construct (4). For this we want to show in the general setting, that given a morphism $f\colon A \to B$ and an injective morphism $g\colon B \to C$, $\operatorname{coker}(g \circ f)$ is an extension of $\operatorname{coker}(f)$ by $\operatorname{coker}(g)$. We have the following diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "\alpha"] \arrow[d, "g"] & \operatorname{coker}(f) \arrow[r] \arrow[d, dashed, "\varphi"] & 0 \\
 & A \arrow[r, "g\circ f"] \arrow[d, "f"] & C \arrow[r, "\beta"] \arrow[d, equal] & \operatorname{coker}(g \circ f) \arrow[r] \arrow[d, dashed, "\psi"] & 0 \\
0 \arrow[r] & B \arrow[r, "g"] & C \arrow[r, "\gamma"] & \operatorname{coker}(g) \arrow[r] & 0.
\end{tikzcd}$$

Here φ and ψ are constructed by the universal property of cokernels. It remains to show, that the sequence

$$0 \to \operatorname{coker}(f) \xrightarrow{\varphi} \operatorname{coker}(g \circ f) \xrightarrow{\psi} \operatorname{coker}(g) \to 0$$

is exact. This is a direct diagram chase, where the most involved part is to show, that φ is injective. So take an $x \in \ker(\varphi)$. Then $x = \alpha(b)$ for some $b \in B$. Furthermore $0 = \varphi(\alpha(b)) = \beta(g(b))$ and exactness of the middle row implies $g(b) = g(f(a))$ for some $a \in A$. By injectivity of g we get $b = f(a)$ and therefore $x = \alpha(f(a)) = 0$. $\square$

Next want to show a duality statement for $\mathcal{E}_n$ and $\mathcal{F}_{2g-2-n}$. Define $\mathcal{K} \in \operatorname{Pic}^0(C)$ by $\mathcal{K} = \omega_C \otimes \xi_{2-2g}$ so the canonical line bundle has the form $\omega_C = \mathcal{K} \otimes \xi_{2g-2}$. Consider the automorphism θ of Pic^0 that is given by $\theta(\mathcal{L}) = \mathcal{K} \otimes \mathcal{L}^*$. This also gives an automorphism of J. It extends to $J \times C$ as $\theta \times id_c$ which we shall denote by $\tilde{\theta}$. We are interested in the pullbacks of the Picard sheaves under θ.

Lemma 4.10. *There are isomorphisms*

$$\theta^*(\mathcal{E}_n) \cong \pi_{J*}(\tilde{\theta}^*\mathscr{P} \otimes \pi_C^*\xi_n) \cong \pi_{J*}(\pi_C^*\mathcal{K} \otimes \mathscr{P}^* \otimes \pi_C^*\xi_n)$$
$$\theta^*(\mathcal{F}_n) \cong R^1\pi_{J*}(\tilde{\theta}^*\mathscr{P} \otimes \pi_C^*\xi_n) \cong R^1\pi_{J*}(\pi_C^*\mathcal{K} \otimes \mathscr{P}^* \otimes \pi_C^*\xi_n)$$

Proof. First we notice that since θ is an endomorphism of J we can use flat base change to get $\theta^*\pi_{J*}\mathscr{F} \cong \pi_{J*}(\tilde{\theta}^*\mathscr{F})$ and $\theta^*R^1\pi_{J*}\mathscr{F} \cong R^1\pi_{J*}(\tilde{\theta}^*\mathscr{F})$. We are left to show

$$\tilde{\theta}^*(\mathscr{P} \otimes \pi_C^*\xi_n) = \tilde{\theta}^*\mathscr{P} \otimes \pi_C^*\xi_n = \pi_C^*\mathcal{K} \otimes \mathscr{P}^* \otimes \pi_C^*\xi_n$$

The first equation follows from $\tilde{\theta}^*\pi_C^* = (\pi_C \circ \tilde{\theta})^* = \pi_C^*$. For the second we see that the line bundles $\tilde{\theta}^*\mathscr{P}$ and $\pi_C^*\mathcal{K} \otimes \mathscr{P}^*$ are equal on every fibre over $x \in J$, where both are $\mathcal{K} \otimes \mathcal{L}_x^*$. Here $\mathcal{L}_x$ denotes the line bundle corresponding to x. At last observe that both line bundles are equal on the fibre of $c \in C$ so by the seesaw principle we are done. □

An important consequence of this is, that applying θ^* to (3) yields an exact sequence, the sequence we get when we replaced $\mathscr{P}$ by $\mathscr{P}^* \otimes \pi_C^*\mathcal{K}$ in the proof of (3).

It will be convenient to prove the duality in a slightly more general setting, as we will need this form to construct the isomorphism to the symmetric product. For this consider a variety X and a morphism $h\colon X \to J$. This defines a diagram

$$\begin{array}{ccc} X \times C & \xrightarrow{\tilde{h}} & J \times C \\ \Big\downarrow{\scriptstyle \pi_X} & & \Big\downarrow{\scriptstyle \pi_J} \\ X & \xrightarrow{h} & J. \end{array}$$

On X we can define the sheaves

$$h\mathcal{E}_n = \pi_{X*}(\tilde{h}^*\mathscr{P} \otimes \pi_C^*\xi_n) \quad \text{and} \quad h\theta^*\mathcal{F}_n = (R^1\pi_{X*})(\tilde{h}^*\tilde{\theta}^*\mathscr{P} \otimes \pi_C^*\xi_n).$$

In order to prove the general duality for these sheaves we will need Grothendieck-Verdier duality in the form of [10, Thm. 3.34]. For a proof see [6].

Theorem 4.11. *Let $f\colon Y \to Z$ be a morphism of smooth schemes. Set $\dim(f) = \dim(Y) - \dim(Z)$ and $\omega_f = \omega_Y \otimes f^*\omega_Z^*$. Then for all $F^\bullet \in D^b(Y), E^\bullet \in D^b(Z)$ there exists a functorial isomorphism*

$$Rf_*R\mathcal{H}om(F^\bullet, Lf^*E^\bullet \otimes \omega_f[\dim(f)]) \cong R\mathcal{H}om(Rf_*F^\bullet, E^\bullet). \tag{5}$$

Proposition 4.12. *There is an isomorphism $\lambda_n\colon h\theta^*\mathcal{F}_{2g-2-n} \xrightarrow{\sim} h\mathcal{E}_n^*$ for all $n > 2g-2$.*

Proof. We want to apply Theorem 4.11 to $\pi_X\colon X \times C \to X$, $E^\bullet$ the complex consisting of $\mathcal{O}_X$ in degree 0 and $F^\bullet$ the complex consisting of $\tilde{h}^*\mathscr{P} \otimes \pi_C^*\xi_n$ in degree 0. Abbreviate $\tilde{h}^*\mathscr{P} \otimes \pi_C^*\xi_n$ to $\mathscr{F}$. Then $L\pi_X^*E^\bullet = \mathcal{O}_{X\times C}$, $\dim(\pi_X) = 1$ and we have to calculate ω_{π_X}. For this use $\omega_{X\times C} = \pi_X^*\omega_X \times \pi_C^*\omega_C$, thus

$$\omega_{\pi_X} = \omega_{X\times C} \otimes \pi_X^*\omega_X^* = \pi_C^*\omega_C = \pi_C^*(\mathcal{K} \otimes \xi_{2g-2}).$$

In total (5) reduces to

$$R\pi_{X*}R\mathcal{H}om(\mathscr{F}, \omega_{\pi_X}[1]) \cong R\mathcal{H}om(R\pi_{X*}(\mathscr{F}), \mathcal{O}_X).$$

Then the left hand side in degree 0 is

$$R^1\pi_{X*}(\mathscr{F}^* \otimes \omega_{\pi_X}) = R^1\pi_{X*}(\pi_C^*\mathcal{K} \otimes \tilde{h}^*\mathscr{P}^* \otimes \pi_C^*\xi_{2g-2-n}) = h\theta^*\mathcal{F}_{2g-2-n}.$$

Since $n > 2g-2$ we know that $R\pi_{X*}(\mathscr{F}) = h\mathcal{E}_n$ is locally free so $R\mathcal{H}om$ can be computed by $\mathcal{H}om$. Hence the right hand side in degree 0 is just $R^0\pi_{X*}(\mathscr{F})^*$. Altogether we have

$$h\theta^*\mathcal{F}_{2g-2-n} = R^1\pi_{X*}(\pi_C^*\mathcal{K} \otimes \tilde{h}^*\mathscr{P}^* \otimes \pi_C^*\xi_{2g-2-n}) \cong R^0\pi_{X*}(\tilde{h}^*\mathscr{P} \otimes \xi_{2g-2})^* = h\mathcal{E}_n^*.$$

□

Remark 4.13. For $n = 2g-2$ there is an analogous isomorphism in the complement of one point. To prove this we use that $R\pi_{X*}(\mathscr{F})$ is locally free in the complement of one point by the considerations of Remark 4.8.

Remark 4.14. It is possible to show an isomorphism $h\mathcal{E}_n \cong h\theta^*\mathcal{F}^*_{2g-2-n}$ for all n when applying Serre duality to the fibres. See [19, Prop. 5] for the idea, but it is quite cumbersome to get the details right.

Theorem 4.15 ([19, Thm. 1])**.** *When $n \geq 2g-1$ then there are isomorphisms*

$$\theta^*\mathcal{F}_{2g-2-n} \cong \mathcal{E}_n^* \quad \text{and} \quad \mathcal{E}_n \cong \theta^*\mathcal{F}^*_{2g-2-n}.$$

Proof. We get the first isomorphism directly by applying Proposition 4.12 for $X = J$ and $h = id_J$. For the second statement we get a dual isomorphism $\lambda_n^* : \mathcal{E}_n^{**} \to \theta^*\mathcal{F}^*_{2g-2-n}$ which we compose with the epimorphism $\mathcal{E}_n \to \mathcal{E}_n^{**}$. The latter is an isomorphism when $\mathcal{E}_n$ is locally free, which by Proposition 4.7 is the case for $n \geq 2g-1$. □

Applying this to Proposition 4.9 allows us to relate the Chern classes of $\mathcal{E}_n$ and $\theta^*\mathcal{E}_n^*$.

Corollary 4.16. *For $r > 2g-2$ and $s > 2g-2$ there is an exact sequence*

$$0 \to \mathcal{E}_r \to \mathcal{M} \to \theta^*\mathcal{E}_s^* \to 0$$

where $\mathfrak{M}$ is a successive extension of $(r+s-2g+2)$ copies of $\mathcal{O}_J$. In particular we get for the total Chern classes that $c(\mathcal{E}_r)c(\theta^*\mathcal{E}_s) = c(\mathcal{M}) = 1$.

4.2 Varieties associated to Picard sheaves and their Chern classes

Given a coherent sheaf E on J we will consider the projective fibred variety associated to E defined as $\mathbb{P}(E) := \mathbf{Proj}(\mathrm{Sym}^\bullet(E))$. For E locally free this is the associated projective bundle. First recall two general statements for the projective fibred variety, for proofs we refer to EGA.

Lemma 4.17 ([4, II, 4.1.2])**.** *Let E and F be coherent sheaves and $u\colon E \to F$ an epimorphism of sheaves. Then it induces a closed immersion $q\colon \mathbb{P}(F) \to \mathbb{P}(E)$ and $q^*\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}(F)}(1)$.*

Lemma 4.18 ([4, II, 4.1.3])**.** *Let $h\colon X \to J$ be a morphism, then*

$$\mathbb{P}(h^*E) \cong \mathbb{P}(E) \times_J X.$$

Furthermore we will need a more concrete lemma which will be used to calculate the Segre classes of $\mathcal{F}_n$.

Lemma 4.19. *If $u\colon E \to F$ is an epimorphism of $\mathcal{O}_J$-modules with ker $u = \mathcal{O}_J$, then the immersion $q\colon \mathbb{P}(F) \to \mathbb{P}(E)$ is represented by the sheaf of ideals $\mathcal{O}_{\mathbb{P}(E)}(-1)$ dual to $\mathcal{O}_{\mathbb{P}(E)}(1)$.*

Proof. The hypothesis gives us a short exact sequence

$$0 \to \mathcal{O}_J \xrightarrow{f} E \xrightarrow{u} F \to 0$$

From this we get another exact sequence

$$0 \to \mathrm{Sym}^\bullet(E)[-1] \xrightarrow{\bar{f}} \mathrm{Sym}^\bullet(E) \xrightarrow{\bar{u}} \mathrm{Sym}^\bullet(F) \to 0.$$

Here $\mathrm{Sym}^\bullet(E)[-1]$ denotes the same sheaf of rings as $\mathrm{Sym}^\bullet(E)$ but with grading shifted by one. On any open set we define $\bar{f}(e_1 \otimes \cdots \otimes e_{n-1}) = e_1 \otimes \cdots \otimes e_{n-1} \otimes f(1)$ respectively $\bar{u}(e_1 \otimes \cdots \otimes e_n) = u(e_1) \otimes \cdots \otimes u(e_n)$ and extend linearly. Both definitions are symmetric in the e_i and hence descend to well-defined morphisms of the symmetric algebras.

Now [4, II 3.6.2] states, that the closed immersion corresponding to $\bar{u}$ is exactly corresponding to $\widetilde{ker}(\bar{u})$. Observing $\widetilde{\mathrm{Sym}^\bullet(E)}[-1] = \mathcal{O}_{\mathbb{P}(E)}(-1)$ yields the claim.

□

As an next step we want to consider the varieties associated to the Picard sheaves.

Definiton 4.20. We define $C_n = \mathbb{P}(\theta^*\mathcal{F}_{2g-2-n})$. Write $\pi_n\colon C_n \to J$ for the projection and $\mathcal{O}_n(1), \mathcal{O}_n(-1)$ for the tautological sheaf and its dual. By Lemma 4.17 we get a closed immersion $C_{n-1} \to C_n$ from applying θ^* to (3), which we denote by q_n.

Let us investigate the structure of the C_n more closely.

Proposition 4.21. *For $n < 0$, C_n is empty and for $n \geq 0$ we have dim $C_n = n$. Moreover for $n \geq 2g-1$, C_n is a projective fibre bundle with fibre $\mathbb{P}^{n-g}$.*

Proof. By Proposition 4.7 we know $\theta^*\mathcal{F}_n$ is zero for $n > 2g-2$ and locally free for $n < 0$. This directly gives that C_n is empty for $n < 0$ and that it is a projective bundle for $n \geq 2n-1$. Furthermore $\mathcal{F}_{2g-2}$ is supported at a single point so C_0 consists of a single point. For $n = 0$ and $n \geq 2g-1$ we thus already have that C_n is irreducible of dimension n. Let $j\colon X \to J$ be the inclusion of a subvariety and consider the sequence

$$j^*\mathcal{O}_J \xrightarrow{\varphi} j^*\theta^*\mathcal{F}_{2g-2-n} \to j^*\theta^*\mathcal{F}_{2g-1-n} \to 0$$

which we get from the sequence (3) by right exactness of j^* and θ^*. First we inductively show, that dim $C_n \leq n$. The induction start $n = 0$ is already done. So assume C_n had a component of dimension bigger than n and let X be its support. By Lemma 4.18 we know $\mathbb{P}(j^*\theta^*\mathcal{F}_{2g-2-n})$ is $\pi_n^{-1}(X)$. The fibre of $\mathbb{P}(j^*\theta^*\mathcal{F}_{2g-2-n})$ at $x \in X$ is just the projective space corresponding to $j^*\theta^*\mathcal{F}_{2g-2-n} \otimes k(x)$. The above sequence tells us therefore, that the dimension of the fibres of $\pi_n^{-1}(X)$ have at most one bigger dimension than the fibres of $\pi_{n-1}^{-1}(X)$. Therefore C_{n-1} would have a component of dimension at least n, contradicting the induction assumption.

On the other hand we can use the same argument from above. Then dim $C_{2g-1} = 2g-1$ yields dim $C_n \geq n$ so altogether we get the desired dimension. □

The idea of the proof can be extended, which will enable us to apply Proposition 4.19:

Proposition 4.22. *If $X = \operatorname{supp} \theta^* \mathcal{F}_{2g-2-n}$ and $j\colon X \to J$ is the inclusion, then*

$$\varphi\colon j^*\mathcal{O}_J \to j^*\theta^*\mathcal{F}_{2g-2-n}$$

given by applying $j^*\theta^*$ to (3), is a monomorphism.

Proof. We want to show that otherwise C_{n-1} had a component of dimension n. For this assume that φ were not a monomorphism, then we have an ideal sheaf $\mathcal{I}$ on Z with

$$0 \to \mathcal{I} \to \mathcal{O}_X \overset{\varphi}{\to} j^*\theta^*\mathcal{F}_{2g-2-n} \to j^*\theta^*.\mathcal{F}_{2g-1-n} \to 0$$

Ideal sheaves are torsion free, so $\operatorname{supp}(\mathcal{I}) = X$. Further $\mathcal{I} \otimes k(x)$ does not vanish so by dimension arguments we know that the fibres of $\pi_n^{-1}(X)$ and π_{n-1}^{-1} have the same dimension, so C_{n-1} has a component of dimension n. This is the desired contradiction. □

We are ready to prove an important theorem over the C_n. It will allow us to calculate the Segre and Chern classes of the Picard sheaves.

Theorem 4.23. *The closed immersion $q_n\colon C_{n-1} \to C_n$ is associated to the sheaf of ideals $\mathcal{O}_n(-1)$. If $\alpha \in \mathrm{CH}^1(C_n)$ is the class of C_{n-1}, then α^r is the class of C_{n-r}.*

Proof. By Proposition 4.22 we may apply Lemma 4.19. Thus $q_n\colon C_{n-1} \to C_n$ is associated to the sheaf of ideals $\mathcal{O}_n(-1)$. Now $q_n^*\mathcal{O}_n(1) = \mathcal{O}_{n-1}(1)$ implies $q_n^*\mathcal{O}_n(-1) = \mathcal{O}_{n-1}(-1)$. So we can express α^2 as $q_n^*\alpha \in \mathrm{CH}^1(C_{n-1})$ which represents the subvariety C_{n-2} of C_{n-1}. Continuing this inductively yields the theorem. □

In order to calculate the Chern classes of the Picard-bundles we will also need their Segre classes and how they are connected with the Chern classes. We will use the definition given in [2].

Definiton 4.24. Let X be a smooth projective variety, $\mathcal{E}$ be a vector bundle of rank r on X and $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ be its projectivization. Set $\zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. The i-th Segre class of $\mathcal{E}$ is the class

$$s_i(\mathcal{E}) = \pi_*(\zeta^{r-i+1}) \in \mathrm{CH}^i(X)$$

and the (total) Segre class of $\mathcal{E}$ is the sum

$$s(\mathcal{E}) = 1 + s_1(\mathcal{E}) + s_2(\mathcal{E}) + \cdots$$

Proposition 4.25. *The Segre class and the Chern class are reciprocals of each other:*

$$c(\mathcal{E})s(\mathcal{E}) = 1$$

Proof. See [2, Prop. 10.3]. □

Finally we have gathered all statements we will need in order to calculate the Chern classes. Let $W_n = \pi_n(C_n)$ and $U_n = \theta(W_n)$ for $0 \le n \le g$ and denote the classes of U_{g-i}, W_{g-i} in the Chow ring of J by u_i respectively w_i. We will later show $C^{(n)} \cong C_n$ as projective bundles, so W_n will coincide with W^n.

Theorem 4.26. *We have $c_i(\mathcal{E}_n) = (-1)^i u_i$ for $n > 2g-2$ and $c_i(\mathcal{F}_n) = w_i$ for $n < g$.*

Proof. Let $n > 2g-2$ and abbreviate $\theta^*\mathcal{F}_{2g-2-n}$ to $\mathscr{F}_n$. Denote the Chern class of $\mathscr{F}_n$ by $\sum d_i$. We want to calculate its Segre class. By Proposition 4.7 we know that $\mathcal{F}_{2g-2-n}$ is of rank $n-g+1$, so we have $s_i(\mathscr{F}_n) = \pi_*(\zeta^{(n-g+1)-1+i}) = \pi_*(\zeta^{n-g+i})$, where ζ is $c_1(\mathcal{O}_{\mathbb{P}(\mathscr{F}_n)}(1)) = c_1(\mathcal{O}_{C_n}(1)) = (-\alpha)$. Thus $s_i(\mathscr{F}_n) = \pi_*((-\alpha)^{n-g+i})$. By Theorem 4.23 we know that α^{n-g+i} represents the class of $C_{n-(n-g+i)} = C_{g-i}$, so $s_i(\mathscr{F}_n) = \pi_*((-\alpha)^{n-g+i})$ is the class of $(-1)^{n-g+i}\pi(C_{g-i}) = (-1)^{n-g+i}w_i$.

Applying Proposition 4.25 for $\mathscr{F}_n$ gives the equation

$$\sum d_i \sum (-1)^i w_i = (-1)^{n-g}.$$

Now if we consider the dual of $\mathscr{F}_n$ using $c_i(\mathscr{F}_n^*) = (-1)^i c_i(\mathscr{F}_n)$ and $s_i(\mathscr{F}_n^*) = (-1)^i s_i(\mathscr{F}_n)$ we get

$$\sum (-1)^i d_i \sum w_i = (-1)^{n-g}.$$

Theorem 4.15 gives us $\mathcal{E}_n \cong \theta^*\mathcal{F}^*_{2g-2-n}$ and hence $c(\mathcal{E}_n) = \sum (-1)^i d_i$. By Corollary 4.16 we have $c(\theta^*\mathcal{E}_n^*) = c(\mathcal{E}_n)^{-1} = (-1)^{n-g}\sum w_i$ for $n > 2g-2$. Using $c_i(\theta^*\mathcal{E}_n^*) = \theta^*((-1)^i c_i(\mathcal{E}_n))$ we conclude

$$c(\mathcal{E}_n) = \sum (-1)^i u_i \quad \text{for } n > 2g-2.$$

In a similar way Proposition 4.9 gives us $c(\mathcal{F}_s) = c(\mathcal{E}_t)^{-1}$ for $s < g$ and $t > 2g-2$, so we conclude

$$c(\mathcal{F}_n) = \sum w_i \quad \text{for } n < g.$$

□

4.3 Structure of fibres

We want to investigate $\pi_n^{-1}(X)$ for subvarieties X of J. In particular we are interested the structure of the fibres of π_n. In order to do so we have to relate $h^*\theta^*\mathcal{F}_n$ and $h\theta^*\mathcal{F}_n$ to each other. Recall that

$$h^*\theta^*\mathcal{F}_n = h^*\theta^* R^1\pi_{J*}(\mathscr{P} \otimes \pi_C^*\xi_n) \qquad h\theta^*\mathcal{F}_n = (R^1\pi_{X*})(\tilde{h}^*\tilde{\theta}^*\mathscr{P} \otimes \pi_C^*\xi_n)$$

We will further need a theorem to connect the fibres of Rf_* and the fibrewise cohomology:

Theorem 4.27. *Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes and assume $y \mapsto \dim_{k(y)} H^p(X_y, \mathscr{F}_y)$ is constant, then*

$$R^{p-1}f_*\mathscr{F} \otimes k(y) \to H^{p-1}(X_y, \mathscr{F}_y)$$

is an isomorphism for all $y \in Y$.

For a proof see [18, 5 Cor. 2].

Proposition 4.28. *Let $h\colon X \to J$ be a morphism, then there is an isomorphism*

$$u_n\colon h^*\theta^*\mathcal{F} \xrightarrow{\sim} h\theta^*\mathcal{F}_n$$

for all n.

Proof. Recall that we are working in the following setting:

$$\begin{tikzcd} X \times C \arrow[r, "\tilde{\theta}\circ\tilde{h}"] \arrow[d, "\pi_X"] & J \times C \arrow[d, "\pi_J"] \\ X \arrow[r, "\theta\circ h"] & J \end{tikzcd}$$

which can be interpreted as a fibre product diagram. Abbreviate $\mathscr{P} \otimes \pi_C^*\xi_n$ by $\mathscr{F}$. Then by [5, III 9.3.1] we get a natural map

$$h^*\theta^* R^1\pi_{J*}(\mathscr{F}) \to (R^1\pi_{X*})(\tilde{h}^*\tilde{\theta}^*\mathscr{F}). \tag{6}$$

We have to show, this map is an isomorphism. First we claim, that for all $x \in X$

$$R^1\pi_X(\tilde{h}^*\tilde{\theta}^*\mathscr{F}) \otimes k(x) \cong H^1(C, j_x^*\tilde{h}^*\tilde{\theta}^*\mathscr{F}).$$

To see this observe, that $\pi_X\colon X \times C \to X$ is proper by base change, as C is smooth and projective hence proper. Applying Theorem 4.27, using $H^2(C, \mathscr{F}_x) = 0$ as C is one dimensional, establishes the claim.

For any point $x \in X$ we thus have

$$R^1\pi_X(\tilde{h}^*\tilde{\theta}^*\mathscr{F}) \otimes k(x) \cong H^1(C, j_x^*\tilde{h}^*\tilde{\theta}^*\mathscr{F}) \cong H^1(C, j_{\theta(h(x))}^*\mathscr{F}) \cong R^1\pi_J\mathscr{F} \otimes k(\theta(h(x))).$$

Here the last isomorphism is shown the same way as above. So in total (6) induces isomorphism on all fibres and thereby is an isomorphism. □

Combining this with Lemma 4.18 yields:

Proposition 4.29. *Let X be a subvariety of J and $h\colon X \to J$ the corresponding inclusion. Then $\pi_n^{-1}(X) \cong \mathbb{P}(h\theta^*\mathcal{F}_{2g-2-n})$ and the closed immersion $\pi_{n-1}^{-1}(X) \to \pi_n^{-1}(X)$ is induced by $h\theta^*\mathcal{F}_{2g-2-n} \to h\theta^*\mathcal{F}_{2g-1-n}$.*

Proof. We have $\pi_n^{-1}(X) = C_n \times_J X \overset{4.18}{=} \mathbb{P}(h^*\theta^*\mathcal{F}_{2g-2-n}) \overset{4.28}{=} \mathbb{P}(h\theta^*\mathcal{F}_{2g-2-n})$. We also get $\pi_{n-1}^{-1}(X) = \mathbb{P}(h\theta^*\mathcal{F}_{2g-1-n})$. □

Applying this to X a closed point x of J furthermore gives us:

Corollary 4.30. *For a closed point $x \in J$, $\pi_n^{-1}(x)$ is isomorphic to the projective space associated to $H^0(C, j_x^*\mathscr{P} \otimes \xi_n)$ and $\pi_{n-1}^{-1}(x)$ is the subspace corresponding to sections vanishing at c.*

Proof. In this case the map is $h\colon \{x\} \to J$ with $\tilde{h} = j_x\colon \{x\} \times C \to J \times C$. We have $h\theta^*\mathcal{F}^*_{2g-2-n} \cong h\mathcal{E}_n$ by Theorem 4.15. Now by breaking down the definitions we observe $h\mathcal{E}_n = \pi_{\{x\}_*}(j_x^*\mathscr{P} \otimes \xi_n) \cong H^0(C, j_x^*\mathscr{P} \otimes \pi_C^*\xi_n)$, which establishes the first claim. For the second part we have to investigate the map $h\theta^*\mathcal{F}_{2g-2-n} \to h\theta^*\mathcal{F}_{2g-1-n}$. This comes from $H^0(C, j_x^*\mathscr{P} \otimes \xi_{n-1}) \to H^0(C, j_x^*\mathscr{P} \otimes \xi_n)$ with image those sections, vanishing at c. □

4.4 Application to the symmetric product

In order to apply our calculations of Chern classes we have to relate C_n and $C^{(n)}$. This section is therefore dedicated to the construction of an isomorphisms $r_n \colon C^{(n)} \to C_n$ over J. By [5, II.7.12] in order to give a map $r_n \colon C^{(n)} \to \mathbb{P}(\theta^* \mathcal{F}_{2g-2-n})$ over J it suffices to give an invertible sheaf $\mathcal{L}$ on $C^{(n)}$ and an epimorphism of sheaves $\varphi_n^*(\theta^* \mathcal{F}_{2g-2-n}) \to \mathcal{L}$. Recall that φ_n is the canonical map corresponding to some basepoint $c \in C$ given by $c_1 + \cdots + c_n \mapsto \mathcal{O}(c_1 + \cdots + c_n - nc)$. So the first step is to construct an appropriate invertible sheaf and an epimorphism.

Consider on $C^{(n)}$ the divisor X consisting of points $c_1 + \cdots + c_{n-1} + c$ containing the base point and on $C^{(n)} \times C$ the divisor X' consisting of points $(c_1 + \cdots + c_n, c_1)$ where the second coordinate is contained in the first. Denote the corresponding line bundles on $C^{(n)}$ respectively $C^{(n)} \times C$ by $\mathcal{L}_n$ respectively $\mathcal{L}_n'$. Moreover we extend φ_n to a map $\tilde{\varphi}_n = \varphi_n \times id_C \colon C^{(n)} \times C \to C^{(n)} \times C$:

$$\begin{array}{ccc} C^{(n)} \times C & \xrightarrow{\tilde{\varphi}_n} & J \times C \\ \Big\downarrow{\scriptstyle \pi_{C^{(n)}}} & & \Big\downarrow{\scriptstyle \pi_J} \\ C^{(n)} & \xrightarrow{\varphi_n} & J. \end{array}$$

First we want to investigate, how the sheaves $\mathcal{L}_n$ and $\mathcal{L}_n'$ are related.

Proposition 4.31. $\mathcal{L}_n' = \pi_{C^{(n)}}{}^* \mathcal{L}_n \otimes \tilde{\varphi}_n^*(\mathscr{P} \otimes \pi_C^* \xi_n)$, *where* $\mathscr{P}$ *is a Poincare sheaf for* C *in the sense of Definition 4.1.*

Proof. We want to apply the seesaw principle, using that C is a projective, hence complete variety. Consider both side on the fibre of $\pi_{C^{(n)}}$ at $A = c_1 + \cdots + c_n$. We have to calculate its pullback under $j_A \colon C \to C^{(n)} \times C$. We do it for each factor separately. First

$$j_A^* \tilde{\varphi}_n^* \pi_C^* \xi_n = (\pi_C \circ \tilde{\varphi}_n \circ j_A)^* \xi_n = id_C^* \xi_n = \mathcal{O}(nc)$$

$$j_A^* \pi_{C^{(n)}}^* \mathcal{L}_n^* = (\pi_{C^{(n)}} \circ j_A)^* \mathcal{L}_n^* = const_A^* \mathcal{L}_n^* = \mathcal{O}_C.$$

$$j_A^* \tilde{\varphi_n}^* \mathscr{P} = (\tilde{\varphi_n} \circ j_A)^* \mathscr{P} = j_{\mathcal{O}(A-nc)}^* \mathscr{P} = \mathcal{O}(c_1 + \cdots + c_n - nc).$$

In the last step we used the universal property of $\mathscr{P}$. Finally for the fibres of $\mathcal{L}_n'$ we have to consider the intersection of X' with $\{A\} \times C$. By definition of X' this intersection consists exactly of the points in A so $j_A^* \mathcal{L}_n' = \mathcal{O}(c_1 + \cdots + c_n)$. In total both sides are $\mathcal{O}_C(c_1 + \cdots + c_n)$ on the fibre.

To complete the proof we must in addition find a point of C over which the fibres of the line bundles agree. The canonical choice is to try the basepoint c. Then for the inclusion $i_c \colon C^{(n)} \to C^{(n)} \times C$ we have:

$$i_c^* \tilde{\varphi}_n^* \pi_C^* \xi_n^* = (\pi_C \circ \tilde{\varphi}_n \circ i_c)^* \xi_n^* = const_c^* \xi_n^* = \mathcal{O}_{C^{(n)}}$$

$$i_c^* \pi_{C^{(n)}}^* \mathcal{L}_n = (\pi_{C^{(n)}} \circ i_c)^* \mathcal{L}_n = id_{C^{(n)}}^* \mathcal{L}_n = \mathcal{L}_n = \mathcal{O}_{C^{(n)}}(X)$$

$$i_c^* \tilde{\varphi_n}^* \mathscr{P} = (\tilde{\varphi_n} \circ i_c)^* \mathscr{P} = (i_c \circ \varphi_n)^* = \varphi_n^* i_c^* \mathscr{P} = \varphi_n^* \mathcal{O}_J = \mathcal{O}_{C^{(n)}}.$$

Lastly we observe, that $\mathcal{L}'$ on the fibre of c corresponds to $X' \cap C^{(n)} \times c$ which is $X \times c$, hence $\mathcal{L}'_{|C^{(n)} \times \{c\}} = \mathcal{O}_{C^{(n)}}(X)$. So in total both line bundles are $\mathcal{O}_{C^{(n)}}(X)$ at the fibre of the basepoint, finishing the proof. $\square$

With this we can construct an epimorphism $\varphi_n^*(\theta^*\mathcal{F}_{2g-2-n}) \to \mathcal{L}_n$. As $\mathcal{L}_n$ is invertible, it is enough to find a nowhere vanishing section of $(\varphi_n^*(\theta^*\mathcal{F}_{2g-2-n}))^* \otimes \mathcal{L}_n$. Consider the above constructed subvariety X' of $C^{(n)} \times C$. Let $s \in \mathcal{O}(X')$ be a section representing X' in its linear system, then s vanishes exactly at the points of X'. By the previous calculations s is a section of $\pi_{C^{(n)}}^*\mathcal{L}_n \otimes \tilde{\varphi}_n^*(\mathscr{P} \otimes \pi_C^*\xi_n)$. If its corresponding section ${\pi_{C^{(n)}}}_*(s)$ of the sheaf

$$ {\pi_{C^{(n)}}}_*(\pi_{C^{(n)}}^*\mathcal{L}_n \otimes \tilde{\varphi}_n^*(\mathscr{P} \otimes \pi_C^*\xi_n)) = \mathcal{L}_n \otimes {\pi_{C^{(n)}}}_*(\tilde{\varphi}_n^*\mathscr{P} \otimes \pi_C^*\xi_n) = \mathcal{L}_n \otimes \tilde{\varphi}\mathcal{E}_n $$

is zero at $A = c_1 + \cdots + c_n \in C^{(n)}$ then X' contains $\pi_n^{-1}(A)$. This is not possible since $\pi_{C^{(n)}}^{-1}(c_1 + \cdots + c_n)$ consists of all tuples $(c_1 + \cdots + c_n, c')$ with $c' \in C$ arbitrary, while X' only contains those where c' appears in $c_1 + \cdots + c_n$.

We have $\tilde{\varphi}_n\mathcal{E}_n \cong (\varphi_n^*\theta^*\mathcal{F}_{2g-2-n})^*$ by Proposition 4.12 and 4.28, so ${\pi_{C^{(n)}}}_*(s)$ gives the desired nowhere vanishing section. Altogether we get a morphism $r_n \colon C^{(n)} \to C_n$ such that $\pi_{C^{(n)}} \circ r_n = \varphi_n$ and $r_n^*\mathcal{O}_n(1) = \mathcal{L}_n$.

Theorem 4.32. *The morphism r_n induces an isomorphism between C_n and the symmetric product $C^{(n)}$ for $n \geq 2g-1$.*

Proof. We want to track, how r_n acts on a fibre $\varphi_n^{-1}(x)$ for $x \in J$. For this let P denote the projective space corresponding to $H^0(C, j_x^*\mathscr{P} \otimes \xi_n)$. Then by Corollary 4.30 we know P is isomorphic to $\pi_n(x)$. By tracing the effect of r_n through the last propositions [19] claims, that r_n maps $c_1 + \cdots + c_n$ to the section of $j_x^*\mathscr{P} \otimes \xi_n$, vanishing exactly at $c_1, \ldots, c_n$. So it is the identification of $\varphi_n^{-1}(x)$ with the complete linear system of x. Be aware that the author did not manage to check the details. In the following we will assume, that r_n has this form on the fibre. Especially r_n is fibre wise an isomorphism. For $n \geq 2g-1$ we know that $C^{(n)}$ is a projective bundle so r_n is an isomorphism. □

As always the result holds for $n = 2g-2$ over J without one point.

Remark 4.33. By combining Theorem 4.32 with Theorem 4.26 we get the main result of this preparatory section, namely that for $n \geq 2g-1$ we can express $C^{(n)}$ as $\mathbb{P}(E)$ for a rank $n-g+1$ vector bundle E on J with Chern classes $c_i(E)$ exactly the class of $\theta(W^{g-i})$ in the Chow ring of J. As mentioned at the beginning of this section, there are other approaches leading to the same result. The advantage of this approach is, that the isomorphism $C^{(n)} \cong C_n$ extends to smaller n where both are only projective fibred varieties. We kind of used this for the case $n = 2g-2$. Moreover one can apply the machinery of Fourier–Mukai transformations to the Picard sheaves. The latter is done by Mukai in [17, §4, §5].

5 The case $d \geq 2g-2$

Assume $g > 2$ and $d \geq 2g-1$. We can return to the proof of Theorem 1.1.

By considerations of the Picard sheaves we have $C^{(d)} \cong \mathbb{P}(E_C), D^{(d)} \cong \mathbb{P}(E_D)$ for rank $d-g+1$ vector bundles E_C, E_D on J. By Theorem 4.26 we have $c_j(E_C) \cong [\theta(W^{g-j})]$ respectively $c_j(E_D) \cong [\theta(V^{g-j})]$ for $1 \leq j \leq g-1$. To relate E_C and E_D we need the following result:

Proposition 5.1 ([2, Cor 9.5]). *Let E and F be vector bundles such that $\mathbb{P}(E) \cong \mathbb{P}(F)$. Then there is a line bundle $\mathcal{L}$ on J with $E \cong F \otimes \mathcal{L}$.*

For the following calculations we will need a result about the behaviour of Chern classes under tensoring with line bundles:

Proposition 5.2 ([2, Prop 5.17]). *Let $\mathcal{E}$ be a rank r vector bundle and $\mathcal{L}$ be a line bundle, then*

$$c_k(\mathcal{E} \otimes \mathcal{L}) = \sum_{l=0}^{k} \binom{r-l}{k-l} c_1(\mathcal{L})^{k-l} c_l(\mathcal{E}) \tag{7}$$

Denote $[\theta(W^{g-1})]$ by α_C and $[\theta(V^{g-1})]$ by α_D. Poincaré's formula states that for $1 \leq j \leq g-1$ we have $[W^{g-j}] = [W^{g-1}]^j/j!$ and $[V^{g-j}] = [V^{g-1}]^j/j!$. Thus we also deduce $[\theta(W^{g-j})] = \alpha_C^j/j!$ and $[\theta(V^{g-j})] = \alpha_D^j/j!$. We claim, that we already get $\alpha_C^{g-1} = \alpha_D^{g-1}$. Let r denote the rank of E_C and E_D. Then since $E_C \cong E_D \otimes \mathcal{L}$ we get by the case $\mathcal{E} = E_D, k = 1$ of Proposition 5.2 that $c_1(E_C) \cong c_1(E_D) + rc_1(\mathcal{L})$, so $c_1(\mathcal{L}) = (\alpha_C - \alpha_D)/r$. Applying Proposition 5.2 again for $k = 2$ we deduce:

$$c_2(E_C) = \binom{r}{2} c_1(\mathcal{L})^2 c_0(E_D) + \binom{r-1}{1} c_1(\mathcal{L}) c_1(E_D) + \binom{r-2}{0} c_2(E_D).$$

Plugging in $c_2(E_C) = \alpha_C^2/2$, $c_2(E_D) = \alpha_D^2/2$ and $c_1(\mathcal{L}) = (\alpha_C - \alpha_D)/r$ leads to

$$\frac{\alpha_C^2}{2} = \frac{r(r-1)}{2} \frac{(\alpha_C - \alpha_D)^2}{r^2} + (r-1)\frac{\alpha_C - \alpha_D}{r}\alpha_D + \frac{\alpha_D^2}{2} = (r-1)\frac{(\alpha_C + \alpha_D)(\alpha_C - \alpha_D)}{2r} + \frac{\alpha_D^2}{2}.$$

Which can be rearrange to

$$\frac{1}{2r}(\alpha_C^2 - \alpha_D^2) = 0.$$

This gives $\alpha_C^2 = \alpha_D^2$. For $g = 3$ we are already done, for higher g we have to consider the third Chern class as well. So we plug in $k = 3$:

$$c_3(E_C) = \binom{r}{3} c_1(\mathcal{L})^3 c_0(E_D) + \binom{r-1}{2} c_1(\mathcal{L})^2 c_1(E_D) + \binom{r-2}{1} c_1(\mathcal{L}) c_2(E_D) + c_3(E_D).$$

Use again our calculation of $c_3(E_i) = \alpha_i^3/3!$ and $c_1(\mathcal{L}) = (\alpha_C - \alpha_D)/r$, then $\frac{\alpha_C^3}{6}$ equals

$$\frac{r(r-1)(r-2)}{6} \frac{(\alpha_C - \alpha_D)^3}{r^3} + \frac{(r-1)(r-2)}{2} \frac{(\alpha_C - \alpha_D)^2 \alpha_D}{r^2} + (r-2)\frac{\alpha_C - \alpha_D}{r} \frac{\alpha_D^2}{2} + \frac{\alpha_D^3}{6}.$$

Collect the different α terms to get

$$\begin{aligned} \frac{\alpha_C^3}{6} = \frac{(r-1)(r-2)}{6r^2}\alpha_C^3 + \frac{-3(r-1)(r-2) + 3(r-1)(r-2)}{6r^2}\alpha_C^2\alpha_D \\ + \frac{3(r-1)(r-2) - 6(r-1)(r-2) + 3r(r-2)}{6r^2}\alpha_C\alpha_D^2 \\ + \frac{-(r-1)(r-2) + 3(r-1)(r-2) - 3r(r-2) + r^2}{6r^2}\alpha_D^3. \end{aligned}$$

Reducing and expanding with $6r^2$ yields

$$0 = (2-3r)\alpha_C^3 + 3(r-2)\alpha_C\alpha_D^2 + 4\alpha_D^3.$$

As a last step we have to use the proven equality $\alpha_C^2 = \alpha_D^2$ to get $0 = 4(\alpha_C^3 - \alpha_D^3)$ which gives $\alpha_C^3 = \alpha_D^3$.

These two equalities are sufficient as we may write $g-1 = 2a+3b$ for some natural numbers a, b. Then $\alpha_C^{g-1} = (\alpha_C^2)^a(\alpha_C^3)^b = (\alpha_D^2)^a(\alpha_D^3)^b = \alpha_D^{g-1}$. One should note, that these calculations fails in the genus two case as we can not directly show $\alpha_C = \alpha_D$.

To finish the proof, we will need Matsusaka's criterion for the Jacobian. First one can observe

Proposition 5.3 ([13, Prop. 3])**.** *Let J be the Jacobian of a complete, non-singular curve C and $\Theta = W^{g-1}(C)$ its theta divisor, then $\deg(\Theta^g) = g!$ and $\Theta^{g-1} \equiv (g-1)!C$ modulo numerical equivalence.*

The result of Matsusaka is that the reverse is true in the sense of:

Theorem 5.4 ([13, Thm. 3])**.** *Let A be an n dimensional abelian variety, X an irreducible divisor on A and C a 1-cycle on A. If $\deg(X^n) = n!$ and $X^{n-1} \equiv (n-1)!C$ modulo numerical equivalence in A, then C is irreducible, A is isomorphic to the Jacobian of C and X is a corresponding theta divisor.*

Denote $[W^{g-1}]$ by θ_C and $[V^{g-1}]$ by θ_D. Then from $\alpha_C^{g-1} = \alpha_D^{g-1}$ we get $\theta_C^{g-1} = \theta_D^{g-1}$ as well. Applying Proposition 5.3 to (J, C) gives us $\theta_C^{g-1} \equiv (g-1)!C$ and applying it to (J, D) gives us $\deg(\theta_D^g) = g!$. Together with $\theta_D^{g-1} = \theta_C^{g-1}$ this gives $\theta_D^{g-1} \equiv (g-1)!C$. Then Theorem 5.4 implies that θ_D is a theta divisor for C and (J, θ_D) is a polarized Jacobian for C. Since the theta divisor is unique up to translation we can conclude $W^{g-1} = V_c^{g-1}$. Applying our proof of the case $d \leq g-1$ gives $C \cong D$. Alternatively one can argue that (J, θ_D) is a polarized Jacobian variety for both C and D, which is the original form of Torelli's theorem.

Extension to $d = 2g-2$

With more care one can extend the calculations to the case $d = 2g-2$ as well. Here $C^{(d)}$ is no longer a projective bundle over J, but it is in the complement of one point. There we still have $C^{(d)} \cong \mathbb{P}(\theta^*\mathcal{F}_{2g-2-d}) = \mathbb{P}(E_C)$ and E_C are locally free. So after removing one point we still find a line-bundle $\mathcal{L}$ relating E_C and E_D. Moreover E_C and E_D have by Theorem 4.26 still the same Chern classes. As removing one point does not change Chern and Segre classes except for the top class we can still do the above calculations. Note that we never needed the top Chern class. For $g = 3$ we only needed the second Chern class and for higher g the second and the third. Thus we get the case $d = 2g-2$ as well, finishing the proof of Theorem 1.1.

6 Extensions to smaller g

In this section we want to discuss extensions of Theorem 1.1.

First of all it extends trivially to the case of genus zero and one. In genus zero there is only one curve namely the projective line. In genus one any curve is itself an abelian variety so $J(C) \cong C$. Moreover we have shown in Lemma 2.3 that $\mathrm{Alb}(C^{(d)}) \cong J(C)$. Altogether we can directly recover the curves C, D from their symmetric products.

Next we want to see, that Theorem 1.1 does not hold in total generality. There are non-isomorphic curves in genus two whose second symmetric products are isomorphic. For this we need to understand how second symmetric power and Jacobian are related.

Lemma 6.1. *The second symmetric power of a smooth curve C of genus 2 is isomorphic to the blow-up of the Jacobian $J(C)$ at one point.*

Proof. Fix a basepoint c of C and consider the canonical map $\varphi_2 \colon C^{(2)} \to J(C)$. We want to compute the fibre of this map at $\mathcal{L} \in J(C)$. This fibre is the linear system associated to $\mathcal{L}(2c) = \mathcal{L} \otimes \mathcal{O}(2c)$ with dimension $h^0(\mathcal{L}(2c)) - 1$. Denote by ω_C the canonical bundle of C. By Riemann–Roch

$$h^0(\mathcal{L}(2c)) = h^0(\omega_C \otimes \mathcal{L}^* \otimes \mathcal{O}(-2c)) + \deg(\mathcal{L}(2c)) - g + 1 = h^0(\omega_C \otimes \mathcal{L}^* \otimes \mathcal{O}(-2c)) + 1.$$

Now $\omega_C \otimes \mathcal{L}^* \otimes \mathcal{O}(-2c)$ is a line bundle of degree zero so $h^0(\omega_C \otimes \mathcal{L}^* \otimes \mathcal{O}(-2c)) = 0$ except if $\omega_C \otimes \mathcal{L}^* \otimes \mathcal{O}(-2c) = \mathcal{O}_C$ then $h^0(\omega_C \otimes \mathcal{L}^* \otimes \mathcal{O}(-2c)) = 1$. So up to one point the fibres all consist of exactly one point. The fibre at $\omega_C(-2c)$ is a complete linear system of dimension one, thus it is isomorphic to $\mathbb{P}^1$. So φ_2 is a birational transformation of surfaces with $\omega(-2c)$ being a fundamental point. Thus by [5, V Prop. 5.3.] we know that φ_2 factors over the blow up of J at $\omega(-2c)$. This is only possible if the induced morphism $C^{(2)} \to \mathbb{B}l_P(J)$ is already an isomorphism.

□

There are indeed non-isomorphic smooth curves C and D of genus two with isomorphic Jacobians. Notice that they are isomorphic as abelian varieties, not as polarized abelian varieties since that would contradict the classical Torelli's theorem. We refer to [7, 8, 9] for examples over $\mathbb{C}$, over $\mathbb{Q}$ and in positive characteristic. Then C and D have isomorphic second symmetric products, possibly after a translation of one blow-up-point to the other. On the other hand by construction the curves themselves are not isomorphic.

Remark. The author does not know whether the theorem is true for genus two and $d > 2$. Note that the reasoning of [3] in this case is not quite correct, which was communicated to and confirmed by Fakhruddin. It may still be possible to use the results of [17]. Another approach for the opposite direction would be to investigate whether one can construct $C^{(d)}$ for some $d > 2$ directly from $C^{(2)}$. With this one would be able to extend the above counter example from the case $d = 2$.

Remark. In another related direction, one can investigate the automorphisms of $C^{(d)}$. Indeed one can prove, that all these automorphisms come from automorphisms of C. For this result we refer to [1].

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