p-adic Integration and Birational Calabi–Yau Varieties

Pablo Magni

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Betreuer: Prof. Dr. Daniel Huybrechts

Zweitgutachter: Dr. Wenhao Ou

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

Zusammenfassung

Die vorliegende Bachelorarbeit befasst sich mit bestimmten topologischen Invarianten von birationalen Calabi–Yau Varietäten. Zentrales Ziel ist es einen Satz von Batyrev detailliert zu beweisen. Dieser besagt, dass zwei birationale projektive Calabi–Yau Varietäten über $\mathbb{C}$ dieselben Bettizahlen haben (siehe Theorem 3.1).

Der vorgestellte Beweis basiert auf Methoden der p-adischen Analysis, insbesondere auf der p-adischen Integration. Die nötigen Grundlagen aus der Zahlentheorie und p-adischen Analysis sowie analytischen Geometrie werden in Abschnitt 1 eingeführt. Dabei wird die Analytifizierung von glatten Varietäten über einem p-adischen Körper detalliert dargestellt. Anschließend wird in Abschnitt 2 ein zentrales Theorem von Weil bewiesen, welches analytische Informationen (genauer das Volumen einer K-analytischen Mannigfaltigkeit) und zahlentheoretische Informationen (genauer die Anzahl der Punkte einer Reduktion einer Varietät) miteinander vergleicht. Dies reduziert den Vergleich von bestimmten lokalen Zeta-Funktionen auf den Vergleich von Volumina von bestimmten K-analytischen Mannigfaltigkeiten. In Abschnitt 3 wird diese Verbindung ausgenutzt, um Batyrevs Theorem zu beweisen, in welchem letztendlich gezeigt wird, dass das Volumen zweier birationaler projektiver Calabi–Yau Varietäten im obigen Sinne gleich ist. Die Gleichheit der Bettizahlen folgt dann aus den Weil-Vermutungen.

Um die oben angedeutete Strategie formal durchzuführen, müssen einige technische Aussagen bewiesen werden. Insbesondere wird in Abschnitt 4 gezeigt, wie die Situation von birationalen Calabi–Yau Varietäten geeignet “ausgebreitet” und “geliftet” werden kann, so dass die oben erwähnten Methoden der p-adischen Analysis anwendbar sind.

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Contents

Introduction

The bachelor thesis at hand deals with certain topological invariants of birationally equivalent Calabi–Yau varieties. The central goal is the detailed presentation of the following theorem by Batyrev (Theorem 3.1 in this text).

Theorem (Batyrev's Theorem, [Bat99, Theorem 1.1])**.** *Let X and Y be two integral, projective Calabi–Yau varieties over $\mathbb{C}$. If X and Y are birationally equivalent, then their Betti numbers coincide, i.e. for all $i \geq 0$ we have*

$$\dim_{\mathbb{C}} \mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{H}^i_{\mathrm{sing}}(Y^{\mathrm{an}}, \mathbb{C}).$$

The proof is based on methods of p-adic analysis, especially p-adic integration. The necessary fundamental concepts from number theory and p-adic analysis as well as analytic geometry are introduced in Section 1. In doing so we present the analytification of smooth varieties over a p-adic field K in detail. Afterwards, in Section 2, we prove a theorem of Weil that plays an important role in the proof of Batyrev's theorem. It enables us to compare analytic informations (more precisely, the volume of a K-analytic manifold associated to the variety under consideration) and arithmetic informations (more precisely, the number of points in a reduction to a finite field of the variety under consideration). This reduces the comparison of local zeta functions to the comparison of volumes of certain K-analytic manifolds. In Section 3 this connection is used to give a proof of Batyrev's theorem, in which it is shown that the volumes, in the above sense, of two birationally equivalent projective Calabi–Yau varieties, considered over a p-adic field K, are equal. Then the equality of Betti numbers follows from the Weil conjectures. In order to realize the indicated strategy formally, we have to prove a few technical propositions. In particular, we show in Section 4 how the situation of two birationally equivalent Calabi–Yau varieties over $\mathbb{C}$ can be "spread out" and "lifted" so that the mentioned methods of p-adic analysis can be applied.

The author always tried to support his arguments by using references to the literature and by working out basic concepts he tried to make the text accessible to readers that are not completely familiar with the used theories. The expert may skip some "obvious" explanations and reference to "standard" propositions. For space reasons we have to refer sometimes to the literature for proofs and details. In many of these cases the proofs are not very difficult and the interested reader is advised to look at the referenced literature. Nevertheless the reader should be familiar with the foundations of algebraic geometry, as they are presented, for example, in the first few chapters of [Liu02].

1 Fundamental p-adic analysis

In this section we develop and recall the foundations of p-adic analysis that we will need in the rest of this text. We present these foundations in a level of generality that highlights the developed objects on their own, but still keeps our goals in mind. In this regard we use, for example, in Section 1.3 the theory explained in Section 1 to classify compact K-analytic manifolds. This result is interesting on its own, but is still connected to our aims, since it uses p-adic integration in a similar way as applied in the proof of Batyrev's theorem.

1.1 p-adic numbers and p-adic fields

We introduce the field of p-adic numbers and explain basic results concerning it briefly. In this way we want to establish a first understanding of the p-adic setting. Further we recall some basic results about p-adic local number fields and their rings of integers. These will be the objects over which our schemes will be defined later.

1.1.1 p-adic numbers

The results of the following subsections and more details may be found in most instances in [Neu07, Chapter II] or [Ser78]. Let us start by fixing some notation. Notation 1.1 is in effect in the entire text.

Notation 1.1. We will denote by p a prime number unless otherwise stated. Similarly, q will denote some power p^k of p.

Definition 1.2 (p-adic integers)**.** The ring of p-adic integers $\mathbb{Z}_p$ is defined as the completion of $\mathbb{Z}$ at the maximal ideal (p), i.e. $\mathbb{Z}_p$ is the projective limit $\varprojlim \mathbb{Z}/(p^n) = \{ (x_n)_n \in \prod_{n\geq 1} \mathbb{Z}/(p^n) \mid x_n \equiv x_{n-1} \mod p^{n-1} \}$.

Proposition 1.3. *Let $p \in \mathbb{Z}$ be a prime number.*

i) *The ring of p-adic integers is an integral domain.*
ii) *The ring of p-adic integers is a compact topological ring with the subspace topology induced by the one of $\prod_{n\geq 1} \mathbb{Z}/(p^n)$, where each $\mathbb{Z}/(p^n)$ is endowed with the discrete topology. We call this topology the profinite topology on $\mathbb{Z}_p$.*
iii) *Each element $0 \neq x \in \mathbb{Z}_p$ can be written uniquely as $x = p^n u$ with $n \geq 0$ and $u \in \mathbb{Z}_p^\times$.*

Proof. See [Ser78, Section II.1.1] and [Ser78, Section II.1.2] for a proof. □

Definition 1.4 (p-adic numbers)**.** The field of p-adic numbers $\mathbb{Q}_p$ is the fraction field $\mathrm{Q}(\mathbb{Z}_p)$ of the ring of p-adic integers $\mathbb{Z}_p$.

Remark 1.5. By the representation in Proposition 1.3.iii) it suffices to invert $p \in \mathbb{Z}_p$ to obtain $\mathbb{Q}_p$, i.e. $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$. In particular, every $0 \neq x \in \mathbb{Q}_p$ can be written uniquely as $p^n u$ with $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^\times$.

Proposition 1.6. *Let $p \in \mathbb{Z}$ be a prime number.*

i) *There exists a discrete valuation ν_p on the field of p-adic numbers $\mathbb{Q}_p$ such that the ring of p-adic integers $\mathbb{Z}_p$ is the discrete valuation ring associated to ν_p. This discrete valuation is given by $\nu_p(p^n u) := n$, where $u \in \mathbb{Z}_p$ as in Proposition 1.3.*iii) and Remark 1.5.
ii) The ring of integers $\mathbb{Z}$ is a subring of the ring of p-adic integers via the homomorphism $x \mapsto (x \mod p^n)_n \in \mathbb{Z}_p$. Furthermore, $\mathbb{Z} \subset \mathbb{Z}_p$ is a dense subspace.

Proof. i) This is a short calculation in consideration of Propostion 1.3.iii) and Remark 1.5.

ii) We show that the map is injective. Let $x \in \mathbb{Z}$ and assume x maps to 0, then $x \equiv 0 \mod p^n$, i.e. $p^n \mid x$, for every $n \geq 1$. This is only possible for $x = 0$.

To see that $\mathbb{Z} \subset \mathbb{Z}_p$ is dense consider an arbitrary element $x \in \mathbb{Z}_p$ and write $x = (x_n)_n$. Consider the x_n as integers and note that $x - x_n \in p^n\mathbb{Z}_p$, since $x - x_n \equiv x_n - x_n \equiv 0 \mod p^n$. This means $\lim_{n\to\infty} x_n = x$. □

Definition 1.7. The p-adic absolute value on $\mathbb{Q}_p$ is defined as $|x|_p := p^{-\nu_p(x)}$ for $x \neq 0$ and $|0|_p := 0$.

Remark 1.8. Using the absolute value $|\cdot|_p$, we can view the ring of p-adic integers as the unit disc in the field of p-adic numbers. That is $\mathbb{Z}_p = \{\, x \in \mathbb{Q}_p \mid |x|_p \leq 1 \,\}$.

Proposition 1.9. *Let $p \in \mathbb{Z}$ be a prime number.*

i) *The p-adic absolute value $|\cdot|_p$ is a non-Archimedean absolute value on $\mathbb{Q}_p$, i.e. $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}_p$.*
ii) *The topology induced by the absolute value $|\cdot|_p$ on $\mathbb{Z}_p$, called the metric topology, is the same as the profinite topology on $\mathbb{Z}_p$.*
iii) *The metric topology on $\mathbb{Q}_p$ is locally compact.*

Proof. i) This follows immediately from the definition $|x|_p = p^{-\nu_p(x)}$ and the fact that ν_p is a discrete valuation (cf. Proposition 1.6.i)).

ii) In the metric topology the balls $\mathrm{B}(0, p^k) = (p^k)$ form a neighborhood basis of 0. We show that it is also a neighborhood basis of 0 in the profinite topology. We write $(p^k) = \{(x_n)_n \mid x_k = 0\} = \mathbb{Z}_p \cap \{0\} \times \cdots \times \{0\} \times \prod_{n\geq k+1} \mathbb{Z}/(p^n)$. Since the profinte topology is induced from the product topology on $\prod_{n\geq 1} \mathbb{Z}/(p^n)$ we have the neighborhood basis $\mathbb{Z}_p \cap U_1 \times \cdots \times U_k \times \prod_{n\geq k+1} \mathbb{Z}/(p^n)$ of 0, where $k \geq 1$, $U_i \subset \mathbb{Z}/(p^i)$ open and $0 \in U_i$. This shows that also the (p^k) form a neighborhood basis of 0 in the profinite topology as desired. Since both topologies are compatible with the group structure on $\mathbb{Z}_p$ we conclude that the two topologies have a common basis and are therefore equal.

iii) Note that, since $\mathrm{im}(|\cdot|_p) \subset \mathbb{R}$ is discrete, we see that $\mathbb{Z}_p = \overline{\mathrm{B}}(0,1) = \mathrm{B}(0,p) \subset \mathbb{Q}_p$ is open. Now, since $\mathbb{Z}_p$ is compact and multiplication by $p \in \mathbb{Q}_p$ is a homeomorphism, we conclude that $\mathrm{B}(0,p^n)$, $n \in \mathbb{Z}$, is a neighborhood basis of 0 consisting of compact and open sets. □

Remark 1.10. The metric induced by the absolute value $|\cdot|_p$ is complete and, in fact, $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the restriction of $|\cdot|_p$ on $\mathbb{Q}$. Indeed, this is a short calculation using Cauchy sequences and the fact that $\mathbb{Z}_p$ is compact and $\mathbb{Z} \subset \mathbb{Z}_p$ is dense. Alternatively see [Neu07, Satz II.2.2] and [Neu07, Satz II.2.3] for a proof.

The following result illustrates some topological properties in the p-adic setting that may be unfamiliar when one is accustomed to the Euclidean or Zariski topology.

Proposition 1.11. *Let $|\cdot|$ be a non-Archimedean absolute value on some field K. Then the following hold.*

i) *Every point of a ball is a midpoint of it, i.e. if $y \in \mathrm{B}(x,\varepsilon)$ then $\mathrm{B}(x,\varepsilon) = \mathrm{B}(y,\varepsilon)$.*
ii) *Tow balls $B_1 = \mathrm{B}(x_1,\varepsilon_1)$ and $B_2 = \mathrm{B}(x_2,\varepsilon_2)$ are either disjoint or one is included in the other.*
iii) *The topology induced by the absolute value is totally disconnected.*

Proof. i) Let $y \in \mathrm{B}(x,\varepsilon)$ and take $z \in \mathrm{B}(x,\varepsilon)$. Then $|y - z| = |(y-x) + (x-z)| \leq \max\{|x-y|, |x-z|\} < \varepsilon$, i.e. $z \in \mathrm{B}(y,\varepsilon)$, and we see that $\mathrm{B}(y,\varepsilon) \subset \mathrm{B}(x,\varepsilon)$. By exchanging the roles of x and y we conclude that $\mathrm{B}(y,\varepsilon) = \mathrm{B}(x,\varepsilon)$.

ii) Assume $B_1 \cap B_2 \neq \emptyset$ and take $x_3 \in B_1 \cap B_2$. Then by i) we can write $B_1 = \mathrm{B}(x_3, \varepsilon_1)$ and $B_2 = \mathrm{B}(x_3, \varepsilon_2)$. Say $\varepsilon_1 \leq \varepsilon_2$, then $B_1 \subset B_2$.

iii) Let $x \in K$ and consider the closed ball $\overline{\mathrm{B}}(x, \varepsilon) = \{y \in K \mid |x - y| \leq \varepsilon\}$. Take $y \in \overline{\mathrm{B}}(x, \varepsilon)$ and note $\mathrm{B}(y, \varepsilon) \subset \overline{\mathrm{B}}(y, \varepsilon) = \overline{\mathrm{B}}(x, \varepsilon)$. This means that $\overline{\mathrm{B}}(x, \varepsilon)$ is open and we have found a neighborhood basis consisting of open and closed sets in a Hausdorff space. □

1.1.2 p-adic local number fields

Definition 1.12 (p-adic field)**.** A p-adic (local number) field K is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$.

Notation 1.13. In the whole text K will denote a p-adic field, unless otherwise stated.

Proposition 1.14. *Let K be a p-adic field. Then the following hold.*

i) *The absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ extends uniquely to a non-Archimedean absolute value $|\cdot|$ on K. Explicitly the extension is given by $|\cdot| = |\mathrm{N}_{K/\mathbb{Q}_p}(\cdot)|_p^{1/[K:\mathbb{Q}_p]}$, where $\mathrm{N}_{K/\mathbb{Q}_p}$ denotes the norm of the field extension.*

ii) *This absolute value makes K into a locally compact topological field. Furthermore, the metric induced by the absolute value is complete.*

Proof. i) See [Neu07, Theorem II.4.8] for a proof of a more general result. Note that in our case the uniqueness follows from the fact that all norms on a finite dimensional vector space over a locally compact field are equivalent (cf. [Kob77, Theorem 10]).

We want to motivate the definition of the absolute value $|\cdot|$ following the exposition in [Kob77, Page 61]. Consider $a \in K$ of degree n over $\mathbb{Q}_p$ and let L be the normal closure of $\mathbb{Q}_p(a)$. Now $L/\mathbb{Q}_p$ is a finite Galois extension and for every conjugate $a_i \in L$ of a there is a $\sigma_i \in \mathrm{Gal}(L/\mathbb{Q}_p)$ with $\sigma_i(a) = a_i$. If $|\cdot|$ is an absolute value on L extending $|\cdot|_p$, then $x \mapsto |\sigma_i(x)|$ is also an absolute value extending $|\cdot|_p$. By the uniqueness of the extension of the absolute value we deduce that for every $x \in L$ we have $|x| = |\sigma_i(x)|$ and in particular $|a| = |a_i|$. Now

$$|\mathrm{N}_{\mathbb{Q}_p(a)/\mathbb{Q}_p}(a)|_p = |\prod_{i=1}^{n} a_i| = |a|^n$$

and hence $|a| = |\mathrm{N}_{\mathbb{Q}_p(a)/\mathbb{Q}_p}(a)|_p^{1/n}$. To conclude note that we have $n = [\mathbb{Q}_p(a) : \mathbb{Q}_p] = [K : \mathbb{Q}_p]/[K : \mathbb{Q}_p(a)]$ and $\mathrm{N}_{K/\mathbb{Q}_p}(a) = (\mathrm{N}_{\mathbb{Q}_p(a)/\mathbb{Q}_p}(a))^{[K:\mathbb{Q}_p(a)]}$.

ii) Endow $\mathbb{Q}_p^n$ with the maximum norm $\|(a_1, \ldots, a_n)\| := \max\{|a_1|_p, \ldots, |a_n|_p\}$. This makes $\mathbb{Q}_p^n$ into a locally compact, complete normed space over $\mathbb{Q}_p$. Using as in i) that all norms on K are equivalent, we deduce that for every basis $v_1, \ldots, v_n$ of K over $\mathbb{Q}_p$ the map $\mathbb{Q}_p^n \to K$, $(a_1, \ldots, a_n) \mapsto a_1 v_1 + \cdots + a_n v_n$ is an isomorphism of topological vector spaces. Hence, also K is a locally compact complete normed space over $\mathbb{Q}_p$. □

Definition 1.15. Let K be a p-adic field. We define its ring of integers as $\mathcal{O}_K := \{\, x \in K \mid |x| \leq 1\}$. We also define $\mathfrak{m}_K := \{\, x \in K \mid |x| < 1\}$.

Proposition 1.16. *Let K be a p-adic field. Then the following hold.*

i) *The ring of integers $\mathcal{O}_K$ is a discrete valuation ring with maximal ideal $\mathfrak{m}_K$.*

ii) *The residue field $\mathcal{O}_K/\mathfrak{m}_K$ is a finite extension of $\mathbb{F}_p$.*

Proof. i) Define $\nu(x) := -\log(|x|)$ for $0 \neq x \in K$ and $\nu(0) := \infty$. This is a valuation, since the absolute value $|\cdot|$ is non-Archimedean. Furthermore, $\mathrm{im}(|\cdot|_p) = \{p^n \mid n \in \mathbb{Z}\}$ implies that $\mathrm{im}(\nu) \subset \{n \log(p)/[K : \mathbb{Q}_p] \mid n \in \mathbb{Z}\}$, i.e. ν is a discrete valuation.

Note that for $x \in K$ we have $|x| \leq 1$ if and only if $\log(|x|) \leq \log(1) = 0$, if and only if $\nu(x) \geq 0$. So $\mathcal{O}_K$ is the discrete valuation ring associated to ν with maximal ideal $\mathfrak{m}_K$.

ii) Compare to [Neu07, Satz II.5.2]. First note that for $x \in \mathbb{Z}_p$ with $|x|_p < 1$ we have $x \in \mathfrak{m}_K$, since $|x| = |x|_p < 1$. This means that $\mathcal{O}_K/\mathfrak{m}_K$ is a $\mathbb{F}_p$ vector space.

Now let $x_1 \ldots, x_n \in \mathcal{O}_K$ be elements such that $\overline{x}_1, \ldots, \overline{x}_n \in \mathcal{O}_K/\mathfrak{m}_K$ are linearly independent over $\mathbb{F}_p$. Assume $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ is a non-trivial linear combination. By dividing by the α_{i_0} with the largest absolute value we can assume that all $\alpha_i \in \mathcal{O}_K$ and $\alpha_{i_0} = 1$. This means that we get a non-trivial linear combination $\overline{\alpha}_1\overline{x}_1 + \cdots + \overline{\alpha}_n\overline{x}_n = 0 \in \mathcal{O}_K/\mathfrak{m}_K$. This is a contradiction and hence $x_1, \ldots, x_n \in K$ are linear independent. It follows that $\dim_{\mathbb{F}_p}(\mathcal{O}_K/\mathfrak{m}_K) \leq \dim_{\mathbb{Q}_p}(K) < \infty$. □

Notation 1.17. We will write $\mathbb{F}_q$ for the residue field $\mathcal{O}_K/\mathfrak{m}_K$, where it is understood that $q = p^k$ for some $k \geq 1$. The (normalized) discrete valuation associated to $\mathcal{O}_K$ is denoted by ν_K and the (normalized) absolute value is defined as $|\cdot|_p := q^{-\nu_K(\cdot)}$.

Remark 1.18. The normalized absolute value $|\cdot|_p$ on K has image $\operatorname{im}(|\cdot|_p) = \{q^k \mid k \in \mathbb{Z}\} \cup \{0\} \subset \mathbb{R}$.

We now recall some technical results that are needed later in the text and are included for completeness and ease of reference.

Proposition 1.19. *Let K be a p-adic field. Then for every $r \geq 1$ there exists an extension of p-adic fields $K^{(r)}/K$ such that $[K^{(r)} : K] = r$ and $[\mathcal{O}_{K^{(r)}}/\mathfrak{m}_{K^{(r)}} : \mathcal{O}_K/\mathfrak{m}_K] = r$. This is called an "unramified" extension of degree r.*

Proof. See [Neu07, Satz II.7.12] for a proof. □

Proposition 1.20.

i) *Let $\mathcal{O}$ be a complete discrete valuation ring with fraction field $K = \mathrm{Q}(\mathcal{O})$ of characteristic zero and finite residue field $\mathbb{F}_{p^n}$. Then K is a p-adic field.*

ii) *Let $\mathcal{O}$ be a discrete valuation ring. Then its completion $\widehat{\mathcal{O}}$ is a complete (in the metric and algebraic sense) discrete valuation ring.*

Proof. i) See [Neu07, Satz II.5.2] for a proof.

ii) See [Neu07, Satz II.4.3] and [Neu07, Satz II.4.5] for a proof. □

1.2 Analytification of smooth schemes over a p-adic field

In this section we recall fundamental concepts from p-adic analysis, develop the notion of K-analytic manifold and show how we can associate such a K-analytic manifold to a smooth scheme over a p-adic field K.

1.2.1 p-adic analysis

The following results serve the purpose to convince the reader that basic concepts from real analysis carry over to the p-adic setting. A more detailed exposition of the concepts of this section may be found in [Sch11, Chapter I]. We restrict ourselves to the case of p-adic fields K, but the statements we will see in this subsection are also valid for complete non-Archimedean valued fields.

Definition 1.21 (Convergent power series)**.**

i) A power series $f = \sum_\alpha a_\alpha x^\alpha \in K[[x_1, \ldots, x_n]]$ is called ε-convergent if we have $\lim_{|\alpha|\to\infty} \varepsilon^{|\alpha|}|a_\alpha|_p = 0$. Here α is a multiindex and $\varepsilon > 0$ is a real number.

ii) The set of ε-convergent power series is denoted by $\mathrm{T}_{n,\varepsilon}(K)$ and the set of power series that are ε-convergent for some $\varepsilon > 0$ is denoted by $K\langle\langle x_1, \ldots, x_n\rangle\rangle$.

Remark 1.22. Indeed, the definition of ε-convergence makes sense, since in the complete Non-Archimedean setting the following holds. Let $a_n \in K$ for $n \geq 0$. The series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$ (cf. [Sch11, Lemma I.3.1]).

Proposition 1.23. *The set $K\langle\langle x_1, \ldots, x_n\rangle\rangle$ becomes a K-algebra with algebraic operations inherited from $K[[x_1, \ldots, x_n]]$. Furthermore, if $f \in K\langle\langle x_1, \ldots, x_n\rangle\rangle$ satisfies $f(0) \neq 0$, then $1/f \in K\langle\langle x_1, \ldots, x_n\rangle\rangle$. Also composition in the following sense is well-defined. Let $g = (g_1, \ldots, g_n) \in \mathrm{T}_{r,\delta}(K)^n$ and write $g_i = \sum_\alpha a_\alpha^{(i)} x^\alpha$. If for all $i = 1, \ldots, n$ we have $\max_\alpha(\delta^{|\alpha|}|a_\alpha^{(i)}|_p) < \varepsilon$, then there is a K-linear map $\mathrm{T}_{n,\varepsilon}(K) \to \mathrm{T}_{r,\delta}(K)$, $f \mapsto f \circ g$.*

Proof. See [Sch11, Page 25], [Sch11, Proposition I.5.3], [Igu00, Corollary 2.1.2] and [Sch11, Proposition I.5.4] for a proof and the definition of the composition map. □

Proposition 1.24. *We can view a power series $f \in \mathrm{T}_{n,\varepsilon}(K)$ as a continuous function on $\mathrm{B}(0, \varepsilon) \subset K^n$ via evaluation, i.e. there is a K-algebra homomorphism $\mathrm{T}_{n,\varepsilon}(K) \to \mathrm{C}^0(\mathrm{B}(0, \varepsilon), K)$, $f \mapsto (a \mapsto f(a))$. Furthermore, this homomorphism is compatible with compositions as in Proposition 1.23.*

Proof. The map is well-defined by Remark 1.22 and the uniform limit theorem [Que01, Satz 1.23]. See [Sch11, Proposition I.5.3] for a proof that the map is a K-algebra homomorphism and [Sch11, Proposition I.5.4] for a proof of the compatibility with compositions. □

Remark 1.25. We can define partial derivatives of ε-convergent power series via formal partial derivatives. Indeed, if $f \in \mathrm{T}_{n,\varepsilon}(K)$, then also $\frac{\partial f}{\partial x_i} \in \mathrm{T}_{n,\varepsilon}(K)$. To see this use Remark 1.22 and that $|\cdot|_p$ is a non-Archimedean absolute value. Let us also remark that for $f \in \mathrm{T}_{n,\varepsilon}(K)$ and $y, a \in \mathrm{B}(0, \varepsilon)$ we have $\frac{\partial(f(x+y))}{\partial x_i}(a) = \frac{\partial f}{\partial x_i}(a + y)$ (cf. [Sch11, Proposition I.5.6]). This will allow us to consider partial derivatives of K-analytic functions (cf. Definition 1.29) later.

Proposition 1.26. *Let $0 \neq f \in \mathrm{T}_{n,\varepsilon}(K)$. Then there exists an $a \in \mathrm{B}(0, \varepsilon)$ such that $f(a) \neq 0$.*

Proof. See [Sch11, Corollary I.5.8] for a proof. □

The following theorems well-known from real analysis also work in the p-adic setting. This will allow us to introduce a concept of manifold, similar to smooth manifolds, in the next section.

Theorem 1.27 (Implicit function theorem)**.** *Let $F_1, \ldots, F_m \in K\langle\langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle\rangle$ with all $F_i(0, 0) = 0$. If $\det((\frac{\partial F_i}{\partial y_j}(0, 0))_{i,j}) \neq 0$, then there exist power series $f_1, \ldots, f_m \in K\langle\langle x_1, \ldots, x_n\rangle\rangle$ as well as open neighborhoods $0 \in U \subset K^n$ and $0 \in V \subset K^m$ such that $f := (f_1, \ldots, f_m)$ converges on U, $F := (F_1, \ldots, F_n)$ converges on $U \times V$ and*

$$\{(x, f(x)) \mid x \in U\} = \{(x, y) \in U \times V \mid F(x, y) = 0\}.$$

Proof. See [Igu00, Theorem 2.1.1] for a proof. □

Corollary 1.28 (Inverse function theorem)**.** *Let $f_1, \ldots, f_n \in K\langle\langle x_1, \ldots, x_n\rangle\rangle$ with all $f_i(0) = 0$. If $\det(\frac{\partial f_i}{\partial x_j}(0))_{ij} \neq 0$, then there exist power series $g_1, \ldots, g_n \in K\langle\langle x_1, \ldots, x_n\rangle\rangle$ and open neighborhoods $0 \in U, V \subset K^n$ such that the map $f = (f_1, \ldots, f_n)\colon U \to V$ is a homeomorphism with inverse $g = (g_1, \ldots, g_n)$.*

Proof. Compare to [Igu00, Corollary 2.1.1]. Apply Theorem 1.27 with $m = n$ and $F_i(x, y) := x_i - f_i(y)$. For a direct proof see [Sch11, Proposition I5.9]. □

1.2.2 K-analytic manifolds

We now define K-analytic manifolds, the analogon of smooth (respectively complex) manifolds, where instead of $\mathbb{R}$ (respectively $\mathbb{C}$) we use a p-adic field K. These are the objects on which we will integrate later. More precisely, we are aiming at taking the volume of the p-adic manifold $\mathfrak{X}(\mathcal{O}_K)$ associated to a Calabi–Yau variety $\mathfrak{X}$ over $\mathcal{O}_K$ (cf. Section 1.2.3).

Definition 1.29 (K-analytic functions)**.** Let K be a p-adic field and $U \subset K^n$ an open subset.

i) A function $f\colon U \to K$ is a K-analytic function if for every $a \in U$ we can write it locally around a as a convergent power series, i.e. $f \in K\langle\langle x_1 - a_1, \dots, x_n - a_n\rangle\rangle$.
ii) A map $g\colon U \to K^m$ is called a K-analytic map if its components are K-analytic functions.

Remark 1.30. The operations considered for convergent power series in the last section also make sense for K-analytic functions.

Remark 1.31.

i) When we define $\mathcal{O}_{K^n}(U) \coloneqq \{\, f\colon U \to K \mid f \text{ is } K\text{-analytic} \,\}$, then $\mathcal{O}_{K^n}$ becomes a sheaf on K^n, where for $V \subset U$ open the restriction map $\rho_{UV}\colon \mathcal{O}_{K^n}(U) \to \mathcal{O}_{K^n}(V)$ is restriction of functions $\rho_{UV}(f) \coloneqq f|_V$.
ii) In this way $(K^n, \mathcal{O}_{K^n})$ is a locally ringed space, since for every $a \in K^n$ we see that $\mathfrak{m}_{K^n,a} \coloneqq \{\, f \in \mathcal{O}_{K^n,a} \mid f(a) = 0 \,\}$ is a maximal ideal (cf. Proposition 1.23). We have $\mathcal{O}_{K^n,a}/\mathfrak{m}_{K^n,a} \simeq K$ via $f \mapsto f(a)$.
iii) We will always consider $(K^n, \mathcal{O}_{K^n})$ as a locally ringed space over (pt, K) via the map $K \to \mathcal{O}_{K^n}(K^n)$, $k \mapsto \mathrm{const}_k \coloneqq (x \mapsto k)$. In other words $(K^n, \mathcal{O}_{K^n})$ is locally ringed in K-algebras. We refer to such locally ringed spaces also by locally K-ringed spaces.

Definition 1.32 (K-analytic Manifold)**.** Let K be a n-dimensional p-adic field. A K-analytic manifold of dimension n is a locally K-ringed Hausdorff space $(X, \mathcal{O}_X)$ such that for every $a \in X$ there exist an open neighborhood $U \subset X$ of a and an open set $V \subset K^n$ such that $(U, \mathcal{O}_X|_U) \simeq (V, \mathcal{O}_{K^n}|_V)$.

Remark 1.33.

i) The morphisms of K-analytic manifolds are the morphisms of locally K-ringed spaces.
ii) Let $(X, \mathcal{O}_X)$ be a K-analytic manifold, and let $U \subset X$ and $V \subset K^n$ be open such that $(f, f^\#)\colon (U, \mathcal{O}_X|_U) \xrightarrow{\sim} (V, \mathcal{O}_{K^n}|_V)$. Then we call (U, f) a chart. A collection $\{(U_i, f_i)\}_{i\in I}$ of charts is called an atlas if $X = \bigcup_{i\in I} U_i$.

Notation 1.34.

i) We denote a K-analytic manifold by X instead of $(X, \mathcal{O}_X)$ when there is no ambiguity.
ii) If (U, f) is a chart on X, then we can write $f(x) = (f_1(x), \dots, f_n(x))$ for $x \in U$. We call the f_i local coordinates and usually denote $f_i(x)$ by x_i.

Example 1.35.

i) The locally ringed space $(K^n, \mathcal{O}_{K^n})$ is a K-analytic manifold.
ii) If X is a K-analytic manifold and $U \subset X$ is an open subset, then $(U, \mathcal{O}_X|_U)$ is a K-analytic manifold.

Remark 1.36. Figure 1 visualizes the compact $\mathbb{Q}_7$-analytic manifold $\mathbb{Z}_7 \sqcup \mathbb{Z}_7 \sqcup \mathbb{Z}_7$. Note that it is indeed a $\mathbb{Q}_7$-analytic manifold, since it is a disjoint union of open subsets $\mathbb{Z}_7 \subset \mathbb{Q}_7$. The depicted discs represent open balls $\mathrm{B}(a, p^{-n})$ ($n = 0, 1, 2$, not all drawn) and some of these balls are partitioned by the smaller balls they contain. We will see in Proposition 1.98 that this is a typical example of a compact K-analytic manifold.

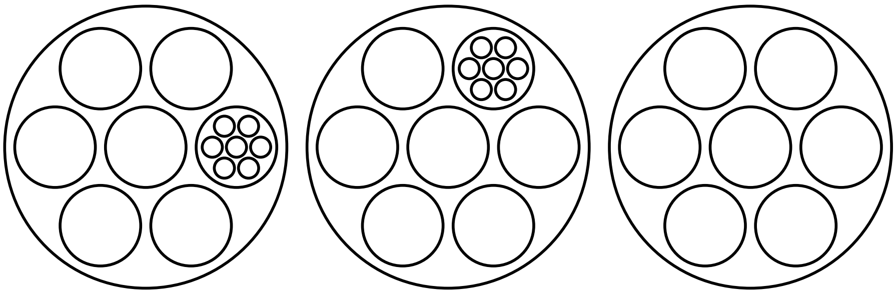


Figure 1: Visualization of a compact $\mathbb{Q}_7$-analytic manifold.

Proposition 1.37. *Let K be a p-adic field, let $U \subset K^n$ and $V \subset K^m$ be open subsets. Then every morphism $(f, f^\#)\colon (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$ of locally K-ringed space satisfies $f^\#_{V'}(g) = g \circ f$ for every $V' \subset V$ open and $g \in \mathcal{O}_V(V')$. In particular, f is a K-analytic map. Conversely, every K-analytic map $f\colon U \to V$ induces a morphism of locally ringed spaces via composition.*

Proof. Let $V' \subset V$ be open, $a \in V'$ arbitrary, and set $U' := f^{-1}(V')$. Consider the commutative diagram below. Since $f^\#_a$ is a local homomorphism it induces $\overline{f^\#_a}$ which corresponds to $\mathrm{id}\colon K \to K$ because the morphism $(f, f^\#)$ is a morphism over (pt, K).

$$\begin{tikzcd}
\mathcal{O}_V(V') \arrow[r] \arrow[d, "f^\#_{V'}"] & \mathcal{O}_{V,f(a)} \arrow[r] \arrow[d, "f^\#_a"] & \mathcal{O}_{V,f(a)}/\mathfrak{m}_{V,f(a)} \arrow[r, "\sim"] \arrow[d, "\overline{f^\#_a}"] & K \arrow[d, "\mathrm{id}"] \\
\mathcal{O}_U(U') \arrow[r] & \mathcal{O}_{U,a} \arrow[r] & \mathcal{O}_{U,a}/\mathfrak{m}_{U,a} \arrow[r, "\sim"] & K
\end{tikzcd}$$

Now we consider some $g \in \mathcal{O}_V(V')$. The commutativity of the diagram implies that $f^\#_{V'}(g)(a) = g(f(a))$. Since $a \in V'$ is arbitrary, we conclude $f^\#_{V'}(g) = g \circ f$.

If we consider the coordinate functions y_i on V in place of g, then we get $f_i = y_i \circ f \in \mathcal{O}_U(U)$. This implies f is K-analytic.

For the converse we note that compositions of K-analytic maps are again K-analytic maps. Since compositions and restrictions of functions are compatible, i.e. we have $(g \circ f)|_{f^{-1}(V')} = g|_{V'} \circ f|_{f^{-1}(V')}$ for $V' \subset V$ open, we get a well defined map of ringed spaces. The induced homomorphisms on stalks are local homomorphisms, since $g(f(a)) = 0$ implies $(g \circ f)(a) = 0$. Clearly it is a morphism of locally ringed spaces over (pt, K), since $\mathrm{const}_k \circ f = \mathrm{const}_k$. □

Remark 1.38.

i) Proposition 1.37 implies that our definition of manifolds is equivalent to the classical definition via charts (cf. [Sch11, Chapter II] or [Nee07, Section 2.1]).
ii) Similarly, Proposition 1.37 allows us to think about $\mathcal{O}_X$ as the sheaf of functions $f\colon X \to K$ that are K-analytic functions in charts (cf. [Nee07, Reminder 2.2.1]). In this interpretation morphisms of p-adic manifolds X and Y are just continuous maps $f\colon X \to Y$ that are K-analytic maps in charts (cf. [Nee07, Example 2.2.13]). This means that our definition of maps between K-analytic manifolds is equivalent to the classical one.

As we have seen already for p-adic fields in Proposition 1.11, also K-analytic manifolds are totally disconnected.

Proposition 1.39. *Let $(X, \mathcal{O}_X)$ be a K-analytic manifold. Then the topological space X is totally disconnected.*

Proof. Since X is Hausdorff it is enough to find for every $a \in X$ a neighborhood basis consisting of open and compact sets. Consider a chart (U, f) centered at a, i.e. $f(a) = 0$. By shrinking U we can assume that f is a homeomorphism onto a ball $\mathrm{B}(0, q^k)$. Since $\mathrm{B}(0, q^l) \subset \mathrm{B}(0, q^k)$ for $l \leq k$ is open and compact, we have found the desired neighborhood basis. □

Proposition 1.39 suggest that K-analytic manifolds are not very well-behaved. This will lead to the observation that there are not many different compact K-analytic manifolds (cf. Theorem 1.99). In rigid analytic geometry (cf. [Bos14]) one starts by requiring that the considered functions converge not only on some open ball but on the whole unit ball (cf. the notion of "Tate algebra"). The resulting theory is deeper than the one of compact K-analytic manifolds.

In the rest of this section we focus our attention on differential forms on K-analytic manifolds. They will be needed in the construction of the Weil measure in Section 2.2. As we have see so far for K-analytic manifolds, their definition follows the same strategy familiar from smooth manifolds. Our presentation is based on the one in [Igu00, Section 2.4].

Definition 1.40 (Tangent and cotangent space)**.** Let X be a K-analytic manifold and $a \in X$.

i) We call a K-linear map $\partial\colon \mathcal{O}_{X,a} \to K$ a derivation at a if it satisfies the Leibniz rule $\forall f, g \in \mathcal{O}_{X,a} : \partial(fg) = (\partial f)g(a) + f(a)(\partial g)$.
ii) The space of derivations at a is denoted by $\mathrm{T}_{X,a}$ and called the tangent space at a.
iii) The cotangent space at a is $\Omega_{X,a} \coloneqq \mathrm{T}_{X,a}^{\vee}$.

Remark 1.41. In a chart $x_1, \ldots, x_n$ around a (recall Notation 1.34) we have a natural isomorphism $\mathcal{O}_{X,a} \simeq K\langle\langle x_1 - a_1, \ldots, x_n - a_n\rangle\rangle$ and $\mathfrak{m}_{X,a} \simeq (x_1 - a_1, \ldots, x_n - a_n)$ (cf. Proposition 1.26).

Notation 1.42. We use the notation $\frac{\partial f}{\partial x_i}|_a \coloneqq \frac{\partial (f \circ \varphi^{-1})(a)}{\partial x_i}$, where $f \in \mathcal{O}_{X,a}$ and $\varphi = (x_1, \ldots, x_n)$ is a chart.

Proposition 1.43. *Let X be a K-analytic manifold and $x_1, \ldots, x_n$ a chart around $a \in X$. Then $\{\frac{\partial}{\partial x_1}|_a, \ldots, \frac{\partial}{\partial x_n}|_a\}$ is a basis of the vector space $\mathrm{T}_{X,a}$.*

Proof. It is clear that the $\frac{\partial}{\partial x_i}|_a$ are K-linear and satisfy the Leibniz rule. They are linear independent, since $\frac{\partial}{\partial x_n}|_a(x_j) = \delta_{ij}$. In order to see that they generate $\mathrm{T}_{X,a}$ we take an arbitrary $f \in \mathcal{O}_{X,a}$. Now f can be viewed as a power series around a and we write

$$f = f(a) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_a x_i + \sum_{i<j} x_i x_j \tilde{f}_{ij}.$$

Using the K-linearity of $\partial \in \mathrm{T}_{X,a}$ and the Leibniz rule, we see that $\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}|_a \partial x_i$. Hence, we conclude $\partial = \sum_{i=1}^{n} \partial x_i \frac{\partial}{\partial x_i}|_a$. □

Definition 1.44. Let X be a K-analytic manifold, let $a \in X$ and let $f \in \mathcal{O}_{X,a}$. We define $(\mathrm{d}f)_a \coloneqq \overline{f - f(a)} \in \mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$.

Remark 1.45.

i) In a chart $x_1, \dots, x_n$ around a the $(\mathrm{d}x_i)_a$ form a K-basis of $\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$. To see this use Remark 1.41 and Nakayama's lemma.

ii) In the situation of i) we have $(\mathrm{d}f)_a = \sum_{i=1}^n \frac{\partial f}{\partial x_i}|_a (\mathrm{d}x_i)_a$ for $f \in \mathcal{O}_{X,a}$. This follows from $\frac{\partial}{\partial x_i}|_a((\mathrm{d}f)_a) = \frac{\partial f}{\partial x_i}|_a$.

Proposition 1.46. *Let X be a K-analytic manifold and let $a \in X$, then $\Omega_{X,a} \simeq \mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$.*

Proof. Consider the K-linear map $\mathfrak{m}_{X,a} \to \Omega_{X,a}$, $f \mapsto (\partial \mapsto \partial(f))$. By the Leibniz rule $\partial(\mathfrak{m}_{X,a}^2) = 0$ and hence we get an induced K-linear map $\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2 \to \Omega_{X,a}$. Now note that in a chart $x_1, \dots, x_n$ centered at a we have $(\mathrm{d}x_i)_a \mapsto (\partial \mapsto \partial(x_i)) = (\frac{\partial}{\partial x_i}|_a)^\vee$. By Remark 1.41 the $(\mathrm{d}x_i)_a$ form a basis of $\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$ and by Proposition 1.43 the $(\frac{\partial}{\partial x_i}|_a)^\vee$ form a basis of $\Omega_{X,a}$. This means our map is an isomorphism. $\square$

Remark 1.47. Let X be a K-analytic manifold and $a \in X$. If $(\mathrm{d}f_1)_a, \dots, (\mathrm{d}f_n)_a$ is a basis of $\mathfrak{m}_{X,a}/\mathfrak{m}_{X,a}^2$, then there exists an open neighborhood U around a on which $\varphi_U := (f_1, \dots, f_n)$ defines a chart. To see this note that the change of basis matrix from $(\mathrm{d}f_1)_a, \dots, (\mathrm{d}f_n)_a$ to $(\mathrm{d}x_1)_a, \dots, (\mathrm{d}x_n)_a$ is the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})_{ij}$, and apply the inverse function theorem (Corollary 1.28).

Notation 1.48. Let X be a n-dimensional K-analytic manifold, let $a \in X$ and let $0 \leq r \leq n$. We introduce the notation $\Omega_{X,a}^r := \bigwedge^r \Omega_{X,a}$.

Remark 1.49. If $x_1, \dots, x_n$ is a chart around a, then $\{(\mathrm{d}x_{i_1})_a \wedge \dots \wedge (\mathrm{d}x_{i_r})_a\}_{i_1 < \dots < i_r}$ is a basis of $\Omega_{X,a}^r$.

Definition 1.50 (Differential form)**.** Let X be a K-analytic manifold of dimension n and let $0 \leq r \leq n$. A map $\omega\colon X \to \bigsqcup_{a \in X} \Omega_{X,a}^r$ is called a differential p-form if

i) for all $a \in X$ one has $\omega(a) \in \Omega_{X,a}^r$, and

ii) we can write $\omega(a) = \sum_{i_1 < \dots < i_r} f_{i_i, \dots, i_r}(a)(\mathrm{d}x_{i_1})_a \wedge \dots \wedge (\mathrm{d}x_{i_r})_a$ in every chart $(U, x_1, \dots, x_n)$, and the $f_{i_i, \dots, i_r}\colon U \to K$ are K-analytic functions.

Definition 1.51. Let X be a K-analytic manifold of dimension n and let $0 \leq r \leq n$. The sheaf of differential r-forms Ω_X^r is defined as

$$\Omega_X^r(U) := \{\omega\colon X \to \bigsqcup_{a \in X} \Omega_{X,a}^r \mid \omega \text{ is differential } r\text{-forms on } U\}$$

for $U \subset X$ open and the restriction maps $\rho_{UV}\colon \Omega_X^r(U) \to \Omega_X^r(V)$ are restriction of functions $\rho_{UV}(\omega) := \omega|_V$, where $V \subset U$ is open.

Remark 1.52. The sheaf Ω_X^r is a sheaf of $\mathcal{O}_X$-modules.

Example 1.53. Let X be a K-analytic manifold, let $U \subset X$ be open and let $f \in \mathcal{O}_X(U)$, then Remark 1.45 shows that $\mathrm{d}f := (x \mapsto (\mathrm{d}f)_x)$ is a differential 1-form on U. This induces a morphism $\mathrm{d}\colon \mathcal{O}_K \to \Omega_X^1$.

Definition 1.54. Let X and Y be K-analytic manifolds, let $f\colon X \to Y$ be a K-analytic map and let $a \in X$.

i) Define $\mathrm{D}_a f\colon \mathrm{T}_{X,a} \to \mathrm{T}_{Y,f(a)}$ as $\partial \mapsto (g \mapsto \partial(g \circ f))$, and

ii) denote the dual map by $f_a^* := (\mathrm{D}_a f)^\vee\colon \Omega_{Y,f(a)}^1 \to \Omega_{X,a}^1$.

Remark 1.55. Explicitly $f_a^*\omega = (\partial \mapsto \omega(\mathrm{D}_a f(\partial)))$ for $\omega \in \Omega^1_{Y,f(a)}$ and under the isomorphism of Proposition 1.46, where $\omega \mathrel{\widehat{=}} \overline{g} \in \mathfrak{m}_{Y,f(a)}/\mathfrak{m}^2_{Y,f(a)}$, this is just $\overline{g} \mapsto \overline{g \circ f}$, since we have $\omega(\mathrm{D}_a f(\partial)) = \mathrm{D}_a f(\partial)(\overline{g}) = \partial(\overline{g \circ f})$.

Definition 1.56. We denote the map $\bigwedge^r f_a^* \colon \Omega^r_{Y,f(a)} \to \Omega^r_{X,a}$ again by f_a^* and define further $f^* \colon \Omega^1_Y(V) \to \Omega^1_X(f^{-1}(V))$ by $f^*\omega(x) := f_x^*(\omega(f(x)))$ for $V \subset Y$ open and $x \in f^{-1}(V)$.

Remark 1.57. The map f^* in Definition 1.56 is well-defined, since in a chart $y_1, \dots, y_m$ on Y around $f(a)$ we have $f_a^*(\mathrm{d}y_i)_{f(a)} = f_a^*(\overline{y_i}) = \overline{y_i \circ f} = (\mathrm{d}f_i)_a$, and hence if $\omega = \sum_{i_1 < \cdots < i_r} g_{i_1 \dots i_r} \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_r}$ we get $f^*\omega(a) = \sum_{i_1 < \cdots < i_r} g_{i_1 \dots i_r}(f(a))(\mathrm{d}f_{i_1})_a \wedge \cdots \wedge (\mathrm{d}f_{i_r})_a$. Using $(\mathrm{d}f_i)_a = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}|_a (\mathrm{d}x_j)_a$, for a chart $x_1, \dots, x_n$ around a, we see that $f^*\omega$ is again a differential p-form.

Example 1.58. If X and Y have the same dimension n and $r = n$, then we can compute $f^*\omega = (g \circ f)\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = (g \circ f) \det(\frac{\partial f_i}{\partial x_j})_{ij} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

1.2.3 Analytification of X/K

One can associate to a variety $\mathfrak{X}_K$ over a p-adic field K a topological space $\mathfrak{X}_K^{\mathrm{an}}$ in such a way that if $\mathfrak{X}_K$ is smooth over K the space $\mathfrak{X}_K^{\mathrm{an}}$ has the structure of a K-analytic manifold. This builds a bridge between the algebraic world of varieties and the analytic world of K-analytic manifolds. We will encounter later Weil's theorem (Theorem 2.17) which shows that in the Calabi–Yau case arithmetic data of $\mathfrak{X}$ over $\mathcal{O}_K$, namely the number of points in the reduction $\mathfrak{X}(\mathbb{F}_q)$, and analytic data, namely the volume of $\mathfrak{X}(\mathcal{O}_K) \subset \mathfrak{X}_K^{\mathrm{an}}$ are closely related. We begin this section by recalling the notion of K-rational and $\mathcal{O}_K$-integral points as well as the reduction map $\mathfrak{X}(\mathcal{O}_K) \to \mathfrak{X}(\mathbb{F}_q)$.

Definition 1.59. A variety is a separated scheme of finite type over a field k or over a discrete valuation ring $\mathcal{O}_K$.

Definition 1.60 (Rational and integral points)**.**

i) Let X be a scheme over a ring S and R an S-algebra, then we define the set $X(R) := \mathrm{Mor}_{\mathbf{Sch}/S}(\mathrm{Spec}(R), X)$.
ii) Let X be a scheme over a field k. We call $X(k)$ the set of k-rational points of X.
iii) Let $\mathfrak{X}$ be a scheme over a discrete valuation ring $\mathcal{O}_K$ with fraction field K. Then we call $\mathfrak{X}(\mathcal{O}_K)$ the set of $\mathcal{O}_K$-integral points of $\mathfrak{X}$, and we call $\mathfrak{X}(K)$ the K-rational points of $\mathfrak{X}$.

Remark 1.61. Note that if $\mathfrak{X}$ is a scheme over $\mathcal{O}_K$ then $X := \mathfrak{X}_K = \mathfrak{X} \times_{\mathcal{O}_K} K$ is a scheme over the field K. By the universal property of fiber products the K-rational points of X and the ones of $\mathfrak{X}$ coincide.

Remark 1.62. Let X be a scheme over a field k. Then we can view the k-rational points $X(k)$ as the subset of points of X with residue field k via the map $X(k) \to X$, $f \mapsto f(\mathrm{Spec(k)})$.

Proposition 1.63 ([Bat99, Remark 2.2])**.** *Let $\mathfrak{X}$ be a variety over a discrete valuation ring $\mathcal{O}_K$. Then the following hold.*

i) *There is a natural inclusion $\mathfrak{X}(\mathcal{O}_K) \hookrightarrow \mathfrak{X}(K)$.*
ii) *If $\mathfrak{X}$ is proper, we have $\mathfrak{X}(\mathcal{O}_K) = \mathfrak{X}(K)$ via the inclusion of* i).
iii) If $\mathfrak{X}$ is affine, we can identify $\mathfrak{X}(\mathcal{O}_K) = \{a \in \mathfrak{X}(K) \mid \forall f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) : f(a) \in \mathcal{O}_K\}$.

Proof. i) First note that an $\mathcal{O}_K$-integral point induces a K-rational point via composition with $\operatorname{Spec}(K) \to \operatorname{Spec}(\mathcal{O}_K)$. Take a K-rational point $f \in \mathfrak{X}(K)$ and consider it as a morphism. By the valuation criterion of separatedness (cf. [Har83, Theorem II.4.3]) there exists at most one diagonal morphism $\overline{f}$ in the following commutative diagram.

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r, "f"] \arrow[d] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec}(\mathcal{O}_K) \arrow[r, "\mathrm{id}"] \arrow[ur, dashed, "\overline{f}"] & \operatorname{Spec}(\mathcal{O}_K) \end{tikzcd}$$

That means for every $f \in \mathfrak{X}(K)$ there is at most one $\overline{f} \in \mathfrak{X}(\mathcal{O}_K)$ inducing f.

ii) Using the notation used in i) we see by the valuation criterion for properness (cf. [Har83, Theorem II.4.7]), that there exists a unique $\overline{f}$ inducing f.

iii) Write $\mathfrak{X} = \operatorname{Spec}(A)$, where $A = \mathcal{O}_K[x_1, \ldots, x_n]/\mathfrak{a}$. Now a K-rational point a corresponds to a map $A \otimes_{\mathcal{O}_K} K = K[x_1, \ldots, x_n]/\mathfrak{a}' \to K$ with kernel $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in K$. The point a is $\mathcal{O}_K$-integral if and only if the a_i are in $\mathcal{O}_K$. When we evaluate $f \in A$ at a we get $f(a) := \overline{f} \in (A \otimes_{\mathcal{O}_k} K)/\mathfrak{m} \simeq K$. We note that $f(a) = f(a_1, \ldots, a_n)$, where on the right hand side we evaluate f as a polynomial at the point $(a_1, \ldots, a_n)$. Hence, all a_i are elements of $\mathcal{O}_K$ if and only if all $x_i \in A$ evaluated at a are in $\mathcal{O}_K$. □

Remark 1.64.

i) Let $x \in \mathfrak{X}(\mathcal{O}_K)$ be an $\mathcal{O}_K$-integral point. By definition x is a morphism $\operatorname{Spec}(\mathcal{O}_K) \to \mathfrak{X}$ and we can consider the composition with $\operatorname{Spec}(\mathbb{F}_q) \to \operatorname{Spec}(\mathcal{O}_K)$. We call this composition the reduction of x modulo $\mathfrak{m}_K$ and write $\overline{x} \in \mathfrak{X}(\mathbb{F}_q)$ or $x \mod \mathfrak{m}_K$. In summery there is a map $\mathfrak{X}(\mathcal{O}_K) \to \mathfrak{X}(\mathbb{F}_q)$ called the reduction map.

ii) By abuse of notation we will sometimes identify a $\mathbb{F}_q$-rational point $\overline{x} \in \mathfrak{X}(\mathbb{F}_q)$ with its image in $\mathfrak{X}$. We will even associate sometimes to a $\mathcal{O}_K$-integral point $x \in \mathfrak{X}(\mathcal{O}_K)$ the point $x(\eta) \in \mathfrak{X}$, where $\eta \in \operatorname{Spec}(\mathcal{O}_K)$ is the generic point.

iii) Note that for $x \in \mathfrak{X}(\mathcal{O}_K)$ the reduction $\overline{x} \in \mathfrak{X}(\mathbb{F}_q)$ is in the closure of x considered as points of $\mathfrak{X}$. This is the case, since otherwise the generic point $\eta \in \operatorname{Spec}(\mathcal{O}_K)$ would be closed.

Remark 1.65. Recall that a Dedekind scheme is an integral, normal, locally Noetherian scheme of dimension 0 or 1. In particular, $\operatorname{Spec}(\mathcal{O}_K)$ is a Dedekind scheme when $\mathcal{O}_K$ is a discrete valuation ring.

Proposition 1.66. *Let X be a scheme over a Dedekind scheme Y with generic point η. Then X is flat over Y if and only if $X_\eta \subset X$ is dense.*

Proof. See [GW10, Proposition 14.14] and [Liu02, Lemma 4.3.7] for a proof. □

We now come to the analytification X^{an} of a smooth variety X. We do not restrict to the case where X is a variety over a p-adic field K, but also consider varieties over $\mathbb{R}$ or $\mathbb{C}$ (see also Remark 1.76). We will actually need the analytification of a variety over $\mathbb{C}$ in the statement of Batyrev's theorem (Theorem 3.1), or more precisely the topological space underlying X^{an}.

We choose to follow the structure and proofs of [Nee07, Chapter 4] loosely so that the reader who prefers a less condensed presentation can look up the proofs in [Nee07, Chapter 4] easily and we have the possibility to cite the proof of Propositio 1.70 in order to save space.

Situation 1. In this section, K denotes a p-adic field or the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Definition 1.67 (Strong topology). Let $X = \mathrm{V}((f_1, \dots, f_r)) \subset \mathbb{A}^n_K$. The strong topology on $X(K)$ is the subspace topology induced by the inclusion $X(K) \subset \mathbb{A}^n_K(K) = K^n$.

Proposition 1.68. *Let X be an affine scheme of finite type over K. Then $X(K)$ can be endowed with a topology such that for every presentation $X \simeq \mathrm{V}((f_1, \dots, f_r)) \subset \mathbb{A}^n_K$ the topology coincides with the strong topology. We call this topology on $X(K)$ again the strong topology.*

Proof. Compare to [Nee07, Lemma 4.5.3]. We choose some presentation of X and endow $X(K)$ with the strong topology of this presentation. We need to show that any other choice of presentation induces the same topology. Take two presentations $\mathrm{Spec}(K[x_1, \dots, x_n]/\mathfrak{a}) \simeq X \simeq \mathrm{Spec}(K[y_1, \dots, y_m]/\mathfrak{b})$. This isomorphism corresponds to an isomorphism of algebras $K[x_1, \dots, x_n]/\mathfrak{a} \xrightarrow{\sim} K[y_1, \dots, y_m]/\mathfrak{b}, x_i \mapsto f_i$, where we consider $f_i \in K[y_1, \dots, y_m]$. By considering the map $K[x_1, \dots, x_n] \to K[y_1, \dots, y_m], x_i \mapsto f_i$ we get the following diagram

$$\begin{tikzcd} & X(K) \arrow[dl, hook] \arrow[dr, hook] & \\ K^m \arrow[rr, "f"] & & K^n \end{tikzcd}.$$

Here f is continuous, since it is given by polynomials. This means the topology induced from K^n is finer then the one induced by K^m. By reversing the roles of the two presentations we conclude that the topologies coincide. □

Proposition 1.69. *Let X be an affine scheme of finite type over K. The strong topology on $X(K)$ is finer than the Zariski topology induced by the inclusion $X(K) \hookrightarrow \mathrm{X}$ from Remark 1.62.*

Proof. Compare to [Nee07, Lemma 4.5.4]. We only need to show that the inclusion $X(K) \hookrightarrow X$ is continuous, where X is endowed with the Zariski topology and $X(K)$ is endowed with the strong topology. Since both $X \subset \mathbb{A}^n_K$ and $X(K) \subset K^n$ have the subspace topologies it is enough to show that $K^n \to \mathbb{A}^n_K$ is continuous. For this recall that the standard open sets $\mathrm{D}(f) = \{\mathfrak{p} \in \mathrm{Spec}(K[x_1, \dots, x_n]) \mid f \notin \mathfrak{p}\}$, where $f \in K[x_1, \dots, x_n]$ form a basis of the Zariski topology of $\mathbb{A}^n_K$. Now we see that $\mathrm{D}(f) \cap K^n = \{x \in K^n \mid f(x) \neq 0\}$ is open in K^n, since polynomials are continuous. □

Proposition 1.70. *Let A be a finite type algebra over K, and let $f \in A$. Then $A \to A[1/f]$ induces an open embedding* $\mathrm{Spec}(A[1/f])(K) \to \mathrm{Spec}(A)(K)$ *onto the open set* $\mathrm{D}(f) \cap \mathrm{Spec}(A)(K)$.

Proof. See [Nee07, Proposition 4.5.10] for a proof. □

Proposition 1.71. *Let X be a scheme of finite type over K. Then there is a topology on $X(K)$ such that for every affine open $U \subset X$ the subspace topology on $U(K) \subset X(K)$ is the strong topology. We call this topology on $X(K)$ again the strong topology.*

Proof. Compare to [Nee07, Lemma 4.6.1]. Cover X by affine open subsets $\{U_i\}_i$ and endow each $U_i(K)$ with the strong topology. We check that the topologies on $U_i(K) \cap U_j(K)$ induced by $U_i(K)$ and $U_j(K)$ coincide in order to glue the topologies. To fix notation take $i = 1$ and $j = 2$. We can find for every $x \in U_1 \cap U_2$ an open neighborhood V of x such that V is a standard open in U_1 and U_2 (cf. [Nee07, Proposition 3.10.9]). Write $V = \mathrm{D}(f_i) \subset \mathrm{Spec}(A_i) = U_i$ for $i = 1, 2$. Then by Proposition 1.70 the strong topology on

$\mathrm{D}(f_i)(K)$ is the subspace topology coming from $\mathrm{Spec}(A_i) = U_i$. Since $\mathrm{D}(f_1) \simeq V \simeq \mathrm{D}(f_2)$ we see that $\mathrm{D}(f_1)(K)$ and $\mathrm{D}(f_2)(K)$ are homeomorphic.

To see that for every $U \subset X$ affine open the subspace topology on $U(K) \subset X(K)$ is the strong topology we note that we can just extend the above covering by including U. $\square$

Remark 1.72. Proposition 1.71 says that the strong topology on $X(K)$ is the weak topology with respect to the system $\{U(K) \hookrightarrow X(K) \mid U \subset X \text{ open affine}\}$, where $U(K)$ is endowed with the strong topology as in Proposition 1.68 (cf. [Nee07, Reminder 4.6.3]).

Notation 1.73. Let X be a variety over K. Then we denote $X(K)$ endowed with the strong topology by X^{an}.

Remark 1.74. Consider a morphism $f\colon \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ between varieties over K. When we embed $\mathrm{Spec}(A) \subset \mathbb{A}_K^N$ and $\mathrm{Spec}(B) \subset \mathbb{A}_K^M$ for some $N, M \geq 1$, then f is specified by polynomials $f_1, \ldots, f_M \in K[x_1, \ldots, x_N]$. This means that we get a continuous (with respect to the strong topology) map $f^{\mathrm{an}}\colon \mathrm{Spec}(A)(K) \to \mathrm{Spec}(B)(K)$. See [Nee07, Lemma 4.5.6] for more details.

As in the affine case a morphism $f\colon X \to Y$ between schemes of finite type over K induces a continuous map $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$.

This assignment is functorial, i.e. $(f \circ g)^{\mathrm{an}} = f^{\mathrm{an}} \circ g^{\mathrm{an}}$ and $\mathrm{id}_X^{\mathrm{an}} = \mathrm{id}_{X^{\mathrm{an}}}$.

Proposition 1.75. *Let X and Y be schemes of finite type over K. Then the following hold.*

i) $(X \times_K Y)^{\mathrm{an}} = X^{\mathrm{an}} \times Y^{\mathrm{an}}$.
ii) *If X is separated, i.e. a variety, then X^{an} is a Hausdorff space.*
iii) *X^{an} is a second-countable space.*

Proof. i) By the universal property of fiber products $(X \times_K Y)(K)$ and $X(K) \times Y(K)$ are equal as sets. To see that the topologies coincide, we look at the situation locally, i.e. we may assume $X = \mathrm{Spec}(A) \subset \mathbb{A}_K^n$ and $Y = \mathrm{Spec}(B) \subset \mathbb{A}_K^m$. Then

$$(X \times_K Y)(K) = \{(x, y) \in K^n \times K^m \mid \forall f \in A, g \in B : f(x) = 0 = g(y)\} = X(K) \times Y(K)$$

and both get their topology inherited from $K^n \times K^m$.

ii) Recall that the topological space X^{an} is Hausdorff if and only if the diagonal $\Delta := \{(x, y) \in X^{\mathrm{an}} \times X^{\mathrm{an}} \mid x = y\}$ is closed. Now X separated means that $\Delta_{X/K}(X) \subset X \times_K X$ is closed in the Zariski-topology. Now by Proposition 1.69 we see that

$$\Delta_{X/K}(X) \cap (X \times_K X)^{\mathrm{an}} \subset (X \times_K X)^{\mathrm{an}} = X^{\mathrm{an}} \times X^{\mathrm{an}}$$

is closed.

We will show that $\Delta = \Delta_{X/K}(X) \cap (X \times_K X)^{\mathrm{an}}$. Take $a \in \Delta$ and consider the situation locally in an affine open neighborhood. There the point a has coordinates $(a_1, \ldots, a_n, a_1, \ldots, a_n)$ and since the point corresponding to the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ is mapped to a under $\Delta_{X/K}$ we see that $a \in \Delta_{X/K}(X) \cap (X \times_K X)^{\mathrm{an}}$. For the converse inclusion take $a \in \Delta_{X/K}(X) \cap (X \times_K X)^{\mathrm{an}}$ and $b \in X$ such that $b \mapsto a$ under $\Delta_{X/K}$. Now locally the K-rational point a has coordinates $(a_1, \ldots, a_n, a'_1, \ldots, a'_n)$. The definition of $\Delta_{X/K}$ means that b verifies $(a_1, \ldots, a_n) \mathrel{\widehat{=}} \mathrm{pr}_1(a) = \mathrm{pr}_2(a) \mathrel{\widehat{=}} (a'_1, \ldots, a'_n)$, and hence $a \in \Delta$.

iii) Since X is of finite type over K it is in particular quasi-compact. Hence, we can cover X by finitely many affine open subschemes U_i. Since each $U_i(K)$ is endowed with the subspace topology from some affine space $\mathbb{A}_K^n(K) = K^n$ it suffices to note that K^n

is second-countable. Since products of second-countable spaces are second-countable and we can view K as a finite dimensional normed vector space over $\mathbb{Q}_q$, we only need to show that $\mathbb{Q}_p$ is second countable. The latter is satisfied, since $\mathbb{Q} \subset \mathbb{Q}_p$ is dense (cf. Proposition 1.6) and hence a separable metric space. □

Remark 1.76. If we would consider a topological field K that is *not* Hausdorff then Proposition 1.75.ii) cannot be true (consider the separated variety $\mathbb{A}^1_K$). It is the proof of Proposition 1.69 that fails. There we used that $\{0\} \subset K$ was closed, but this is true if and only if K is Hausdorff (cf. [Que01, Satz 16.17]). See [Con12] and [LS14] for a more general discussion of endowing the $X(R)$ with a topology, where X is a scheme of finite type over a topological ring R.

Theorem 1.77 (Jacobian criterion)**.** *Let k be a field and $X = \mathrm{V}((f_1, \dots, f_r)) \subset \mathbb{A}^n_k$. Then X is smooth at $x \in X(k)$ if and only if $\operatorname{rank}((\frac{\partial f_i}{\partial x_j}(x))_{ij}) = n - \dim \mathcal{O}_{X,x}$, where the x_i are coordinates on $\mathbb{A}^n_k$.*

Proof. See [Liu02, Theorem 4.2.19] and [Liu02, Exercise 4.3.20] for a proof. □

Remark 1.78.

i) On a locally Noetherian regular scheme the irreducible components and the connected components coincide, since regular local rings are integral domains (cf. [Liu02, Proposition 4.2.11]).
ii) Let X be an integral scheme of finite type over a field k and $x \in X$ a closed point. Then $\dim \mathcal{O}_{X,x} = \dim X$. See [Liu02, Proposition 2.5.23].

Proposition 1.79. *Let X be a smooth variety over K. Then X^{an} can be endowed with the structure of a K-analytic manifold*[1].

Proof. Since X is smooth and hence regular we can consider the irreducible components independently by Remark 1.78.i). Hence, we may assume that X is irreducible. Let $n = \dim X$ and note that we can cover X by open affine subschemes U_i of the form $\mathrm{V}((f_1^{(i)}, \dots, f_{N_i-n}^{(i)})) \subset \mathbb{A}^{N_i}_K$, since X is a local complete intersection (cf. [Har83, Theorem II.8.17]). Since X is quasi-compact, finitely many of the U_i cover X already.

First we assume $X = \mathrm{V}((f_1, \dots, f_{N-n})) \subset \mathbb{A}^N_K$ is affine and of the form described above. Now take a K-rational point $x \in X(K)$ and note that by Remark 1.78.ii) $n = \dim X = \dim \mathcal{O}_{X,x}$. So the Jacobian criterion (Theorem 1.77) implies that we have $\operatorname{rank}((\frac{\partial f_i}{\partial x_j}(x))_{ij}) = N - n$. We can assume that the minor $(\frac{\partial f_i}{\partial x_j}(x))_{i,j=1,\dots,N-n}$ is invertible by relabeling the coordinates. The implicit function theorem (Theorem 1.27) now tells us that there exist open subsets $U \subset K^n$, $V \subset K^{N-n}$, and a K-analytic map $g = (g_1, \dots, g_{N-n})$ on U such that $x \in V \times U$ and $\{(g(y), y) \in K^N \mid y \in U\} = \{(x, y) \in V \times U \mid f_i(x, y) = 0 \text{ for } i = 1, \dots, N-n\}$. Define $\varphi_x(y) \coloneqq (g(y), y)$. For another point $y \in X(K)$ the transition map $\varphi_x \circ \varphi_y^{-1}$ is K-analytic, since φ_y^{-1} is just a projection onto some coordinates and the components of φ_x are K-analytic functions.

We define $\mathcal{O}_X(U) \coloneqq \{f \colon U \to K \mid \forall \varphi_x : f \circ \varphi_x \text{ is } K\text{-analytic if it is defined}\}$ for $U \subset X^{\mathrm{an}}$ open. This defines indeed a sheaf $\mathcal{O}_X$ locally isomorphic to $\mathcal{O}_{K^n}|_V$ for suitable open sets $V \subset K^n$, since the transition maps are K-analytic.

We consider the general case now. We have just seen that every U_i is a K-analytic manifold. Since their pairwise intersections are isomorphic[2] as schemes, say $f_{ij} \colon U_i \xrightarrow{\sim} U_j$, and such isomorphisms are described locally by polynomial functions we see that the K-analytic manifolds U_i^{an} glue to a K-analytic manifold structure on X^{an} via the transition functions f_{ij}^{an}. □

[1] If $K = \mathbb{R}$ or $K = \mathbb{C}$ we consider smooth, respectively complex, manifolds instead.

[2] They are equal, but become just isomorphic when we consider the U_i as subsets of affine space.

Our next goal is to define the analytification ω^{an} of a Kähler differential form ω on a smooth variety X. This will be needed in the construction of the Weil measure (cf. Section 2.2). First we consider local parameters at a K-rational point $x \in X$ and see that they induce a chart on X^{an} around x. This and the characterization of local parameters in Proposition 1.82 will be handy later in the text.

Definition 1.80 (Local parameter)**.** Let X be a smooth variety over a field k or a discrete valuation ring $\mathcal{O}_K$ and let $x \in X$. We call $t_1, \dots, t_n \in \mathcal{O}_{X,x}$ local parameters at x if $\mathfrak{m}_{X,x} = (t_1, \dots, t_n)$ and $n = \dim \mathcal{O}_{X,x}$.

Proposition 1.81. *Let X and Y be schemes of finite type over a locally Noetherian scheme S, and let $f\colon X \to Y$ be a morphism over S. Then the canonical homomorphism $(f^*\Omega^1_{Y/S})_x \to (\Omega^1_{X/S})_x$ is an isomorphism if f is étale at x. The converse holds if X and Y are smooth at x and $y = f(x)$ respectively.*

Proof. See [Liu02, Proposition 6.2.10] for a proof[3]. □

Proposition 1.82. *Let X be a smooth variety of dimension n over a perfect field k, and let $f_1, \dots, f_n \in \mathcal{O}_X(U)$ for some open set $U \subset X$. Then the following are equivalent.*

i) *For all closed points $x \in U$ the $t_i := \overline{f_i - f_i(x)}$ generate $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$.*

ii) $\Omega^1_{X/k}|_U \simeq \bigoplus_{i=1}^n \mathcal{O}_X|_U \cdot \mathrm{d}f_i$.

iii) *The morphism $f_1 \times \dots \times f_n\colon U \to \mathbb{A}^n_k$ is étale.*

Proof. Compare to [Mum99, Theorem III.§6.1]. i) ⇔ ii) By [Liu02, Lemma 6.2.1] we know that for $x \in U$ closed the canonical homomorphism $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \to \Omega^1_{X/k,x} \otimes \mathrm{k}(x)$ is an isomorphism. So t_i generate $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ if and only if $\mathrm{d}f_i$ generate $\Omega_{X/k,x} \otimes \mathrm{k}(x)$. By Nakayama this is equivalent to $\mathrm{d}f_i$ generate $\Omega^1_{X/k}|_V$ for some open neighborhood V of x. Since by smoothness $\Omega^1_{X/k}$ is locally free we conclude that is is true if and only if the $\mathrm{d}f_i$ form a basis.

ii) ⇒ iii) Since the problem is local we can assume U is affine. Set $A := \mathcal{O}_U(U)$. Then $f := f_1 \times \dots \times f_n$ corresponds to the homomorphism $k[x_1, \dots, x_n] \to A, x_i \mapsto f_i$. Now the canonical map $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ looks like (cf. [Liu02, Proposition 6.1.8])

$$\begin{aligned} \bigoplus_{i=1}^n k \cdot \mathrm{d}x_i \otimes_{k[x_1,\dots,x_n]} A &\longrightarrow \bigoplus_{i=1}^n A \cdot \mathrm{d}f_i \\ \mathrm{d}x_i \otimes a &\longmapsto a\mathrm{d}f_i \end{aligned}$$

This means that we can apply Proposition 1.81 and conclude that f is étale.

iii) ⇒ ii) If f is étale then Proposition 1.81 implies that the $\mathrm{d}f_i$ generate $\Omega^1_{X/k,x}$ for all $x \in U$. As in the proof of "i) ⇔ ii)" we see that $\Omega^1_{X/k}|_U \simeq \bigoplus_{i=1}^n \mathcal{O}_X|_U \cdot \mathrm{d}f_i$. □

Remark 1.83. Proposition 1.82 (and its proof) tells us that if $t_1, \dots, t_n$ are local parameters at a closed point x of a variety X over a field k, then there is an open neighborhood U of x such that $t_1 \times \dots \times t_n\colon U \to \mathbb{A}^n_k$ is étale.

Proposition 1.84. *Let X be a smooth variety of dimension n over K, $x \in X(K)$ and $t_1, \dots, t_n$ local parameters at x. Then the t_i, viewed as K-analytic functions on an open neighborhood of x in X^{an}, define a chart $t := (t_1, \dots, t_n)$ centered at x.*

[3] Consult also the errata for [Liu02].

Proof. The problem is local, so we may assume that X is affine, i.e. $X \subset \mathbb{A}_K^m$ for some $m \geq 1$. By Proposition 1.82 we know that the $\mathrm{d}t_i$ form a basis of $\Omega^1_{X/K}|_V$ as an $\mathcal{O}_X|_V$-module for some open neighborhood V of x. Since X is affine we can write the t_i as polynomials in the coordinates x_j of $\mathbb{A}_K^m$. Now we have $\mathrm{d}t_i = \sum_{i=1}^m \frac{\partial t_i}{\partial x_j}\mathrm{d}x_j$ and we see that the matrix $(\frac{\partial t_i}{\partial x_j})_{ij}$ has full rank n on V.

Similarly to the proof of Proposition 1.79 we can use the smoothness hypothesis and the implicit function theorem (Theorem 1.27) to get a map $\varphi\colon U \to K^m$ defined on some open subset $U \subset K^n$. This map is the inverse of a chart and its derivative $(\frac{\partial \varphi_i}{\partial x_j}(0))_{ij}$ has rank n. It follows that t has an invertible derivative at 0 in the chart φ^{-1} and the inverse function theorem (Theorem 1.28) tells us that t defines a chart centered at x. □

Proposition 1.85. *Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$, let $\overline{x} \in \mathfrak{X}(\mathbb{F}_q)$ and let $\varpi, t_1, \ldots, t_n$ be local parameters at $\overline{x}$, where $\varpi \in \mathcal{O}_K$ is a uniformizing parameter. Then for every point $x \in \mathfrak{X}(\mathcal{O}_K)$ that reduces to $\overline{x}$ the map $t := (t_1, \ldots, t_n)$ defines a chart at x.*

Proof. Note that $\overline{x}$ is in the closure of x (cf. Remark 1.64.iii)), so every open neighborhood of $\overline{x}$ also contains x. In order to argue as in the step "i) $\Rightarrow$ ii)" of the proof of Proposition 1.82 we have to check that $\Omega^1_{\mathrm{k}(\overline{x})/\mathcal{O}_K} = 0$, since then $\mathfrak{m}_{\mathfrak{X},\overline{x}}/\mathfrak{m}^2_{\mathfrak{X},\overline{x}} \to \Omega^1_{\mathfrak{X}/\mathcal{O}_K,\overline{x}} \otimes \mathrm{k}(\overline{x})$ is surjective (cf. [Liu02, Proposition 6.1.8]). When we have done this the mentioned proof shows that $\Omega^1_{X/\mathcal{O}_K}|_U$ is generated by the basis $\mathrm{d}t_i$, where U is an open neighborhood of $\overline{x}$. Note that we have used $\mathrm{d}\varpi = 0$ here. Base changing form $\mathcal{O}_K$ to K we can use Proposition 1.84.

Now we compute that $\Omega^1_{\mathrm{k}(\overline{x})/\mathcal{O}_K} = 0$. Since $\overline{x}$ is a $\mathbb{F}_q$-rational point we can write locally $\mathfrak{m}_{\overline{x}} = \mathfrak{m}_K + (x_1 - a_1, \ldots, x_N - a_N)$, where $a_i \in \mathcal{O}_K$. It follows that $\mathrm{k}(\overline{x}) = \mathcal{O}_K/\mathfrak{m}_K$ and hence $\Omega^1_{\mathrm{k}(\overline{x})/\mathcal{O}_K} = \Omega^1_{(\mathcal{O}_K/\mathfrak{m}_K)/\mathcal{O}_K} = 0$. □

Proposition 1.86. *Let X be a smooth variety over K. Then there exists a natural map of $\mathcal{O}_X$-modules $\Omega^1_{X/K} \to \Omega^1_{X^{\mathrm{an}}}, \omega \mapsto \omega^{\mathrm{an}}$.*

Proof. Note that we can consider $\Omega^1_{X^{\mathrm{an}}}$ as an $\mathcal{O}_X$-module, since every Zariski-open set is also open in the strong topology and regular functions can be considered as K-analytic functions.

Denote the operator d of Example 1.53 now by d^{an}. The map $\Omega^1_{X/K} \to \Omega^1_{X^{\mathrm{an}}}$ is defined as the $\mathcal{O}_X$-linear extension of $\mathrm{d}f \mapsto \mathrm{d}^{\mathrm{an}}f$, where we consider the regular function f as a K-analytic function on the right hand side. This is indeed well-defined, since on every affine open subscheme $U \subset X$ the map $\mathcal{O}_X(U) \to \mathcal{O}_{X^{\mathrm{an}}}(U) \xrightarrow{\mathrm{d}^{\mathrm{an}}} \Omega^1_{X^{\mathrm{an}}}(U)$ is a K-derivation of $\mathcal{O}_X(U)$ into $\Omega^1_{X^{\mathrm{an}}}(U)$. So by the universal property of Kähler differentials we have the following commutative diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r, "\mathrm{d}^{\mathrm{an}}"] \arrow[d, "\mathrm{d}"] & \Omega^1_{X^{\mathrm{an}}}(U) \\ \Omega^1_{\mathcal{O}_X(U)/K} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

which guarantees that our definition makes sense. □

Remark 1.87.

i) By taking exterior powers we get a map $\Omega^r_{X/K} \to \Omega^r_{X^{\mathrm{an}}}, \omega \mapsto \omega^{\mathrm{an}}$.

ii) Consider local parameter $t_1, \ldots, t_n$ at $a \in X(K)$ and $\omega \in \Omega^n_{X/K}$. By Proposition 1.82 we can write $\omega|_U = f\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$ for some open neighborhood U of a. Now if $\omega(a) \neq 0$ we have $f(a) \neq 0$ and hence $\omega^{\mathrm{an}}(a) = f(a)(\mathrm{d}^{\mathrm{an}}t_1)_a \wedge \cdots \wedge (\mathrm{d}^{\mathrm{an}}t_n)_a \neq 0$.

1.3 Haar measure on p-adic fields

We recall the notion of Haar measure. For a more detailed exposition we refer to [Els11]. The general construction of Haar measures provides a measure on a p-adic field K and is a first step in the definition of the Weil measure, since it endows every chart of a K-analytic manifold with a 'good' measure.

Definition 1.88 (Haar measure)**.** Let G be a locally compact Hausdorff topological group. Denote by $\mathrm{Bo}(G)$ the σ-algebra of Borel sets, generated by the open subsets of G. A Haar measure on G is a measure $\mu\colon \mathrm{Bo}(G) \to [0, \infty]$ such that

i) it is non-trivial, i.e. $\mu \neq 0$,
ii) it is locally finite, i.e. for every $g \in G$ there exists $g \in U \subset G$ open such that $\mu(U) < \infty$,
iii) it is inner regular, i.e. for every $B \in \mathrm{Bo}(G)$ we have $\mu(B) = \sup\{\,\mu(K) \mid K \subset B \text{ compact}\,\}$,
iv) it is (left) translation-invariant, i.e. for every $g \in G$ and $B \in \mathrm{Bo}(G)$ we have $\mu(g \cdot B) = \mu(B)$.

Theorem 1.89 (Existence and uniqueness of Haar measures)**.** *Let G be a locally compact Hausdorff topological group, then there exists a Haar measure on G. This measure is unique up to a positive factor.*

Proof. See [Els11, Theorem VIII.3.12] for a proof. □

Notation 1.90. When a locally compact Hausdorff topological group G and a normalization (i.e. the positive factor in Theorem 1.89) are fixed we denote the Haar measure on G by μ_{Haar}.

Remark 1.91.

i) On $\mathbb{R}^n$ the Haar measure with normalization $\mu_{\mathrm{Haar}}([0,1]^n) = 1$ is the Lebesque measure λ^n.
ii) Let G and H be two locally compact Hausdorff topological groups with Haar measures μ_G and μ_H, respectively. Then the product measure $\mu_G \otimes \mu_H$ is a Haar measure on $G \times H$.[4] For a finite product $G^{\times n}$ we denote the product measure also by μ_G^n, or μ_{Haar}^n if the group G is clear from the context (cf. Notation 1.90).

Proposition 1.92. *Let G be a locally compact Hausdorff topological group with Haar measure μ_{Haar}. Then the following hold.*

i) *For every compact set $K \subset G$ we have $\mu_{\mathrm{Haar}}(K) < \infty$.*
ii) *For every open set $\emptyset \neq U \subset G$ we have $\mu_{\mathrm{Haar}}(U) > 0$.*

Proof. i) Since μ_{Haar} is locally finite we find for every $a \in K$ an open neighborhood $U_a \subset G$ of a with $\mu_{\mathrm{Haar}}(U_a) < \infty$. By compactness of K finitely many of those cover K, say $\{U_{a_i}\}_{i=1}^n$, and we conclude $\mu_{\mathrm{Haar}}(K) \leq \sum_{i=1}^n \mu_{\mathrm{Haar}}(U_{a_i}) < \infty$.

ii) Assume $\mu_{\mathrm{Haar}}(U) = 0$ and consider an arbitrary compact set $K \subset G$. Then K can be covered by finitely many $a_i \cdot U$ $(i = 1, \ldots, n)$ with $a_i \in K$ and hence $\mu_{\mathrm{Haar}}(K) \leq \sum_{i=1}^n \mu_{\mathrm{Haar}}(a_i \cdot U) = 0$. So all compact subsets of G have measure 0, but by inner regularity this means μ_{Haar} is trivial. This is a contradiction. □

Notation 1.93. Recall that every p-adic field K is a locally compact Hausdorff topological field with compact open ring of integers $\mathcal{O}_K$ (cf. Proposition 1.9.iii) and its proof). In the following we denote by μ_{Haar} the Haar measure on K with normalization $\mu_{\mathrm{Haar}}(\mathcal{O}_K) = 1$.

[4] This can be deduced using the Riesz–Markov representation theorem [Els11, Theorem VIII.2.5].

Proposition 1.94. *Let K be a p-adic field, where $\mathcal{O}_K$ has residue field $\mathbb{F}_q$.*
i) *For all $i \geq 1$ we have $\mu_{\mathrm{Haar}}(\mathfrak{m}_K^i) = q^{-i}$.*
ii) *The measure of a point $a \in K$ is zero with respect to the Haar measure.*

Proof. i) Note that $\mathfrak{m}_K \subset \mathcal{O}_K$ as a subgroup has index $(\mathcal{O}_K : \mathfrak{m}_K) = q$, since $\mathcal{O}_K/\mathfrak{m}_K \simeq \mathbb{F}_q$. The translation invariance of the Haar measure and the normalization $\mu_{\mathrm{Haar}}(\mathcal{O}_K) = 1$ imply that $\mu_{\mathrm{Haar}}(\mathfrak{m}_K) = 1/q$. Similarly, using induction, we see that $\mu_{\mathrm{Haar}}(\mathfrak{m}_K^i) = q^{-i}$.

ii) By translation invariance of the Haar measure we can assume $a = 0$. Now consider the balls $\mathrm{B}(0, q^{-i}) = \mathfrak{m}_K^i$ around 0 for $i \geq 0$. These have measure q^{-i} as seen in i). We note that $\{0\} = \bigcap_{i=0}^{\infty} \mathrm{B}(0, q^{-i})$ and hence $\mu_{\mathrm{Haar}}(\{0\}) = \lim_{i\to\infty} \mu_{\mathrm{Haar}}(\mathrm{B}(0, q^{-i})) = \lim_{i\to\infty} q^{-i} = 0$. □

Remark 1.95. Note that, since K is non-Archimedean every compact open subset $U \subset K^n$ can be written as a disjoint union of balls $\mathrm{B}(a_i, \varepsilon_i)$, where $a_i \in U$ and $\varepsilon_i > 0$ (cf. Proposition 1.11.ii)). Also note that the Borel σ-algebra is generated by the compact open sets, in fact the balls $\mathrm{B}(a_i, \varepsilon_i)$ are compact and open and form a basis of the topology. This means that the calculation in Proposition 1.94 characterized the Haar measure on K (cf. [Els11, Satz II.4.5]).

We have seen already that the implicit function theorem (Theorem 1.27) holds in the p-adic setting. The following theorem tells us that the transformation formula known from the Lebesque integral also works in the p-adic setting. This will allow us to define the Weil measure in analogy to the definition of measures one associates to volume forms on smooth manifolds.

Theorem 1.96 (Transformation formula)**.** *Let K be a p-adic field and $U, V \subset K^n$ be two open subsets. If $\varphi\colon U \to V$ is K-bianalytic, then for every integrable function $f\colon V \to \mathbb{R}$ the function $(f \circ \varphi) \cdot |\det(\frac{\partial \varphi_i}{\partial x_j})_{ij}|_p$ is integrable on U and we have*

$$\int_{\varphi(U)} f \,\mathrm{d}\mu_{\mathrm{Haar}}^n = \int_U (f \circ \varphi) \cdot |\det\left(\frac{\partial \varphi_i}{\partial x_j}\right)_{ij}|_p \,\mathrm{d}\mu_{\mathrm{Haar}}^n.$$

Proof. See [Igu00, Proposition 7.4.1] for a proof. □

Lemma 1.97. *Let K be a p-adic field and let V be a proper linear subspace of K^n. Then we have $\mu_{\mathrm{Haar}}^n(V) = 0$.*

Proof. Since linear maps are K-analytic the transformation formula (Theorem 1.96) allows us to assume $Y \subset \mathrm{span}(\mathrm{e}_1, \ldots, \mathrm{e}_{n-1})$. Here $(\mathrm{e}_i)_i$ is the standard basis of K^n. Now Remark 1.91.ii) tells us that

$$\mu_{\mathrm{Haar}}^n(V) \leq \mu_{\mathrm{Haar}}(K) \cdots \mu_{\mathrm{Haar}}(K) \cdot \mu_{\mathrm{Haar}}(\{0\}).$$

Since the measure of a point is zero (cf. Proposition 1.94.ii)), we conclude[5] $\mu_{\mathrm{Haar}}^n(V) = 0$. □

The rest of this section is devoted to the classification of compact K-analytic manifolds. An understanding of the following propositions is not necessary for comprehending the rest of the text. Nevertheless they demonstrate the usage of p-adic integrals in a simpler context than considered in Section 2.3. The reader may wish to read Section 2.2, especially the construction of Weil's measure (Construction 2.7), before looking at the proof of Theorem 1.99.

[5] Recall that in the context of measures a product is 0 if one factor is 0, even if some factors are ∞.

Proposition 1.98 ([Ser65, Théorème 1.a]). *Let K be a p-adic field with residue field $\mathbb{F}_q$ and let X be a compact K-analytic manifold of dimension $n \geq 1$. Then X is bianalytic to $\bigsqcup_{i=1}^{r} \mathcal{O}_K^n$ for some $0 < r < q$.*

Proof. Compare to [Igu00, Lemma 7.5.1]. Take an atlas $\{(U_i, \varphi_i)\}$ such that every U_i is compact. By compactness of X we can assume that we have only finitely many U_i and by replacing $U_1, U_2, \dots$ by $U_1, U_2 \setminus U_1, U_3 \setminus (U_1 \cup U_2), \dots$ we can assume the U_i to be pairwise disjoint. Since the U_i are compact, we can write $\varphi_i(U_i) = \bigsqcup_{j=1}^{N_i} \mathrm{B}(x_{ij}, q^{n_{ij}})$ for certain N_i, x_{ij} and n_{ij}. Each of these balls is bianalytic to $\mathcal{O}_K^n$ via translation and rescaling. We conclude that X is bianalytic to $\bigsqcup_{i=1}^{r} \mathcal{O}_K^n$ for some $r \geq 1$.

Now take a set of representatives $a_1, \dots, a_q \in \mathcal{O}_K$ of $\mathbb{F}_q$ and decompose $\mathcal{O}_K = \bigsqcup_{i=1}^{q} a_i + \varpi_K \mathcal{O}_K$, where $\varpi_K \in \mathcal{O}_K$ is a uniformizing parameter. This means that $\mathcal{O}_K^n \simeq \mathcal{O}_K^{n-1} \times \mathcal{O}_K \simeq \mathcal{O}_K^{n-1} \times \bigsqcup_{i=1}^{q} \mathcal{O}_K \simeq \bigsqcup_{i=1}^{q} \mathcal{O}_K^n$ and we can reduce r until we have $0 < r < q$. □

Theorem 1.99 ([Ser65, Théorème 1.b, Théorème 2]). *Let K be a p-adic field with residue field $\mathbb{F}_q$ and let X be a compact K-analytic manifold of dimension $n \geq 1$. Then there is a bianalytic invariant $0 < i(X) < q$ of X such that $X \simeq \bigsqcup_{i=1}^{i(X)} \mathcal{O}_K^n$. Moreover, for every $r > 0$ we have $i(\bigsqcup_{i=1}^{r} \mathcal{O}_K^n) \equiv r \mod q-1$.*

Proof. Compare to [Igu00, Theorem 7.5.1]. Consider a nowhere vanishing differential form $\omega \in \Omega_X^n(X)$ and associate to it a measure μ_ω as in Construction 2.7. We claim that if $\omega' \in \Omega_X^n(X)$ is another nowhere vanishing differential form then $\mu_\omega(X) \equiv \mu_{\omega'}(X) \mod q-1$. By compactness of X and the proof of Proposition 1.39 we may assume that we have an atlas $\{(U_i, \varphi_i)\}_i$ consisting of finitely many pairwise disjoint open sets U_i. In a chart $(U_i, \varphi_i) = (U, x_1, \dots, x_n)$ we can write $\omega|_U = f_U \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$ and $\omega'|_U = f'_U \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$, where f_U and f'_U are K-analytic functions without zeros. Since $\mathrm{im}(|\cdot|_p) = \{q^n \mid n \in \mathbb{Z}\} \subset \mathbb{R}$ is discrete, we may assume by subdividing U that $f_U(x) = q^{n_U}$ and $f'_U(x) = q^{m_U}$ for every $x \in U$ and fixed $n_U, m_U \in \mathbb{Z}$. Now

$$\mu_\omega(X) = \sum_i \int_{\varphi_i(U_i)} |f_{U_i} \circ \varphi_i^{-1}|_p \mathrm{d}\mu_{\mathrm{Haar}}^n = \sum_i q^{n_{U_i}} \mu_{\mathrm{Haar}}^n(\varphi_i(U_i))$$

and we see

$$\mu_\omega(X) - \mu_{\omega'}(X) = \sum_i (q^{n_{U_i}} - q^{m_{U_i}}) \mu_{\mathrm{Haar}}^n(\varphi_i(U_i)).$$

Since the U_i are compact (they are closed subsets of a compact space), we deduce that $\varphi_i(U_i)$ is a finite union of disjoint balls B_{ij}. Recall that $\mu_{\mathrm{Haar}}^n(B_{ij}) \in \{q^n \mid n \in \mathbb{Z}\}$ and hence we see that $\mu_{\mathrm{Haar}}^n(\varphi_i(U_i)) = \sum_{j=1}^{N_i} q^{k_{ij}}$ for suitable N_i and k_{ij}. This means that

$$\mu_\omega(X) - \mu_{\omega'}(X) \equiv 0 \mod q-1.$$

By Proposition 1.98 we have a bianalytic map $f\colon X \xrightarrow{\sim} \bigsqcup_{i=1}^{r} \mathcal{O}_K^n$ for some $0 < r < q$. We find a nowhere vanishing differential n-form ρ on $\bigsqcup_{i=1}^{r} \mathcal{O}_K^n$ by defining $\rho|_{\mathcal{O}_K^n} := \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$ on each copy of $\mathcal{O}_K^n$. We can pull back this form to get a nowhere vanishing differential n-form $\omega = f^*\rho$ on X. The transformation formula (Theorem 1.96) and Example 1.58 tell us that $\mu_\omega(X) = \mu_{f^*\rho}(X) = \mu_\rho(\bigsqcup_{i=1}^{r} \mathcal{O}_K^n)$. Using the definition of ρ, we calculate $\mu_\rho(\bigsqcup_{i=1}^{r} \mathcal{O}_K^n) = r\mu_{\mathrm{Haar}}^n(\mathcal{O}_K^n) = r$.

We define $0 < i(X) < q$ as the number that satisfies $i(X) \equiv \mu_\omega(X) \mod q-1$ for some (and hence all) nowhere vanishing differential n-form on X. □

Corollary 1.100. *Let K be a p-adic field with residue field $\mathbb{F}_q$ and let $n, r, k \geq 1$ be natural numbers. Then the K-analytic manifolds $\bigsqcup_{i=1}^{r} \mathcal{O}_K^n$ and $\bigsqcup_{i=1}^{k} \mathcal{O}_K^n$ are bianalytic if and only if $r \equiv k \mod q-1$.*

Proof. This follows directly from Theorem 1.99 and the proof of Proposition 1.98. □

2 p-adic integration on Calabi–Yau varieties

This section is devoted to the Weil measure on the analytification of a Calabi–Yau variety and Weil's theorem. We will finally be able to construct the Weil measure based on the foundations presented in Section 1.

2.1 Calabi–Yau varieties

Before we denote our attention to the Weil measure, let us define the notion of Calabi–Yau varieties and consider some examples, in order to get a feeling for the varieties we will be concerned with in the rest of this text.

Definition 2.1 (Calabi–Yau variety). A Calabi–Yau variety X is a proper, smooth variety over a field k such that its canonical bundle is trivial, i.e. $\det \Omega^1_{X/k} := \Omega^{\dim(X)}_{X/k} \simeq \mathcal{O}_X$. We call X a *strict* Calabi–Yau variety if in addition $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim(X)$.

Remark 2.2.

i) We will be concerned with not necessarily strict Calabi–Yau varieties.

ii) A strict Calabi–Yau variety of dimension two is also called a K3-surface.

Proposition 2.3. *Let us give a few examples of Calabi–Yau varieties.*

i) *Non-Example: The projective space $\mathbb{P}^n_k$ is* not *a Calabi–Yau variety, since its canonical bundle is* $\det \Omega^1_{\mathbb{P}^n_k/k} \simeq \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$.

ii) *A smooth hypersurface X in $\mathbb{P}^{d-1}_k$ of degree d is a strict Calabi–Yau variety.*

iii) *A smooth complete intersection X of dimension m of k hypersurfaces of degree $d_1, \ldots, d_k$ in $\mathbb{P}^{m+k}_k$ is a Calabi–Yau variety if and only if $\sum_{i=1}^k d_i = m + k + 1$.*

iv) *Abelian varieties are Calabi–Yau varieties. In dimension ≥ 2 they are* not *strict.*

v) *Integral Calabi–Yau varieties of dimension one with a k-rational point are exactly elliptic curves.*

Proof. i) Using the Euler sequence $0 \to \Omega^1_{\mathbb{P}^n_k/k} \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n_k} \to 0$ (cf. [Har83, Theorem II.8.13]) and taking determinants we get $\det \Omega^1_{\mathbb{P}^n_k/k} \simeq \det \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus n+1} = \mathcal{O}_{\mathbb{P}^n_k}(-(n+1))$.

ii) The short exact sequence

$$0 \to \mathcal{O}(-d) \to \mathcal{O}_{\mathbb{P}^{d-1}_k} \to \mathcal{O}_X \to 0 \tag{2.1}$$

induced by $1 \mapsto F$, where $X = \mathrm{V}_+(F)$ implies $\vartheta_X \simeq \mathcal{O}(-d)$. Since X is smooth, we also have the short exact sequence $0 \to \vartheta_X/\vartheta_X^2 \to \Omega^1_{\mathbb{P}^{d-1}_k/k}|_X \to \Omega^1_{X/k} \to 0$ (cf. [Har83, Theorem II.8.17]). Applying determinants yields

$$\mathcal{O}(-d)|_X = \det \Omega^1_{\mathbb{P}^{d-1}_k/k}|_X \simeq \det \Omega^1_{X/k} \otimes \det \vartheta_X/\vartheta_X^2 \simeq \det \Omega^1_{X/k} \otimes \det \mathcal{O}(-d)|_X$$

and we see $\det \Omega^1_{X/k} \simeq \mathcal{O}_X$.

In order to deduce the strict Calabi–Yau condition, we use the long exact sequence of cohomology (cf. [Har83, Theorem III.1.1A]) associated to the short exact sequence (2.1) and the fact that $\mathrm{H}^i(\mathbb{P}^{d-1}_k, \mathcal{O}(l)) = 0$ for $0 < i < d-1$ and all l (cf. [Har83, Theorem III.5.1]).

iii) See [Har83, Excercise II.8.4] for a proof.

iv) In fact the cotangent sheaf $\Omega^1_{X/k}$ is a free $\mathcal{O}_X$-module for any group scheme X over a field k (cf. [Stacks, Tag 047I] or [Mum08, Section II.4]). Since X is smooth as an abelian variety, we know that $\operatorname{rank} \Omega^1_{X/k} = \dim(X)$. This means that the canonical bundle $\det \Omega^1_{X/k}$ is trivial.

An abelian variety of dimension $n \coloneqq \dim(X) \geq 2$ is not a strict Calabi–Yau variety, since $\dim_k \mathrm{H}^r(X, \mathcal{O}_X) = \binom{n}{r}$ (cf. [Mum08, Corollary III.13.2])[6].

v) An elliptic curve X is defined as an integral, proper, smooth curve over a field k with geometric genus $\mathrm{p_g}(X) = 1$ and a chosen k-rational point. Now every elliptic curve is an abelian variety (cf. [Liu02, Page 298] or [Har83, Proposition IV.4.8]) and thus by iv) its canonical bundle $\Omega^1_{X/k}$ is trivial. Note that for dimension reasons it is a strict Calabi–Yau variety. Conversely, if $\Omega^1_{X/k} \simeq \mathcal{O}_X$, then $\mathrm{p_g}(X) = \dim_k \mathrm{H}^0(X, \Omega^1_{X/k}) = \dim_k \mathrm{H}^0(X, \mathcal{O}_X) = 1$. For the last equality apply [Liu02, Corollary 3.3.21] using that X is proper, reduced and connected and has a k-rational point. □

2.2 Weil measure and canonical measure

Now we come to the construction of the Weil measure as promised before. We will also encounter the canonical measure, which will be useful for proving a strengthening of Batyrev's theorem (cf. Theorem 3.19). We use the following notation in this subsection.

Notation 2.4. Let K be a p-adic field with ring of integers $\mathcal{O}_K$. For a smooth variety $\mathfrak{X}$ of relative dimension n over $\mathcal{O}_K$ we denote by $\Omega^n_{\mathfrak{X}/\mathcal{O}_K}$ the n-th exterior power of the sheaf of relative differentials $\Omega^1_{\mathfrak{X}/\mathcal{O}_K}$. We denote $\mathfrak{X} \times_{\mathcal{O}_K} K$ by X.

Definition 2.5 (Gauge form)**.** Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$. A gauge form is a differential form $\omega \in \mathrm{H}^0(\mathfrak{X}, \Omega^n_{\mathfrak{X}/\mathcal{O}_K})$ that vanishes nowhere.

Remark 2.6.

i) A choice of a gauge ω form is equivalent to a choice of a trivialization $\mathcal{O}_{\mathfrak{X}} \xrightarrow{\sim} \Omega^n_{\mathfrak{X}/\mathcal{O}_K}$, $1 \mapsto \omega$.
ii) Since $\Omega^n_{\mathfrak{X}/\mathcal{O}_K}$ is locally free of rank one, we can always find locally a gauge form. More precisely, for every $x \in \mathfrak{X}$ there exists an open neighborhood $\mathfrak{U} \subset \mathfrak{X}$ around x and a section $\omega_{\mathfrak{U}} \in \mathrm{H}^0(\mathfrak{U}, \Omega^n_{\mathfrak{X}/\mathcal{O}_K})$ such that $1 \mapsto \omega_{\mathfrak{U}}$ defines an isomorphism $\mathcal{O}_{\mathfrak{X}} \simeq \Omega^n_{\mathfrak{X}/\mathcal{O}_K}$.

Construction 2.7. Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$. Consider a gauge form $\omega \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/\mathcal{O}_K})$ on $\mathfrak{X}$. Since $\mathfrak{X}$ is smooth, X^{an} and ω_X^{an} are well-defined. Here, ω_X is the pullback of ω to X. Now take an atlas $\{(U_i, \varphi_i)\}_i$. On a chart (U, φ) write $\varphi = (x_1, \dots, x_n)$ and $\omega^{\mathrm{an}}|_U = f_U \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$, where $f_U \colon U \to K$ is K-analytic. Define $\mu_U(A) \coloneqq \int_{\varphi_U(A)} |f_U \circ \varphi_U^{-1}|_p \mathrm{d}\mu^n_{\mathrm{Haar}}$ for $A \in \mathrm{Bo}(U)$.

Proposition 2.8. *In the situation of Construction 2.7 let U and V be charts of the atlas. Then $\mu_U(A) = \mu_V(A)$ for every $A \in \mathrm{Bo}(U) \cap \mathrm{Bo}(V)$ and the measures glue to a Borel-measure μ on X^{an} independent of the choice of the atlas.*

Proof. We begin by fixing notation. Write $\varphi_U = (t_1, \dots, t_n)$, $\varphi_V = (s_1, \dots, s_n)$ and denote the coordinates on $\varphi_U(U \cap V)$ by $x_1, \dots, x_n$, and on $\varphi_V(U \cap V)$ by $y_1, \dots, y_n$. Define $\tilde{f}_U \coloneqq f_U \circ \varphi_U^{-1}$, $\tilde{f}_V \coloneqq f_V \circ \varphi_V^{-1}$ and denote the transition map by $\varphi_{UV} \coloneqq \varphi_V \circ \varphi_U^{-1}$. In this notation we have $\varphi_{UV}(\varphi_U(A)) = \varphi_V(A)$ and $\tilde{f}_V \circ \varphi_{UV} = f_V \circ \varphi_U^{-1}$.

Now by the transformation formula (Theorem 1.96) we have

$$\begin{aligned}\mu_V(A) = \int_{\varphi_V(A)} &|\tilde{f}_V|_p \,\mathrm{d}\mu^n_{\mathrm{Haar}}(y_1, \dots, y_n) \\ &= \int_{\varphi_U(A)} |(\tilde{f}_V \circ \varphi_{UV}) \det \left(\frac{\partial(\varphi_{UV})_i}{\partial x_j}\right)_{ij} |_p \,\mathrm{d}\mu^n_{\mathrm{Haar}}(x_1, \dots, x_n).\end{aligned}$$

[6] In characteristic 0 this can be deduced from Hodge symmetry.

By Remark 1.45 we have that $\mathrm{d}s_i = \sum_{j=1}^n \frac{\partial s_i}{\partial t_j}\mathrm{d}t_j$ and it follows that $\mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n = \det(\frac{\partial s_i}{\partial t_j})\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n$. Recall that in the notation introduced in Section 1.2.2 we have $\frac{\partial s_i}{\partial t_j} = \frac{\partial(s_i \circ \varphi_U^{-1})}{\partial x_j} \circ \varphi_U = \frac{\partial(\varphi_{UV})_i}{\partial x_j} \circ \varphi_U$. Since $f_U \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n = f_V \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n$ we see that $f_U = f_V \cdot \det(\frac{\partial(\varphi_{UV})_i}{\partial x_j} \circ \varphi_U)_{ij}$ and hence $\tilde{f}_U = (f_V \circ \varphi_U^{-1}) \det(\frac{\partial(\varphi_{UV})_i}{\partial x_j})_{ij}$. So we can conclude $\mu_V(A) = \mu_U(A)$.

Now we glue the measures μ_U to a measure μ on X^{an}. Since the topological space X^{an} is Hausdorff per definition and every point has a neighborhood basis consisting of open and closed sets (cf. the proof of Proposition 1.39), it follows that it is a regular topological space. We have also seen that X^{an} is second-countable (cf. Proposition 1.75.iii)) and hence paracompact (cf. [Que01, Korollar 10.7]). Thus we may choose a partition of unity $\{f_i\}_i$ subordinate to the covering $\{U_i\}_i$ (cf. [Que01, Theorem 10.3]). Now define for $A \in \mathrm{Bo}(X^{\mathrm{an}})$ the measure

$$\mu(A) := \sum_i \int_{U_i \cap A} f_i \,\mathrm{d}\mu_{U_i}.$$

This is indeed a measure, since $f_i \geq 0$ and hence the involved limits are in fact suprema and we can change their order freely. We remark that for $A \in \mathrm{Bo}(U_{i_0})$ we have $\mu(A) = \mu_{U_{i_0}}(A)$, since we have the following chain of equalities by the pairwise compatibility of the μ_{U_j}, the monotone convergence theorem (cf. [Els11, Theorem IV.2.7]) and the fact $\mathrm{supp}(f_j) \subset U_j$.

$$\begin{aligned}\mu(A) = \sum_j \int_{A \cap U_j} f_j \,\mathrm{d}\mu_{U_j} &= \sum_j \int_{A \cap U_{i_0}} f_j \,\mathrm{d}\mu_{U_{i_0}} \\ &= \int_{A \cap U_{i_0}} \sum_j f_j \,\mathrm{d}\mu_{U_{i_0}} = \int_{A \cap U_{i_0}} 1 \,\mathrm{d}\mu_{U_{i_0}} = \mu_{U_{i_0}}(A)\end{aligned}$$

In order to check the independence of the atlas and the partition of unity we take another atlas $\{(V_j, \psi_j)\}$ and a partition of unity g_j subordinate to the cover $\{V_j\}_j$. We calculate

$$\begin{aligned}\sum_i \int_{A \cap U_i} f_i \,\mathrm{d}\mu_{U_i} &= \sum_i \int_{A \cap U_i} (\sum_j g_j) f_i \,\mathrm{d}\mu_{U_i} = \sum_i \sum_j \int_{A \cap U_i \cap V_j} g_j f_i \,\mathrm{d}\mu_{U_i} \\ &= \sum_i \sum_j \int_{A \cap U_i \cap V_j} g_j f_i \,\mathrm{d}\mu_{V_j} = \sum_j \sum_i \int_{A \cap U_i \cap V_j} g_j f_i \,\mathrm{d}\mu_{V_j} = \cdots = \sum_j \int_{A \cap V_j} g_j \,\mathrm{d}\mu_{V_j}\end{aligned}$$

using monotone convergence, interchangeability of suprema, and pairwise compatibility of the μ_{U_i} and μ_{V_j}. □

Definition 2.9 (Weil measure)**.** We call the measure constructed in Proposition 2.8 the Weil measure on X^{an} and denote it by μ_{Weil} when it is clear which variety $\mathfrak{X}$ we are considering. Otherwise we also use the notation $\mu_{\mathfrak{X}}$ or μ_ω when we want to stress the variety or gauge form.

Proposition 2.10. *Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$. If $\omega, \rho \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/\mathcal{O}_K})$ are two gauge forms on $\mathfrak{X}$, then the Weil-measures associated to ω and ρ coincide on $\mathfrak{X}(\mathcal{O}_K) \subset X^{\mathrm{an}}$.*

Proof. Consider the isomorphisms $\omega \colon \mathcal{O}_{\mathfrak{X}} \xrightarrow{\sim} \Omega^n_{\mathfrak{X}/\mathcal{O}_K}, 1 \mapsto \omega$ and $\rho \colon \mathcal{O}_{\mathfrak{X}} \xrightarrow{\sim} \Omega^n_{\mathfrak{X}/\mathcal{O}_K}, 1 \mapsto \rho$. Then the composition $\rho^{-1} \circ \omega$ is determined by $1 \mapsto f$ for some $f \in \mathcal{O}_{\mathfrak{X}}^\times(\mathfrak{X})$ and we can write $\omega = f\rho \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/\mathcal{O}_K})$. Now in a chart $(U, x_1, \ldots, x_n)$ on X^{an} we can write

$\omega^{\mathrm{an}} = g\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ and $\rho^{\mathrm{an}} = h\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. We see that $g = fh$, where we consider f as a K-analytic function, and hence $|g(x)|_p = |f(x)|_p |h(x)|_p$ for every $x \in U$.

Since $f \in \mathcal{O}_{\mathfrak{X}}^{\times}(\mathfrak{X})$ we see that for $x \in \mathfrak{X}(\mathcal{O}_K)$ we have $f(x) \in \mathcal{O}_K$, i.e. $|f(x)|_p \leq 1$. When we denote by $\overline{x}$ the reduction modulo $\mathfrak{m}_K$ of x, then we see $\overline{f(x)} = f(\overline{x}) \neq 0$ in $\mathbb{F}_q$, since f vanishes nowhere. This means that $|f(x)|_p \geq 1$ and hence $|f(x)|_p = 1$. □

Construction 2.11. Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$. Take a finite cover $\{\mathfrak{U}_i\}_{i=1,\ldots,k}$ of $\mathfrak{X}$ such that on each $\mathfrak{U}_i$ we have $\mathcal{O}_{\mathfrak{U}_i} \simeq \Omega^n_{\mathfrak{X}/\mathcal{O}_K}|_{\mathfrak{U}_i}$ with $1 \mapsto \omega_i$ for some nowhere vanishing differential forms ω_i. We can associate to each ω_i a Weil measure μ_{ω_i} on $(\mathfrak{U}_i \times_{\mathcal{O}_k} K)^{\mathrm{an}}$. By Proposition 2.10 these measures coincide pairwise on $(\mathfrak{U}_i \cap \mathfrak{U}_j)(\mathcal{O}_K) = \mathfrak{U}_i(\mathcal{O}_K) \cap \mathfrak{U}_j(\mathcal{O}_K) \subset (\mathfrak{U}_i \times_{\mathcal{O}_k} K)^{\mathrm{an}}$. As in the proof of Proposition 2.8, these measures glue to a measure on $\mathfrak{X}(\mathcal{O}_K) = \bigcup_{i=1}^k \mathfrak{U}_i(\mathcal{O}_K)$.

Definition 2.12 (Canonical Measure)**.** The measure defined in Construction 2.11 is called the canonical measure on $\mathfrak{X}(\mathcal{O}_K)$ and is denoted by μ_{can}.

Remark 2.13. If $\mathfrak{X}$ admits a gauge form, then $\mu_{\mathrm{can}} = \mu_{\mathrm{Weil}}$. Indeed, Proposition 2.10 allows us to take the trivial cover $\{\mathfrak{X}\}$ of $\mathfrak{X}$ in Construction 2.11. Hence, by definition $\mu_{\mathrm{can}} = \mu_{\mathrm{Weil}}$.

As one would expect sets of codimension greater or equal to one are null sets with respect to the canonical measure (respectively Weil measure).

Lemma 2.14 ([Bat99, Theorem 2.8])**.** *Let $\mathfrak{X}$ be a smooth, integral variety over $\mathcal{O}_K$ for some p-adic field K and let $\mathfrak{Y} \subset \mathfrak{X}$ be a closed reduced subscheme of codimension* $\mathrm{codim}_{\mathfrak{X}}(\mathfrak{Y}) \geq 1$.

i) *For the canonical measure μ_{can} on $\mathfrak{X}(\mathcal{O}_K)$ we have $\mu_{\mathrm{can}}(\mathfrak{Y}(\mathcal{O}_K)) = 0$.*

ii) *If $\mathfrak{X}$ admits a gauge form, then we have $\mu_{\mathrm{Weil}}(\mathfrak{Y}(K)) = 0$.*

Proof. ii) We base change to K and denote $X = \mathfrak{X} \times_{\mathcal{O}_K} K$ and $Y = \mathfrak{Y} \times_{\mathcal{O}_K} K$. Since $\mathfrak{X}$ is flat over $\mathcal{O}_K$, we have $\mathrm{codim}_X(Y) \geq 1$ because otherwise by integrality[7] of X we would have $X = Y$ and hence $\mathfrak{X} = \mathfrak{Y}$ by Proposition 1.66.

Note that Y is a finite union of its irreducible components, since X is Noetherian as a scheme of finite type over K. By the additivity of measures we can assume that Y is irreducible.

We will now stratify Y into smooth schemes and proof the result by induction on $k = \dim(Y)$. The case $k = 0$ follows from Proposition 1.94.ii) and the definition of the Weil measure via the Haar measure (cf. Construction 2.7).

Now assume $k > 0$. The smooth locus $Y_{\mathrm{sm}} \subset Y$ is open and non-empty (cf. [Liu02, Lemma 4.2.21, Proposition 4.2.24, Remark 4.2.25, Corollary 4.3.33] and [Stacks, Tag 020I]). Since Y is irreducible, we have $\dim(Y \setminus Y_{\mathrm{sm}}) < k$ and hence by the induction hypothesis we get $\mu_{\mathrm{Weil}}((Y \setminus Y_{\mathrm{sm}})(K)) = 0$. Note that $\imath \colon Y_{\mathrm{sm}} \hookrightarrow X$ is unramified as an immersion (cf. [Liu02, Proposition 4.3.22]) and hence for all $y \in Y_{\mathrm{sm}}$ the canonical map $\mathrm{T}_{Y_{\mathrm{sm}},z} \to \mathrm{T}_{X,z}$ is injective (cf. [Stacks, Tag 0B2G]). This means that $\imath^{\mathrm{an}} \colon Y_{\mathrm{sm}}^{\mathrm{an}} \to X^{\mathrm{an}}$ is an immersion (in the sense of manifolds, cf. [Ser92, Section II.III.10.1]), i.e. X^{an} is covered by charts in which $\imath^{\mathrm{an}}$ looks like the standard embedding $K^k \hookrightarrow K^n$. Since X^{sm} is second-countable, we can assume the cover is countable and using Lemma 1.97 and the countable additivity of measures we conclude $\mu_{\mathrm{Weil}}(Y_{\mathrm{sm}}(K)) = \mu_{\mathrm{Weil}}(Y_{\mathrm{sm}}^{\mathrm{an}}) = 0$.

i) We can cover $\mathfrak{X}$ by open affine subschemes on which μ_{can} is a Weil measure (cf. Construction 2.11). Since $\mathfrak{X}$ is of finite type over $\mathcal{O}_K$ and in particular quasi-compact, we can take the cover to be finite. This means by additivity of measures that we can assume $\mathfrak{X}$ is one of the open affine subschemes, in particular we have $\mu_{\mathrm{can}} = \mu_{\mathrm{Weil}}$. Now we can apply statement i). □

[7] X is smooth and in fact connected.

2.3 Weil's theorem

This section is concerned with Weil's theorem. It is a central ingredient in the proof of Batyrev's theorem, where it will allow us to calculate certain local zeta functions of the reductions module $\mathfrak{m}_K$ of the varieties under consideration, by calculating the volumes of K-analytic manifolds associated to the varieties. In itself this connection between analysis and arithmetic provided by Weil's theorem is beautiful and constitutes the first climax of this text.

Lemma 2.15 (Hensel)**.** *Let K be a p-adic field and $f_1, \dots, f_n \in \mathcal{O}_K[T_1, \dots, T_n]$. If $x \in \mathcal{O}_K^{\oplus n}$ satisfies that all $f_i(x) \equiv 0 \mod \mathfrak{m}_K$ and the Jacobian $\det(\frac{\partial f_i}{\partial T_j}(x))$ is a unit in $\mathcal{O}_K$, then there exists a unique $x' \in \mathcal{O}_K^{\oplus n}$ with all $f_i(x') = 0$ and $x' \equiv x \mod \mathfrak{m}_K$.*

Proof. See [Mum99, Page 177] for a proof. □

Lemma 2.16. *Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$ for some p-adic field K with residue field $\mathbb{F}_q$. Then for every $\overline{x} \in \mathfrak{X}(\mathbb{F}_q)$ the fiber $F_{\overline{x}} := \{\, x \in \mathfrak{X}(\mathcal{O}_K) \mid \overline{x} \equiv x \mod \mathfrak{m}_K \,\}$ is bianalytic to $\mathfrak{m}_K^{\oplus n}$ via local parameters at $\overline{x}$.*

Proof. First recall that $\overline{x}$ (considered as a point of $\mathfrak{X}$ as in Remark 1.62) is in the closure of each $x \in F_{\overline{x}}$ (cf. Remark 1.64.iii)). So every open neighborhood of $\overline{x}$ in $\mathfrak{X}$ will contain $F_{\overline{x}}$. Take local parameters $\varpi_K, g_1, \dots, g_n \in \mathcal{O}_{\mathfrak{X},\overline{x}}$ at $\overline{x}$, where $\varpi_K \in \mathcal{O}_K$ is a uniformizing parameter, and let U be an affine open neighborhood of $\overline{x}$ such that all g_i are defined on U. Now write $U \simeq \mathrm{V}((f_1, \dots, f_r)) \subset \mathbb{A}^m_{\mathcal{O}_K} = \mathrm{Spec}(\mathcal{O}_K[x_1, \dots, x_m])$ and note that $(\frac{\partial g_i}{\partial x_j}(\overline{x}))_{i,j}$ has rank n, where we view the g_i as polynomials (cf. Proposition 1.82). Since $\mathfrak{X}$ is smooth and hence a local complete intersection, we can assume $r = m - n$ by shrinking U if necessary.

The Jacobian criterion tells us that $\mathrm{rk}(\frac{\partial f_i}{\partial x_j}(\overline{x})) = m - n$. We consider the matrix

$$M_x = \begin{pmatrix} \frac{\partial g_i}{\partial x_j}(x) \\ \hline \frac{\partial f_i}{\partial x_j}(x) \end{pmatrix} \in \mathrm{Mat}(m \times m, \mathrm{k}(x))$$

for $x \in F_{\overline{x}}$. In order that $M_{\overline{x}}$ has rank m we have to check that the rows are linear independent. Assume not, then there is a nontrivial linear combination

$$\sum_{i=1}^{n} \overline{\alpha}_i \begin{pmatrix} \frac{\partial g_i}{\partial x_1}(\overline{x}) \\ \vdots \\ \frac{\partial g_i}{\partial x_m}(\overline{x}) \end{pmatrix} = \sum_{j=1}^{m-n} \overline{\beta}_j \begin{pmatrix} \frac{\partial f_j}{\partial x_1}(\overline{x}) \\ \vdots \\ \frac{\partial f_j}{\partial x_m}(\overline{x}) \end{pmatrix}$$

for $\alpha_i, \beta_j \in \mathcal{O}_K$. Since the f_j vanish on U, we see that $\varpi_K, (g_1 - \sum_{j=1}^{m-n} \beta_j f_j), g_2, \dots, g_n$ are local parameters at $\overline{x}$, but their Jacobian matrix evaluated at $\overline{x}$ does not have rank n. This is a contradiction (cf. Proposition 1.82) and we conclude that $\det(M_{\overline{x}}) \neq 0 \in \mathbb{F}_q$. Hence, for every $x \in F_{\overline{x}}$, we have $\det(M_x)$ is a unit in $\mathcal{O}_K$.

We apply the generalized Hensel Lemma 2.15 and conclude that for arbitrary $(\alpha_i)_i \in \mathfrak{m}_K^{\oplus n}$ there exists a unique $x \in \mathcal{O}_K^{\oplus n}$ such that all $f_i(x) = 0$ and $g_i(x) - \alpha_i = 0$ and $x \equiv \overline{x} \mod \mathfrak{m}_K$. This means that $g_1, \dots, g_n$ define a bijection $F_{\overline{x}} \xrightarrow{\sim} \mathfrak{m}_K^{\oplus n}$ and, since $\varpi_K, g_1, \dots, g_n$ are local parameters, we see that this map is in fact bianalytic (cf. Proposition 1.84). □

Theorem 2.17 (Weil, [Wei82, Theorem 2.2.5])**.** *Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$ for some p-adic field K with residue field $\mathbb{F}_q$. Assume that $\omega \in$*

$\mathrm{H}^0(\mathfrak{X}, \Omega^n_{\mathfrak{X}/\mathcal{O}_K})$ *is a gauge form and denote by* μ_{Weil} *the corresponding Weil measure on* $\mathfrak{X}(K)$*. Then*

$$\mu_{\mathrm{Weil}}(\mathfrak{X}(\mathcal{O}_K)) = \frac{\#\mathfrak{X}(\mathbb{F}_q)}{q^n}.$$

Proof. Denote for $\overline{x} \in \mathfrak{X}(\mathbb{F}_q)$ the fiber over $\overline{x}$ by $F_{\overline{x}}$ as in Lemma 2.16. Since

$$\mu_{\mathrm{Weil}}(\mathfrak{X}(\mathcal{O}_K)) = \sum_{\overline{x} \in \mathfrak{X}(\mathbb{F}_q)} \mu_{\mathrm{Weil}}(F_{\overline{x}})$$

it is enough to see that $\mu_{\mathrm{Weil}}(F_{\overline{x}}) = \frac{1}{q^n}$.

By Lemma 2.16 we know that $F_{\overline{x}} \simeq \mathfrak{m}_K^{\oplus n}$ via local parameters $x_1, \dots, x_n$. Write $\omega = f \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$ on some open affine neighborhood $\mathfrak{U}$ around $\overline{x}$, where $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ form a basis of $\Omega^1_{\mathfrak{U}/\mathcal{O}_K}$ (cf. the proof of Proposition 1.85). Now $\mathfrak{U} \simeq \mathrm{Spec}(\mathcal{O}_K[x_1, \dots, x_N]/\mathfrak{a})$ and we can consider f as a polynomial in the variables $x_1, \dots, x_N$. This means that $f(a_1, \dots, a_N) \in \mathcal{O}_K$, i.e. $\nu_K(f(a_1, \dots, a_N)) \leq 1$, when all $a_i \in \mathcal{O}_K$. Also note that $f(\overline{x}) \neq 0$ implies for every $x \in F_{\overline{x}}$ that $\overline{f(x)} \neq 0 \in \mathbb{F}_q$, i.e. $\nu_K(f(x)) \geq 1$. This means that for every $x \in F_{\overline{x}}$ we have $|f^{\mathrm{an}}(x)|_p = 1$. Using the definition of Weil's measure we conclude

$$\mu_{\mathrm{Weil}}(F_{\overline{x}}) = \int_{\mathfrak{m}_K^{\oplus n}} |f|_p \mathrm{d}\mu^n_{\mathrm{Haar}} = \mu^n_{\mathrm{Haar}}(\mathfrak{m}_K^{\oplus n}) = \frac{1}{q^n}.$$ □

Corollary 2.18. *Let $\mathfrak{X}$ be a smooth variety of relative dimension n over $\mathcal{O}_K$ for some p-adic field K with residue field $\mathbb{F}_q$. Denote by μ_{can} the canonical measure on $\mathfrak{X}(\mathcal{O}_K)$. Then*

$$\mu_{\mathrm{can}}(\mathfrak{X}(\mathcal{O}_K)) = \frac{\#\mathfrak{X}(\mathbb{F}_q)}{q^n}.$$

Proof. Compare to [Bat99, Theorem 2.7]. Since $\mathfrak{X}$ is smooth over $\mathcal{O}_K$, the sheaf $\Omega^n_{\mathfrak{X}/\mathcal{O}_K}$ is locally free. Hence we can find an open cover $\mathfrak{U}_1, \dots, \mathfrak{U}_r \subset \mathfrak{X}$ on which $\Omega^n_{\mathfrak{X}/\mathcal{O}_K}$ trivializes. Then we have on each $\mathfrak{U}_i$ the identity $\mu_{\mathrm{can}} = \mu_{\mathrm{Weil}}$, where the Weil measure is induced by some trivializing section ω_i over $\mathfrak{U}_i$ (cf. Remark 2.13).

We apply the inclusion-exclusion principle for the canonical measure as well as for the counting measure and obtain the following equations.

$$\begin{aligned}\mu_{\mathrm{can}}(\mathfrak{X}(\mathcal{O}_K)) = \sum_{i_1} \mu_{\mathrm{can}}(\mathfrak{U}_{i_1}(\mathcal{O}_K)) - \sum_{i_1<i_2} \mu_{\mathrm{can}}((\mathfrak{U}_{i_1} \cap \mathfrak{U}_{i_2})(\mathcal{O}_K)) + \dots \\ + (-1)^{r+1} \mu_{\mathrm{can}}((\mathfrak{U}_1 \cap \dots \cap \mathfrak{U}_r)(\mathcal{O}_K))\end{aligned}$$

$$\begin{aligned}\#\mathfrak{X}(\mathbb{F}_q) = \sum_{i_1} \#\mathfrak{U}_{i_1}(\mathbb{F}_q) - \sum_{i_1<i_2} \#(\mathfrak{U}_{i_1} \cap \mathfrak{U}_{i_2})(\mathbb{F}_q) + \dots \\ + (-1)^{r+1} \#(\mathfrak{U}_1 \cap \dots \cap \mathfrak{U}_r)(\mathbb{F}_q)\end{aligned}$$

Now by Theorem 2.17 the right hand sides are termwise equal up to the factor q^n. □

3 Batyrev's theorem

The central theorem presented in this text is Batyrev's theorem. We will follow Batyrev's proof using p-adic integration and the Weil conjectures closely. A difference of our presentation here is that we separated the passage from the field of complex numbers $\mathbb{C}$ to a p-adic field K from the rest of the proof. Thus we will assume later in this section that our varieties are already defined over $\mathcal{O}_K$. The technical step of "spreading out" our varieties and "lifting" them to a p-adic field are presented in Section 4.

Theorem 3.1 (Batyrev's Theorem, [Bat99, Theorem 1.1])**.** *Let X and Y be two integral projective Calabi–Yau varieties over $\mathbb{C}$. If X and Y are birationally equivalent, then their Betti numbers coincide, i.e. for all $i \geq 0$ we have*

$$\dim_{\mathbb{C}} \mathrm{H}^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{H}^i_{\mathrm{sing}}(Y^{\mathrm{an}}, \mathbb{C}).$$

Remark 3.2. Recall that singular cohomology only considers the underlying topological spaces. This means Theorem 3.1 makes an assertion about a topological invariant.

Remark 3.3. Batyrev's theorem is in general not true if we drop the assumption that the canonical bundles are trivial. For example consider $X = \mathbb{P}^2_{\mathbb{C}}$ and $Y = \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$. Then X and Y are integral, smooth, projective varieties over $\mathbb{C}$ that are birationally equivalent. But $\Omega^2_{Y/\mathbb{C}} \simeq \det \Omega^1_{Y/\mathbb{C}} \simeq \det(p_1^* \Omega^1_{\mathbb{P}^1_{\mathbb{C}}/\mathbb{C}} \oplus p_2^* \Omega^1_{\mathbb{P}^1_{\mathbb{C}}/\mathbb{C}}) \simeq p_1^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$ is not trivial and also $\Omega^2_{X/\mathbb{C}} \simeq \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3)$ is not trivial. And indeed $\dim_{\mathbb{C}} \mathrm{H}^2_{\mathrm{sing}}(\mathbb{CP}^2, \mathbb{C}) = 1$ and $\dim_{\mathbb{C}} \mathrm{H}^2_{\mathrm{sing}}(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{C}) = 2$.

We recall a few well-known propositions, that will be used in the proof of Lemma 3.8 and later in the text.

Proposition 3.4. *Let X and Y be projective varieties over a field k. Assume that X is normal. Then every birational map $f\colon X \dashrightarrow Y$ is defined in codimension 1, i.e. f is defined on an open set $U \subset X$ such that $\operatorname{codim}_X(X \setminus U) \geq 2$.*

Proof. See [Har83, Lemma IV.5.1] for a proof. □

Proposition 3.5. *Let X be a normal, locally Noetherian scheme and let $A \subset X$ be a closed subset of codimension $\operatorname{codim}_X(A) \geq 2$. Then the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(X \setminus A)$ is an isomorphism.*

Proof. See [Liu02, Theorem 4.1.14] for a proof. □

Theorem 3.6 (Zariski's Main Theorem)**.** *Let $f\colon X \to Y$ be a separated birational morphism of finite type into an integral, normal, locally Noetherian scheme Y. If f is quasi-finite, then f is an open immersion.*

Proof. See [Liu02, Corollary 4.4.6] for a proof. □

Remark 3.7. We will use Theorem 3.6 only for varieties over a field k. So the conditions Y is Noetherian and f is separated and of finite type are automatically satisfied.

Lemma 3.8 ([Bat99, Proposition 3.1])**.** *Let X and Y be two integral, projective Calabi–Yau varieties of dimension n over $\mathbb{C}$. If X and Y are birationally equivalent then they are isomorphic in codimension one, i.e. there exist open subsets $U \subset X$ and $V \subset Y$ such that $U \simeq V$ and $\operatorname{codim}_X(X \setminus U) \geq 2$ as well as $\operatorname{codim}_Y(Y \setminus V) \geq 2$.*

Proof. Denote by $\varphi\colon X \dashrightarrow Y$ a birational map from X to Y. Since X and Y are projective and X is normal as a smooth variety, we see by Proposition 3.4 that the maximal open subset U, where φ is regular satisfies $\operatorname{codim}_X(X \setminus U) \geq 2$.

Now take trivializing sections $\omega_X \in \Omega^n_{X/\mathbb{C}}(X)$ and $\omega_Y \in \Omega^n_{Y/\mathbb{C}}(Y)$. We get[8] $\varphi|_U^*\omega_Y = h\omega_X|_U$ for some $h \in \mathcal{O}_X(U)$. By Proposition 3.5 we can extend h to a regular function $h \in \mathcal{O}_X(X)$. Since X is integral and projective, we have $\mathrm{H}^0(X, \mathcal{O}_X) = \mathbb{C}$ (cf. Proposition 4.17.i)). As φ is birational, it is an isomorphism between some open subsets of X and Y. This implies $h \neq 0 \in \mathbb{C}$.

We want to apply Proposition 1.81. Consider some $x \in U$ and bases $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ of $\Omega^1_{X/\mathbb{C},x}$ and $\{\mathrm{d}y_1, \ldots, \mathrm{d}y_n\}$ of $\Omega^1_{Y/\mathbb{C},\varphi(x)}$. Nakayama's Lemma tells us that it is enough to check that the natural map (cf. [Liu02, Proposition 6.1.24]) $\alpha\colon (\varphi^*\Omega^1_{Y/\mathbb{C}})_x \otimes_{\mathcal{O}_{X,x}} \mathrm{k}(x) \twoheadrightarrow (\Omega^1_{X/\mathbb{C}})_x \otimes_{\mathcal{O}_{X,x}} \mathrm{k}(x)$ is surjective, since then we get a surjection $(\varphi^*\Omega^1_{Y/\mathbb{C}})_x \twoheadrightarrow (\Omega^1_{X/\mathbb{C}})_x$ between free modules of rank n and again by Nakayama's Lemma this gives us the desired isomorphism. Now note that $(\varphi^*\mathrm{d}y_i)(x) := \varphi^*\mathrm{d}y_i \otimes 1 \in \varphi^*\Omega^1_{Y/\mathbb{C}} \otimes_{\mathcal{O}_{X,x}} \mathrm{k}(x)$ considered via α as an element of $\Omega^1_{X/\mathbb{C}} \otimes_{\mathcal{O}_{X,x}} \mathrm{k}(x)$ is just $\mathrm{d}(\varphi_x^{\#}(y_i))(x) := \mathrm{d}(\varphi_x^{\#}(y_i)) \otimes 1$. These latter elements are indeed linearly independent, otherwise $\mathrm{d}(\varphi_x^{\#}(y_1))(x) \wedge \cdots \wedge \mathrm{d}(\varphi_x^{\#}(y_n))(x) = 0$, and hence $0 = \varphi^*\omega_Y(x) = h\omega_X(x)$. This is a contradiction, since ω_X does not vanish anywhere and $h \neq 0$. Hence, we can apply Proposition 1.81 and get that $\varphi|_U$ is étale and in particular quasi-finite.

Zariski's Main Theorem (Theorem 3.6) tells us that $\varphi|_U$ is in fact an open immersion into the maximal open subset V, where φ^{-1} is regular. Exchanging the roles of X and Y we conclude that $\varphi|_U\colon U \to V$ and $\varphi^{-1}|_V\colon V \to U$ are open immersions. Since φ is a birational map, we have $(\varphi^{-1} \circ \varphi)|_{U'} = \mathrm{id}_{U'}$ for some open subset $U' \subset U$. Now we conclude $(\varphi^{-1} \circ \varphi)|_U = \mathrm{id}_U$ by using that X is integral and separated and applying [Liu02, Proposition 3.3.11]. This implies that $\varphi|_U\colon U \to V$ is an isomorphism. $\square$

We will see in Section 4 how to reduce the situation of Theorem 3.1 to the situation of Lemma 3.9 below, where we can apply Weil's Theorem (Theorem 2.17). The following diagram sketches the connection of the situations.

$$\begin{tikzcd}
X/\mathbb{C} \arrow[d, rightsquigarrow, "\text{spreading out}"'] & \\
X_0/A_0 \arrow[r, rightsquigarrow, "\text{lift}"'] & \mathfrak{X}/\mathcal{O}_K
\end{tikzcd}$$

Situation 2. In the diagram $A_0 \subset \mathbb{C}$ is a finitely generated $\mathbb{Z}$-algebra, X_0 is projective and smooth over A_0 and there is a prime ideal $\mathfrak{p} \subset A_0$ such that $A_0/\mathfrak{p} = \mathbb{F}_q$ and $(X_0)_\mathfrak{p}$ is geometrically connected. Furthermore, there is a p-adic field K and a morphism $\operatorname{Spec}(\mathcal{O}_K) \to \operatorname{Spec}(A_0)$ sending $\mathfrak{m}_K$ to $\mathfrak{p}$. We have $X_0 \times_{A_0} \mathbb{C} = X$ and $\mathfrak{X} = X_0 \times_{A_0} \mathcal{O}_K$. In particular, $(X_0)_\mathfrak{p} \simeq \mathfrak{X}_{\mathfrak{m}_K}$.

Lemma 3.9 ([Bat99, Theorem 1.1])**.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be proper, smooth varieties of relative dimension n over $\mathcal{O}_K$ such that the generic fibers $\mathfrak{X}_K$ and $\mathfrak{Y}_K$ are geometrically connected. Let $\emptyset \neq \mathfrak{U} \subset \mathfrak{X}$ and $\emptyset \neq \mathfrak{V} \subset \mathfrak{Y}$ be open subschemes. Assume that there is an isomorphism $\varphi\colon \mathfrak{U} \xrightarrow{\sim} \mathfrak{V}$ and that there exist $h \in \mathcal{O}_K^\times$ as well as gauge forms $\omega_\mathfrak{X} \in \Omega^n_{\mathfrak{X}/\mathcal{O}_K}(\mathfrak{X})$ and $\omega_\mathfrak{Y} \in \Omega^n_{\mathfrak{Y}/\mathcal{O}_K}(\mathfrak{Y})$ such that $\varphi^*(\omega_\mathfrak{Y}|_\mathfrak{V}) = h\omega_\mathfrak{X}|_\mathfrak{U}$. Then for every $r \geq 1$ we have*

$$\#\mathfrak{X}(\mathbb{F}_{q^r}) = \#\mathfrak{Y}(\mathbb{F}_{q^r}).$$

[8] We denote by $\varphi|_U^*$ the composition of natural maps $\Omega^1_{V/\mathbb{C}}(V) \to \varphi|_U^*\Omega^1_{V/\mathbb{C}}(U) \to \Omega^1_{U/\mathbb{C}}(U)$.

Proof. In the following we denote the Weil measure associated to a gauge form ω by μ_ω. We also use the notation $\omega_{\mathfrak{U}} := \omega_{\mathfrak{X}}|_{\mathfrak{U}}$ and $\omega_{\mathfrak{V}} := \omega_{\mathfrak{Y}}|_{\mathfrak{V}}$. The assumption $\varphi^*\omega_{\mathfrak{V}} = h\omega_{\mathfrak{U}}$ means that $\mu_{\varphi^*\omega_{\mathfrak{V}}} = \mu_{\omega_{\mathfrak{U}}}$ on $\mathfrak{U}(K)$, since h satisfies $|h|_p = 1$ as it is a unit in $\mathcal{O}_K$. Hence, by the transformation formula (Theorem 1.96, see also the formulation in [Pop11, Theorem 3.3.8])

$$\begin{aligned}\mu_{\omega_{\mathfrak{X}}}(\mathfrak{U}(K)) = \mu_{\omega_{\mathfrak{U}}}(\mathfrak{U}(K)) = \int_{\mathfrak{U}(K)} 1 \,\mathrm{d}\mu_{\omega_{\mathfrak{U}}} = \int_{\mathfrak{U}(K)} 1 \,\mathrm{d}\mu_{\varphi^*\omega_{\mathfrak{V}}} \\ = \int_{\mathfrak{V}(K)} 1 \,\mathrm{d}\mu_{\omega_{\mathfrak{V}}} = \cdots = \mu_{\omega_{\mathfrak{Y}}}(\mathfrak{V}(K)).\end{aligned}$$

Since $\mathfrak{X}_K$ (respectively $\mathfrak{Y}_K$) is connected and smooth it is in particular integral. Now $\mathfrak{X}$ (respectively $\mathfrak{Y}$) is irreducible, since by flatness $\mathfrak{X}_K \subset \mathfrak{X}$ is dense (cf. Proposition 1.66). This means that $\operatorname{codim}_{\mathfrak{X}}(\mathfrak{X} \setminus \mathfrak{U}) \geq 1$ and $\operatorname{codim}_{\mathfrak{Y}}(\mathfrak{Y} \setminus \mathfrak{V}) \geq 1$ because $\mathfrak{U}, \mathfrak{V} \neq \emptyset$. So we can apply Lemma 2.14 to see that[9]

$$\mu_{\omega_{\mathfrak{X}}}(\mathfrak{X}(K)) = \mu_{\omega_{\mathfrak{Y}}}(\mathfrak{Y}(K)).$$

Note that by properness we have $\mathfrak{X}(K) = \mathfrak{X}(\mathcal{O}_K)$ and $\mathfrak{Y}(K) = \mathfrak{Y}(\mathcal{O}_K)$ (cf. Proposition 1.63.ii)). Now Weil's theorem (Theorem 2.17) gives us $\#\mathfrak{X}(\mathbb{F}_q) = \#\mathfrak{Y}(\mathbb{F}_q)$.

For $r > 1$ consider an unramified extension $K^{(r)}$ as in Proposition 1.19 and base change our situation to $\mathcal{O}_{K^{(r)}}$. Since $\mathcal{O}_{K^{(r)}}/\mathfrak{m}_{K^{(r)}} = \mathbb{F}_{q^r}$, the above argument yields $\#\mathfrak{X}(\mathbb{F}_{q^r}) = \#\mathfrak{Y}(\mathbb{F}_{q^r})$ as desired. □

We can now reach the conclusion of Batyrev's theorem using the Weil Conjectures, proven by Deligne in [Del73]. See [Mus11] for an introduction to the statement of the Weil Conjectures. We recall only the statement of the parts of the Weil Conjectures that are relevant for Batyrev's theorem (cf. [Mus11, Section 4.3, Page 39]).

Definition 3.10. Let X' be a variety over the finite field $\mathbb{F}_q$. The Hasse–Weil zeta function, or local zeta function, of X' is the power series

$$\mathrm{Z}(X', t) := \exp\left(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m\right).$$

Remark 3.11. The conclusion of Lemma 3.9 is that the local zeta functions of the reduction modulo $\mathfrak{m}_K$ of $\mathfrak{X}$ and $\mathfrak{Y}$ are equal, i.e. $\mathrm{Z}(\mathfrak{X}_{\mathfrak{m}_K}, t) = \mathrm{Z}(\mathfrak{Y}_{\mathfrak{m}_K}, t)$.

Theorem 3.12 (Weil Conjectures)**.** *Let X' be a smooth, projective, geometrically connected variety of dimension n over the finite field $\mathbb{F}_q$. Then*

i) *the local zeta function of X' is a rational function of the form*

$$\mathrm{Z}(X', t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)}$$

, where $P_i \in \mathbb{Z}[t]$ and all roots α_{ij} of P_i have Euclidean absolute value $|\alpha_{ij}| = q^{-i/2}$.

ii) *Under the assumptions of Situation 2, and $X' := (X_0)_{\mathfrak{p}}$, we have*

$$\deg(P_i) = \dim_{\mathbb{C}} \mathrm{H}^i_{\mathrm{sing}}((X_0 \times_{A_0} \mathbb{C})^{\mathrm{an}}, \mathbb{C}).$$

[9] Note that here it is used that we consider K-rational points, in order to have $\mathfrak{X}(K) = \mathfrak{U}(K) \cup (\mathfrak{X} \setminus \mathfrak{U})(K)$. This in turn is the reason, why we want that h is constant, so that our measures $\mu_{\varphi^*\omega_{\mathfrak{V}}}$ and $\mu_{\omega_{\mathfrak{U}}}$ are indeed equal on the set of K-rational points of $\mathfrak{U}$ and not just on the $\mathcal{O}_K$-integral points of it.

Remark 3.13. Now if X and Y are varieties over $\mathbb{C}$, spread out to X_0 and Y_0 as in Situation 2, then $\mathrm{Z}((X_0)_\mathfrak{p}, t) = \mathrm{Z}((Y_0)_\mathfrak{p}, t)$ implies

$$\dim_\mathbb{C} \mathrm{H}^i_{\mathrm{sing}}((X_0 \times_{A_0} \mathbb{C})^{\mathrm{an}}, \mathbb{C}) = \dim_\mathbb{C} \mathrm{H}^i_{\mathrm{sing}}((Y_0 \times_{A_0} \mathbb{C})^{\mathrm{an}}, \mathbb{C}).$$

Indeed, using the Weil Conjectures (Theorem 3.12) we can write

$$\mathrm{Z}((X_0)_\mathfrak{p}, t) = \frac{P_1(t)P_3(t)\dots P_{2n-1}(t)}{P_0(t)P_2(t)\dots P_{2n}(t)} \quad \mathrm{Z}((Y_0)_\mathfrak{p}, t) = \frac{Q_1(t)Q_3(t)\dots Q_{2n-1}(t)}{Q_0(t)Q_2(t)\dots Q_{2n}(t)},$$

where $P_i, Q_i \in \mathbb{Z}[t]$. The condition on the absolute value of the roots implies $P_i = aQ_i$ for some $0 \neq a \in \overline{\mathbb{Q}}$. So $\deg(P_i) = \deg(Q_i)$, which implies by the Weil Conjectures the desired equality Betti numbers[10].

Proof of Theorem 3.1 (Batyrev's theorem). In conclusion this proves Batyrev's theorem modulo the spreading out and lifting mentioned in Situation 2. These remaining steps will be performed in the next section (cf. Lemma 4.14 and Lemma 4.23). □

3.1 Batyrev's theorem for K-equivalent varieties

Using the methods developed so far Batyrev proved a more general version of Theorem 3.1 concerning K-equivalent varieties instead of birationally equivalent Calabi–Yau varieties. We want to give a sketch of the proof of this generalization. For space reasons we will refer to the literature for details.

In this section we will denote the canonical bundle of a smooth variety X over a field k by ω_X. This notation should not be confused with the usage of ω before, where it denoted a differential form. We also want to remark that the "K" in "K-equivalence" does not refer to a p-adic field.

Definition 3.14 (K-equivalent varieties)**.** Let X and Y be integral, smooth, projective varieties over $\mathbb{C}$. Then X and Y are called K-equivalent if there exist a smooth projective variety Z over $\mathbb{C}$ and proper birational morphisms $f\colon Z \to X$, $g\colon Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$, where ω_X (respectively ω_Y) is the canonical bundle on X (respectively Y).

The next proposition tells us that the notion of K-equivalence is trivial in the case of curves or surfaces. In view of Proposition 3.16 this implies that Batyrev's theorem is clear for curves and surfaces. We refer to the introduction of [Bat99] for a remark on the three dimensional case.

Proposition 3.15.

i) *If X and Y are smooth integral projective birationally equivalent curves over k, then they are isomorphic.*
ii) *If X and Y are K-equivalent surfaces, then they are isomorphic.*

Proof. i) The category of smooth projective curves together with dominant morphisms is equivalent to the category of function fields of dimension 1 over k together with k-algebra homomorphisms (cf. [Har83, Corollary I.6.12]). On the level of objects this equivalence associates the function field $\mathrm{K}(C)$ to a curve C. Now X and Y are isomorphic, since they have the same function field.

ii) See [Pop11, Proposition 4.1.4] for a proof. □

[10] Note that for $i > 2n$ the respective singular cohomology groups are zero for dimension reasons.

Proposition 3.16. *Let X and Y be projective Calabi–Yau varieties over $\mathbb{C}$. If X and Y are birationally equivalent, then they are K-equivalent.*

Proof. Since both canonical bundles ω_X and ω_Y are trivial, it suffices to find a smooth projective variety Z over $\mathbb{C}$ and proper birational morphisms $f\colon Z \to X$, $g\colon Z \to Y$. Since X and Y are birationally equivalent, there exists an isomorphism $\varphi\colon U \to V$ between open subschemes $U \subset X$ and $V \subset Y$. Now consider the graph $\Gamma_\varphi \subset U \times_{\mathbb{C}} V$. We desingularize (cf. [Hir64]) its closure $\overline{\Gamma_\varphi} \subset X \times_{\mathbb{C}} Y$ endowed with the reduced induced scheme structure. That is we get a smooth variety Z over $\mathbb{C}$ together with a birational proper morphism $Z \to \overline{\Gamma_\varphi}$. Postcomposition with the projection $\mathrm{pr}_X\colon X \times_{\mathbb{C}} Y \to X$ (respectively $\mathrm{pr}_Y\colon X \times_{\mathbb{C}} Y \to Y$) gives the desired morphism f (respectively g). □

Lemma 3.17. *Let X and Y be K-equivalent varieties. Then X and Y are isomorphic in codimension one, i.e. there exist open subschemes $U \subset X$ and $V \subset Y$ such that $U \simeq V$ and $\operatorname{codim}_X(X \setminus U) \geq 2$ as well as $\operatorname{codim}_Y(Y \setminus V) \geq 2$.*

Proof. Compare to [Pop11, Lemma 4.1.6]. Let Z, f and g be as in the definition of K-equivalence (cf. Definition 3.14). By the ramification formula (cf. [Iit82, Theorem 5.5]) there exist effective divisors R_f and R_g on Z such that $\omega_Z \simeq f^*\omega_X \otimes_{\mathcal{O}_Z} \mathcal{O}(\mathrm{R}_f)$ and $\omega_Z \simeq g^*\omega_Y \otimes_{\mathcal{O}_Z} \mathcal{O}(\mathrm{R}_g)$. Furthermore, R_f (respectively R_g) is supported on the exceptional locus of f (respectively g) (cf. [Pop11, Chapter 3, Page 14]). Now we write $\mathrm{R}_f = \sum_{i=1}^k a_i E_i$ and $\mathrm{R}_g = \sum_{j=1}^l a'_j E'_j$ with $a_i, a'_j \geq 0$. The ramification formula together with $f^*\omega_X \simeq g^*\omega_Y$ imply $\sum_{i=1}^k a_i E_i \sim \sum_{j=1}^l a'_j E'_j$ are linearly equivalent. We can assume in the above linear equivalence that the E_i, E'_j are pairwise distinct by subtracting common terms. Now there exists a $h \in \mathrm{K}(Z)$ such that $\operatorname{div}(h) = \sum_{i=1}^k a_i E_i - \sum_{j=1}^l a'_j E'_j$. Since Z and Y are birationally equivalent, we can view h as an element of $\mathrm{K}(Y)$. Now the divisor $\operatorname{div}(h)$ on Y cannot have any poles, since $\operatorname{codim}_Y(g(E'_j)) \geq 2$ and away from R_g the morphism g is an isomorphism (cf. [Iit82, Proposition 5.8]). This means $h \in \mathrm{H}^0(Y, \mathcal{O}_Y) = \mathbb{C}$ and we conclude $\operatorname{div}(h) = 0$ on Z. In conclusion we see that X and Y are isomorphic in codimension 1. □

Remark 3.18. Note that Proposition 3.16 together with Lemma 3.17 provide an alternative proof of Lemma 3.8. The second proof is conceptually easier but depends on desingularization.

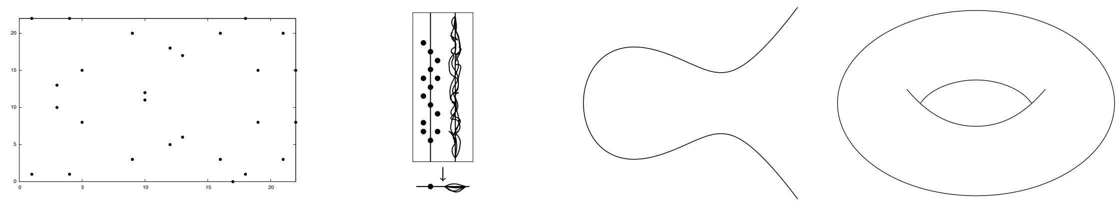
Theorem 3.19 ([Bat99, Theorem 4.2])**.** *Let X and Y be K-equivalent varieties. Then their Betti numbers coincide.*

Sketch. See [Ito03, Section 3.3–3.4] or [Pop11, Theorem 4.3.1] for details. First the situation is spread out and lifted to a p-adic field K similarly as in the proof of Batyrev's theorem (Theorem 3.1). Now we have $f\colon \mathfrak{Z} \to \mathfrak{X}$ and $g\colon \mathfrak{Z} \to \mathfrak{Y}$ satisfying $f^*\omega_{\mathfrak{X}} \simeq g^*\omega_{\mathfrak{Y}}$. We can 'pull back' the canonical measure on $\mathfrak{X}(\mathcal{O}_K)$ (respectively $\mathfrak{Y}(\mathcal{O}_K)$) to a measure $\mu_{\mathfrak{Z}/\mathfrak{X}}$ (respectively $\mu_{\mathfrak{Z}/\mathfrak{Y}}$) on $\mathfrak{Z}(\mathcal{O}_K)$ (cf. [Ito03, Section 3.3] for details). The condition $f^*\omega_{\mathfrak{X}} \simeq g^*\omega_{\mathfrak{Y}}$ and the equality of ramification divisors, seen in the proof of Proposition 3.17, imply that $\mu_{\mathfrak{Z}/\mathfrak{X}} = \mu_{\mathfrak{Z}/\mathfrak{Y}}$. Using the transformation formula (Theorem 1.96) and Lemma 2.14 we deduce $\mu_{\mathrm{can}}(\mathfrak{X}(\mathcal{O}_K)) = \mu_{\mathfrak{Z}/\mathfrak{X}}(\mathfrak{Z}(\mathcal{O}_K))$. Using a similar argument for $\mathfrak{Y}$ instead of $\mathfrak{X}$ we conclude $\mu_{\mathrm{can}}(\mathfrak{Y}(\mathcal{O}_K)) = \mu_{\mathfrak{Z}/\mathfrak{Y}}(\mathfrak{Z}(\mathcal{O}_K)) = \mu_{\mathfrak{Z}/\mathfrak{X}}(\mathfrak{Z}(\mathcal{O}_K)) = \mu_{\mathrm{can}}(\mathfrak{X}(\mathcal{O}_K))$. Now we can apply Corollary 2.18 and get $\#\mathfrak{X}(\mathbb{F}_q) = \#\mathfrak{Y}(\mathbb{F}_q)$. As in the proof Lemma 3.9 and using the Weil Conjectures (Theorem 3.12) we see that the the Betti numbers of X and Y coincide. □

4 Spreading out $X/\mathbb{C}$ and lifting to $\mathfrak{X}/\mathcal{O}_K$

In this section we demonstrate how we can spread out varieties over the field of complex numbers and subsequently lift them to a p-adic field. These technical steps fill in the missing details in our presentation of the proof of Batyrev's theorem in Section 3, where we assumed that we have already spread out and lifted our varieties.

Figure 2 depicts some of the Situations considered in the proof of Batyrev's theorem. In the figure X denotes a smooth curve over the field of complex numbers, X^{an} the associated complex manifold (Subfigure (d)), X_0 a spread out model over some finitely generated $\mathbb{Z}$-algebra A_0 (Subfigure (c)), $\mathfrak{X}$ a lift of X_0 to the ring of integers $\mathcal{O}_K$ of some p-adic field (Subfigure (b)), and $\mathfrak{X}_{\mathfrak{m}_K}$ the reduction modulo $\mathfrak{m}_K$ of $\mathfrak{X}$ (Subfigure (a)).



(a) $\mathfrak{X}_{\mathfrak{m}_K}/\mathbb{F}_q$ (b) $\mathfrak{X}/\mathcal{O}_K$ (c) X_0/A_0 (d) X^{an}

Figure 2: Some situations considered in the proof of Batyrev's theorem.

4.1 Projective systems of schemes

We will use the general theory of projective systems of schemes to perform the spreading out. One can certainly try to do the spreading out in an ad hoc way by looking at the defining equations of a variety and consider a field that includes all coefficients needed to write down the equations. But by not using the general theory of projective systems of schemes one is doomed to relate properties of a spread out model X_0 with the original variety X by hand instead of using the notion of "compatibility of properties with limits" (cf. Definition 4.11).

Nevertheless the basic idea behind the theory of projective systems of schemes as presented in this section is to assume suitable finiteness conditions (e.g. quasi-compact, finite presentation, quasi-coherence, finitely generated) so that the situation at hand is represented faithfully by finitely many "data" and then use this "data" to define a spread out model over one of the rings of the inductive system of rings over which the considered projective system lives (cf. Situation 3).

The theory is develop in detail and greater generality in [EGA IV$_3$, §8] and subsequent sections. We recall the basic propositions needed to use the theory of projective systems of schemes over an inductive system of rings for spreading out varieties. The presentation in this section is a summery of the statements in [GW10, Chapter 10]. In hindsight of our application we will assume our schemes to be Noetherian as this simplifies the finiteness assumptions in the propositions.

Before we begin, let us mention some applications other than "spreading out".

i) Replacing the field of definition of a variety by one, that is finitely generated over its prime field.
ii) Starting from a scheme defined over a fraction field $\mathrm{Q}(A)$, find a model that is defined over some localization A_f of A.
iii) Spreading out varieties defined over a field to a model over a finitely generated $\mathbb{Z}$-algebra. This can be understood as a combination of i) and ii).

iv) Eliminating Noetherian hypotheses. This is a variant of iii), where a ring A is written as the inductive limit of its finitely generated $\mathbb{Z}$-subalgebras (which are Noetherian by Hilbert's basis theorem).

v) "Expanding" properties valid at a stalk to an open neighborhood. Here we consider the stalk $A_{\mathfrak{p}}$ as the inductive limit over the open neighborhoods A_f with $f \notin \mathfrak{p}$.

The presentation in this section is quite formal. The following remark stresses the application we have in mind.

Remark 4.1. We can write $\mathbb{C}$ as the inductive limit $\varinjlim A_\lambda$, where $A_\lambda \subset \mathbb{C}$ are finitely generated algebras over $\mathbb{Z}$.

Let us fix some notation used in this section.

Situation 3.

i) Denote by Λ a filtered partially ordered set with unique minimal element 0.

ii) Let $(A_\lambda)_{\lambda \in \Lambda}$ be an inductive system of rings. Denote the transition maps by $\sigma_{\lambda\mu} \colon A_\lambda \to A_\mu$, define $A := \varinjlim A_\lambda$ and denote the natural maps $A_\lambda \to A$ by σ_λ.

iii) Let X_0 be a scheme over A_0 and set $X_\lambda := X_0 \times_{A_0} A_\lambda$, $X := X_0 \times_{A_0} A$. Denote the morphisms induced by $\sigma_{\lambda\mu}$ and σ_λ by $x_{\lambda\mu}$ and x_λ. For Y_0 we define Y_λ, Y, $y_{\lambda\mu}$ and y_λ similarly.

iv) If $f_0 \colon X_0 \to Y_0$ is a morphism, we denote the morphisms obtained by the base changes in iii) by $f_\lambda \colon X_\lambda \to Y_\lambda$ and $f \colon X \to Y$.

v) We assume the rings A_λ and schemes X_λ, Y_λ to be Noetherian[11].

Definition 4.2. Let X be a Noetherian topological space. A subset $C \subset X$ is called constructible if it is a finite union of locally closed sets in X.

Notation 4.3. Let T be a topological space. We denote by $\mathrm{Open}(T)$ the family of open subsets of T, i.e. the topology of T, and by $\mathrm{Con}(T)$ the family of constructible sets in T.

Remark 4.4. Since the morphisms $x_{\lambda\mu} \colon X_\mu \to X_\lambda$ and $x_\lambda \colon X \to X_\lambda$ are continuous, they induce maps $\mathrm{Open}(X_\lambda) \to \mathrm{Open}(X_\mu)$ and $\mathrm{Open}(X_\lambda) \to \mathrm{Open}(X)$ by taking inverse images of open sets. This gives us a map $\varinjlim \mathrm{Open}(X_\lambda) \to \mathrm{Open}(X)$. Similarly, we get a map $\varinjlim \mathrm{Con}(X_\lambda) \to \mathrm{Con}(X)$.

Proposition 4.5. *The maps* $\varinjlim \mathrm{Open}(X_\lambda) \to \mathrm{Open}(X)$ *and* $\varinjlim \mathrm{Con}(X_\lambda) \to \mathrm{Con}(X)$ *are bijective.*

Proof. See [GW10, Theorem 10.57] for a proof. □

Proposition 4.6. *Assume X is irreducible. Then all X_λ are irreducible if one of the following conditions is satisfied.*

i) *The morphisms $x_{\lambda\mu} \colon X_\mu \to X_\lambda$ are dominant.*

ii) *X_0 is flat over A_0 and the homomorphisms $\sigma_{\lambda\mu} \colon A_\lambda \to A_\mu$ are injective.*

Proof. i) See [GW10, Exercise 10.26].

ii) Locally, on affine open subschemes, the assumptions imply that the ring homomorphisms corresponding to the $x_{\lambda\mu}$ are injective. This in turn means that the $x_{\lambda\mu}$ are dominant and we can apply i). □

Remark 4.7. Let $\mathcal{F}_0$ and $\mathcal{G}_0$ be $\mathcal{O}_{X_0}$-modules. Define $\mathcal{F}_\lambda := x_{0\lambda}^*\mathcal{F}_0$ as well as $\mathcal{G}_\lambda := x_{0\lambda}^*\mathcal{G}_0$ for each $\lambda \in \Lambda$, and define $\mathcal{F} := x_\lambda^*\mathcal{F}_0$, $\mathcal{G} := x_\lambda^*\mathcal{G}_0$. Since the pull-back $x_{0\lambda}^*$ is a functor, we get a natural homomorphism $\varinjlim \mathrm{Hom}_{\mathbf{Mod}(X_\lambda)}(\mathcal{F}_\lambda, \mathcal{G}_\lambda) \to \mathrm{Hom}_{\mathbf{Mod}(X)}(\mathcal{F}, \mathcal{G})$.

[11] For a more general situation, without Noetherian hypotheses, see [GW10, Chapter 10]

Proposition 4.8. *If $\mathcal{F}_\lambda$ and $\mathcal{G}_\lambda$ are quasi-coherent, and $\mathcal{F}_\lambda$ is finite for some λ, then the natural map $\varinjlim \mathrm{Hom}_{\mathbf{Mod}(X_\lambda)}(\mathcal{F}_\lambda, \mathcal{G}_\lambda) \to \mathrm{Hom}_{\mathbf{Mod}(X)}(\mathcal{F}, \mathcal{G})$ is bijective.*

Proof. See [GW10, Theorem 10.58] for a proof, recalling that we assume our schemes to be Noetherian (Situation 3). □

Proposition 4.9. *Assume that X_0 and Y_0 are of finite type over A_0. Then*
i) *the natural map $\varinjlim \mathrm{Mor}_{\mathbf{Sch}/A_\lambda}(X_\lambda, Y_\lambda) \to \mathrm{Mor}_{\mathbf{Sch}/A}(X, Y)$ is bijective, and*
ii) *f is an isomorphism if and only if f_λ is an isomorphism for some λ.*

Proof. See [GW10, Theorem 10.63] and [GW10, Corollary 10.64] for a proof. □

Proposition 4.10. *Let Z be a scheme of finite type over A. Then there exists a λ, and a scheme Z_λ of finite type over A_λ such that $Z \simeq Z_\lambda \times_{A_\lambda} A$.*

Proof. See [GW10, Theorem 10.66] for a proof. □

Definition 4.11. Assume that X_0 and Y_0 are of finite type over A_0. We say a property $\mathbf{P}$ of morphisms of schemes is compatible with projective limits if the morphism $f\colon X \to Y$ satisfies $\mathbf{P}$ if and only if there is a $\lambda_0 \in \Lambda$ such that $f_\lambda\colon X_\lambda \to Y_\lambda$ satisfies $\mathbf{P}$ for all $\lambda \geq \lambda_0$ (cf. [GW10, Appendix C]).

Proposition 4.12. *The following properties of morphisms of schemes are compatible with projective limits.*
i) *"open immersion",*
ii) *"closed immersion",*
iii) *"flat", and*
iv) *"smooth".*

Proof. See [GW10, Proposition 10.75], [EGA IV_3, Théorème 11.2.6] and [EGA IV_4, Proposition 17.7.8] for a proof. □

Remark 4.13. For our applications, we could also use the generic flatness theorem ([EGA IV_2, Théorème 6.9.1]), and respectively for smoothness [EGA IV_3, Théorème 12.2.4].

4.2 Spreading out birationally equivalent Calabi–Yau varieties

We now apply the theory of projective systems of schemes recalled in Section 4.1 to the situation of two birationally equivalent Calabi–Yau varieties. Furthermore, we prove a technical lemma concerning the generic geometric connectedness of the spread out model X_0 over A_0, as required in Situation 2 and Lemma 3.9. In the following we use the inductive system of rings from Remark 4.1.

Lemma 4.14. *Let X and Y be projective Calabi–Yau varieties of relative dimension n over $\mathbb{C}$ and let $U \subset X$, $V \subset Y$ be open subsets such that $U \simeq V$ and $\mathrm{codim}_X(X \setminus U) \geq 2$ as well as $\mathrm{codim}_Y(Y \setminus V) \geq 2$. Then there exist a finitely generated $\mathbb{Z}$-algebra $A_0 \subset \mathbb{C}$, schemes X_0, Y_0 over A_0 and open subschemes $U_0 \subset X_0$, $V_0 \subset Y_0$ such that*
i) *$X_0 \times_{A_0} \mathbb{C} \simeq X$ and $Y_0 \times_{A_0} \mathbb{C} \simeq Y$,*
ii) *X_0 and Y_0 are projective and smooth of relative dimension n over A_0,*
iii) *$\Omega^n_{X_0/A_0} \simeq \mathcal{O}_{X_0}$ and $\Omega^n_{Y_0/A_0} \simeq \mathcal{O}_{Y_0}$,*
iv) *$U_0 \simeq V_0$, and*
v) *$\mathrm{codim}_{X_0}(X_0 \setminus U_0) \geq 2$ and $\mathrm{codim}_{Y_0}(Y_0 \setminus V_0) \geq 2$.*

Proof. In the following we will increase $\lambda \in \Lambda$ several times, i.e. base change to larger $\mathbb{Z}$-algebras. We will suppress this in the notation and will just redefine λ. This will not be a problem, since assertions i)–iv) are stable under base change.

i) We use Remark 4.1 together with Proposition 4.10 to get a finite type $\mathbb{Z}$-algebra $A_\lambda \subset \mathbb{C}$ and schemes X_λ, Y_λ of finite type over A_λ such that $X_\lambda \times_{A_\lambda} \mathbb{C} \simeq X$ and $Y_\lambda \times_{A_\lambda} \mathbb{C} \simeq Y$.

ii) Since closed immersions and smothness is compatible with projective limits (cf. Proposition 4.12) we deduce that (after increasing λ if necessary) X_λ and Y_λ are projective and smooth over A_λ. To see that the relative dimension is n we can use that for morphism locally of finite type the relative dimension is stable under base change (cf. [Stacks, Tag 02NK]). Alternatively we can use assertion iii) by noting that if the relative dimension is not n, then $\Omega^n_{X_\lambda/A_\lambda}$, respectively $\Omega^n_{Y_\lambda/A_\lambda}$, cannot have rank one (cf. [Liu02, Proposition 6.2.5]).

iii) Note that $\Omega^n_{X_\lambda/A_\lambda}$ and $\mathcal{O}_{X_\lambda}$ pull back to $\Omega^n_{X/\mathbb{C}}$ and $\mathcal{O}_X$ under the base change to $\mathbb{C}$ (cf. [Liu02, Proposition 6.1.24]). Since $\Omega^n_{X_\lambda/A_\lambda}$ and $\mathcal{O}_{X_\lambda}$ are both coherent sheaves (cf. [Liu02, Proposition 6.1.20]), we can apply Proposition 4.8 and see that (after increasing λ if necessary) $\Omega^n_{X_\lambda/A_\lambda} \simeq \mathcal{O}_{X_\lambda}$. Similarly, we have $\Omega^n_{Y_\lambda/A_\lambda} \simeq \mathcal{O}_{Y_\lambda}$.

iv) By Propostion 4.5 there exist open subsets $U_\lambda \subset X_\lambda$ and $V_\lambda \subset Y_\lambda$ (after increasing λ if necessary) such that $U_\lambda \times_{A_\lambda} \mathbb{C} \simeq U$ and $V_\lambda \times_{A_\lambda} \mathbb{C} \simeq V$. Now $U_\lambda \simeq V_\lambda$ (after increasing λ if necessary) follows from $U \simeq V$ using Proposition 4.9.

v) We define $C := X \setminus U$. The condition $\operatorname{codim}_X(C) \geq 2$ means that there are irreducible closed sets $C^{(0)}$, $C^{(1)}$, $C^{(2)}$ in X such that $C \subset C^{(0)} \subsetneq C^{(1)} \subsetneq C^{(2)} \subset X$. Using Proposition 4.5 we get (after increasing λ if necessary) closed sets $C_\lambda \subset C^{(0)}_\lambda \subsetneq C^{(1)}_\lambda \subsetneq C^{(2)}_\lambda \subset X_\lambda$ and we can assume $C_\lambda = X_\lambda \setminus U_\lambda$. Noting that the $C^{(i)}$ are flat over the field $\mathbb{C}$, we can use Proposition 4.12.iii) to see that the $C^{(i)}_\lambda$ are flat over A_λ (after increasing λ if necessary). Applying Proposition 4.6 we can assume that the $C^{(i)}_\lambda$ are irreducible (after increasing λ if necessary). This shows that $\operatorname{codim}_{X_\lambda}(X_\lambda \setminus U_\lambda) \geq 2$. Similarly, we get $\operatorname{codim}_{Y_\lambda}(Y_\lambda \setminus V_\lambda) \geq 2$.

In conclusion, by writing "$0 := \lambda$", we have found the desired $\mathbb{Z}$-algebra A_0 and schemes X_0, Y_0, U_0 and V_0 over A_0. □

Remark 4.15. We can assume that A_0 in Lemma 4.14 is a regular algebra. Indeed, it suffices to note that there is an open regular subscheme $\emptyset \neq \mathrm{D}(f) \subset \operatorname{Spec}(A_0)$, since we can achieve this localization by replacing A_0 by a suitable A_λ. Note that the set of regular points $\operatorname{Reg}(A_0) \subset \operatorname{Spec}(A_0)$ is open, since A_0 is an algebra of finite type over $\mathbb{Z}$ and the latter is an excellent ring (cf. [Liu02, Corollary 8.2.40.(c)]). In fact the proof of [Liu02, Corollary 8.2.40.(c)] also shows that if A_0 is integral, then $\operatorname{Reg}(A_0)$ is non-empty.

We want to remark that we are following the sketch of the spreading out argument in Batyrev's original proof of his theorem. One can be slightly more efficient by using the codimension condition and Proposition 3.5 before spreading out. This would mean that we don't have to control the codimension of $X_0 \setminus U_0$ (respectively $Y_0 \setminus V_0$) in Lemma 4.14. It would also make Proposition 4.17.ii) in the next section superfluous.

Proposition 4.16. *Let $X \to S$ be a projective, flat morphism of Noetherian schemes and assume S is integral. If the generic fiber X_η is geometrically connected and geometrically reduced, then there exists an open subset $\emptyset \neq U \subset S$ such that all fibers over U are geometrically connected.*

Proof. Using [Liu02, Corollary 3.3.21] we see that $\mathrm{H}^0(X_\eta, \mathcal{O}_{X_\eta}) \simeq \mathrm{K}(S)$. Now we use the semicontinuity theorem ([Har83, Theorem III.12.8]) to get that the function $s \mapsto$

$\mathrm{h}^0(s, \mathcal{O}_X) := \dim_{\mathrm{k}(s)} \mathrm{H}^0(X_s, \mathcal{O}_{X_s})$ is upper semi-continuous on S. Hence, we see that the set $\{s \in S \mid \mathrm{h}^0(s, \mathcal{O}_X) \leq 1\}$ is open and non-empty, since η lies inside it. Since $\mathrm{h}^0(s, \mathcal{O}_X) \geq 1$, we see that $U := \{s \in S \mid \mathrm{h}^0(s, \mathcal{O}_X) = 1\}$ is open and non-empty.

Note that for $s \in U$, i.e. $\mathrm{H}^0(X_s, \mathcal{O}_{X_s}) \simeq \mathrm{k}(s)$, we have by the flat base change theorem ([Liu02, Corollary 5.2.27]) that

$$\mathrm{H}^0(X_{\overline{s}}, \mathcal{O}_{X_{\overline{s}}}) \simeq \mathrm{H}^0(X_s, \mathcal{O}_{X_s}) \otimes_{\mathrm{k}(s)} \overline{\mathrm{k}(s)} \simeq \overline{\mathrm{k}(s)}.$$

This means that X_s is geometrically connected. □

4.3 Lift to p-adic fields

The last step we have to perform is to lift our spread out situation to a p-adic field. This will be done in this section.

Proposition 4.17. *Let K be a p-adic field.*

i) *Let X be a proper variety over a field k. If X is geometrically reduced and geometrically connected then $\mathrm{H}^0(X, \mathcal{O}_X) = k$.*

ii) *Let $\mathfrak{X}$ be a proper, flat variety over $\mathcal{O}_K$. If $\mathrm{H}^0(\mathfrak{X}_K, \mathcal{O}_{\mathfrak{X}_K}) = K$ then $\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \mathcal{O}_K$.*

Proof. i) See [Liu02, Corollary 3.3.21] for a proof.

ii) Recall that a module M over a principal ideal domain A is flat if and only if it is torsion-free over A (cf. [Liu02, Corollary 1.2.5]). Consider $M = \mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and $A = \mathcal{O}_K$. For every $f \neq 0 \in \mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ there exists an open affine subscheme $\mathrm{Spec}(B) = \mathfrak{U} \subset \mathfrak{X}$ such that $f|_{\mathfrak{U}} \neq 0 \in \mathrm{H}^0(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}}) = B$. Now $\mathfrak{X}$ flat over $\mathcal{O}_K$ implies that B is flat over $\mathcal{O}_K$ and hence the latter is torsion-free over $\mathcal{O}_K$. This means that f cannot be a torsion element and we conclude that $\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is torsion-free and hence flat over $\mathcal{O}_K$.

By Serre's Theorem (cf. [Liu02, Theorem 5.3.2]) we know that $\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a finitely generated $\mathcal{O}_K$-module. This means that $\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is in fact free of finite rank, say r, over the local ring $\mathcal{O}_K$ (cf. [Liu02, Theorem 1.2.16]).

Now we consider the generic fiber $\mathfrak{X}_K$, i.e. the base change to $\mathrm{Spec}(K)$. Since this base change is flat as a localization, we conclude $\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_K} K \simeq \mathrm{H}^0(\mathfrak{X}_K, \mathcal{O}_{\mathfrak{X}_K})$ by [Liu02, Corollary 5.2.27]. By assumption $\dim_K \mathrm{H}^0(\mathfrak{X}_K, \mathcal{O}_{\mathfrak{X}_K}) = 1$ and hence

$$1 = \dim_K(\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \otimes_{\mathcal{O}_K} K) = \dim_K(\mathcal{O}_K^r \otimes_{\mathcal{O}_K} K) = r.$$

This means $\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \mathcal{O}_K$. □

Remark 4.18. In particular, Proposition 4.17 says that for a smooth, proper variety $\mathfrak{X}$ over $\mathcal{O}_K$ with $\mathfrak{X}_K$ geometrically connected we have $\mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \mathcal{O}_K$.

Proposition 4.19. *Let A be a finitely generated algebra over $\mathbb{Z}$ of characteristic 0 and let $\mathfrak{p} \in \mathrm{Spec}(A)$ be a regular point with $\mathrm{k}(\mathfrak{p}) \simeq \mathbb{F}_q$. Then there exist a p-adic field K with residue field $\mathbb{F}_q$ and a morphism $\mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(A)$ mapping $\mathfrak{m}_K$ to $\mathfrak{p}$.*

Proof. Compare to [Pop11, Proposition 4.2.3]. Let $n = \dim(A_{\mathfrak{p}})$ and write $\mathfrak{p}A_{\mathfrak{p}} = (s_1, \ldots, s_n)$. Since the s_i form a regular sequence, only one s_i divides p, say s_1. By localizing A such that all $s_i \in A$ and by dividing out $s_2, \ldots, s_n$ we can assume that $n = 1$ (cf. Krull's principal ideal theorem).

Now $A_{\mathfrak{p}}$ is a discrete valuation ring and hence its completion $\widehat{A_{\mathfrak{p}}}$ is a complete discrete valuation ring (cf. Proposition 1.20.ii)). By Proposition 1.20.i) $\mathcal{O}_K := \widehat{A_{\mathfrak{p}}}$ is the ring of integers of a p-adic field K. The residue field of K is $\widehat{A_{\mathfrak{p}}}/\mathfrak{p}\widehat{A_{\mathfrak{p}}} \simeq A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \simeq \mathbb{F}_q$ and the desired morphism is the composition $A \to A_{\mathfrak{p}} \to \widehat{A_{\mathfrak{p}}}$. □

Remark 4.20. Let A be a finitely generated $\mathbb{Z}$-algebra and let $\mathfrak{p} \in \operatorname{Spec}(A)$ be a closed point, i.e. $\mathfrak{p} \subset A$ is a maximal ideal, then $\mathrm{k}(\mathfrak{p})$ is a finite field. Indeed, by Hilbert's Nullstellensatz A is a Jacobson ring as a finitely generated $\mathbb{Z}$-algebra and hence $A/\mathfrak{p}$ is a finite extension of a finite field.

The following theorem by Cassels can also be used to perform the "lift". It is rather general and can be applied to make methods of p-adic analysis available to questions on finitely generated fields of characteristic zero.

Theorem 4.21 (Embedding Theorem, Cassels)**.** *Let k be a finitely generated field extension of $\mathbb{Q}$ and let $C \subset k^\times$ be a finite set. Then there are infinitely many prime numbers p for which there exists an embedding $\iota_p\colon k \hookrightarrow \mathbb{Q}_p$ such that $|\iota_p(c)|_p = 1$ for all $c \in C$.*

Proof. See [Cas86, Chapter 5] for a proof. □

Remark 4.22. When we apply Cassel's Embedding Theorem (Theorem 4.21) to the fraction field K of a finitely generated $\mathbb{Z}$-algebra A and choose C to be the set of generators of A over $\mathbb{Z}$, we get for infinitely many prime numbers p a morphism $\operatorname{Spec}(\mathbb{Z}_p) \to \operatorname{Spec}(A)$.

Lemma 4.23. *Let A_0 be a finitely generated, integral, regular $\mathbb{Z}$-algebra of characteristic 0 and X_0, Y_0 schemes over A_0. Then the following hold.*

i) *There exist a p-adic field K and a homomorphism $A_0 \to \mathcal{O}_K$. In particular, if $\mathfrak{m}_K$ contracts to the prime ideal $\mathfrak{p}$ under this homomorphism, then $\mathrm{k}(\mathfrak{p})$ is a finite field. Define $\mathfrak{X} := X_0 \times_{A_0} \mathcal{O}_K$ and $\mathfrak{Y} := Y_0 \times_{A_0} \mathcal{O}_K$.*

ii) *If X_0 (respectively Y_0) is projective and smooth of relative dimension n over A_0, then $\mathfrak{X}$ (respectively $\mathfrak{Y}$) is a variety over $\mathcal{O}_K$ that is projective and smooth of relative dimension n.*

iii) *If in addition to the assumptions in* ii) we have that the generic fiber $(X_0)_{\mathrm{Q}(A_0)}$ (respectively $(Y_0)_{\mathrm{Q}(A_0)}$) is geometrically connected, then we may assume that in i) the p-adic field K and homomorphism $\mathcal{O}_K \to A_0$ are chosen such that $\mathfrak{X}_K$ (respectively $\mathfrak{Y}_K$) and $\mathfrak{X}_{\mathfrak{m}_K}$ (respectively $\mathfrak{Y}_{\mathfrak{m}_K}$) are geometrically connected.

iv) If $U_0 \subset X_0$ and $V_0 \subset Y_0$ are isomorphic open subschemes, then there exist isomorphic open subschemes $\mathfrak{U} \subset \mathfrak{X}$ and $\mathfrak{V} \subset \mathfrak{Y}$, say $\varphi\colon \mathfrak{U} \xrightarrow{\sim} \mathfrak{Y}$.

v) If in addition to the assumptions in ii), iii) and iv) the open subset U_0 in iv) satisfies $\operatorname{codim}_{X_0}(X_0 \setminus U_0) \geq 2$ and there exist nowhere vanishing forms $\omega_{X_0} \in \Omega^n_{X_0/A_0}(X_0)$ and $\omega_{Y_0} \in \Omega^n_{Y_0/A_0}(Y_0)$, then there exist gauge forms $\omega_{\mathfrak{X}} \in \Omega^n_{\mathfrak{X}/\mathcal{O}_K}(\mathfrak{X})$, $\omega_{\mathfrak{Y}} \in \Omega^n_{\mathfrak{Y}/\mathcal{O}_K}(\mathfrak{Y})$ such that $\varphi^*\omega_{\mathfrak{Y}}|_{\mathfrak{V}} = h\omega_{\mathfrak{X}}|_{\mathfrak{U}}$ for some $h \in \mathcal{O}_K^\times$

Proof. i) Take some maximal ideal $\mathfrak{p} \subset A_0$. Using Remark 4.20 we can apply Proposition 4.19 to get the desired p-adic field K and homomorphism $A_0 \to \mathcal{O}_K$.

ii) This is clear, since smoothness and projectiveness are compatible with base change. The proof that the relative dimension is n is similar to the argument in Lemma 4.14.

iii) By Proposition 4.16 there is an open subset $\mathrm{D}(f) \subset \operatorname{Spec}(A_0)$ such that all fibers over $\mathrm{D}(f)$ are geometrically connected. Now we can just replace A_0 by $(A_0)_f$ in the proof of i).

iv) This is clear, since open immersions and isomorphisms are compatible with base change.

v) Write $\varphi_0\colon U \xrightarrow{\sim} V$. Since $\Omega^n_{X_0/A_0}$ is by assumption trivial we have $\varphi_0^*\omega_{Y_0}|_V = h_0\omega_{X_0}|_U$ for some $h_0 \in \mathrm{H}^0(U, \mathcal{O}_{X_0})$. Note that X_0 is Noetherian as a finite type scheme over $\mathbb{Z}$, and it is normal as a smooth scheme. So we can apply Proposition 3.5 to see that $h_0 \in \mathrm{H}^0(X_0, \mathcal{O}_{X_0})$.

Since isomorphisms of $\mathcal{O}_{X_0}$-modules (respectively $\mathcal{O}_{Y_0}$-modules) are compatible with base change and $\Omega^n_{X_0/A_0}$ (respectively $\Omega^n_{Y_0/A_0}$) is preserved under base change (cf. [Liu02, Proposition 6.1.24]), it follows that the base changes $\omega_{\mathfrak{X}}$ and $\omega_{\mathfrak{Y}}$ of ω_{X_0} and ω_{Y_0} are gauge forms. Note that the base change of the global section h_0 is now a global section $h \in \mathrm{H}^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Hence, by Proposition 4.17 $h \in \mathcal{O}_K$. Since $\omega_{\mathfrak{X}}$ and $\omega_{\mathfrak{Y}}$ are gauge forms, we conclude $h \in \mathcal{O}_K^{\times}$. $\square$

At this point we have shown in Lemma 4.14, Remark 4.15 and Lemma 4.23 how to perform the spreading out and lifting needed in the proof of Batyrev's theorem. This concludes our presentation.

5 Conclusion and further work

To conclude our presentation on K-analytic manifolds, p-adic integration and Batyrev's theorem, let us reflect on the methods and results encountered in the text.

First let mention some generalizations and strengthenings of Batyrev's theorem, as well as further applications of p-adic integration to questions about invariants in birational situations.

- In [Wan98] Wang weakens the requirements on the canonical bundle needed to conclude the equality of Betti numbers.
- In [Ito03] Ito and in [Wan02] Wang prove that two K-equivalent varieties X and Y have equal Hodge numbers $\mathrm{H}^r(X, \Omega^s_X) = \mathrm{H}^r(Y, \Omega^s_Y)$, using methods of p-adic integration together with p-adic Hodge theory.
- In [Ito04] Ito gives a proof of the well-definedness of Batyrev's stringy Hodge numbers using p-adic integration and p-adic Hodge theory.

We want to consider now some positive and some negative aspects of the theory of p-adic integration as applied in the proof of Batyrev's theorem. Certainly a positive aspect is the concreteness of p-adic integration and basic p-adic analysis. Many fundamental results are familiar from real analysis and one doesn't have to develop an intuition for the new objects encountered from scratch. Also p-adic analysis is a very powerful tool and in fact the first of the Weil Conjectures (rationality of local zeta functions) was proved by Dwork using methods from p-adic analysis. In contrast we have seen that compact K-analytic manifolds are not very rich objects. This shows that the naive definition of manifold in the p-adic setting may not be the "right" notion. This reflects the fact that despite their similarities, the p-adic "world" and the real "world" can be quite different. Another negative aspect of p-adic integration encountered in the text is that one has to first translate the problem at hand into the p-adic setting. This technical translation can be rather tedious and in most cases is difficult to reuse in other arguments. In our case the theory of projective systems of schemes mitigates this problem somewhat. Last but not least the proof of Batyrev's theorem presented in this text and the proofs of the generalizations and strengthenings referenced above rely heavily on difficult machinery like the Weil Conjectures, étale cohomology and, as mentioned above, p-adic Hodge theory.

Let us mention an alternative theory that solve some of the problems just observed. Kontsevich introduced in [Kon95] the theory of motivic integration to give a proof of the strengthening of Batyrev's theorem regarding Hodge numbers (Kontsevich's proof predates the proofs using p-adic Hodge theory cited above). In this theory the part where one performs "integration" is separated from the part where one compares invariants like the Betti or Hodge numbers. This makes the application of the theory easier, since one can reuse the general theory and doesn't have to resort to ad hoc methods. Kontsevich's theory of motivic integration is more geometric in comparison to the arithmetic character of p-adic integrals and no detour through finite or p-adic fields is needed. We refer to [CNS11, Chapter 6] or [Pop11, Chapter 5–6] for an introduction to motivic integration. After the introduction by Kontsevich the theory of motivic integration was developed further and nowadays incorporates ideas from model theory. Some variants of the theory specialize to p-adic integrals. See [CNS11, Chapter 1] for an overview of the connections between the different theories.

In conclusion after studying p-adic integration and Batyrev's theorem it is also worth to look at motivic integration.

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