GEOMETRIES ON K3 SURFACES AND THEIR SYMMETRIES

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Abstract. The theory of K3 surfaces combines in a fascinating way complex, algebraic, and symplectic geometry. A good deal is known about this important class of complex surfaces, but many questions remain open. (Even one version of the Global Torelli theorem as formulated by A. Weil is still not proven.)

Starting with a few historical comments, we will sketch some of the more recent developments related to K3 surfaces. In particular, we will touch upon mirror symmetry for K3 surfaces and its formulation using generalized Calabi–Yau structures introduced by Hitchin. Although it has been clear for a long time that special symplectic structures on K3 surfaces (so called hyperkähler structures) provide a powerful tool in the theory, it is only now that the true nature of the interplay between complex and symplectic structures becomes apparent.

More specifically, we will explain how to view symplectic and complex structures on the differentiable manifold M underlying the quartic $Z(x_0^4+x_1^4+x_2^4+x_3^4)\subset\mathbb{P}^3$ as special examples of generalized Calabi–Yau structures. In particular, both types of geometries are parametrized by the same moduli space. It turns out that this moduli space has a natural symplectic structure with respect to which the symplectic structures on M form a Lagrangian.

This is a write-up of a colloquium talk delivered at Humboldt-University, Berlin, and later again in a modified form at Kyoto University.

The talk begins with a few historical remarks and then passes on to more recent developments. The aim was to convey my own excitement about the subject and to convince the audience that, despite the fact that an almost complete theory of K3 surfaces seems available for almost twenty years, some of the fundamental problems still await a solution and many new challenges are ahead.

0. Let me begin with of the name K3. A poetic explanation for it was given in André Weil's remarks on his only article on the subject (which was in fact a report for an U.S. Air Force grant). He says

... il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira, et de la belle montagne K2 au Cachemire.[1]

[1]A nice picture can be found on J. Milne's homepage.

But these comments were made many years after the original paper [1] was written in which the name was used without any further explanation. In particular, it seems unclear to me who actually coined the name. The origins become even more uncertain when one browses through old papers e.g. by Steiner, where names like K_2 were used for certain hypersurfaces. [2] What seems however completely justified is to relate K3 surfaces to the three big **K**'s: Kummer, Kähler, and Kodaira.

Many of the results in [1] were stated, as theorems or conjectures, but unfortunately Weil refrains from attributing the results to the corresponding author (explicitly stating so himself). Thus, not only the origin of the name K3 remains in the dark but also who is actually responsible for making the first step towards the ascension of K3.

1. What is a K3 surface? For the moment let us just consider it as the set M of zeroes of the homogeneous equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0,$$

viewed as a subset of $\mathbb{P}^3_{\mathbb{C}}$.[3] What interests us in this talk are the various geometries the set M can endowed with. What is meant by this will become clear by the following list. At the same time, realizing M as the subset of $\mathbb{P}^3_{\mathbb{C}}$ induces special examples of such geometrical structures.

i) Topological structures. The classical topology on $\mathbb{P}^3_{\mathbb{C}}$ induces the structure of a topological manifold on M.

ii) Differentiable structures. In fact, the differentiable structure on $\mathbb{P}^3_{\mathbb{C}}$ endows the topological manifold M with a differentiable atlas.

iii) Symplectic structures, i.e. real two forms ω with $d(\omega) = 0$ and $\omega \wedge \omega > 0$ (an orientation of M has tacitly been fixed). The restriction of the Fubini–Study form $\omega = \frac{i}{2\pi}\partial\bar{\partial}\sum \|x_i\|^2|_M$ is an example of a symplectic structure on M.

iv) Complex structures, which can be thought of complex structures $I : T_X M \to T_x M$, $I^2 = -1$, for each of its tangent spaces satisfying an integrability condition. Such a structure naturally exists on $\mathbb{P}^3_{\mathbb{C}}$ and, as M is defined by a holomorphic function, also on M.

v) Algebraic structures. Our specific M is indeed given by a polynomial.

One more comment on complex structures.

Remark 0.1. One more comment on complex structures: Weil mentions in his report an alternative description of them which is most likely due to Andreotti. Namely, any complex two-form σ on M satisfying $d\sigma = 0$,

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[2] In the early age of mirror symmetry physicists would call K_4 some hypersurfaces in $\mathbb{P}^4_{\mathbb{C}}$, apparently mistaking '3' in K3 as the dimension of the ambient projective space of the K3 surface isomorphic to a quartic hypersurface.

[3] You can find pictures of it on the web. Often, the zero set of a slightly deformed equation is depicted which defines a Kummer surface (singular in 16 points). It exhibits more symmetries and thus gives rise to a more appealing shape.

$\sigma \wedge \bar{\sigma} > 0$, and $\sigma \wedge \sigma = 0$ defines a complex structure I by setting as the subspaces of eigenvectors with eigenvalues $-\sqrt{-1}$ the annihilator of σ, i.e. $T^{0,1}M = \mathrm{Ann}(\sigma)$. With respect to this complex sructure σ becomes a holomorphic form. Clearly, any nonzero constant scalar $\lambda\sigma$ defines the same complex structure. More recent results prove furthermore that any I arises in this way. [4] Thus, there is a one-to-one correspondence

$$\{I = \text{complex structure}\} \leftrightarrow \{\sigma = \text{complex two} - \text{form as above}\}/\mathbb{C}^*$$

2. These geometries are studied via their periods. To explain how this works, we shall first recall the exact form of the cohomology of M. Let us suppose we have chosen on M the topology induced by the embedding $M \subset \mathbb{P}^3_{\mathbb{C}}$. Then the cohomology $H^*(M,\mathbb{Z})$ is a lattice of rank 24. More precisely, $H^*(M,\mathbb{Z})$ is the orthogonal sum of the two sublattices H^2 and $H^0 \oplus H^4$, where the latter is just a copy of the hyperbolic plane U and H^2 is an even unimodular indefinite lattice of index $(3, 19)$. Thus, using the classification theory of such lattices one finds $H^*(M,\mathbb{Z}) = 2(-E_8) \oplus 4U$.[5]

Remark 0.2. i) Freedman's results imply that any topological structure on M realizing the same intersection pairing is in fact equivalent to the one we have considered so far. In other words, the cohomology $H^*(M,\mathbb{Z})$ determines the topological manifold M.

ii) In contrast, the differentiable structure on this topological manifold is not unique. There are in fact exotic differentiable structures on M, see e.g. work of Gompf and Mrowka [2]. For the rest of this talk, we will not worry about these exotic structures and consider only the natural differentiable structure on M obtained by viewing it as a submanifold of $\mathbb{P}^3_{\mathbb{C}}$. So from now on M will be fixed as a differentiable manifold (again, the orientation is tacitly assumed to be fixed).

We now turn to the two structures that interest us most here: symplectic and complex structures. We have explained that both structures are described in terms of real respectively complex closed two-forms satisfying certain additional conditions and we will now proceed to study their periods.

We shall begin with symplectic structures. Associating to a symplectic form ω its cohomology class $[\omega] \in H^2(M,\mathbb{Z})$ defines a map from the set of all symplectic structures $\{\omega\}$ to $H^2(M,\mathbb{R})$. This map takes values in the open subset

$$\mathcal{C}_M := \{\alpha \in H^2(M,\mathbb{R}) \mid \alpha^2 > 0\},$$

which is the *symplectic period domain.* Thus, $\mathcal{C}_M$ is an open subset of the 22-dimensional real vector space $H^2(M,\mathbb{R})$. On the other hand, the set of symplectic structures is clearly infinite-dimensional, a fact due to the natural

[4] See [4]. Every complex surface diffeomorphic to a K3 surface is actually a K3 surface.

[5] In fact, for most things it is convenient to change the sign on the hyperbolic plane $H^0 \oplus H^4$ (it remains isomorphic to U). The induced pairing on $H^*(M,\mathbb{Z})$ is called the Mukai pairing.

action of the diffeomorphism group. As a symplectic structure ω and its pull-back $f^*\omega$ under a diffeomorphism $f : M \to M$ should be considered as isomorphic, we shall divide $\{\omega\}$ by $\mathrm{Diff}(M)$. However, in order to use the above map to the symplectic period domain, we will in a first step only divide out by the subgroup $\mathrm{Diff}_*(M) := \{f \in \mathrm{Diff}(M) \mid f^* = \mathrm{id} \text{ on } H^*(M,\mathbb{R})\}$. Thus, if we define the symplectic moduli space as

$$\mathrm{Sympl}(M) := \{\omega\}/\mathrm{Diff}_*(M),$$

then we may still exploit the *symplectic period map*

$$\mathcal{P}_s : \mathrm{Sympl}(M) \to \mathcal{C}_M,\ \omega \mapsto [\omega].$$

That this map is indeed well suited to study the $\mathrm{Sympl}(M)$ can be seen already by the following classical result

Moser 1965: *The symplectic period map $\mathcal{P}_s$ is a local isomorphism.*

Obtaining any further information about this map in this purely symplectic context is extremely difficult, for four-dimensional differentiable topology is a notorious difficult subject. In fact, only in conjunction with the study of complex structures on M anything global can be proved about the symplectic period map. So, before stating any of the known global results, we will set the stage of the moduli space of complex structures. This will be done in complete analogy by associating to a complex structure I defined in terms of a complex two-form σ satisfying $d\sigma = 0$, $\sigma \wedge \bar{\sigma} > 0$, and $\sigma \wedge \sigma = 0$, the cohomology class $[\sigma] \in H^2(M,\mathbb{C})$. There is, however, one difference here, for σ and $\lambda\sigma$, $\lambda \in \mathbb{C}^*$, define the same complex structure. Thus, we associate to the complex structure I the cohomology line $[\sigma]\mathbb{C} \in \mathbb{P}(H^2(M,\mathbb{C}))$. This leads to the *complex period domain*

$$Q_M := \{x \in \mathbb{P}(H^2(M,\mathbb{C}) \mid x^2 = 0,\ x\bar{x} > 0\},$$

in which all these classes $[\sigma]$ are contained.

If we let now $\mathrm{Compl}(M)$ be the quotient $\{\sigma\mathbb{C}\}/\mathrm{Diff}_*(M)$ of all complex two-forms σ defining complex structures on M modulo non-zero scalar factors by the restricted diffeomorphism group as before, then we may study this space via the *complex period map*

$$\mathcal{P}_c : \mathrm{Compl}(M) \to Q_M,\ \sigma\mathbb{C} \mapsto [\sigma]\mathbb{C}.$$

The analogue of Moser's result is the

Local Torelli theorem 1953: *The complex period map $\mathcal{P}_c$ is a local isomorphism.*

This result had be stated in [1], but, again, it might actually be due to Andreotti. In any case, Weil goes on and asks for a global version of this statement. In fact, Weil phrases two possible generalizations:

Conjecture. 1. (**Global Torelli, first version**) Let $\mathcal{P}_c(\sigma) = \mathcal{P}_c(\sigma')$, i.e. σ and σ' are cohomologous up to scalar factors. Then there exists a diffeomorphism $f \in \mathrm{Diff}(M)$ isotopic to the identity, i.e. f is contained in the identity component $\mathrm{Diff}_{\mathrm{o}}(M)$ of the full diffeomorphism group $\mathrm{Diff}(M)$, such that $\sigma = f^*\sigma'$.

2. (**Global Torelli, second version**) If $[\sigma] = g[\sigma']$ for a lattice isometry $g \in \mathrm{O}(H^2(M, \mathbb{Z}))$, then there exists a diffeomorphism $f \in \mathrm{Diff}(M)$ such that $\sigma = f^*\sigma'$.

Before stating the main results about the two period maps $\mathcal{P}_c$ and $\mathcal{P}_s$ let me make the following two remarks:

Remark 0.3. i) Clearly, $\mathrm{Diff}_{\mathrm{o}}(M)$ is a subgroup of $\mathrm{Diff}_*(M)$, but to the best of my knowledge it is unknown whether the two groups coincide.

ii) The picture we have developed so far seems to indicate a striking analogy between the symplectic and the complex world. There is, however, one major difference between the two quotients $\{\omega\}/\mathrm{Diff}_*(M)$ and $\{\sigma\mathbb{C}\}/\mathrm{Diff}_*(M)$ that seemingly outbalances the situation completely: The stabilizer $\mathrm{Stab}(\sigma) \subset \mathrm{Diff}_*(M)$ of any symplectic structure ω is an infinite dimensional group, whereas $\mathrm{Stab}(\sigma) \subset \mathrm{Diff}_*(M)$ is known to be trivial for any σ. We will comment on this phenomenon later and shall show how to rescue the analogy.

Here are the main facts concerning the two period maps. The history of these results is interwoven and the important articles are listed in the references.

Theorem 0.4. *1. The complex period map*

$$\mathcal{P}_c : \mathrm{Compl}(M) \to Q_M$$

is essentially [6] *an isomorphism.*

2. The symplectic period map

$$\mathcal{P}_s : \mathrm{Sympl}(M) \to \mathcal{C}_M$$

is surjective.

So again, there seems to be an almost complete analogy between the two structures when we pass from local to the global properties of the period map. But, as a matter of fact, neither of the two statements could we proved independently from the other. The key to understand the relation between both moduli spaces is the following diagram

$$\begin{tikzcd} & \{(\omega,\sigma) \mid \omega \wedge \sigma = 0,\ \omega \wedge \omega = \lambda \sigma \wedge \bar{\sigma}\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \{\omega\} & & \{\sigma\} \end{tikzcd}$$

[6]More explicitly, $\mathcal{P}_c$ is surjective and generically injective. In fact, the only trouble is caused by the non-Hausdorffness of $\mathrm{Compl}(M)$. We will neglect this technical point here.

which can be thought of as an incidence variety. Deep results due to Yau and Siu show that the second projection π_2 is in fact surjective.

Going through the list of references one sees that the theory appears rather complete since the beginning of the eighties. But what about the Global Torelli theorem?

Remark 0.5. 1. If one assumes one further result saying thet the natural representation $\mathrm{Diff}(M) \to \mathrm{O}(H^2(M,\mathbb{Z}))/\pm$ is surjective then the bijectivity of$\mathcal{P}_c$ yields the Global Torelli theorem in its second version.

2. As we divided out by the a priori bigger group $\mathrm{Diff}_*(M)$ and not by the identity component $\mathrm{Diff}_o(M)$ the above result does not say anything about the Global Torelli theorem in its first form. It appears to be equivalent to the equality $\mathrm{Diff}_*(M) \stackrel{?}{=} \mathrm{Diff}_o(M)$. **Thus, the Global Torelli theorem (in its first form) as formulated by Weil is still open!**

3. What about the bijectivity of the symplectic period map $\mathcal{P}_s$, i.e. a Global Torelli theorem or symplectic structures? This question remains open for the time being. One can prove that an affirmative answer is is equivalent to the surjectivity of the first projection π_0. The latter holds if every symplectic structure on M is (hyper)kähler with respect to some complex structure. on M.

Leaving these two questions aside the theory has achieved an astonishing degree of completeness already 20 years ago. But the short selection of more recent articles on K3 surfaces shows that the theory seems to attract every growing interest and, moreover, branches out in various directions.

In the following I shall just allude to one aspect, which is also responsible for the title of this talk.

3. Before actually talking about symmetries we will explain how Hitchin's invention of generalized Calabi–Yau structures allows to unify the symplectic and the complex picture.

Let us first consider this at the level of the two period domains $\mathcal{C}_M$ and Q_M. For this purpose we define the subset $\widetilde{Q}_M \subset \mathbb{P}(H^*(M,\mathbb{C}))$ in analogy to the complex period domain Q_M by replacing $H^2(M,\mathbb{C})$ by the full cohomology:

$$\widetilde{Q}_M := \{x \ \in \mathbb{P}(H^*(M,\mathbb{C})) \mid x^2 = 0,\ x\bar{x} > 0\}.$$

We recall that $H^*(M,\mathbb{C})$ is endowed with the Mukai pairing, a slight modification of the usual intersection pairing. In particular, $\widetilde{Q}_M$ is an open subset of a non-degenerate quadric in the 23-dimensional complex projective space $\mathbb{P}(H^*(M,\mathbb{C}))$. Intersecting with the codimensional two linear subspace $\mathbb{P}(H^2(M,\mathbb{C}))$ yields the natural inclusion $Q_M \subset \widetilde{Q}_M$. On the other hand, also $\mathcal{C}_M$ can naturally be embedded via the map $\alpha \mapsto 1 + i\alpha - \alpha^2/2$. Thus,

$$Q_M \subset \widetilde{Q}_M \supset \mathcal{C}_M.$$

On the level of forms these inclusion can be given a geometrical explanation by viewing $\widetilde{Q}_M$ as the period domain for generalized Calabi–Yau structures.

Definition 0.6. (Hitchin) A generalized Calabi–Yau structure on M is an even complex form φ satisfying $d\varphi = 0$, $\langle\varphi, \bar{\varphi}\rangle > 0$, and $\langle\varphi, \varphi\rangle = 0$.

Here, $\langle\ ,\ \rangle$ denotes the Mukai pairing on the level of forms, $\langle\varphi, \varphi\rangle = \varphi_2^2 - 2\varphi_0\varphi_4$.

The two types of examples one should be aware of are the following:

i) The form $\varphi = \sigma$ is a generalized Calabi–Yau structure, where σ is a complex two-form defining a complex structure on M. Since σ defines a holomorphic volume form on the complex manifold it defines and thus is a Calabi–Yau form, this explains the name. In view of the next example and could have doubted them with the same right generalized Kähler structures.

ii) The form $\varphi = \exp(i\omega) = 1 + i\omega - \omega^2/2$ is a generalized Calabi–Yau structure for any symplectic form ω.

Furthermore, if φ is a generalized Calabi–Yau structure and B is a real closed two-form, then $\exp(B)\cdot\varphi$ is again a generalized Calabi–Yau structure.

Let us again consider the moduli space of all such structures:

$$\mathrm{CY}(M) := \{\varphi\mathbb{C} \mid \varphi = \text{generalized Calabi} - \text{Yau structure}\}/\widetilde{\mathrm{Diff}}_*(M),$$

where $\widetilde{\mathrm{Diff}}_*(M)$ is the semi-direct product of $\mathrm{Diff}_*(M)$ and the space of real exact two-forms acting by $\exp(B)$.

Why dividing out by this bigger group? Again one reason is that changing by an exact B-field B in this way cannot be detected by the generalized period map

$$\widetilde{\mathcal{P}} : \mathrm{CY}(M) \to \widetilde{Q}_M,\ \ \varphi\mathbb{C} \mapsto [\varphi]\mathbb{C}.$$

There are two remarkable things about this moduli space. Firstly, $\mathrm{CY}(M)$ contains $\mathrm{Compl}(M)$ and also $\mathrm{Sympl}(M)$ via $\omega \mapsto \exp(i\omega)$. Secondly, this approach seems to cure the difference between $\mathrm{Stab}(\omega)$ and $\mathrm{Stab}(\sigma)$ inside $\mathrm{Diff}_*(M)$, because the stabilizer of the actions of the bigger group $\widetilde{\mathrm{Stab}}(\exp(i\omega)) \subset \widetilde{\mathrm{Diff}}_*(M)$ (which actually equals $\mathrm{Stab}(\omega)$) and $\widetilde{\mathrm{Stab}}(\sigma \subset \widetilde{\mathrm{Diff}}_*(M)$ (which consists of the set of all exact two forms B of type $(1,1)$) are of 'equal' size, i.e. they are both infinite dimensional.

An analogous version of the Global Torelli theorem can in fact be proved [6]:

$$\widetilde{\mathcal{P}} : \mathrm{CY}(M) \to \widetilde{Q}_M$$

is essentially an isomorphism on at least one connected component of $\mathrm{CY}(M)$.

Remark 0.7. The old problem about the injectivity of $\mathcal{P}_s$ did not get resolved by this approach. We do not know whether $\mathrm{CY}(M)$ is actually connected. Roughly the surjectivity of $\widetilde{\mathcal{P}}$ on every connected component is equivalent to this.

4. Let us go one step back and consider the period map $\mathcal{P}_c : \mathrm{Compl}(M) \to Q_M$. Clearly, Q_M is invariant under the natural action of the orthogonal group $\mathrm{O} := \mathrm{O}(H^2(M,\mathbb{Z}))$ and, moreover, this action can be lifted in a geometric way to an action on the moduli space. More precisely, the group $\mathrm{Diff}(M)$ acts naturally on the set of all complex structures, as was explained, and this action is compatible with the natural representation $\mathrm{Diff}(M) \to \mathrm{O}$. That the action of the diffeomorphism group explains the action of the full group O is due to a result of Borcea saying that the induced map $\mathrm{Diff}(M) \to O/\pm$ is surjective.

Passing on to the generalized period domain $\tilde{Q}_M$ we again find a natural group action. Indeed, the orthogonal group $\tilde{\mathrm{O}} := \mathrm{O}(H^*(M,\mathbb{Z}))$ acting in the natural way on $\mathbb{P}(H^*(M,\mathbb{C}))$ leaves invariant $\tilde{Q}_M$. Is there a geometric/mathematical explanation for this action? In fact, this action is responsible for mirror symmetry phenomena for K3 surfaces. Although mirror symmetry in a deeper sense still eludes us even for K3 surfaces, some glimpses of it can be spotted already.

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