Jacobians of Curves on Surfaces

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Contents

1. Introduction

Take a smooth projective surface S over $\mathbb{C}$ and let L be a very ample line bundle on S. We want to study the structure of the Jacobians of smooth curves in the linear system $|L|$. A natural question to ask is whether the Jacobians are simple abelian varieties, i.e. have no proper non-trivial abelian subvarieties. Unfortunately, the next lemma shows that this requires $\mathrm{Alb}(S) \sim \mathrm{Jac}(C)$ or $\mathrm{Alb}(S) = 0$.

Lemma 1.1. *Let C be a smooth curve in $|L|$ and $K(C, S)$ the kernel of the natural map* $\mathrm{Jac}(C) \to \mathrm{Alb}(S)$. *Then* $\mathrm{Jac}(C) \sim K(C, S) \times \mathrm{Alb}(S)$.

Thus, a more interesting question turns out to be:

Question 1.2. For which curves C in $|L|$ is the abelian variety $K(C, S)$ simple?

This was answered by Ciliberto and van der Geer in [6]. Namely, they proved the following theorem:

Theorem 1.3. *Let S and L be as above. Then for a very general curve $C \in |L|$, $K(C, S)$ has only trivial endomorphisms. In particular, $K(C, S)$ is simple.*

Which gives us the following corollary for the case of a regular surface S (i.e. $\mathrm{Alb}(S) = 0$):

Corollary 1.4. *Let S be a regular smooth projective surface over $\mathbb{C}$ and L some very ample line bundle on S. Then for a very general curve $C \in |L|$, $\mathrm{Jac}(C)$ is a simple abelian variety.*

In the first half of this thesis we work out a detailed proof of the theorem, based on the original proof given by Ciliberto and van der Geer. We begin by giving the necessary background in abelian varieties and curves in very ample linear systems on surfaces in section 2, focusing on the relationship between abelian subvarieties and endomorphisms and the structure of the discriminant divisor of $|L|$.

The actual proof of the theorem starts in section 3, with the case of S being regular. We use the representability of the endomorphism functor of abelian schemes to reduce the proof to showing that the Jacobian of the generic curve of $|L|$ has only trivial endomorphisms and then, using degenerations given by the discriminant divisor, show that this is equivalent to the Jacobian of the generic fibre being simple.

In sections 4 and 5 we finish the proof of theorem 1.3 in the regular case by giving two very different proofs of the fact that the Jacobian of the generic fibre is simple. The original proof of Ciliberto and van der Geer, given in section 4, focuses on the relationship of endomorphisms of $\mathrm{Jac}(C)$ and divisors on $C \times C$, explicitly computing that the latter can only be trivial. The proof given in section 5 on the other hand, uses the irreducibility of the monodromy action on the middle vanishing cohomology of hyperplane sections, which in our case will just be $H^1(\mathrm{Jac}(C), \mathbb{Q})$.

We complete the proof of theorem 1.3 in section 6 by generalizing the arguments of the earlier sections to non-regular surfaces, showing that the techniques we used on $\mathrm{Jac}(C)$ also work for $K(C, S)$.

The latter half of this thesis then focuses on possible generalizations of theorem 1.3. In section 7 we investigate if the theorem holds for more general line bundles. We show that in general, at least birationality of the induced map by $|L|$ from S to $\mathbb{P}^n$ is needed and give a counterexample in the case of L being ample but not very ample.

We then look at the case of arbitrary algebraically closed ground fields k. For k uncountable, there is the following generalization of theorem 1.3, due to Banerjee:

Theorem 1.5. *Let S be a smooth projective surface over some uncountable algebraically closed field k and L a very ample line bundle on S. Assume that the embedding induced by L is a Lefschetz embedding. Then for C a very general curve in $|L|$, $K(C, S)$ is absolutely simple.*

After reviewing the necessary results from étale cohomology, we will give a proof of this theorem in section 8. The proof is similar to the one given in section 5, using the geometry and étale monodromy of Lefschetz pencils to proof that the Jacobian of the generic curve is absolutely simple.

The final section focuses on the case of the ground field being $\overline{\mathbb{Q}}$, which is not covered by Banerjee's theorem. Using a consequence of the Hilbert irreducibility theorem proven by Terasoma in [15], we will show that nevertheless, still the following holds:

Theorem 1.6. *Let S be a smooth projective surface over $\overline{\mathbb{Q}}$ and L a very ample line bundle on S. Then there exists a smooth curve $C \in |L|$, defined over $\overline{\mathbb{Q}}$, such that $K(C, S)$ is absolutely simple.*

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Convention All maps between schemes are morphisms. An abelian variety over k is a complete, separated, geometrically integral group scheme over k. A homomorphism of abelian varieties is a morphism between abelian varieties which respects the group structure. An isogeny between two abelian varieties A and B is a surjective homomorphism $\varphi\colon A \to B$ such that $\ker(\varphi)$ is finite. If such an isogeny exists, we call A and B isogenous and denote this by $A \sim B$. As we will see below, this is an equivalence relation.

A general (closed) point in a scheme X is a closed point lying in some dense open subset, i.e. if some result holds for a general point, it holds for all closed points in some dense open subset $U \subset X$.

A closed point x in a scheme X is called very general, if it lies in the complement of a countable union of nowhere dense closed subsets, i.e. $x \in X \setminus \bigcup_{i\in\mathbb{N}} Z_i$, where the Z_i are closed subsets of X such that $X \setminus Z_i$ is dense in X.

2. Preliminaries

2.1. **Simple Abelian Varieties.** We begin with some basic definitions and results about abelian varieties over arbitrary fields.

Definition 2.1. Let A be an abelian variety over a field k. Then A is called simple (or k-simple), if there are no proper non-trivial abelian subvarieties of A over k and absolutely simple, if $A_{\overline{k}}$ is simple over $\overline{k}$.

Similar to abelian groups, we can split an abelian variety into a product of its simple abelian subvarieties.

Lemma 2.2. *Let A be an abelian variety and $B \subset A$ a proper abelian subvariety. Then there exists an abelian variety $C \subset A$ such that the map*

$$B \times C \longrightarrow A$$

$$(b, c) \longmapsto b + c$$

is an isogeny.

Proof. [12, Proposition 12.1] □

Lemma 2.3. *Let A and B be simple abelian varieties and $\varphi \colon A \to B$ a homomorphism of abelian varieties. Then φ is either an isogeny or the 0-morphism.*

Proof. The identity component of $\ker(\varphi)$ is an abelian subvariety of A, thus either $\ker(\varphi) = A$ or $|\ker(\varphi)| < \infty$. On the other hand, $\operatorname{im}(\varphi)$ is an abelian subvariety of B, so either $\operatorname{im}(\varphi) = 0$ or $\operatorname{im}(\varphi) = B$. If φ is not the 0-morphism, we thus must have $|\ker(\varphi)| < \infty$ and $\operatorname{im}(\varphi) = B$ and hence φ is an isogeny. □

Lemma 2.4. *Let A be an abelian variety. Then $A \sim \prod A_i^{d_i}$, where the A_i are simple abelian varieties and $A_i \not\sim A_j$ for $i \neq j$. Furthermore, the d_i are uniquely determined and the A_i are unique up to isogeny.*

Proof. Let A be an abelian variety. Either A is simple, or there exists a proper non-trivial simple abelian subvariety A_1 of A. Then by lemma 2.2, we find another proper non-trivial abelian subvariety B_1 such that $A \sim A_1 \times B_1$. Either B_1 is simple, or we can find a proper non-trivial simple abelian variety $A_2 \subset B_1$ and a proper non-trivial abelian subvariety $B_2 \subset B_1$ such that $A \sim A_1 \times A_2 \times B_2$. As the dimension of the B_i's shrinks with every iteration, we can only do this finitely many times before we have a product of simple abelian varieties. The uniqueness then easily follows from lemma 2.3. □

The simplicity of an abelian variety is closely linked with its endomorphism, as the next two lemmas show.

Lemma 2.5. *Let $\varphi \colon A \to B$ be an isogeny of abelian varieties. Then there exists an isogeny $\psi \colon B \to A$ such that $\psi \circ \phi = [n]_A$ and $\phi \circ \psi = [n]_B$ for some $n \in \mathbb{N}$.*

Proof. The proof for arbitrary fields is analogous to the one for $k = \mathbb{C}$ given in [4, Proposition 1.2.6]. □

Lemma 2.6. *An abelian variety is simple if and only if all of its endomorphisms are either isogenies or the* 0*-morphism.*

Proof. The 'only if-direction' follows immediately from lemma 2.3.

For the converse, assume A is not simple, i.e. there exists a proper non-trivial abelian subvariety $B \subset A$. Then by lemmas 2.2 and 2.5, we find an abelian variety $C \subset A$ and an isogeny $\phi\colon B \times C \to A$, as well an isogeny $\psi\colon A \to B \times C$. Consider $\mathrm{id}_B \times 0_C \in \mathrm{End}(B \times C)$, then $\psi \circ \mathrm{id}_B \times 0_C \circ \phi \in \mathrm{End}(A)$ is (for dimension reasons) neither an isogeny nor the 0-morphism. $\square$

Knowing this, we are interested in the structure of the endomorphism ring of an abelian variety. Namely, we want to show that it is discrete.

Lemma 2.7. *Let A and B be simple abelian varieties. Then* $\mathrm{Hom}(A, B)$ *is torsion free.*

Proof. [12, Lemma 12.2] $\square$

Hence, it will be enough to prove that for an abelian variety A, $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is discrete. To do this, we need the following proposition:

Proposition 2.8. *The function $F\colon \mathrm{End}^0(A) \to \mathbb{Q}$, sending an endomorphism φ to its degree, is a homogenous polynomial function of degree $2g$ on* $\mathrm{End}^0(A)$*.*

Proof. [12, Proposition 12.4] $\square$

Corollary 2.9. $\mathrm{End}^0(A)$ *is discrete.*

Proof. Combining lemmas 2.3 and 2.4, we see that $\mathrm{End}^0(A) = \prod \mathrm{End}(A_i^{d_i})$. Furthermore, we have that $\mathrm{End}^0(A_i^{d_i}) \cong M_{d_i}(\mathrm{End}^0(A_i))$, hence it is enough to prove the statement for a simple abelian variety A.

In this case, we know by lemma 2.6 that the only degree 0 endomorphism of A is the 0-morphism. By lemma 2.8, the function $F\colon \mathrm{End}^0(A) \to \mathbb{Q}$ is continuous in the real topology and thus $U := \{\varphi \in \mathrm{End}^0(A) \mid F(\varphi) < 1\}$ is an open set in $\mathrm{End}^0(A)$. However, as A is simple, $U = \{0\}$ and hence $\mathrm{End}^0(A)$ is discrete. $\square$

Remark 2.10. In the case of the Jacobian of a curve, we can describe the endomorphism of $\mathrm{Jac}(C)$ by looking at correspondences of $C \times C$. Let $N := \{L_1 \otimes L_2 \in \mathrm{Pic}(C \times C) \mid L_1 \in \mathrm{im}(\pi_1^*),\ L_2 \in \mathrm{im}(\pi_2^*)\}$ for π_i the canonical projections. Then we get the following theorem:

Proposition 2.11. *There exists a canonical isomorphism of abelian groups*

$$\mathrm{Pic}(C \times C)/N \longrightarrow \mathrm{End}(\mathrm{Jac}(C)) \tag{2.1}$$

Proof. The proof for arbitrary fields is analogous to the one for $k = \mathbb{C}$ given in [4, Theorem 11.5.1]. □

2.2. **Curves on Surfaces.** Let now S be a projective surface over $\mathbb{C}$ and L be a very ample line bundle on S. Then the set U of all smooth divisors in the linear system $|L|$ is open and we define the discriminant divisor D as the complement of U in $|L|$.

Lemma 2.12. *Let $D := |L| \setminus U$, then the following assertions hold:*

(i) *D is a divisor.*

(ii) *Let $V \subset D$ be the set of curves with one single ordinary double point. Then V is open and non-empty.*

(iii) *The general curve $C \in V$ is irreducible.*

Proof. Let $Z \subset S \times |L|$ be the algebraic subset defined by $Z = \{(p, C) \in S \times |L| \mid p \in C \text{ and } C \text{ is singular at } p\}$ and $p_1 \colon Z \to S$ and $p_2 \colon Z \to |L|$ the projections. Then for every $s \in S$, $p_1^{-1}(s)$ is the projective space of dimension $\dim|L| - 3$ of curves singular at s. Thus, $\dim Z = \dim|L| - 1$ and $D = p_2(Z)$, but a priori we could have $\dim(D) < \dim(Z)$.

Claim. $\dim(D) = \dim(Z) = \dim(|L|) - 1$ *if* $(S, L) \ncong (\mathbb{P}^2, \mathcal{O}(1))$

Proof. [16, Example 7.5] □

This proves (i). For (ii), use the following statement:

Claim. *The curve corresponding to the generic point of D has only one single ordinary double point if* $\dim(D) = \dim(Z)$.

Proof. [17, Page 45] □

This ends the proof of (ii), as the set of curves with one single ordinary double point is certainly open.

Let $s \in S$ be a sufficiently general point. Then by the above, we see that $p_1^{-1}(s)$ is a linear system on S with a single base point. The irreducibility then follows by Bertini's 2nd theorem, see [11, Theorem 5.3]. □

As we are interested in the Jacobians of the curves in the linear system, let us compute the Picard group of an irreducible curve with one single ordinary double point (i.e. a general curve in D) for later use.

Lemma 2.13. *Let C be an irreducible curve with a single ordinary double point as its only singularity and $\pi\colon \tilde{C} \to C$ its normalization. Then we get a short exact sequence of abelian groups:*

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(\tilde{C}) \longrightarrow 0$$

Proof. Let $p \in C$ be the ordinary double point and $p_1, p_2 \in \tilde{C}$ the two points lying over p. Let $\tilde{L} \in \mathrm{Pic}(\tilde{C})$ be a line bundle and $\varphi\colon \tilde{L}(p_1) \to \tilde{L}(p_2)$ an isomorphism of the fibres. Then by identifying $\tilde{L}(p_1)$ and $\tilde{L}(p_2)$ via φ, we obtain a line bundle L on C.

On the other hand, for a line bundle $L \in \mathrm{Pic}(C)$, we have canonical isomorphisms $\pi^*L(p_1) \cong \pi^*L(p_2)$ of the fibres. As these two construction are inverse to each other, we see that there is a $1:1$ correspondence between line bundles L on C and pairs $(\tilde{L}, \varphi)$, where $\tilde{L} \in \mathrm{Pic}(\tilde{C})$ and $\varphi\colon \tilde{L}(p_1) \to \tilde{L}(p_2)$ an isomorphism.

Thus, $\ker(\pi^*)$ corresponds to the set of isomorphisms from $\mathcal{O}_{\tilde{C}}(p_1)$ to $\mathcal{O}_{\tilde{C}}(p_2)$. Both $\mathcal{O}_{\tilde{C}}(p_1)$ and $\mathcal{O}_{\tilde{C}}(p_2)$ are canonically isomorphic to $\mathbb{C}$ and hence we can identify $\ker(\pi^*)$ with $\mathbb{C}^*$. $\square$

For $C \in U$, the inclusion $C \subset S$ induces a natural map $\phi\colon\ \mathrm{Jac}(C) = \mathrm{Alb}(C) \to \mathrm{Alb}(S)$. We denote its kernel by $K(C, S)$. This yields the following result, which was used in the introduction.

Lemma 2.14. $\mathrm{Jac}(C) \sim \mathrm{Alb}(S) \times K(C, S)$.

Proof. We know by Lemma 2.2 that there exists an abelian subvariety $A \subset \mathrm{Jac}(C)$ such that $\mathrm{Jac}(C) \sim K(C, S) \times A$, so it suffices to prove that $A \sim \mathrm{Alb}(S)$, which follows immediately if ϕ is surjective. To see that this holds, consider the natural morphism $i^*\colon\ \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(C) = \mathrm{Jac}(C)$ induced by i, which is, up to isogeny, dual to ϕ. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{2.2}$$

By Kodaira vanishing, $H^1(S, \mathcal{O}_S(-C)) = 0$, so $H^1(S, \mathcal{O}_S) \to H^1(C, \mathcal{O}_C)$ is injective and thus i^* has finite kernel. Hence, i^* is injective up to isogeny and therefore its dual ϕ must be surjective. $\square$

3. Reduction to the generic fibre

We are now ready to begin the proof of theorem 1.3. Following Ciliberto and van der Geer, we will first do this for S being a regular surface. The proof will be done in two steps. First, we will reduce the proof to showing that the Jacobian of the generic curve of $|L|$ is simple, which we will then prove in the second step. If the Jacobian of the generic curve in $|L|$ is trivial, there will be a non-empty open subset of $|L|$ in which all curves have trivial Jacobian, thus proving the theorem. So without loss of generality, we can assume that the generic curve has genus greater than 0.

Let $x \in S$ be a general point and define $|L|_x$ to be the linear subsystem of all curves in $|L|$ running through x. Let $U \subset |L|_x$ be the open set of smooth curves in $|L|_x$ and $p\colon \mathcal{C} \to U$ the universal family of smooth curves over U. To prove the theorem, it will suffice to show that a very general curve $C \in |L|_x$ has only trivial endomorphisms. Considering $|L|_x$ instead of $|L|$ has the advantage that now $\mathcal{C} \to U$ has a section given by x, and hence the relative Picard scheme of $\mathcal{C}$ over U exists. Thus, we get a projective family $\mathcal{J} \to U$ of Jacobians over U, given by $\underline{\mathrm{Pic}}^0_{\mathcal{C}/U}$.

3.1. **The functor** $\mathrm{End}_{\mathcal{J}/U}$**.** Consider the functors

$$\mathrm{Hilb}_{\mathcal{J}\times_U\mathcal{J}/U}\colon (Sch/U)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Hilb}_{\mathcal{J}\times_U\mathcal{J}/U}(T)$$

and

$$\mathrm{End}_{\mathcal{J}/U}\colon (Sch/U)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Hom}_T(\mathcal{J}_T, \mathcal{J}_T).$$

Lemma 3.1. *There is an injective natural transformation* $i\colon \mathrm{End}_{\mathcal{J}/U} \to \mathrm{Hilb}_{\mathcal{J}\times_U\mathcal{J}/U}$ *sending an endomorphism* $f \in \mathrm{Hom}_T(\mathcal{J}_T, \mathcal{J}_T)$ *to its graph* Γ_f*.*

Proof. Let T be some U-scheme and $f \in \mathrm{Hom}_T(\mathcal{J}_T, \mathcal{J}_T)$. First note that $\mathcal{J} \to U$ is separated, hence f will be separated and thus Γ_f will be closed. Furthermore, the first projection induces an isomorphism $\Gamma_f \to \mathcal{J}_T$ and thus Γ_f will be proper and flat over U, which shows that $[\Gamma_f] \in \mathrm{Hilb}_{\mathcal{J}\times_U\mathcal{J}/U}(T)$. For $T' \to T$ a morphism of U-schemes, we have that $(\Gamma_f)_{T'} \cong \Gamma_{f_{T'}}$ and hence i defines a natural transformation. The injectivity follows from the fact that $\Gamma_f \sim \Gamma_g$ in $\mathrm{Hilb}_{\mathcal{J}\times_U\mathcal{J}/U}(T)$ if and only if $f = g$. □

Proposition 3.2. $\mathrm{End}_{\mathcal{J}/U}$ *is representable by a disjoint union of projective* U*-schemes.*

Proof. (We follow ideas given in [9, Lemma 3.4.4])

To prove this, we will show that $\mathrm{End}_{\mathcal{J}/U}$ is represented by a closed subscheme of $\underline{\mathrm{Hilb}}_{\mathcal{J}\times_U\mathcal{J}/U}$. Consider the functor

$$\mathrm{Mor}_{\mathcal{J}/U}\colon (Sch/U)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Mor}_T(\mathcal{J}_T, \mathcal{J}_T)$$

Then the natural transformation $i'\colon \mathrm{Mor}_{\mathcal{J}/U} \to \mathrm{Hilb}_{\mathcal{J}\times_U\mathcal{J}/U}$ sending f to Γ_f is representable by an open embedding by [14, Theorem 5.23]. Let T be some U-scheme and $Z \subset (\mathcal{J}\times_U \mathcal{J})_T$ a family representing an element of $\mathrm{Hilb}_{\mathcal{J}\times_U\mathcal{J}/U}(T)$. Let $V \subset U$ be the open set such that $s \in V$ if and only if $Z_s \subset \mathrm{Jac}(\mathcal{C}_s) \times \mathrm{Jac}(\mathcal{C}_s)$ is a graph of a morphism. Let $s \in V$, then Z_s is the graph of a homomorphism of abelian varieties if and only if the corresponding morphism sends the unit of $\mathrm{Jac}(\mathcal{C}_s)$ to the unit. This is a closed condition, so $\mathrm{End}_{\mathcal{J}/U}$ is represented by a locally closed subscheme $\underline{\mathrm{End}}_{\mathcal{J}/U}$ of $\underline{\mathrm{Hilb}}_{\mathcal{J}\times_U\mathcal{J}/U}$.

It remains to show that $\underline{\mathrm{End}}_{\mathcal{J}/U}$ is a closed subscheme. To do this, we will prove that $i\colon \underline{\mathrm{End}}_{\mathcal{J}/U} \to \underline{\mathrm{Hilb}}_{\mathcal{J}\times_U\mathcal{J}}$ is proper using the valuative criterion of properness. Consider a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \underline{\mathrm{End}}_{\mathcal{J}/U} \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathrm{Spec}(A) & \longrightarrow & \underline{\mathrm{Hilb}}_{\mathcal{J}\times_U\mathcal{J}/U}, \end{array}$$

for K some field and A a DVR with quotient field K. Then i is proper if and only if for each such diagram there exists a unique morphism $s\colon \mathrm{Spec}(A) \to \underline{\mathrm{End}}_{\mathcal{J}/U}$ that makes the diagram commute. However, for a U-scheme T, a morphism $T \to \underline{\mathrm{End}}_{\mathcal{J}/U}$ corresponds to an endomorphism of $\mathcal{J}_T$, the base change of $\mathcal{J}$ by $T \to U$. So proving the valuative criterion corresponds to proving that every endomorphism of $\mathcal{J}_K$ can be extended to a unique endomorphism of $\mathcal{J}_A$. This follows immediately from the fact that abelian schemes over Dedekind schemes are Néron models of their generic fibres, see [5, Proposition 1.2.8]. Thus, i is proper and its image is closed, hence $\underline{\mathrm{End}}_{\mathcal{J}/U}$ is a closed subscheme. □

Therefore, $\pi\colon \underline{\mathrm{End}}_{\mathcal{J}/U} \to U$ is of the form $\coprod_{i\in\mathbb{N}} \underline{\mathrm{End}}^i_{\mathcal{J}/U}$ with the $\mathrm{End}^i_{\mathcal{J}/U}$ being projective schemes over U. For a point $u \in U$ and $x \in \pi^{-1}(u)$, x corresponds to an endomorphism of $\mathrm{Jac}(\mathcal{C}_u)$ over $k(x)$.

Lemma 3.3. *We have $\underline{\mathrm{End}}_{\mathcal{J}/U} \cong \coprod_{d\in\mathbb{N}} \underline{\mathrm{End}}_{\mathcal{J}/U}(d)$, where the $\underline{\mathrm{End}}_{\mathcal{J}/U}(d)$ parametrize the endomorphisms of degree d.*

Proof. By lemma 2.8, we find a continuous map $f\colon \underline{\mathrm{End}}_{\mathcal{J}/U} \to \mathbb{Z}$ sending a point x to the degree of the endomorphism it corresponds to. As $\underline{\mathrm{End}}_{\mathcal{J}/U}(d) = f^{-1}(d)$, this proves the lemma. □

3.2. **The very general construction.** Let $\overline{\eta}$ be a fixed geometric generic point of U and assume $\mathrm{End}(\mathcal{J}_{\overline{\eta}}) = \mathbb{Z}$. We want to show that this implies the existence of a countable number of closed sets Z_i, such that $\dim(Z_i) < \dim(U)$ and for every

$[C] \in U \setminus \bigcup Z_i$, $\mathrm{End}(\mathrm{Jac}(C)) = \mathbb{Z}$. Looking at the construction above, the natural idea is to take $Z_i = \pi(\underline{\mathrm{End}}^i_{\mathcal{J}/U})$. Unfortunately, some of the $\pi_i := \pi_{|\underline{\mathrm{End}}^i_{\mathcal{J}/U}}$ will be dominant, as $\mathrm{End}(\mathcal{J}_\eta)$ always contains the homomorphisms given by multiplication with an integer. So, to get our closed sets Z_i, we will have to shrink the $\underline{\mathrm{End}}^i_{\mathcal{J}/U}$'s.

Let $n \in \mathbb{Z}$ and consider $[n]_{\mathcal{J}}$, the endomorphism of $\mathcal{J}$ given by multiplication by the integer n in the fibres. By definition of $\underline{\mathrm{End}}_{\mathcal{J}/U}$, this corresponds to a morphism $s_n \colon U \to \underline{\mathrm{End}}_{\mathcal{J}/U}$, which is a section of π. Let $\underline{\mathrm{End}}^i_{\mathcal{J}/U}$ be the connected component containing the image of s_n. Let $\underline{\mathrm{End}}^i_{\mathcal{J}/U} = \bigcup V_j$ be the irreducible components of $\underline{\mathrm{End}}^i_{\mathcal{J}/U}$ and V_k the irreducible component containing $\mathrm{im}(s_n)$. As s_n is proper, we have that $\mathrm{im}(s_n)$ is a closed irreducible subset of V_k. By corollary 2.9, every fibre of π is discrete. Hence, $\dim(V_k) \leq \dim(U)$ and thus $\mathrm{im}(s_n) = V_k$. Conversely, every dominant V_k will be of this form, as $\mathrm{End}(\mathcal{J}_{\overline{\eta}}) = \mathbb{Z}$.

Define $J_i := \{j \mid \pi_{|V_j} \text{ is not dominant}\}$ and $Z_i := \pi(\bigcup_{j \in J_i} V_j)$. Then the Z_i are non-dominant by definition and for all curves $[C] \in U \setminus \bigcup Z_i$, we have $\mathrm{End}(\mathrm{Jac}(C)) = \mathbb{Z}$, as the points we removed from the $\underline{\mathrm{End}}^i_{\mathcal{J}/U}$'s corresponded to multiplication with an integer.

3.3. **Simplicity of $\mathcal{J}_{\overline{\eta}}$.** We have shown that to prove the theorem, it is enough to prove that $\mathrm{End}(\mathcal{J}_{\overline{\eta}}) = \mathbb{Z}$. Our next goal is to prove that this is in fact equivalent to $\mathcal{J}_\eta$ being simple. For this, we need the following technical lemmas.

Lemma 3.4. *Let Y be a projective irreducible curve with generic point η and $V \subset Y$ a non-empty open subset. Let $\mathcal{X} \to V$ be an irreducible flat family and $\overline{\mathcal{X}} \to Y$ a family such that $\mathcal{X} \subset \overline{\mathcal{X}}$ is dense. Then for a closed point $o \in Y \setminus V$, we have* $\dim(\overline{\mathcal{X}}_o) = \dim(\mathcal{X}_\eta)$.

Proof. By the semi-continuity of fibre dimension, we know that $\dim(\mathcal{X}_\eta) \leq \dim(\overline{\mathcal{X}}_o)$, so it suffices to show that $\dim(\overline{\mathcal{X}}_o) \leq \dim(\mathcal{X}_\eta)$. Assume $\dim(\overline{\mathcal{X}}_o) > \dim(\mathcal{X}_\eta)$. As $\mathcal{X} \subset \overline{\mathcal{X}}$ is dense and $\mathcal{X} \to V$ flat, we have $\dim(\overline{\mathcal{X}}) = \dim(\mathcal{X}) = \dim(\mathcal{X}_\eta) + 1$. So if $\dim(\overline{\mathcal{X}}_o) > \dim(\mathcal{X}_\eta)$, it follows that $\dim(\overline{\mathcal{X}}_o) \geq \dim(\overline{\mathcal{X}})$, which is impossible as $\overline{\mathcal{X}}$ is irreducible. $\square$

Lemma 3.5. *Let R be a DVR with maximal ideal $\mathfrak{m}$. Let A and B be two R-algebras such that $\mathfrak{m}A$ and $\mathfrak{m}B$ are prime ideals in A and B. Let $f \colon A \to B$ be an injective R-algebra homomorphism. Assume $f_{\mathfrak{m}} \colon A/\mathfrak{m}A \to B/\mathfrak{m}B$ is injective and the induced map $\tilde{f} \colon Q(A) \to Q(B)$ is an isomorphism. Then $\tilde{f}_{\mathfrak{m}} \colon Q(A/\mathfrak{m}A) \to Q(B/\mathfrak{m}B)$ is an isomorphism as well.*

Proof. We need to show that $\tilde{f}_{\mathfrak{m}}$ is surjective. So, let $\frac{\overline{a}}{\overline{s}} \in Q(B/\mathfrak{m}B)$ and $\frac{b}{t} \in Q(A)$ such that $\frac{f(b)}{f(t)} \sim \frac{a}{s}$ in $Q(B)$. As long as $f(t) \notin \mathfrak{m}B$, $\frac{\overline{b}}{\overline{t}} \in Q(A/\mathfrak{m}A)$ will be sent to $\frac{\overline{a}}{\overline{s}}$ by $\tilde{f}_{\mathfrak{m}}$. Assume $f(t) \in \mathfrak{m}B$. As $a \cdot f(t) = f(b) \cdot s$ and $s \notin \mathfrak{m}B$, we see that $f(b) \in \mathfrak{m}B$ as $\mathfrak{m}B$ is a prime ideal. But then $b, t \in \mathfrak{m}A = \pi A$, so there exist $b', t' \in A$ such that $b = b' \cdot \pi$ and $t = t' \cdot \pi$. Then $\frac{b}{t} \sim \frac{b'}{s'}$ and after repeating this finitely many times, we can assume that $t' \notin \mathfrak{m}A$, finishing the proof. ☐

Using these lemmas, we can now prove that $\mathrm{End}(\mathcal{J}_{\overline{\eta}}) = \mathbb{Z}$ is equivalent to $\mathcal{J}_\eta$ being absolutely simple.

Proposition 3.6. *If $\mathcal{J}_\eta$ is absolutely simple, then* $\mathrm{End}(\mathcal{J}_{\overline{\eta}}) = \mathbb{Z}$.

Proof. Let $\mathcal{J}_\eta$ be absolutely simple, then the only degree 0 endomorphism of $\mathcal{J}_\eta$ is $[0]_{\mathcal{J}_\eta}$. Repeating the steps in 3.2 for $\underline{\mathrm{End}}_{\mathcal{J}/U}(0)$ instead of $\underline{\mathrm{End}}_{\mathcal{J}/U}$, we find countably many non-dominant closed sets $Z'_i \subset U$ such that the Jacobian of every curve $[C] \in U \setminus \bigcup Z'_i$ is absolutely simple. Thus, using Lemma 2.12, we can find a Lefschetz pencil $(C_t)_{t \in \mathbb{P}^1}$ in $|L|_x$, such that the Jacobian of its generic fibre is absolutely simple.

Let $\mu \in \mathbb{P}^1$ be the generic point and $\overline{\mu}$ a geometric point over μ. The idea of the proof is to construct a homomorphism $\phi\colon \mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}})) \to \mathrm{End}(\mathbb{G}_m) \simeq \mathbb{Z}$, which then needs to be injective, hence proving that $\mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}})) = \mathbb{Z}$. To do this, consider $f \in \mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}}))$, an endomorphism of $\mathrm{Jac}(C_\mu)$ defined over some finite field extension of K of $k(\mu)$. Using the equivalence of categories [1, Theorem 0BY1], we find a non-constant morphism of smooth projective curves $P \to \mathbb{P}^1$, such that $k(\mu_P) = K$ for $\mu_P \in P$ the generic point. Let $D \subset |L|_x$ be the discriminant divisor, $V \subset P$ be an open subset with $V \cap D = p$ and call the base change of $(C_t)_{t \in \mathbb{P}^1}$ to V again $(C_t)_{t \in V}$. Considering $\underline{\mathrm{Pic}}^0(C/V)$, we find a family over V, the fibres of which are all abelian varieties except for the fibre at p, which is an extension of an abelian variety by the group $\mathbb{G}_m$ by lemma 2.13.

Let $\Gamma_f \subset \mathrm{Jac}(C_\mu) \times \mathrm{Jac}(C_\mu)$ be the graph of f. Taking its closure, we get a closed set $\overline{\Gamma}_f \subset \underline{\mathrm{Pic}}^0(C/V) \times_V \underline{\mathrm{Pic}}^0(C/V)$. As Γ_f is the graph of a morphism, $p_1\colon \overline{\Gamma}_f \to \underline{\mathrm{Pic}}^0(C/V)$ is generically finite of degree 1. Using lemma 3.4 on $\overline{\Gamma}_f$ and the scheme theoretic image of p_1, we see that $\dim(\underline{\mathrm{Pic}}^0(C_p)) = \dim((\overline{\Gamma}_f)_p)$ and $\tilde{p}_1 \coloneqq (p_1)_{|(\overline{\Gamma}_f)_p}\colon (\overline{\Gamma}_f)_p \to \underline{\mathrm{Pic}}^0(C_p)$ is surjective. Therefore, $\tilde{p}_1$ is also generically finite. Furthermore, using lemma 3.5 on $\mathcal{O}_{C,p}$, we see that $\tilde{p}_1$ is in fact generically finite of degree 1.

By Lemma 2.13, we have a short exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \underline{\mathrm{Pic}}^0(C_p) \longrightarrow \mathrm{Jac}(\tilde{C}_p) \longrightarrow 0 \tag{3.1}$$

for $\tilde{C}_p$ the normalization of C_p. Thus, we find a $\mathbb{G}_m$ in $\underline{\mathrm{Pic}}^0(C_p)$ such that $(\tilde{p}_1)_{|\tilde{p}_1^{-1}(\mathbb{G}_m)}\colon \mathbb{G}_m \to \mathrm{Pic}^0(C_p)$ is generically finite of degree 1. As $\mathbb{G}_m$ is a curve, there exists a non-empty open set $W \subset \mathbb{G}_m$ such that $(\tilde{p}_1)_{|\tilde{p}_1^{-1}(W)}$ is an isomorphism onto its image. We want $\tilde{p}_1^{-1}(W)$ to be the graph of a homomorphism $W \to \mathbb{G}_m$, which will extend to an endomorphism $\mathbb{G}_m \to \mathbb{G}_m$. To prove this, we need $p_2(\tilde{p}_1^{-1}(W)) \subset \ker(\underline{\mathrm{Pic}}^0(C_p) \to \mathrm{Jac}(\tilde{C}_p))$. If this were not the case, we would get a non-trivial map $W \to \mathrm{Jac}(\tilde{\mathcal{C}}_p)$, which would extend to a non-trivial map from $\mathbb{P}^1$ to $\mathrm{Jac}(\tilde{C}_p)$, but such a map does not exist.

Let us check that the map defined by sending f to the homomorphism of $\mathbb{G}_m$ we just constructed is indeed a homomorphism. First consider $[n]_{\mathrm{Jac}(C_{\overline{\mu}})} \in \mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}}))$ for some $n \in \mathbb{Z}$. Then the closure of the graph will just be the graph of $[n]_{\underline{\mathrm{Pic}}^0(C/V)}$ and thus $[n]_{\mathrm{Jac}(C_{\overline{\mu}})}$ will be sent to $[n]_{\mathbb{G}_m}$. Furthermore, if $f, g \in \mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}}))$, $\overline{\Gamma}_{f\circ g}$ is a closed subset of the contraction of $\overline{\Gamma}_f$ and $\overline{\Gamma}_g$. So, there exists an open set $W \subset \mathbb{G}_m$ such that $(\overline{\Gamma}_{f\circ g})_{|W}$ is the contraction of $(\overline{\Gamma}_f)_{|W}$ and $(\overline{\Gamma}_g)_{|W}$ and thus $\phi(f\circ g) = \phi(f)\circ\phi(g)$. Lastly, for $f, g \in \mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}}))$, $f+g$ is given by $s \circ (f,g) \circ \Delta$, where Δ is the diagonal morphism and s is the group structure on $\mathrm{Jac}(C_{\overline{\mu}})$. Then by again considering contraction, we see that $\phi(f+g) = \phi(f)+\phi(g)$. As every non-zero element in $\mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}}))$ is an isogeny and ϕ is a homomorphism, we see by lemma 2.5 that $\mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}})) \to \mathbb{G}_m = \mathbb{Z}$ is injective. Thus, $\mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}}) = \mathbb{Z}$.

Consider an endomorphism f of $\mathcal{J}_{\overline{\eta}}$, which by definition corresponds to a $k(\overline{\eta})$-point in $\underline{\mathrm{End}}_{\mathcal{J}/U}$. Let $Z \subset \underline{\mathrm{End}}_{\mathcal{J}/U}$ be the irreducible component containing this $k(\overline{\eta})$-point, then $Z \to U$ must be dominant. The construction in 3.2 implies, that if f is not multiplication by an integer, we find an open set in $V \subset U$ such that all points of Z lying over V correspond to endomorphisms which are not multiplication by an integer. Therefore, we would find a Lefschetz pencil as above such that $\mathrm{End}(\mathrm{Jac}(C_{\overline{\mu}})) \neq \mathbb{Z}$, which is a contradiction. Thus, also $\mathrm{End}(\mathcal{J}_{\overline{\eta}}) = \mathbb{Z}$. □

We end the reduction steps by showing that $\mathcal{J}_\eta$ being simple already implies that it is absolutely simple.

Lemma 3.7. *$\mathcal{J}_\eta$ is simple if and only if it is absolutely simple.*

Proof. The "if-direction" holds by definition. Conversely, assume $\mathcal{J}_\eta$ is simple but not absolutely simple. Then by assumption, there exists a finite field extension $K/k(\eta)$ and a proper non-trivial simple abelian subvariety $B \subset \mathcal{J}_K$ over K.

Claim. *$\mathcal{J}_K \sim \prod_{\sigma\in G} B_\sigma$, where $G \subset \mathrm{Gal}(K/k(\eta))$ is a subgroup with order greater or equal two and the B_σ are the Galois conjugates of B.*

Proof. Let $s\colon \mathcal{J}_K \times \mathcal{J}_K \to \mathcal{J}_K$ be the group operation. Define morphisms s_n from $\prod_{i=1}^n \mathcal{J}_K \to \mathcal{J}_K$ recursively by setting $s_n = s \circ (s_{n-1} \times \mathrm{id}_{\mathcal{J}_K})$ and $s_1 = s$. For

$n = [K : k(\eta)]$, consider the morphism $s_n : \prod_{\sigma \in \mathrm{Gal}(K/k(\eta))} B_\sigma \to \mathcal{J}_K$. By definition, $\mathrm{im}(s_n)$ will be invariant under the action of $\mathrm{Gal}(K/k(\eta))$ and hence defined over $k(\eta)$. As $\mathcal{J}_\eta$ is simple, this shows that s_n must be surjective. If $\ker(s_n)$ is finite, s_n is an isogeny and we are done. Assume not. Then there exists $\tau \in \mathrm{Gal}(K/k(\eta))$, such that for $s_{n-1} : \prod_{\sigma \in \mathrm{Gal}(K/k(\eta)) \setminus \{\tau\}} B_\sigma \to \mathcal{J}_K$, $\mathrm{im}(s_{n-1}) \cap B_\tau$ is non-finite. As B_τ is simple, we have that $\mathrm{im}(s_{n-1}) \cap B_\tau = B_\tau$ and thus s_{n-1} must already be surjective. After repeating this step a finite number of times, we find a subgroup $G \subset \mathrm{Gal}(K/k(\eta))$ of order i such that $s_i : \prod_{\sigma \in G} B_\sigma \to \mathcal{J}_K$ is an isogeny. As B is not defined over $k(\eta)$, we see that i must be greater or equal to two. $\square$

Let φ be the endomorphism of $\mathcal{J}_K$ with image B. By [1, Theorem 0BXN], there exists a normal variety U' with function field K and a dominant morphism $g : U' \to U$. Using [14, Theorem 5.22 (b)], we can spread φ to an endomorphism over an open subset of U'. Doing the same for the automorphisms of $\mathcal{J}_K$ associated with $\mathrm{Gal}(K/k(\eta))$, we get an action of $\mathrm{Gal}(K/k(\eta))$ defined over some open set of U'. Thus, we can find a Lefschetz pencil $(C_t)_{t \in \mathbb{P}^1}$ in $|L|_x$ with geometric generic point $\overline{\mu} \in \mathbb{P}^1$, such that φ and the action of $\mathrm{Gal}(K/k(\eta))$ are defined on $\mathrm{Jac}(C_{\overline{\mu}})$. Then for $\tilde{B} = \mathrm{im}(\varphi_{\overline{\mu}})$, we have $\mathrm{Jac}(C_{\overline{\mu}}) \sim \prod_{\sigma \in G} \tilde{B}_\sigma$. As we showed above, there is a homomorphism ϕ: $\mathrm{End}(\mathcal{J}_{\overline{\mu}}) \to \mathbb{G}_m$. For $\sigma \in G$, define f_σ to be endomorphism corresponding to $\sigma \circ (\mathrm{id} \times 0 \times \ldots \times 0) \circ \sigma^{-1}$. Then for $\sigma, \tau \in G$ with $\sigma \neq \tau$, we have $f_\sigma \circ f_\tau = 0$, so there exists $\sigma \in G$ such that $\phi(f_\sigma) = 0$. Therefore, $\phi(f_\sigma) = 0$ for all $\sigma \in G$, as the f_σ are conjugates and ϕ respects composition. Thus, $\phi(\sum\limits_{\sigma \in G} f_\sigma) = 0$, which is a contradiction as $\sum\limits_{\sigma \in G} f_\sigma$ is an isogeny. $\square$

4. Simplicity of the generic fibre I

To finish the proof in the regular case, it is enough to show that $\mathcal{J}_\eta$ is simple. We will give two different proofs of this fact, the first being the original proof by Ciliberto and van der Geer and the second using the geometry and monodromy of Lefschetz pencils. The first proof relies on the correspondence given by proposition 2.11 and the fact that simplicity of $\mathcal{J}_\eta$ is equivalent to all endomorphisms of $\mathcal{J}_\eta$ over $k(\eta)$ being isogenies or the 0-morphism. We will actually prove something slightly stronger, namely that the only endomorphisms of $\mathcal{J}_\eta$ over $k(\eta)$ are the ones given by multiplication with an integer. Looking at the construction of the correspondence in proposition 2.11, it is easy to see that this is equivalent to $\mathrm{Pic}(\mathcal{C}_\eta \times \mathcal{C}_\eta)/N$ being equal to $\mathbb{Z} \cdot \Delta$, for $\Delta \subset \mathcal{C}_\eta \times \mathcal{C}_\eta$ the diagonal.

Let T be a divisor in $\mathcal{C}_\eta \times \mathcal{C}_\eta$. Using T, we will construct a rational map $\phi : S \to \mathrm{Pic}(S)$, which will have to be constant as S is regular. This fact, combined with the construction of the map itself, will allow us to show that $\mathcal{O}(T) \sim n \cdot \Delta$ in $\mathrm{Pic}(\mathcal{C}_\eta \times \mathcal{C}_\eta)/N$.

4.1. **Monodromy of curves.** Before getting into the construction of ϕ, we will look at the monodromy on the intersection of curves in our linear system, as this is needed for the proof. Let S and L be as above and let $y, z \in S$ be two general points. Define $|L|_{y,z}$ as the linear subsystem of $|L|$ of all curves passing through y and z and let C be a smooth curve in $|L|_{y,z}$. A curve $D \in |L|_{y,z}$ and C intersect transversely at a point p, if D is smooth at p and $T_pC + T_pD = T_pS$. We call D transverse to C, if C and D meet transversely at every point p in $C \cap D$ and let V be the open set of curves in $|L|_{y,z}$ which are transverse to C.

Define $I := \{(p, H) \in C \times V \mid p \in C \cap H,\ p \notin \{y, z\}\} \subset C \times V$. Then $p_2 \colon I \to V$ is a topological covering, as all points in the intersection of two transverse curves have intersection multiplicity one, and we get the following lemma.

Lemma 4.1. *Let $D \in V$ and $C \cap D = \{y, z, x_1, ..., x_d\}$. Then the image of the monodromy map*

$$\pi_1(V, D) \longrightarrow \mathrm{Aut}(\{x_1, ..., x_d\})$$

is the full symmetric group.

Proof. The proof is analogous to the one of the lemma on page 111 in [2]. We simply replace the linear system of all hyperplane sections with the one of hyperplane sections passing through y and z and consider the monodromy action on the points in the intersection which are not y or z. □

4.2. **Constructing ϕ.** Suppose we are given a divisor T in $\mathcal{C}_\eta \times \mathcal{C}_\eta$. Taking the closure of T in $\mathcal{C} \times_{|L|_x} \mathcal{C}$ gives us a divisor $\overline{T}$ in $\mathcal{C} \times_{|L|_x} \mathcal{C}$. Let Σ be a general two-dimensional linear subsystem of $|L|_x$ whose generic member is smooth and irreducible. We want to construct a rational map $\phi_{\Sigma,T} \colon S \to \mathrm{Pic}(S)$.

We pull back $\mathcal{C}$ to Σ and call it again $\mathcal{C}$. Let $V \subset S$ be an open set in S such that for all $y \in V$, Σ_y, the linear subsystem of all curves in Σ running through y, is a Lefschetz pencil. Consider the map $p \colon \mathcal{C} \times_\Sigma \mathcal{C} \to S \times S$ given by projection in both factors. Set $W := p^{-1}(V \times S)$ and let T' be the pull back of $\overline{T}$ to W. Denote by $\mathcal{D}$ the line bundle corresponding to $(p_{|W})_*[T']$. Then by the universal property of the picard variety,

$$\phi_{T,\Sigma} \colon V \longrightarrow \mathrm{Pic}(S), \quad y \longrightarrow (y \times S)^*\mathcal{D}$$

defines a morphism from V to $\mathrm{Pic}(S)$.

Remark 4.2. (i) Let $D \subset |L|_x$ be the discriminant divisor (see lemma 2.12) and $\tilde{D} = D \cap \Sigma$. Looking at the cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{i} & \mathcal{C} \\ \downarrow & & \downarrow \\ \tilde{D} & \longrightarrow & \Sigma, \end{array}$$

we see $p \circ i(\tilde{\mathcal{C}})$ is closed of dimension smaller or equal to three in $S \times S$. Define $\tilde{V}$ to be the complement of $p \circ i(\tilde{\mathcal{C}})$ in $V \times S$ and $\tilde{W}$ to be $p^{-1}(\tilde{V})$. Then $\tilde{W}$ is smooth and $\tilde{p} \coloneqq p_{|\tilde{W}} \colon \tilde{W} \to \tilde{V}$ is proper.

(ii) Let $C \in |L|_x$ be a general curve and y a general closed point in C. As the pull-back of $\mathcal{C}$ to Σ_y is a Lefschetz pencil, $\mathcal{C}_{\Sigma_y}$ is the blow-up of S in the base points of Σ_y. Thus, $p_{|p^{-1}(y \times C)}$ is an isomorphism outside of the base points of Σ_y and hence $y \times C \setminus \tilde{V}$ is contained in the base points of Σ_y.

4.3. **Calculations with intersection points.** Let T be a divisor in $\mathcal{C}_\eta \times \mathcal{C}_\eta$. Our goal is to show that there exists $n \in \mathbb{Z}$, such that $\mathcal{O}(T)$ is equivalent to $n \cdot \Delta$ in $\mathrm{Pic}(\mathcal{C}_\eta \times \mathcal{C}_\eta)/N$. For this, consider a general curve $C \in |L|_x$, a general closed point $y \in C$ and choose a general two-dimensional linear subsystem Σ containing C. Then define D_y to be $p_*([T'])_{y \times S}$, which is the divisor corresponding to $\phi_{T,\Sigma}(y)$.

Lemma 4.3. *Let E_y be the intersection of D_y and C. This is a divisor of the form*

$$E_y = \alpha x + \beta y + \gamma B_{x,y} + T_C(y) \tag{4.1}$$

for fixed $\alpha, \beta, \gamma \in \mathbb{Z}$ and $B_{x,y}$ the divisor of base points of Σ_y different from x and y.

Proof. Intersecting D_y and C in S is the same as intersecting $p_*[T']$ and $y \times C$ in $V \times S$. Let $\tilde{T} \coloneqq T' \cap \tilde{W}$ and $\tilde{C} \coloneqq y \times C \cap \tilde{V}$. Then $\tilde{p}_*[\tilde{T}] \cap [\tilde{C}]$ is the equal to the pull-back of $p_*[T'] \cap [y \times C]$ to $\tilde{V}$. By remark 4.2, $\tilde{W}$ and $\tilde{V}$ are smooth and $\tilde{p}$ is proper. Thus, by the projection formula, $\tilde{p}_*[\tilde{T}] \cap [\tilde{C}] = \tilde{p}_*([\tilde{T}] \cap \tilde{p}^*[\tilde{C}])$. Outside of the base points of Σ_y, $\tilde{p}_{|p^{-1}(\tilde{C})}$ is an isomorphism. Hence, outside of the base points of Σ_y, $[\tilde{T}] \cap \tilde{p}^*[\tilde{C}]$ is just $\tilde{T}_{\tilde{C}}(y)$ and then so is $\tilde{p}_*[\tilde{T}] \cap [\tilde{C}]$. As $C \setminus \tilde{C}$ is contained in the base points of Σ_y by remark 4.2, the intersection of $p_*[T']$ and $y \times C$ in $V \times S$ is given by $T_C(y)$ plus a sum of base points of Σ_y.

Therefore, E_y is of the form $\alpha x + \beta y + \gamma B_{x,y} + T_C(y)$ for some $\alpha, \beta, \gamma \in \mathbb{Z}$ and by continuity, α, β, γ must be the same for general $y \in C$. ☐

Lemma 4.4. *All base points in $B_{x,y}$ appear with the same multiplicity*

Proof. For general Σ and y, the curves spanning Σ_y are transversal, hence all base points in Σ_y have multiplicity 1. If we vary Σ, the only term of E_y that will change is $B_{x,y}$, as x, y and $T_C(y)$ only depend on y and C. Hence, we get a monodromy action on $B_{x,y}$ which is just the action from lemma 4.1. But by the lemma, monodromy acts as the full symmetric group and hence all base points in $B_{x,y}$ must have the same multiplicity. □

As S is regular, $\mathrm{Pic}(S)$ is discrete and hence $\phi_{T,\Sigma}\colon S \to \mathrm{Pic}(S)$ is constant. Thus, for general points $y, y' \in C$, D_y and $D_{y'}$ are linearly equivalent and then so are E_y and $E_{y'}$. Therefore,

$$\alpha x + \beta y + \gamma B_{x,y} + T_C(y) \sim \alpha x + \beta y' + \gamma B_{x,y'} + T_C(y') \tag{4.2}$$

by lemma 4.3.

As Σ_y and Σ_y' are both linear subsystems of Σ, the intersection of the two spanning curves must be linearly dependent and hence

$$x + y + B_y \sim x + y' + B_{y'}. \tag{4.3}$$

Combining the two equations, we get

$$(\beta - \gamma)y + T_C(y) \sim (\beta - \gamma)y' + T_C(y'). \tag{4.4}$$

This implies that $T \sim (\beta - \gamma)\Delta$, finishing the proof that $\mathcal{J}_\eta$ is simple and thus the proof of theorem 1.3 in the regular case.

5. Simplicity of the generic fibre II

As promised above, we will give another argument for the simplicity of $\mathcal{J}_\eta$, using the irreducibility of monodromy. This time the proof will be by contradiction, so let us assume that $\mathcal{J}_\eta$ is non-simple.

5.1. **Facts about monodromy.** Let X be a smooth projective variety of dimension n and $X \subset \mathbb{P}^m$ a closed embedding. Let $U \subset (\mathbb{P}^m)^*$ the open set parametrizing the smooth hyperplane sections of X and $f\colon \chi_U \to U$ the corresponding universal family. Let $(X_t)_{t\in\mathbb{P}^1}$ be a Lefschetz pencil and $o \in \mathbb{P}^1$ a regular value of $(X_t)_{t\in\mathbb{P}^1}$. For $j\colon X_o \hookrightarrow X$ the inclusion, we define $H^{n-1}(X_o, \mathbb{Q})_{van} := \ker(j_*\colon H^{n-1}(X_o, \mathbb{Q}) \to H^{n+1}(X, \mathbb{Q}))$.

Remark 5.1. The monodromy representation $\rho\colon \pi_1(U, o) \to \mathrm{Aut}(H^{n-1}(X_o, \mathbb{Q}))$ given by $R^{n-1}f_*\mathbb{Q}$ leaves $H^{n-1}(X_o, \mathbb{Q})_{van}$ stable.

Theorem 5.2. *The monodromy representation*

$$\rho\colon \pi_1(U,o) \longrightarrow \mathrm{Aut}(H^{n-1}(X_o,\mathbb{Q})_{van})$$

is irreducible.

Proof. [17, Theorem 3.27] ☐

5.2. **Irreducible monodromy.** We want to use the theorem in our original setting, so let $X = S$ and $\chi_U = \mathcal{C}$. We have two families over U:

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "p"'] & & \mathcal{J} \arrow[dl, "q"] \\ & U & \end{tikzcd}$$

Lemma 5.3. $R^1p_*\mathbb{Q} \cong R^1q_*\mathbb{Q}$

Proof. As we chose U to be in $|L|_x$, the map $\mathcal{C} \to U$ has a section s and thus we get a unique U-morphism $\phi_s\colon \mathcal{C} \to \mathcal{J}$ (see [13] page 5 and 6). Then for $A \subset U$ an affine open subset, we have $\phi_p^*\colon H^1(\mathcal{J}_A,\mathbb{Q}) \xrightarrow{\sim} H^1(\mathcal{C}_A,\mathbb{Q})$ and these glue into an isomorphism $R^1q_*\mathbb{Q} \xrightarrow{\sim} R^1p_*\mathbb{Q}$. ☐

Let $u \in U$ be a general closed point. Then by lemma 2.12, we can find a Lefschetz pencil through u. We assumed that S is regular, so $H^3(S,\mathbb{Q}) = 0$ and thus $H^1(\mathcal{C}_u,\mathbb{Q})_{van} = H^1(\mathcal{C}_u,\mathbb{Q})$. By theorem 5.2, the monodromy representation $\rho\colon \pi_1(U,u) \to \mathrm{Aut}(H^1(\mathcal{C}_u,\mathbb{Q}))$ is irreducible and from the above lemma and the equivalence of local systems and monodromy representations we conclude that $R^1q_*\mathbb{Q}$ is irreducible.

Lemma 5.4. *There exists a non-empty open subset $V \subset U$ such that $R^1q_*\mathbb{Q}_{|V}$ has a non-trivial local subsystem.*

Proof. We assumed $\mathcal{J}_\eta$ to be non-simple, so there exists a non-trivial degree 0 endomorphism $\tilde{\phi} \in \mathrm{End}(\mathcal{J}_\eta)$. Using again [14, Theorem 5.22 (b)], we find $V' \subset U$ open and an endomorphism ϕ of $\mathcal{J}_{V'}$ such that $\phi_\eta = \tilde{\phi}$. Then $G := \mathrm{im}(\phi)$ is a group scheme over V' and we set $s' := q_{|G}$. There exists a non-empty open subset $V \subset V'$ such that $s := s'_{|V}$ is smooth and proper. For $A \subset V$ an affine open, we have $\phi^*\colon H^1(G_A,\mathbb{Q}) \to H^1(\mathcal{J}_A,\mathbb{Q})$, which glues to a map $\psi\colon R^1s_*\mathbb{Q} \to R^1q_*\mathbb{Q}_{|V}$ of local systems on V.

Then $\dim(\mathrm{im}(\psi)_\eta) = \dim(\mathrm{im}(\tilde{\phi})) < \dim(\mathcal{J}_\eta) = \dim(R^1q_*\mathbb{Q}_\eta)$, so $\mathrm{im}(\psi)$ is the non-trivial local subsystem we are looking for. ☐

Let $p \in V$ closed. As $U \setminus V$ has real codimension 2, the map $\pi_1(V,p) \to \pi_1(U,p)$ is surjective and thus the monodromy representation $\pi_1(V,p) \to H^1(\mathcal{C}_p, \mathbb{Q})$ is irreducible as well. But then $R^1 q_* \mathbb{Q}_{|V}$ must be irreducible, which is a direct contradiction to lemma 5.4. Thus, we conclude that $\mathcal{J}_\eta$ must be simple.

6. The non-regular case

The proof closely mirrors that for regular surfaces. We only have to check that all steps of the proof work in the more general case. First, we need a family to replace our $\mathcal{J}$. For this, consider the inclusion of U-schemes $j\colon \mathcal{C} \hookrightarrow S \times U$, given by the inclusion of the curves of $\mathcal{C}$ into S. Then j induces a homomorphism $\tilde{j}\colon \mathcal{J} \to \mathrm{Alb}(S) \times U$ of abelian schemes and we define $\mathcal{K} \coloneqq \ker(\tilde{j})$.

There exists an open subset $\tilde{U}$ of U such that $s\colon \mathcal{K} \to \tilde{U}$ is smooth and for all $u \in \tilde{U}$ we have $\mathcal{K}_u = K(\mathcal{C}_u, S)$. Thus, $\mathcal{K} \to \tilde{U}$ is the family we are looking for. If $\mathcal{K}_\eta = 0$, then $\mathcal{K}_u = 0$ for all $u \in \tilde{U}$ by flatness and the theorem holds. Hence, we can again assume that $\mathcal{K}_\eta \neq 0$.

6.1. **Reduction Steps.**

To prove the representability of the endomorphism functor and construct the closed sets Z_i, we only used the fact that $\mathcal{J} \to U$ is an abelian scheme. Thus, we can simply replace $\mathcal{J}$ by $\mathcal{K}$ in the arguments of section 3.1 and 3.2 to reduce the proof to showing that $\mathrm{End}(\mathcal{K}_\eta) = \mathbb{Z}$. We did, however, use the fact that $\mathcal{J}$ is a family of Jacobians in section 3.3, so we need to check that proposition 3.6 and lemma 3.7 still hold if we replace $\mathcal{J}$ by $\mathcal{K}$.

To do this, consider a general curve C in the discriminant divisor D, which by 2.12 will have a single ordinary double point as its only singularity. The inclusion $C \subset S$ induces a map $\underline{\mathrm{Pic}}^0(C) \to \mathrm{Alb}(S)$. As $\mathrm{Alb}(S)$ is an abelian variety, we cannot have a non-trivial rational map $\mathbb{G}_m \to \mathrm{Alb}(S)$, as this would extend to a non-trivial map $\mathbb{P}^1 \to \mathrm{Alb}(S)$. Looking again at

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \underline{\mathrm{Pic}}^0(C) \longrightarrow \mathrm{Jac}(\tilde{C}) \longrightarrow 0 \tag{6.1}$$

for $\tilde{C}$ the normalization of C, we see that the $\mathbb{G}_m \to \underline{\mathrm{Pic}}^0(C) \to \mathrm{Alb}(S)$ must be trivial. Thus, $\mathbb{G}_m \to \underline{\mathrm{Pic}}^0(C)$ factors through $K(C,S)$ and we see that $K(C,S)$ is an extension of an abelian variety by $\mathbb{G}_m$.

Using lemma 2.12, we can find a family over some open set $V \subset \mathbb{P}^1$ with all fibres abelian varieties except one, which is an extension of an abelian variety by $\mathbb{G}_m$ and proposition 3.6 and lemma 3.7 hold for $\mathcal{K}$.

6.2. **Irreducible Monodromy.**

We have again reduced the proof to showing that $\mathcal{K}_\eta$ is simple. Both proofs for the simplicity of the generic fibre we did above can be generalized to non-regular surfaces. We are going to demonstrate how to do it for the second one.

The inclusion $j\colon \mathcal{C} \to S \times U$ induces a morphism $j_*\colon R^1p_*\mathbb{Q} \to R^3\mathrm{pr}_{2*}\mathbb{Q}$. For $u \in U$ a closed point, the morphism on the stalk is just $j'_*\colon H^1(\mathcal{C}_u, \mathbb{Q}) \to H^3(S, \mathbb{Q})$, where $j'\colon \mathcal{C}_u \to S$ is the inclusion. So the kernel of j_* is a local subsystems with fibre $H^1(\mathcal{C}_u, \mathbb{Q})_{van}$ at a point $u \in \tilde{U}$.

Claim. $\ker(j_*) \cong R^1s_*\mathbb{Q}$ *on* $\tilde{U}$.

Proof. Consider the maps $\varphi\colon \mathcal{K} \to \mathcal{J}$ and $\psi\colon \mathcal{J} \to \mathrm{Alb}(S) \times \tilde{U}$ and let $t\colon \mathrm{Alb}(S) \times \tilde{U} \to \tilde{U}$ be the second projection. The map $\varphi^*\colon R^1q_*\mathbb{Q} \to R^1s_*\mathbb{Q}$ of local system induced by φ is surjective, as φ itself is injective and $\psi^*\colon R^1t_*\mathbb{Q} \to R^1q_*\mathbb{Q}$ is injective, as ψ is surjective. Thus, as $\phi^* \circ \psi^* = 0$, $R^1s_*\mathbb{Q} \cong R^1q_*\mathbb{Q}/R^1t_*\mathbb{Q}$. On the other hand, there is an orthogonal decomposition $H^1(\mathcal{C}_x, \mathbb{Q}) = H^1(\mathcal{C}_x, \mathbb{Q})_{van} \oplus H^1(S, \mathbb{Q})$ for x a closed point by [17, Proposition 2.27] and hence $\ker(j_*) \cong R^1s_*\mathbb{Q}$. □

Thus, $R^1s_*\mathbb{Q}$ is irreducible and analogous to the proof of lemma 5.4, we can now show that this cannot be the case if $\mathcal{K}_\eta$ is non-simple. This finishes the proof of the theorem.

7. Counterexample

Having proven theorem 1.3, a natural question to ask is if it holds in broader generality. First, let us consider the case of L not being very ample. In the original paper, Ciliberto and van der Geer actually proved the theorem, while putting some restrictions on the surface S, for all globally generated line bundles L, such that the map $i\colon S \to \mathbb{P}^n$ induced by L is birational. It is easy to see that for $\dim(\mathrm{im}(i)) = 2$, this is in most cases a necessary condition.

Lemma 7.1. *Let* S, L, i *as above,* $\dim(\mathrm{im}(i)) = 2$ *and* i *not birational. Let* $C \in |L|$ *be a general curve and assume* $g(i(C)) > 0$*, then* $\mathrm{Jac}(C)$ *is non-simple.*

Proof. As $\dim(\mathrm{im}(i)) = 2$, i must be generically finite and hence $i' := i_{|C}$ is generically finite as well. Let $D' := \mathrm{im}(i')$ and $n\colon D \to D'$ be the normalization of D'. Then by the universal property of normalization, there exists a morphism $f\colon C \to D$ such that $n \circ f = i'$. Furthermore, the genus of a curve is a birational invariant and D is birational to D', hence $g(D) = g(D') > 0$. As i is not birational, $\deg(i') > 1$ and hence by the Hurwitz formula $g(C) > g(D)$. Thus, $\dim(\mathrm{Jac}(D)) < \dim(\mathrm{Jac}(C))$ and the image of the induced homomorphism $\tilde{f}\colon \mathrm{Jac}(D) \to \mathrm{Jac}(C)$ is a proper non-trivial abelian subvariety of $\mathrm{Jac}(C)$. □

Example 7.2. Let $\varphi\colon X \to \mathbb{P}^2$ be the cyclic covering corresponding to a smooth section of $\mathcal{O}(4)^{\otimes 2}$. Then φ is smooth and finite of degree 2 and $H^1(X, \mathbb{C}) = H^1(\mathbb{P}^2, \mathbb{C}) = 0$ by [18, Proposition 1.1 (d)]. Thus, X is a smooth regular surface. Consider the line bundle $L \coloneqq \varphi^*(\mathcal{O}(3))$ on X. By [18, Proposition 1.1 (b)], $\varphi_*\mathcal{O}_X \simeq \mathcal{O} \oplus \mathcal{O}(-4)$ and hence $\varphi_*L \simeq \mathcal{O}(3) \otimes \varphi_*\mathcal{O}_X \simeq \mathcal{O}(3) \oplus \mathcal{O}(-1)$ by the projection formula. Then $H^0(X, L) \simeq H^0(\mathbb{P}^2, \mathcal{O}(3))$ and the map i induced by L is given by $X \xrightarrow{\varphi} \mathbb{P}^2 \to \mathbb{P}^{\binom{5}{3}-1}$, where $\mathbb{P}^2 \to \mathbb{P}^{\binom{5}{3}-1}$ is the embedding corresponding to $\mathcal{O}(3)$. So i is finite but not birational and by the genus degree formula the general curve in $\mathcal{O}(3)$ has genus 1. Thus, lemma 7.1 implies that the general curve in L has non-simple Jacobian.

The example shows that theorem 1.3 can already fail for ample line bundles on regular surfaces, so the very ample assumptions is really necessary.

8. Generalization to other fields

Another possible generalization would be to consider algebraically closed fields other than $\mathbb{C}$. Looking at theorem 1.3 and its proof, we immediately notice two possible problems with generalizing the statement to arbitrary algebraically closed fields. First, we used a lot of singular cohomology and monodromy in the proof, which does not exist over other fields. As usual, the tactic here will be to replace singular cohomology and monodromy with étale cohomology and étale monodromy, which will allow us to use similar proof strategies over arbitrary algebraically closed fields. The second problem is the more serious one, already occurring in the statement itself. Namely, the notion of very general is not well-behaved over countable fields, as there might not be a single closed point in the complement of a countable union of non-dominant closed subsets.

As stated in the introduction, there is a version of theorem 1.3 in the case of k being uncountable, due to Banerjee. Working out the ideas given by him in [3, Lemma 2.2], we will give a proof of theorem 1.5. This will also show that even for countable fields, at least the Jacobian of the generic curve in the linear system is absolutely simple.

8.1. The Tate module.

We collect some definitions and results about étale cohomology and monodromy before getting into the proof of theorem 1.5.

Recall 8.1. Let X be a scheme and denote by $X_{\acute{e}t}$ the étale site on X. Then the category of sheaves of abelian groups on $X_{\acute{e}t}$ has enough injectives and for a sheaf $\mathcal{F}$ of abelian groups on $X_{\acute{e}t}$, $H^r(X_{\acute{e}t}, \mathcal{F})$ is defined as the r-th derived functor of the global section functor.

Definition 8.2. Let X be a variety over some field k. Then we define $H^r_{ét}(X, \mathbb{Z}_l) = \varprojlim H^r(X_{ét}, \mathbb{Z}/l^n\mathbb{Z})$ and $H^r_{ét}(X, \mathbb{Q}_l) = H^r_{ét}(X, \mathbb{Z}_l) \otimes \mathbb{Q}_l$.

We are especially interested in the étale cohomology of abelian varieties. So let k be an algebraically closed fields and A an abelian variety over k. For n not divisible by $\mathrm{char}(k)$, we define

$$A_n(k) := \ker([n]_A \colon A(k) \longrightarrow A(k)),$$

which is a group of order n^{2g} by [12, Theorem 8.2].

Definition 8.3. Let $l \neq \mathrm{char}(k)$ be a prime. Then we define the Tate module of A as

$$T_l A := \varprojlim A_{l^n}(k)$$

Theorem 8.4. *Let A, k and l be as above. Then there is a canonical isomorphism $H^1_{ét}(A, \mathbb{Z}_l) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_l}(T_l A, \mathbb{Z}_l)$.*

Proof. [12, Theorem 15.1]. □

8.2. **The Étale fundamental group.** To prove a generalization of theorem 1.3, we will need a statement similar to theorem 5.2 in the étale case. To formulate this, we will need a replacement of the topological fundamental group.

Definition 8.5. Let X be a scheme and $\overline{x}$ a geometric point of X.

(i) A pointed $(X, \overline{x})$ scheme is a scheme Y together with a geometric point $\overline{y}$ of Y and a morphism $f \colon Y \to X$ such that $f \circ \overline{y} = \overline{x}$.

(ii) A pointed $(X, \overline{x})$-scheme $f \colon (Y, \overline{y}) \to (X, \overline{x})$ is called a pointed covering space if f is finite and étale.

(iii) A pointed covering space $f \colon (Y, \overline{y}) \to (X, \overline{x})$ is called Galois if $\deg(f) = |\mathrm{Aut}(Y/X)|$.

(iv) The fundamental group of X at the geometric point $\overline{x}$ is the profinite group

$$\pi_1(X, \overline{x}) := \varprojlim_{(Y, \overline{y})} \mathrm{Aut}(Y/X),$$

where $(Y, \overline{y})$ runs over the category of pointed Galois covering spaces.

Proposition 8.6. *Let X be a connected scheme, $\overline{x}$ a geometric point of X and $\mathcal{G}$ a locally constant sheaf of $\mathbb{Q}_l$ vector spaces on X. Then $\pi_1(X, \overline{x})$ acts continuously on the stalk $\mathcal{G}_{\overline{x}}$ and the functor $\mathcal{G} \to \mathcal{G}_{\overline{x}}$ is an equivalence between the category of locally constant sheaves of $\mathbb{Q}_l$ vector spaces and the category of continuous representations of $\pi_1(X, \overline{x})$ on finite-dimensional $\mathbb{Q}_l$ vector spaces.*

Proof. [8, Proposition A I.8]. □

8.3. **The tame fundamental group.** Unfortunately, this is not quite the group we need to formulate the étale analogues of the Picard–Lefschetz formula and theorem 5.2. For this, we need to define the tame fundamental group. From here on out, we fix an algebraically closed field k. Consider a smooth k-variety T and $U \subset T$ open such that $T \setminus U$ has pure codimension 1. Let $X \to U$ be a Galois covering space of U. Let $\overline{X}$ be the normalization of T in the function field of X and η be the generic point of an irreducible component of $T \setminus U$. Then $X \to U$ is called a tamely ramified Galois covering space if for all geometric points $\alpha\colon \operatorname{Spec}(\Omega) \to \overline{X}$ lying over η, the order e_α of the ramification group $G_\alpha \coloneqq \{\sigma \in \operatorname{Aut}(\overline{X}/T) \mid \sigma \circ \alpha = \alpha\}$ is invertible in $\mathcal{O}_{\overline{X},\operatorname{im}(\alpha)}$.

Definition 8.7. Let $u \in U$ be a closed point. The profinite group

$$\pi_1^t(U,u) \coloneqq \varprojlim_{(X,x)} \operatorname{Aut}(X/U),$$

with (X,x) running over all tamely ramified pointed Galois covering spaces of (U,u), is called the tame fundamental group of U at the point u. It is a quotient group of the étale fundamental group $\pi_1(U,u)$.

Remark 8.8. Let T and U be as above and consider a tamely ramified pointed Galois covering space X of U. Let $s\colon \operatorname{Spec}(k) \to \overline{X}$ be a closed point and define $G_s \coloneqq \{\sigma \in \operatorname{Aut}(\overline{X}/T) \mid \sigma \circ s = s\}$ and $e_s = |G_s|$ as above. Then for $\mu_{e_s}(k)$, the group of e_s-th roots of unity of k, there exists an isomorphism

$$\phi^s\colon \mu_{e_s}(k) \longrightarrow G_s,$$

which is characterized by $\phi(\xi) \cdot x = \tilde{s}(\xi) \cdot x$, where x is a generating element of the maximal ideal of $\mathcal{O}_{\overline{X},s}$ and $\tilde{s}(\xi)$ is an inverse image of ξ under the map $s^\#\colon \mathcal{O}_{\overline{X},s} \to k$.

Proof. [8, Lemma A.I.12]. □

Let $\overline{X}, T, U$ and u be as above. Let $s\colon \operatorname{Spec}(k) \to T \setminus U$ be a closed point and $n \in \mathbb{N}$ be a natural number such that all prime numbers $p \nmid n$ are invertible in $\mathcal{O}_{T,s}$ and define $\hat{\mathbb{Z}}^{(n)}(1) = \varprojlim_{(e,n)=1} \mu_e(k)$. Let $t \in \varprojlim_{(X,x)} \operatorname{Hom}_T(\operatorname{Spec}(k), \overline{X})$ and define $\tilde{\phi}^t$ to be the projective limit of the homomorphisms $\phi^{t(X,x)}$. Then the conjugacy class of $\tilde{\phi}^t$ does not depend on t and we denote by

$$\gamma_s\colon \hat{\mathbb{Z}}^{(n)}(1) \longrightarrow \pi_1^t(U,u)$$

any element of this conjugacy class.

Remark 8.9. There is a natural map $\hat{\mathbb{Z}}^{(p)}(1) = \varprojlim_{(e,p)=1} \mu_e(k) \to \varprojlim_n \mathbb{Z}/l^n\mathbb{Z} = \mathbb{Z}_l(1)$ induced by sending $\mu_{l^n}(k)$ to $\mathbb{Z}/l^n\mathbb{Z}$.

We are only really interested in the case of T being $\mathbb{P}^1_k$ and $\overline{X}$ being a Lefschetz pencil. So let $U \subset \mathbb{P}^1_k$ be a non-empty open subset, $\mathbb{P}^1_k \setminus U = \{s_o, ..., s_n\}$ and $u \in U$ a closed point.

Proposition 8.10. *Let $p = 1$ if $\mathrm{char}(k) = 0$, otherwise let $p = \mathrm{char}(k)$. For a suitable choice of the homomorphisms*

$$\gamma_{s_i} \colon \hat{\mathbb{Z}}^{(p)}(1) \longrightarrow \pi_1^t(U, u) \quad i = 0, ..., n$$

in their conjugacy class, the images generate a dense subgroup.

Proof. [8, Proposition A I.15]. □

8.4. **The Picard–Lefschetz formula.** We are now finally ready to state an analogue of theorem 5.2 in the étale case. Let $\tilde{\mathcal{C}} \to \mathbb{P}^1$ be a Lefschetz pencil of curves. Let $\{s_0, ..., s_n\}$ be the set of points corresponding to curves with ordinary double point and $V := \mathbb{P}^1 \setminus \{s_0, ..., s_n\}$. Then we can look at the tame fundamental group $\pi_1^t(V, v)$ for some closed point $v \in V$.

Theorem 8.11. *Let $f \colon \tilde{\mathcal{C}} \to \mathbb{P}^1, V$ and $\{s_0, ..., s_n\}$ be as above. By theorem 8.6, the locally constant sheaf $R^1 f_* \mathbb{Q}_l$ induces a continuous representation*
$\rho \colon \pi_1(V, v) \to \mathrm{Aut}(H^1_{\acute{e}t}(\tilde{\mathcal{C}}_v, \mathbb{Q}_l))$, for which the following holds:

(i) *The monodromy representation $\rho \colon \pi_1(V, v) \to \mathrm{Aut}(H^1_{\acute{e}t}(\tilde{\mathcal{C}}_v, \mathbb{Q}_l))$ factors through $\pi_1^t(V, v)$.*

(ii) *For each s_i, there exists a vanishing cycle δ_{s_i} in $H^1_{\acute{e}t}(\tilde{\mathcal{C}}_v, \mathbb{Q}_l) \otimes \mathbb{Q}_l(1)$, which depends up to conjugation only on s_i and not on v. Furthermore, all the vanishing cycles δ_{s_i} are conjugate up to sign.*

(iii) *For $a \in H^1_{\acute{e}t}(\tilde{\mathcal{C}}_v, \mathbb{Q}_l)$ and $u \in \hat{\mathbb{Z}}^{(p)}(1)$ we have*

$$\rho(\gamma_{s_i}(u))(a) = a + \tilde{u}\langle a, \delta_{s_i} \rangle \delta_{s_i},$$

where $\tilde{u}$ is the natural image of u in $\mathbb{Z}_l(1) \subset \mathbb{Q}_l(1)$ under the map defined in remark 8.9.

Proof. [8, Theorem 3.7.1]. □

Corollary 8.12. *Let $\mathrm{Ev}(\mathcal{C}_v) = \sum_{i, \sigma \in \pi_1^t(V,v)} \mathbb{Q}_l(-1)\sigma(\delta_{s_i})$ be the space of vanishing cycles. Then the induced representation of $\pi_1^t(V, v)$ on $\mathrm{Ev}(\mathcal{C}_v)$ is irreducible.*

Proof. By [8, Corollary 3.7.4], the action of $\pi_1^t(V, v)$ on $\mathrm{Ev}(\mathcal{C}_v)/(\mathrm{Ev}(\mathcal{C}_v) \cap \mathrm{Ev}(\mathcal{C}_v)^\perp)$ is irreducible. Furthermore, [7, Corollaire 4.3.9] shows that $\mathrm{Ev}(\mathcal{C}_v) \cap \mathrm{Ev}(\mathcal{C}_v)^\perp = 0$. □

8.5. **Lefschetz Pencils.** Before we can use the description of the monodromy of Lefschetz pencils we gave above in the proof, we have to check that Lefschetz pencils exist in the linear system $|L|$, as we did in the complex case in lemma 2.12. Unfortunately, this need not be true over arbitrary fields, which leads us to the following definition.

Definition 8.13. Let X be an irreducible variety with a closed embedding $i\colon X \hookrightarrow \mathbb{P}_k^n$. Then i is called a Lefschetz embedding, if Lefschetz pencils form an open dense subset in $\mathrm{Gr}(1, \mathbb{P}_k^{n^*})$.

While not every closed embedding is a Lefschetz embedding, we do however have the following.

Proposition 8.14. *Let X be a smooth irreducible variety with a closed embedding $i\colon X \hookrightarrow \mathbb{P}_k^n$ and denote by $(d)\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$ the d-th Veronese embedding. Then the following holds:*

(i) *For all $d \geq 2$, $(d) \circ i$ is a Lefschetz embedding.*

(ii) *If* $\mathrm{char}(k) = 0$*, i is a Lefschetz embedding.*

Proof. [10, Théorème 2.5] □

Remark 8.15. The condition $d \geq 2$ is really necessary, i.e. there exist embeddings of smooth varieties into projective space which are not Lefschetz pencils. For an example, check [10, Exemple 3.4].

Corollary 8.16. *Let X be a smooth irreducible variety and L a very ample line bundle on X. Then the general line in the linear system $|L^{\otimes 2}|$ is a Lefschetz pencil.*

8.6. **Generalization of the theorem.** We go back to our original situation, i.e. S is a smooth projective surface over an algebraically closed field k, L is a very ample line bundle on S and $x \in S$ is a general point. Then $\mathcal{C} \to U$, the universal family of smooth curves in $|L|_x$, is a smooth family of curves with a section given by x. Hence, $\mathcal{J} := \underline{\mathrm{Pic}}^0(\mathcal{C}/U)$ is defined and gives us a smooth family $\mathcal{J} \to U$ of Jacobians of curves. By considering the kernel of $\mathcal{J} \to \mathrm{Alb}(S) \times U$, we find a smooth family of abelian varieties $\mathcal{K} \to W$ over some open subset $W \subset U$ with $\mathcal{K}_u \cong K(\mathcal{C}_u, S)$ for all $u \in W$ as in section 6.

Theorem 8.17. *Let k be an uncountable algebraically closed field, S, L, U as above and C a very general curve in U. Assume that the embedding induced by L is a Lefschetz embedding, then $K(C,S)$ is absolutely simple.*

Proof. The construction in 3.2 works for any abelian scheme over any uncountable field. Thus, we again conclude that it is enough to prove that the generic fibre of $\mathcal{K} \to U$ is absolutely simple. Assume it were not. By lemma 2.6, there exists an endomorphism φ of the generic fibre $\mathcal{K}_\eta$, defined over some finite field extension L of $k(\eta)$, which is neither an isogeny nor the 0-morphism. By [1, Theorem 0BXN], there exists a normal variety U' with function field L and a dominant morphism $g\colon U' \to U$. Let $\eta' \in U'$ be the generic point, then the φ we found above is an endomorphism of $(\mathcal{K}_{U'})_{\eta'}$. By the usual trick, this will spread to an endomorphism $\tilde{\varphi}$ of $\mathcal{K}_{U'}$ over an open set $V' \subset U'$. Using this, we can find a Lefschetz pencil $\tilde{\mathcal{C}} \to \mathbb{P}^1$ in $|L|_x$ with generic point $\mu \in \mathbb{P}^1$, such that $\tilde{\varphi}$ induces an endomorphism of $K(\tilde{\mathcal{C}}_\mu, S)$, defined over some finite field extension K of $k(\mu)$, which is neither an isogeny nor the 0-morphism. By [1, Theorem 0BY1], there exists a normal projective curve D with function field K and a surjective morphism $\phi\colon D \to \mathbb{P}^1$. Let μ' be the generic point of D. Then $\mathcal{K}_{\mu'}$ is non-simple by lemma 2.6. Let $A \subset \mathcal{K}_{\mu'}$ be a non-trivial proper abelian subvariety and $\tilde{U} \subset \mathbb{P}^1$ the open set of smooth curves in the Lefschetz pencil. Define $V \subset D$ to be the inverse image of $\tilde{U}$ under ϕ. We pull back $\mathcal{K}$ along ϕ to get a family of abelian varieties over V, which we again call $\mathcal{K}$. By lemmas 2.2 and 2.5, there exists an endomorphism ψ of $\mathcal{K}_{\mu'}$ with A as its image. Then by spreading ψ and defining $\mathcal{A}$ to be the image, we get a family of abelian varieties over some open subset of V. After possibly shrinking V further, we can assume that $\mathcal{A} \to V$ and $\mathcal{K} \to V$ are smooth. Then $f\colon \mathcal{K} \to V$ and $g\colon \mathcal{A} \to V$ are abelian schemes, so in particular proper.

Let $x \in V$ be a closed point and consider the étale local systems $R^1 f_* \mathbb{Z}_l$ and $R^1 g_* \mathbb{Z}_l$. By theorem 8.6, $\pi_1(V, v)$ acts on $H^1_{ét}(\mathcal{K}_x, \mathbb{Z}_l)$ and $H^1_{ét}(\mathcal{A}_x, \mathbb{Z}_l)$. Consider the surjective homomorphism $\psi_x\colon \mathcal{K}_x \to \mathcal{A}_x$. This induces a surjection $T_l\mathcal{K}_x \to T_l\mathcal{A}_x$ of Tate modules, which by theorem 8.4 corresponds to an injection $H^1_{ét}(\mathcal{A}_x, \mathbb{Z}_l) \to H^1_{ét}(\mathcal{K}_x, \mathbb{Z}_l)$. Furthermore, as this is induced by a morphism over V, the map is one of π_1-modules. Using the injection of Tate modules given by $i\colon \mathcal{K}_x \hookrightarrow \mathcal{J}_x$, we get a surjection $i^*\colon H^1_{ét}(\mathcal{C}_x, \mathbb{Q}_l) \simeq H^1_{ét}(\mathcal{J}_x, \mathbb{Q}_l) \twoheadrightarrow H^1_{ét}(\mathcal{K}_x, \mathbb{Q}_l)$. On the other hand, looking at the map of Tate modules induced by $\psi\colon \mathcal{J}_x \to \mathrm{Alb}(S)$, we see that $H^1_{ét}(\mathrm{Alb}(S), \mathbb{Q}_l) \twoheadrightarrow H^1_{ét}(\mathrm{im}(\psi), \mathbb{Q}_l) \hookrightarrow H^1_{ét}(\mathcal{J}_x, \mathbb{Q}_l)$. There is a commutative diagram

$$\begin{array}{ccc} H^1_{\acute{e}t}(S,\mathbb{Q}_l) & \xrightarrow{j^*} & H^1(\mathcal{C}_x,\mathbb{Q}_l) \\ \downarrow\wr & & \downarrow\wr \\ H^1_{\acute{e}t}(\mathrm{Alb}(S),\mathbb{Q}_l) & \xrightarrow{\psi_*} & H^1_{\acute{e}t}(\mathcal{J}_x,\mathbb{Q}_l). \end{array}$$

Thus, $H^1_{\acute{e}t}(\mathcal{K}_x,\mathbb{Q}_l) \cong H^1_{\acute{e}t}(\mathcal{J}_x,\mathbb{Q}_l)/H^1_{\acute{e}t}(\mathrm{im}(\psi),\mathbb{Q}_l) \cong H^1_{\acute{e}t}(\mathcal{C}_x,\mathbb{Q}_l)/H^1_{\acute{e}t}(\mathrm{im}(j^*),\mathbb{Q}_l)$. As all these isomorphisms are induced by maps of étale local systems, this is an isomorphism of π_1-modules. After possibly shrinking V again, we can assume that $\phi(V)$ is open in $\mathbb{P}^1$. Every automorphism of a pointed covering space (Y,y) of $(\tilde{U},\phi(x))$ will induce an automorphism of the pullback $(Y_{\phi(V)},y)$ over $(\phi(V),\phi(x))$, hence $\pi_1(\phi(V),\phi(x)) \to \pi_1(\tilde{U},\phi(x))$ is surjective. As the tame fundamental group is a factor group of the étale fundamental group, we see that in fact, $\pi_1^t(\phi(V),\phi(x)) \to \pi_1^t(\tilde{U},\phi(x))$ is surjective. Furthermore, $H^1_{\acute{e}t}(\mathcal{K}_x,\mathbb{Q}_l) \cong H^1_{\acute{e}t}(\mathcal{C}_x,\mathbb{Q}_l)/H^1_{\acute{e}t}(\mathrm{im}(j^*),\mathbb{Q}_l) \cong \mathrm{Ev}(\mathcal{C}_x)$ as π_1-modules by [7, Corollaire 4.3.9]. Then corollary 8.12 implies that the monodromy action of $\pi_1^t(\phi(V),\phi(x))$ on $H^1_{\acute{e}t}(\mathcal{K}_{\phi(x)},\mathbb{Q}_l)$ $= H^1_{\acute{e}t}(\mathcal{K}_x,\mathbb{Q}_l)$ is irreducible.

Consider $\phi'\colon \pi_1^t(V,x) \to \pi_1^t(\phi(V),\phi(x))$. Since ϕ is finite, $\mathrm{im}(\phi')$ is a finite index subgroup of $\pi_1^t(\phi(V),\phi(x))$. By what we did above, we see that $\sigma(H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)) \subset H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)$ for all $\sigma \in \mathrm{im}(\phi')$. Let $\sigma \in \pi_1^t(\phi(V),\phi(x))$ be arbitrary and assume there exists $t \in H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)$ such that $\sigma(t) \notin H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)$. Then by proposition 8.10 and theorem 8.11, there exists $s \in \mathbb{P}^1 \setminus \tilde{U}$ such that $\langle t,\delta_s\rangle \neq 0$ and $\delta_s \notin H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)$. Using theorem 8.11 again, this implies $\rho(\gamma_s(u))(t) = t + \tilde{u}\langle t,\delta_s\rangle\delta_s \notin H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)$ for all $u \neq 0$. However, as $\mathrm{im}(\phi')$ is a finite index subgroup that leaves $H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)$ stable, this cannot be the case. Thus, $H^1_{\acute{e}t}(\mathcal{A}_x,\mathbb{Q}_l)$ is $\pi_1^t(\phi(V),\phi(x))$-stable, which is a contradiction, as the monodromy action of $\pi_1^t(\phi(V),\phi(x))$ on $H^1_{\acute{e}t}(\mathcal{K}_x,\mathbb{Q}_l)$ is irreducible. Hence, $\mathcal{K}_\eta$ must be absolutely simple and we are done. □

9. The case of $\overline{\mathbb{Q}}$

As we pointed out above, other than the Jacobian of the generic curve being simple, Banerjee's paper and theorem 8.17 unfortunately tell us very little about the case of countable fields. In fact, for a countable field k, there might not be a single curve in $|L|$ defined over k whose Jacobian is simple. We want to show that at least in the case of k being $\overline{\mathbb{Q}}$, this does not happen. Questions of this type, i.e. finding objects over $\overline{\mathbb{Q}}$ with properties that hold for very general objects over $\mathbb{C}$, have been studied before by Terasoma. In [15], he proved the existence of a complete intersection with middle Picard number one over $\mathbb{Q}$. Using the same techniques, we want to prove the existence of a curve with simple Jacobian in $|L|$ defined over $\overline{\mathbb{Q}}$.

9.1. **Paths in étale cohomology.** To prove theorem 1.6, we need to introduce the notion of paths between geometric points.

Definition 9.1. Let Y be a scheme and $\overline{s}$ and $\overline{t}$ two geometric points of Y. Then $\overline{s}$ and $\overline{t}$ induce functors from the category of covering spaces of Y to (Sets) by taking the respective fibres and a path from $\overline{s}$ to $\overline{t}$ is defined to be an isomorphism of these functors.

Remark 9.2. (i) Let X be a covering space of Y. For a geometric point $\overline{s}$ of Y, there exists a natural continuous $\pi_1(Y, \overline{s})$ action on $X_{\overline{s}}$ and the functor $X \mapsto X_{\overline{s}}$ establishes an equivalence between the category of covering spaces of Y and finite continuous $\pi_1(Y, \overline{s})$-sets (See [8, Proposition A I.5]). Thus, a path between two geometric points $\overline{s}$ and $\overline{t}$ induces an isomorphism from $\pi_1(Y, \overline{s})$ to $\pi_1(Y, \overline{t})$.

(ii) Every locally constant étale sheaf $\mathcal{G}$ on Y is representable by a covering space X of Y. The monodromy action on the stalks of $\mathcal{G}$ is by definition just given by the action on the geometric fibres of X and thus a path from $\overline{s}$ to $\overline{t}$ induces a commutative diagram

$$\begin{array}{ccc} \pi_1(Y, \overline{s}) & \xrightarrow{\sim} & \pi_1(Y, \overline{t}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Aut}(\mathcal{G}_{\overline{s}}) & \xrightarrow{\sim} & \mathrm{Aut}(\mathcal{G}_{\overline{t}}) \end{array} \tag{9.1}$$

9.2. **Hilbert's irreducibility theorem.** The key idea in the proof of theorem 1.6 is to use Hilbert's irreducibility theorem, which allows us to prove the following statement. Let K be a number field and A a subring of $K(T)$ that is generated by T and $f(T)^{-1}$ for some polynomial $f(T) \in K[T]$. Let $\overline{\eta}$ be a fixed geometric generic point of $\mathrm{Spec}(A)$. Let $t \in \mathrm{Spec}(A)$ be a closed point and $\overline{t}$ a geometric point over t. Let γ be a path from $\overline{\eta}$ to $\overline{t}$ and γ_* the induced isomorphism of fundamental groups. Consider the étale fundamental group $\pi_1(\mathrm{Spec}(k(t)), \overline{t}) = \mathrm{Gal}(k(\overline{t})/k(t))$. The closed point $t\colon \mathrm{Spec}(k) \to A$ induces a map

$$\alpha_t\colon \mathrm{Gal}(k(\overline{t})/k(t)) = \pi_1(\mathrm{Spec}(k(t)), \overline{t}) \longrightarrow \pi_1(\mathrm{Spec}(A), \overline{t}).$$

Given an l-adic continuous representation of $\pi_1(\mathrm{Spec}(A), \overline{\eta})$

$$\phi\colon \pi_1(\mathrm{Spec}(A), \overline{\eta}) \twoheadrightarrow G \subset \mathrm{GL}(n, \mathbb{Q}_l),$$

we get the composite homomorphism $\beta(t, \gamma, \phi) = \phi \circ \gamma_* \circ \alpha_t$.

Theorem 9.3. *Let $A, \overline{\eta}$ and ϕ be as above. Then there exists a sequence of distinct K-rational points $(t_i)_{i\in\mathbb{N}}$ and for each i a path γ_i connecting a geometric point $\bar{t}_i$ over t_i and $\overline{\eta}$, such that the homomorphism*

$$\beta(t_i,\gamma_i,\phi)\colon \operatorname{Gal}(k(\bar{t}_i)/k(t_i)) \longrightarrow G$$

is surjective.

Proof. The proof for general number fields is analogous to the one for the case of $K=\mathbb{Q}$, given by Terasoma in [15, Theorem 2]. $\square$

9.3. **Jacobians over number fields.** Let K again be a number field, we want to prove the following theorem.

Theorem 9.4. *Let S be a smooth projective surface over K and L a very ample line bundle on S. Then there exists a smooth curve $C \in |L|$, defined over K, such that $K(C,S)$ is absolutely simple.*

Proof. Let $x \in S$ be a general point and $U, \mathcal{C}$ and $\mathcal{K}$ as in 8.6. We again want to use the étale monodromy of Lefschetz pencils to derive our conclusion. So, let $\tilde{\mathcal{C}} \to \mathbb{P}^1$ be a Lefschetz pencil in $|L|_x$, $\eta \in \mathbb{P}^1$ be the generic point and let $V \subset \mathbb{P}^1$ be the open set of smooth curves of the pencil. After possibly shrinking V, we can assume that $\mathcal{K}$ is defined and smooth over V. Let $u \in V$ be a closed point and consider the base change

$$\begin{array}{ccc} \mathcal{K}_u & \xrightarrow{i'} & \mathcal{K} \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ u & \xrightarrow{i} & V. \end{array}$$

Then $R^1f'_*\mathbb{Q}_l \cong R^1f'_*i'^*\mathbb{Q}_l \cong i^*R^1f_*\mathbb{Q}_l$ by the proper base change theorem. Let $\overline{u}$ be a geometric point lying over u. As the map $\pi_1(\operatorname{Spec}(k(u)),\overline{u}) \to \pi_1(V,\overline{u})$ is induced by pulling back covering spaces of V, we get a commutative diagram

$$\begin{array}{ccc} \pi_1(\operatorname{Spec}(k(u)),\overline{u}) & \longrightarrow & \pi_1(V,\overline{u}) \\ \big\downarrow & & \big\downarrow \\ \operatorname{Aut}(H^1(\mathcal{K}_{\overline{u}},\mathbb{Q}_l)) & \xrightarrow{\sim} & \operatorname{Aut}(H^1(\mathcal{K}_{\overline{u}},\mathbb{Q}_l)) \end{array} \tag{9.2}$$

Let $\overline{\eta}$ be a geometric generic point of V and assume there is a path from $\overline{u}$ to $\overline{\eta}$. Combining the diagrams 9.1 and 9.2, we get a commutative diagram

$$\begin{array}{ccccc} \pi_1(\operatorname{Spec}(k(u)),\overline{u}) & \longrightarrow & \pi_1(V,\overline{u}) & \xrightarrow{\sim} & \pi_1(V,\overline{\eta}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \operatorname{Aut}(H^1_{\acute{e}t}(\mathcal{K}_{\overline{u}},\mathbb{Q}_l)) & \xrightarrow{\sim} & \operatorname{Aut}(H^1_{\acute{e}t}(\mathcal{K}_{\overline{u}},\mathbb{Q}_l)) & \xrightarrow{\sim} & \operatorname{Aut}(H^1_{\acute{e}t}(\mathcal{K}_{\overline{\eta}},\mathbb{Q}_l)). \end{array} \tag{9.3}$$

By theorem 9.3, we find a sequence of K-rational points $(t_i)_{i\in\mathbb{N}}$ in V together with paths from geometric points $\overline{t_i}$ over the t_i to $\overline{\eta}$, such that the maps

$$\pi_1(\mathrm{Spec}(k(t_i)), \overline{t_i}) \longrightarrow \mathrm{im}(\pi_1(V, \overline{\eta}) \longrightarrow \mathrm{Aut}(H^1_{\acute{e}t}(\mathcal{K}_{\overline{\eta}}, \mathbb{Q}_l)))$$

are surjective. Let $\overline{V}$ be the base change of V to $\overline{\mathbb{Q}}$. We get a map $\pi_1(\overline{V}, \overline{t_i}) \to \pi_1(V, \overline{t_i})$ which again commutes with the monodromy action. Thus, arguing just as in the proof of theorem 8.17, we can use corollary 8.12 to conclude that the monodromy action of $\pi_1(\mathrm{Spec}(k(t_i)), \overline{t_i})$ on $H^1_{\acute{e}t}(\mathcal{K}_{t_i}, \mathbb{Q}_l)$ is irreducible.

Now let A be an abelian subvariety of $\mathcal{K}_{t_i}$, defined over some finite field extension L of K. Then $\pi_1(\mathrm{Spec}(L), \overline{t_i}) = \mathrm{Gal}(k(\overline{t_i})/L)$ is a finite index subgroup of $\pi_1(\mathrm{Spec}(k(t_i)), \overline{t_i})$. Arguing again just like in the proof of theorem 8.17, we see that the Picard–Lefschetz formula implies that A must already be trivial. Hence, the $\mathcal{K}_{t_i}$ are absolutely simple and we are done. □

This theorem immediately implies the following result over $\overline{\mathbb{Q}}$, which we mentioned in the introduction.

Corollary 9.5. *Let S be a smooth projective surface over $\overline{\mathbb{Q}}$ and L a very ample line bundle on S. Then there exists a smooth curve $C \in |L|$, defined over $\overline{\mathbb{Q}}$, such that $K(C, S)$ is absolutely simple.*

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