GEOMETRY OF K3 SURFACES AND HYPERKÄHLER MANIFOLDS – OPEN PROBLEMS AND NEW PERSPECTIVES –

D. HUYBRECHTS

Abstract. These are notes of my plenary talk with the same title at the annual meeting of the DMV, Chemnitz 2020. We keep the informal style of the oral presentation. Usually the results are not stated with all their technical assumptions or in full generality, but we do provide detailed references.

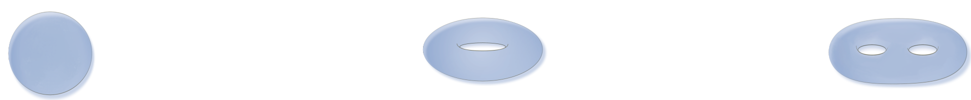
I wish to thank the program committee for the invitation to speak at this year's annual meeting of the DMV which gives me the opportunity to make some publicity for a fascinating research direction in algebraic geometry, which has a long history with roots going back several decades and which, at the same time, has been driven by constant innovation. The type of geometries I will talk about, K3 surfaces and hyperkähler manifolds, has often served as a testing ground for new techniques and ideas. Very much like the classical theory of abelian varieties, deep conjectures have been and still are first tested in the K3 and hyperkähler setting, and new techniques are developed and fine tuned there, before they are used to approach more general situations. The best example is probably Deligne's proof of the Weil conjecture for K3 surfaces [De72], a couple of years before his proof of the conjecture in complete generality. Not only was the conjecture verified in a meaningful situation, but it was achieved by applying Grothendieck's philosophy of motives to a concrete geometric situation. Other more recent examples include the proof of the Tate conjecture in degree two [Ma14, Ch13, Ma15, Mo17] or the verification of certain aspects of conjectures due to Bloch, Beilinson, and Kimura.

Research funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC-2047/1 - 390685813 and the ERC Synergy Grant HyperK.

1. From elliptic curves to K3 surfaces and beyond

1.1. Let me begin by recalling the classification of compact Riemann surfaces (of which as an algebraic geometer I usually refer to as algebraic curves).

Topologically every compact Riemann surface looks like this: Either it is a sphere, or a torus, or has more than one whole. The number of wholes is called the genus.



$g = 0$ $\qquad$ $g = 1$ $\qquad$ $g > 1$

Once the topological structure is fixed, we would like to enhance it to actually do geometry. There are various ways of achieving this. We could fix a Riemannian metric (or a conformal structure) on the surface, we can endow the surface with a complex structure, or we describe the surface algebraically as the zero set of polynomials. In dimension two, all three options eventually lead to equivalent notions.

For example, putting any of the three structures on a topological surface of genus $g = 1$ leads to the concept of elliptic curves which of course is a central notion in mathematics.

• A complex geometer would describe an elliptic curve as a quotient $\mathbb{C}/\Gamma$ of the complex plane by a lattice which we may assume to be of the form $\Gamma = \mathbb{Z} \oplus \mathbb{Z} \cdot \tau \subset \mathbb{C}$ with the parameter τ being an element in the upper half-plane $\mathbb{H}$.

• Algebraically, an elliptic curve is the zero set of a cubic equation say of the form

$$y^2 = x^3 + ax + b. \tag{1.1}$$

The solutions are taken in $\mathbb{C}$ and are conveniently compactified by a point at infinity to get a compact surface. To be specific, we could consider the Fermat cubic equation $x^3 + y^3 = 1$.

• More arithmetically, one could look at the zeroes of the equation (1.1) in $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{F}_p$.

Note that in all three settings we have completely ignored the group structure of the elliptic curve. It is, of course, responsible for much of the appeal of elliptic curves, but geometrically it is not central for what follows. However, many people would call an elliptic curve without its group structure simply a curve of genus one.

1.2. Now, venturing into higher dimensions, say complex dimension two (so real dimension four), there exists more than one natural generalization of the notion of elliptic curves.

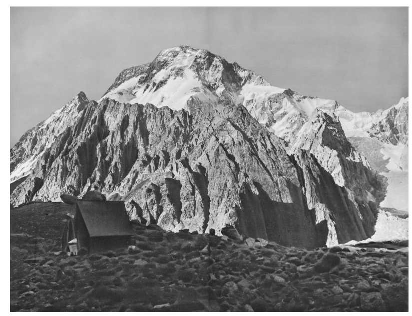
• From the complex geometric point of view, two-dimensional complex tori suggest themselves, so manifolds of the form $\mathbb{C}^2/\Gamma$ with now $\mathbb{Z}^4 \simeq \Gamma \subset \mathbb{C}^2$ a lattice of rank four. Unlike the curve case, usually a torus of this form cannot be described by polynomial equations. Moreover, even when it can be, determining the polynomial equations is a difficult task.

• If an elliptic curve is described by a cubic equation (1.1), for example the Fermat equation, it is naturally to look at similar equations in one more variable, for example the Fermat equation

$$x^4 + y^4 + z^4 = 1. \tag{1.2}$$

Here, the degree is raised by one in order to keep some properties of the curvature. However, for manifolds described in this way there is no (obvious) uniformization as for tori of dimension one or two as above.

Surfaces of this form have been dubbed K3 surfaces by André Weil [We79], to honor the three mathematicians **K**ähler, **K**odaira, and **K**ummer *et la belle montagne K2 au Cachemire.* There is in fact also a mountain top called K3 or 'Broad Peak'. (Photo: Vittorio Sella 1909)



I would like the reader to think of K3 surfaces (and hyperkähler manifolds, their higher dimensional versions) as the more interesting alternative to replacing an elliptic curve $\mathbb{C}/\Gamma$ by just higher-dimensional tori $\mathbb{C}^n/\Gamma$. They come without a group structure, but their geometry is more interesting, for example they are not flat.

The formal definition goes as follows:

A *K3 surface* is a compact complex manifold of dimension two, so small open sets in $\mathbb{C}^2$ glued together to a global structure by biholomorphic maps. It is required to be simply connected, so every closed loop can be contracted continuously to a point, and to carry a holomorphic symplectic structure, i.e. a form σ which locally looks like $f dz_1 \wedge dz_2$ with f holomorphic and nowhere vanishing. For example, for the Fermat quartic the symplectic form σ can be given explicitly as the residue

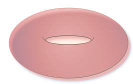
$$\sigma = \mathrm{Res}\left(\frac{dx \wedge dy \wedge dz}{x^4 + y^4 + z^4 - 1}\right)$$

A *hyperkähler manifold* (in this talk) is a compact complex manifold, which is simply connected, again to simplify the topology as much as possible, and which comes with a (unique, up to scaling) holomorphic two-form σ, non-degenerate at every point. In addition one requires the manifold to be Kähler, which in dimension two is automatic and in any case holds whenever the structure can be described algebraically.

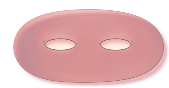
2. What do we want to know

2.1. The first question one would like to understand is in how many ways can we do geometry on a given topological structure. Let us briefly go back to the case of Riemann surfaces. Here we know that for $g = 0$ there is only one geometry, that of the round Riemann sphere (or, more algebraically, the complex projective line $\mathbb{P}^1_{\mathbb{C}}$), and for $g = 1$ there is a one-dimensional family, reflected by the choice of the parameter $\tau \in \mathbb{H}$. However, with increasing genus $g > 1$ there are more and more possibilities which eventually amount to a family, or a moduli space as we say, of dimension $3g-3$.

unique! $\rightsquigarrow \mathbb{P}^1$



1-dim family $\approx \mathbb{C}$

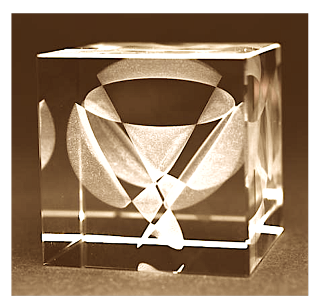


$(3g-3)$-dim family $\approx \mathcal{M}_g$

2.2. For K3 surfaces, and similarly for hyperkähler manifolds, something more intriguing happens: It turns out that on the topological structure underlying the Fermat quartic (1.2) there is a 20-dimensional family of possible complex structures. However, the result is not a nice moduli space, nothing at all like the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. The situation becomes a bit better when one restricts to those complex structures that can be described as zero sets of polynomials. If in addition certain numerical invariants like the degree of the polynomials are fixed, then there is only a 19-dimensional family M_d depending on this numerical invariant d which can be any even integer. The spaces M_d are very similar to Riemann's classical space $\mathcal{M}_g$ and in recent years people who have studied the global structure of $\mathcal{M}_g$ (and its compactification $\bar{\mathcal{M}}_g$) are now turning to these spaces M_d. They seem as interesting and intricate as their famous cousins $\mathcal{M}_g$.

2.3. In the above, we always first fixed a topology and then tried to introduce a geometric structure on it. For Riemann surfaces the topology was unique once the genus was fixed, $g=1$ for elliptic curves. For K3 surfaces and hyperkähler manifolds that is actually not so clear.

- Elliptic curves, abelian varieties, and complex tori are all topologically uniquely determined (by their dimension). They are just tori $S^1\times\cdots\times S^1$.



- K3 surfaces are also topologically uniquely determined, i.e. two K3 surfaces are always homeomorphic, see [Hu16, Thm. 7.1] for references. Being of real dimension four, K3 surfaces are difficult to visualize. The glass cube on the right is the real cut of a (singular) quartic K3 surface [Labs].

- The situation is much more interesting for hyperkähler manifolds of higher dimensions and in fact some of the most urgent questions in the theory concern exactly this point. In any even complex dimension >2, there are at least two distinct topologies that can carry a hyperkähler structure (in contrast to just having one torus $S^1\times\cdots\times S^1$) [Be83]. Moreover, we know two further topologies, one in dimension six and another one in dimension ten [O'G99, O'G03], but otherwise the topological classification is wide open.

2.4. Once we have agreed on the geometric structure, either in terms of a complex structure, a hyperkähler metric, or algebraic equations describing it, there are natural questions one would like to answer. Here is a selection:

• For example, one may want to know whether a K3 surface can be 'uniformized', which in a loose sense could mean that there is at least a holomorphic map $\mathbb{C}^2 \longrightarrow S$ with a two-dimensional image. This is not known in general [BL00].

• Or, weaker, one may ask whether there exist entire curves, so non-constant holomorphic maps $\mathbb{C} \longrightarrow S$. In fact, this is true but not completely trivial to prove.

• More algebraically, one maybe wants to know whether there exist rational curves, i.e. holomorphic maps $\mathbb{P}^1 \longrightarrow S$, and how many. By now we do have a satisfactory answer to this question [BJT11, LL11, CGL19], there are always countably many, but we do not know a useful criterion to decide when there is a rational curve through a given fixed point. This question can be given an arithmetical flavor.

• Or, to make a connection to elliptic curves, is there an elliptic curve through any given point? Yes, there is. But what about fixing two points?

• Other types of geometric objects include bundles and sheaves, everything in the algebraic or holomorphic category. It is a classical problem, for Riemann surfaces going back to Andé Weil, to ask for a parametrization of all bundles and to study the geometry of their moduli spaces.

• In fact, already the existence of bundles, e.g. for given Chern classes (in various cohomology theories), is a highly non-trivial problem, especially in higher dimensions.

3. Linear algebra: the most efficient tool

No matter how we describe the geometry, via algebraic methods or more transcendental ones, we need the appropriate tools to tackle interesting geometric problem. One of the most efficient tools (and almost the only universal one at our disposal) is Hodge theory, which was developed and promoted by Deligne and Griffiths (and his students) in the seventies. It is essentially a piece of linear algebra, most of which one could teach in a first year course, but, somewhat surprisingly, it is able to capture and encode intricate geometric information. So, let me first explain the linear algebra. I will restrict to Hodge structures of weight one and two, before showing how those arise in our geometric context and how they are used there.

For the following we fix a free $\mathbb{Z}$-module or a finite-dimensional $\mathbb{Q}$-vector space Λ and its naturally associated $\mathbb{C}$-vector space $\Lambda_{\mathbb{C}}$:

$$\Lambda = \mathbb{Z}^n \text{ or } \Lambda = \mathbb{Q}^n \rightsquigarrow \Lambda_{\mathbb{C}} = \mathbb{C}^n.$$

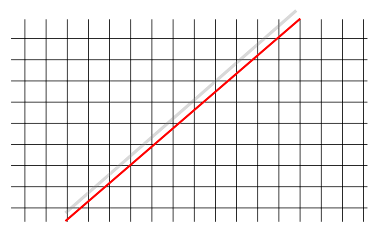
Fixing Λ essentially corresponds to fixing the underlying topological structure in the previous section.

3.1. A Hodge structure of *weight one* on Λ consists of writing $\Lambda_{\mathbb{C}}$ as a direct sum of two subspaces

$$\Lambda_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1},$$

which satisfy the additional condition of being conjugate to each other, i.e.

$$v \in \Lambda^{1,0} \text{ if and only if } \bar{v} \in \Lambda^{0,1}.$$



Note that complex conjugation makes only sense if one starts with a $\mathbb{Z}$-module, a $\mathbb{Q}$-vector space or at least a $\mathbb{R}$-vector space. What makes the notion of a Hodge structure so powerful is the interplay between the decomposition of the complex vector space and the position of the original lattice Λ with respect to it. Here, the vertices of the grid are the elements of the lattice and the red line is one of the complex sub-vector spaces. Also note that the notion is not rigid. Once a decomposition is given, small changes of $\Lambda^{1,0}$, inducing small changes of its complex conjugate $\Lambda^{0,1}$, again describe Hodge structures of weight one.

A Hodge structure of weight two is defined similarly. This time one considers decompositions in three sub-vector spaces[1]

$$\Lambda_{\mathbb{C}} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2},$$

satisfying the two conditions:

$$\text{(i) } v \in \Lambda^{2,0} \text{ if and only if } \bar{v} \in \Lambda^{0,2} \text{ and (ii) } v \in \Lambda^{1,1} \text{ if and only if } \bar{v} \in \Lambda^{1,1}.$$

Maybe it is worth stressing again, that what makes the notion of a Hodge structure so effective is the mix of this direct sum of complex vector spaces and the relative position of the lattice. For example, the intersection of the lattice Λ with the complex vector space given by its $(1,1)$-part $\Lambda^{1,1}$ is a sublattice of Λ whose rank is an important invariant of the Hodge structure.

There is a very easy way of transforming a Hodge structure of weight one into one of weight two: Pass from Λ to $\bigwedge^2 \Lambda$ and define the $(2,0)$-component of the associated complex vector space $\bigwedge^2 \Lambda_{\mathbb{C}}$ simply as $\bigwedge^2 \Lambda^{1,0}$ and, similarly, the $(1,1)$-component as $\Lambda^{1,0} \otimes \Lambda^{0,1}$, etc.

[1] In the situation we are interested in, one additionally has that $\Lambda^{2,0}$ (and hence also $\Lambda^{0,2}$) is of dimension one.

There is also a procedure to go back, at least for Hodge structure of weight two with one-dimensional $\Lambda^{2,0}$.

$$\{ \text{ weight one } \} \xrightarrow{\Lambda \mapsto \bigwedge^2 \Lambda} \{ \text{ weight two } \}$$

$$\{ \text{ weight two } \} \dashrightarrow \{ \text{ weight one } \} \quad \text{Kuga–Satake}$$

This was applied to K3 surfaces with great success already in [De72, PSS75] and we shall come back to this shortly. Roughly, one could think of the following diagram as saying that there is a left adjoint construction (Kuga–Satake) to taking $\bigwedge^2$, cf. [vG00] or [Hu16, Ch. 3].

3.2. Let us now turn back to geometry and explain how Hodge structures come up there. We start again with tori and Hodge structures of weight one. Two instances shall be discussed.

• Assume $T = \mathbb{C}^2/\Gamma$ is a two-dimensional complex torus. Then we let Λ be its first cohomology which is just the dual lattice Γ^*:

$$\Lambda \coloneqq H^1(T, \mathbb{Z}) \simeq \Gamma^*,$$

which comes with a natural Hodge structure of weight one

$$\Lambda_{\mathbb{C}} = H^1(T, \mathbb{C}) = \underbrace{H^{1,0}(T)}_{=\langle dz_1, dz_2\rangle} \oplus H^{0,1}(T), \tag{3.1}$$

where the $(1,0)$-part is spanned by the two constant one-forms dz_1 and dz_2.

This seems almost too naive to be meaningful. However, describing this Hodge structure for the Fermat elliptic curve given by $x^3 + y^3 = 1$ is already not trivial.

• Consider a Riemann surface (or, equivalently, a complex algebraic curve) C. Its first cohomology $\Lambda \coloneqq H^1(C, \mathbb{Z})$, which is generated by the $2g$ loops up to homotopy is also naturally endowed with a Hodge structure of weight one

$$H^1(C, \mathbb{Z}) = \underbrace{H^{1,0}(C)}_{\dim = g} \oplus H^{0,1}(C). \tag{3.2}$$

The amazing fact now is that this linear algebra structure associated to an a priori complicated geometric situation captures the geometry faithfully. This is known as the Torelli theorem which really is the prototype of results one aims for in Hodge theory. We state two examples, the first one being elementary while the second one certainly is not.

Theorem 3.1 (Torelli theorem)**.** (i) *There exists a biholomorphic map $T_1 \simeq T_2$ between two complex tori $T_1 = \mathbb{C}^n/\Gamma_1$ and $T_2 = \mathbb{C}^n/\Gamma_2$ if and only if there exists an isomorphism of Hodge structures $H^1(T_1, \mathbb{Z}) \simeq H^1(T_2, \mathbb{Z})$, i.e. an isomorphism $\Gamma_1^* \simeq \Gamma_2^*$ compatible with the decompositions (3.1):*

$$T_1 \simeq T_2 \;\Leftrightarrow\; H^1(T_1, \mathbb{Z}) \simeq H^1(T_2, \mathbb{Z}) \;\&\; \textit{Hodge} .$$

(ii) *There exists biholomorphic map $C_1 \simeq C_2$ between two Riemann surfaces C_1 and C_2 if and only if there exists an isometry of Hodge structures $H^1(C_1,\mathbb{Z}) \simeq H^1(C_2,\mathbb{Z})$, i.e. an isomorphism $\Gamma_1^* \simeq \Gamma_2^*$ compatible with the decompositions (3.2) and the intersection pairing (.):*

$$C_1 \simeq C_2 \;\Leftrightarrow\; H^1(C_1,\mathbb{Z}) \simeq H^1(C_2,\mathbb{Z}) \;\&\; \textit{Hodge} \;\&\; (\;.\;).$$

Despite its classical nature, there are aspects of this result that are poorly understood. For example, what happens if in (ii) the compatibility with the intersection product (.) is dropped and how often does this happen? This can be rephrased in more geometric terms by asking when do two curves have isomorphic unpolarized Jacobian varieties $\mathrm{Jac}(C_1) \simeq \mathrm{Jac}(C_2)$.

A similar theory has been developed for K3 surfaces and hyperkähler manifolds. But maybe it is instructive to treat again the case of a two-dimensional torus $T = \mathbb{C}^2/\Gamma$ first, this time using its second cohomology. So, we consider the lattice $\Lambda \coloneqq H^2(T,\mathbb{Z}) \simeq \bigwedge^2 \Gamma^*$, which can be expressed in terms of the lattice Γ. Then the Hodge structure of weight two in this case is described by the linear algebra procedure of taking the second exterior power of the Hodge structure of weight one that was mentioned before:

$$\Lambda_{\mathbb{C}} = H^2(T,\mathbb{C}) = \bigwedge^2 H^1(T,\mathbb{C}) = \underbrace{H^{2,0}(T)}_{=\langle dz_1\wedge dz_2\rangle} \oplus H^{1,1}(T) \oplus H^{0,2}(T).$$

The $(2,0)$-part is spanned by the constant two form $dz_1 \wedge dz_2$.

By definition, the second cohomology $\Lambda \coloneqq H^2(S,\mathbb{Z})$ of a K3 surface or a hyperkähler manifold is also naturally endowed with such a Hodge structure of weight two

$$\Lambda_{\mathbb{C}} = H^2(S,\mathbb{C}) = \underbrace{H^{2,0}(S)}_{=\langle \sigma = f dz_1\wedge dz_2\rangle} \oplus H^{1,1}(S) \oplus H^{0,2}(S), \tag{3.3}$$

where in general the $(2,0)$-part is more complicated to describe.

The second cohomology comes with a natural intersection product, i.e. an integral symmetric quadratic form (.), and the decomposition is orthogonal with respect to it.

As alluded to before, there is a purely algebraic way of going back, from the Hodge structure of weight two $H^2(S,\mathbb{Z})$ to a Hodge structure of weight one [KS67]. It is one of the hardest problems in this area to interpret this linear algebra construction geometrically. Only a few examples have been worked out, e.g. in [Pa88].

3.3. Before saying more about the weight two Hodge structure associated with a K3 surface and how it is used to describe geometric phenomena, let us stay for with the lattice $\Lambda = H^2(S,\mathbb{Z})$ itself (without the Hodge structure) a little longer.

The lattice Λ is a unimodular, even lattice of surprisingly large rank $\mathrm{rk}(\Lambda) = 22$. The full cohomology $\widetilde{\Lambda} \coloneqq H^*(S,\mathbb{Z})$ of a K3 surface is also an even, unimodular lattice, now of rank $\mathrm{rk}(\widetilde{\Lambda}) = 24$.

For everyone with an interest in lattice theory this 24 will ring a bell, it is the rank of the famous Leech lattice and its cousins, the other 23 Niemeier lattices. Although there is no direct link between the definite Leech and Niemeier lattices and the indefinite K3 lattices Λ and $\widetilde{\Lambda}$, there seems to exist a mysterious link between both worlds that has been observed again and again in the theory of K3 surfaces.

The classical example is the following result first proved in [Mu88], see als [Ko06, Ko20] or [Hu16, Ch.15].

Theorem 3.2. *For a finite group G the following conditions are equivalent:*
(i) *There exists an algebraic K3 surface S with a faithful action of G that fixe the symplectic structure σ.*
(ii) *There exists an injection of G into the Mathieu group M_{23} with at least five orbits.*

Recall that the Mathieu group M_{23} is the stabilizer subgroup of the Mathieu group M_{24} which in turn is linked to the orthogonal group of one of the Niemeier lattices N by $\mathrm{O}(N) = M_{24} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\oplus 24}$.

There are other links that have been observed. For example between groups of auto-equivalences of the derived category and the Conway group [Hu13] and the larger Mathieu group and symmetries of conformal field theories [CDH14, GV12, TW13].

The classification of the H^2-lattice of higher-dimensional hyperkähler manifolds is largely open. It essentially corresponds to a topological classification and indeed once the lattice is fixed the topology is determined up to finite ambiguity [Hu03]. Here I have listed the ranks in all the known examples and it is not unlikely that we have found all of them already, but just to demonstrate our ignorance, at this point we do not even know how to exclude the smallest possible value, namely rank three.

3.4. Let us come back to complex geometry. We have seen that to any K3 surface S_0 one naturally associates a Hodge structure of weight two:

$$H^2(S_0, \mathbb{C}) = H^{2,0}(S_0) \oplus H^{1,1}(S_0) \oplus H^{0,2}(S_0).$$

The first natural question one should ask is what happens if the purely linear algebra structure is modified slightly. Does it correspond to deforming the complex structure defining S_0? Locally this is not difficult to check, this is what is called the local Torelli theorem, but it turns out to be true globally, see [To80, Hu99] and [Hu16, Ch. 7] for more on the history of the result.

Theorem 3.3 (Surjectivity of the period map)**.** *Any admissible Hodge structure on the K3 lattice Λ is the Hodge structure associated with some K3 surface.*

Here, admissible means that the decomposition is orthogonal and its $(2,0)$-part is of dimension one.

Note, however, that unlike the case of complex tori it is not possible to simply write down a K3 surface to a given admissible Hodge structure. In fact, also going the other way is highly non-trivial. For example, describing the Hodge structure associated with the Fermat quartic is a non-trivial matter. So both arrows in this diagram are hard to realize in practice.

Nevertheless Hodge theory often allows one to deal with elusive geometric properties in an effective way. For example, we alluded already to the fact that as for complex tori many K3 surfaces cannot be described by polynomial equations. Those that can, which are called algebraic, are characterized by a specific property of their Hodge structure. For K3 surfaces this follows from general surface theory and for hyperkähler manifolds see [Hu99].

Theorem 3.4. *A K3 surface or a hyperkähler manifold S is algebraic if and only if*

$$\exists\, \alpha \in H^2(S,\mathbb{Z}) \cap H^{1,1}(S) : \ (\alpha.\alpha) > 0.$$

Of course, finding the equation is a completely different matter.

Clearly, the most important single result in the theory is the next result. For K3 surfaces this goes back to [PSS71] in the algebraic situation and to [BuRa75] in the non-algebraic, see [Hu16, Ch. 7] for further references and historical comments. For hyperkähler manifolds of higher dimensions the result is [Ve13], see also [Hu11a, Lo19, Ma11].

Theorem 3.5 (Torelli theorem)**.** *There exists a biholomorphic map $S_1 \simeq S_2$ between two K3 surfaces if and only if there exists an isometry of Hodge structures $H^2(S_1,\mathbb{Z}) \simeq H^2(S_2,\mathbb{Z})$. For hyperkähler manifolds the same assertion is true if the Hodge isometry is required to satisfy an additional technical condition $(*)$:*

$$S_1 \simeq S_2 \ \Leftrightarrow \ H^2(S_1,\mathbb{Z}) \simeq H^2(S_2,\mathbb{Z}) \ \& \ \textit{Hodge} \ \& \ (\ .\ ) \ \& \ (**).$$

Despite of having been proved early in the history of the theory of K3 surfaces (the higher dimensional case is more recent), there are aspects that we do not fully understand. For example, what happens if the compatibility with the intersection form is dropped?

$$H^2(S_1,\mathbb{Z}) \simeq H^2(S_2,\mathbb{Z}) \ \& \ \text{Hodge} \ \ (\text{but not } (\ .\ )) \ \Rightarrow \ S_1 \overset{?}{\dashleftarrow\dashrightarrow} S_2$$

There is no geometric object that would play the role of the Jacobian in the case of curves. Also, it is not really clear how often this does happen.

3.5. Instead of dropping the intersection form one could instead relax the coefficients and take $\mathbb{Q}$ instead of $\mathbb{Z}$. This leads us the Hodge conjecture for K3 surfaces. The Hodge conjecture for a K3 surface S itself is the question whether for any class $\gamma \in H^{1,1}(S) \cap H^2(S,\mathbb{Z})$ there exists in fact a curve $C \hookrightarrow S$ such that the topological paring with the class γ amounts to integrating over the curve C, i.e. for all classes $\alpha \in H^2(S,\mathbb{C})$ we have

$$(\gamma.\alpha) = \int_C \alpha$$

This is in fact well known in general as the Lefschetz $(1,1)$-theorem. Note however, that an arithmetic analogue of this, the Tate conjecture, has been proved only recently and essentially only for K3 surfaces [Ma14, Ch13, Ma15, Mo17].

More interesting is the Hodge conjecture for products of K3 surfaces $S_1 \times S_2$. It could be phrased in terms of integral $(2,2)$-class on the product or using Hodge morphisms.

Conjecture 3.6. *Assume* $\varphi\colon H^2(S_1,\mathbb{Q}) \longrightarrow H^2(S_2,\mathbb{Q})$ *is a linear map with* $\varphi(H^{p,q}(S_1)) \subset H^{p,q}(S_2)$. *Then there exists a complex surface*

$$Z \hookrightarrow S_1 \times S_2$$

(or rather a $\mathbb{Q}$*-linear combination of surfaces) such that for an orthonormal basis* α_i *we have*

$$(\varphi(\alpha_i).\alpha_j) = \int_Z (\alpha_i \times \alpha_j).$$

Alternatively, one requires that the cohomology class $[\varphi] \in H^4(S_1 \times S_2)$ determined by φ we have $([\varphi].\alpha) = \int_Z \alpha$ for all $\alpha \in H^4(S_1 \times S_2)$.

The conjecture, despite being one of the easiest non-trivial cases of the general Hodge conjecture, is wide open. It has been proved for Hodge morphisms φ which are compatible with the quadratic form.

Theorem 3.7 (Hodge conjecture for CM K3 surfaces)**.** *Assume* $\varphi\colon H^2(S_1,\mathbb{Q}) \longrightarrow H^2(S_2,\mathbb{Q})$ *is compatible with the Hodge structure and the intersection product. Then the corresponding class* $[\varphi] \in H^{2,2}(S_1 \times S_2) \cap H^4(S_1 \times S_2, \mathbb{Q})$ *is algebraic, i.e. there exists a surface* Z *as above.*

The original proof [Bu19], heavily inspired by [Ma20], is an ingenious application of twistor theory. Another proof, also providing an isomorphisms between certain derived categories and Chow motives, was given in [Hu18, Hu19]. Note that the theorem in particular shows that the full Hodge conjecture holds for $S \times S$ whenever S is a K3 surface with complex multiplication, cf. [Za80] or [Hu16, Ch. 3].

4. New ways in geometry

The language of standard algebraic geometry seems too limited to capture all aspects of the situation that are detected by Hodge theory. We will demonstrate how to use more abstract notions to explain certain observations for which the geometric picture seems blurred. The two approaches we will discuss below could both be understood as a way of half-linearizing geometry.

4.1. The Chow group of a K3 surface can be viewed as a replacement for the group structure of an elliptic curve. A priori there is no way of forming the sum of two points on a K3 surface as we are used to do on elliptic curves, but completely abstractly one could form a group of all finite formal linear combinations:

$$\mathrm{CH}_0(S) = \left\{ \sum n_i x_i \mid x_i \in S,\ n_i \in \mathbb{Z} \right\} / \sim .$$

This becomes useful only after dividing out by a certain equivalence relation $\sim$ taking into account the geometry of the surface. The equivalence relation is called 'rational equivalence' and it is generated by, for example, declaring two points x_1 and x_2 to be rationally equivalent, $x_1 \sim x_2$, if they are both contained in the same rational curve, i.e. in the image of a map $f\colon \mathbb{P}^1 \longrightarrow S$. The actual definition is a bit more involved, but this is the gist of it.

For complex K3 surfaces or hyperkähler manifolds $\mathrm{CH}_0(S)$ is an absolutely enormous group [Mu68, Vo02], it a divisible group, very much like the group attached to an elliptic curve, and it is torsion free [Ro80], unlike an elliptic curve. And it certainly does not have any reasonable geometric structure.

By definition, the group $\mathrm{CH}_0(S)$ depends very much on the surface, but it is a little unwieldy and hard to pin down. There is a general set of conjectures due to Bloch and Beilinson that clarify the role of Chow groups, but they seem out of reach for the moment. However, as often in the past, studying these conjectures for K3 surfaces is interesting and rewarding: The situation is far from being trivial but at the same time certain things can be proved. To illustrate one aspect of the expected behavior, we mention the following result which combines [Mu68] for the 'only if' and [HK13, Hu12, Vo12] for the converse.

Theorem 4.1. *Consider an automorphism* $g\colon S \xrightarrow{\sim} S$ *of finite order of a K3 surface* S*. Then*

$$g = \mathrm{id} \text{ on } \mathrm{CH}_0(S) \ \Leftrightarrow\ g = \mathrm{id} \text{ on } H^{2,0}(S).$$

The guiding principle here is that although $\mathrm{CH}_0(S)$ only uses points in the surface, which directly do not leave any trace in the cohomology $H^*(S,\mathbb{Z})$, the behavior of certain natural operations on $\mathrm{CH}_0(S)$ is nevertheless governed by their behavior on the cohomology (and, moreover, on parts that a priori are not geometric at all).

Note, however, that even for K3 surfaces not all questions raised by the Bloch–Beilinson conjectures have been answered yet. For example, for g of infinite order the theorem has not been proved and there is not even an example of a K3 surface over a number field for which the expectation that $\mathrm{CH}_0(S) = \mathbb{Z}$ has been checked [Hu11b]. Also, Chow groups of K3 surfaces and hyperkähler manifolds show some features that are not expected in general. This has been attracted a lot of attention over the last 15 years or so. It started with [BV04] for K3 surfaces, a more general set of conjectures for hyperkähler manifolds in [Be07, Vo08], with partial results e.g. in [Ri16] linking them to expectations in mirror symmetry.

4.2. Chow groups are also used in the linearization of the K3 surface itself. Of course, there is a category whose objects are K3 surfaces, hyperkähler manifolds, or just arbitrary varieties. But the morphisms in this categories just form sets without much additional structures. Grothendieck suggested to build a linear category Mot (which is almost as good as could wish for, it is a pseudo-abelian category), see [MNP13]. Each K3 surface still occurs there as an object $\mathfrak{h}_S \in \mathrm{Mot}$, but the set of morphisms has been enlarged: Instead of actual maps, one defines the set of morphisms between two K3 surfaces S_1, S_2 in Mot as the Chow group $\mathrm{CH}_2(S_1 \times S_2)$ of their product. Here, instead of looking at linear combinations of points, one considers linear combinations of surfaces modulo rational equivalence. One of the hardest questions in this area asks whether the category Mot really behaves like the category of finite-dimensional (super)vector spaces [Ki05]. Even for K3 surfaces this has been settled only in very few cases. A geometric explanation of the Kuga–Satake construction could solve this problem. More geometrically, it would be enough to show that any K3 surface is dominated by a product of two curves, although many experts think that this should not hold.

The next result can be seen as a motivic explanation of the Hodge conjecture, see Theorem 4.3, or as a rational global Torelli theorem 3.5.

Theorem 4.2 (Motivic Hodge/Torelli)**.** *Two K3 surfaces S_1 and S_2 have isomorphic multiplicative Chow motives if and only if there exists a Hodge isometry of their rational Hodge structures:*

$$\mathfrak{h}_{S_1} \simeq \mathfrak{h}_{S_2} \ (mult) \Leftrightarrow \ H^2(S_1, \mathbb{Q}) \simeq H^2(S_2, \mathbb{Q}) \ \& \ Hodge \ \& \ (\ .\ ).$$

The isomorphism of Chow motives of isogenous K3 surfaces has been proved in [Hu19, Hu18]. The multiplicative nature of $\mathfrak{h}_S$ has been worked out in [FV19]. Dropping the compatibility with the intersection pairing on the right hand side should correspond to isomorphisms of Chow motives on the left hand side not necessarily respecting the multiplicative structure.

4.3. Another linearization of the geometry of a K3 surface or a hyperkähler manifold is realized by instead of studying vector bundles taking all complexes of vector bundles into consideration. This leads to the notion of the derived category, which is a linear category, so morphisms between two objects are vector spaces. It is not abelian but triangulated. There are twisted versions which I do not have the time to go into here.

These categories also allow one to express the relation between K3 surfaces with the same rational Hodge structure. The following result is the main input in the proof of Theorem 4.2 and provides a categorical interpretation of [Bu19].

Theorem 4.3 (Derived Hodge/Torelli)**.** *Two K3 surfaces S_1 and S_2 are linked by a chain of equivalences of twisted derived categories if and only if there exists a Hodge isometry of their rational Hodge structures:*

$$\mathrm{D}(S_1, \alpha_1) \simeq \cdots \simeq \mathrm{D}(S_2, \alpha_2) \ \Leftrightarrow \ H^2(S_1, \mathbb{Q}) \simeq H^2(S_2, \mathbb{Q}) \ \& \ Hodge \ \& \ (\ .\ ).$$

The result with integral coefficient goes back to the early days of Fourier–Mukai transforms [Mu84, Or97] and in the above form it is [Hu18].

5. Unexpected findings

I would like to end my talk by mentioning a few instances where K3 surfaces and hyperkähler manifolds occur at places in algebraic geometry where a priori one would not expect them to play any role. The example I chose which is arguably the most prominent case is that of cubic fourfolds, it has become a hot topic as it connects various points of view and promises applications to the Lüroth problem, cf. [Hu20, Ch. 6] for more details and references.

A cubic fourfold is defined by a cubic polynomial in five variable, e.g. the Fermat polynomial

$$x_1^3 + \cdots + x_5^3 = 1.$$

(This is the affine case. In homogenous coordinates one would of course have six variables.) While elliptic curves, tori, K3 surfaces, and hyperkähler manifolds all have trivial Ricci curvature, cubic fourfolds have positive curvature and a priori rather behave like projective spaces, so a completely different geometry.

Nevertheless, hyperkähler aspects crop up, geometrically and Hodge theoretically.

• For example, the Fano variety of lines, i.e. the set

$$F(X) = \{\ L \subset X \mid \text{line}\ \},$$

which itself is a manifold, in fact a smooth algebraic variety, turns out to be a hyperkähler manifold of dimension four [BD85].

• The middle cohomology $H^4(X, \mathbb{Z})$ of a cubic fourfold has a Hodge decomposition of the form

$$H^4(X, \mathbb{C}) = H^{3,1}(X) \oplus H^{2,2}(X) \oplus H^{1,3}(X)$$

with $H^{3,1}(X)$ of dimension one [Ra72]. This looks suspiciously like the Hodge structure of a K3 surface S_X and, as worked out in [Ha00], it frequently is.

The geometric relation, however, between the Fano variety of lines $F(X)$ (a hyperkähler manifold of dimension four), the K3 surface S_X (which is only determined via its Hodge structure and not uniquely determined), and the cubic fourfold X itself is not apparent at all. This has been studied intensively over the last years. Partially this is well understood, e.g. $F(X)$ and X are linked by the Fano correspondence and $F(X)$ can also be interpreted as a moduli space of certain bundles on X. A geometric interpretation of S_X, when it exists, is not known and probably does not always exist. It seems that a more conceptual way of understanding the situation is provided by more abstract notions like Chow motives and derived categories as explained before.

Very roughly, the Chow motive of the Fano variety $F(X)$ is built out of the Chow motive of S_X (when it exists) by using basic linear algebra constructions like taking the symmetric power in the linear category Mot:

$$\mathfrak{h}_{F(X)} \approx S^2\mathfrak{h}_{S_X}.$$

Similarly, the essential (transcendental) parts of the Chow motives of S_X and X are isomorphic (up to Tate twist):

$$\mathfrak{h}^{\mathrm{tr}}_{S_X} \simeq \mathfrak{h}^{\mathrm{tr}}_X.$$

A similar picture exists on the level of derived categories [AT14, Ku10]. Whenever Hodge theory links a cubic fourfold X to a K3 surface S_X, then the derived category $\mathrm{D}(S_X)$ can be realized as a full subcategory of the derived category (in fact, its essential part) of the derived category $\mathrm{D}(X)$:

$$\mathrm{D}(S_X) \hookrightarrow \mathrm{D}(X).$$

One should think of the derived category $\mathrm{D}(F(X))$ of the Fano variety as the symmetric power of $\mathrm{D}(S_X)$.

One of the main motivation for the flurry of activities in this area is the Lüroth problem which asks whether an easy variety like a cubic fourfold X is always rational, so isomorphic to $\mathbb{P}^4$ on a big open set. For cubic threefolds the question has been answered to the negative, but maybe more interestingly than settling the Lüroth problem was the innovation it triggered. Fundamental concepts like the intermediate Jacobian have been introduced to answer the question which subsequently has of course become a standard tool in algebraic geometry [CG72]. Something similar is currently happening for cubic fourfolds. The existence of S_X is conjectured to be linked to the question whether X is rational (Lüroth problem) [Ha00, Ku10].

6. Conclusion

K3 surfaces and hyperkähler manifolds should be seen as central building blocks in algebraic geometry. Their geometry is extremely rich and yet, or therefore, more accessible than completely arbitrary algebraic varieties. There have been many instances where strong evidence for deep general conjectures has been provided by studying them for K3 surfaces and hyperkähler manifolds: Weil conjecture [De72, PSS71], Tate conjecture [Ma14, Ch13, Ma15], Grothendieck standard conjectures [CM13], finite dimensionality of Chow motives [Yi15], Hodge conjecture [Bu19], etc.

References

[AT14] N. Addington, R. Thomas *Hodge theory and derived categories of cubic fourfolds.* Duke Math. J. 163 (2014), 1885–1927.

[Be83] A. Beauville *Variétés Kähleriennes dont la première classe de Chern est nulle.* J. Diff. Geom. 18 (1983), 755–782.

[BD85] A. Beauville, R. Donagi *La variété des droites d'une hypersurface cubique de dimension 4.* C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), 703–706.

[BV04] A. Beauville, C. Voisin *On the Chow ring of a K3 surface.* JAG 13 (2004), 417–426.

[Be07] A. Beauville *On the splitting of the Bloch–Beilinson filtration.* In *Algebraic Cycles and Motives* Vol. 2, 38–53. LMS Lecture Note Series 344. Cambridge University Press, 2007.

[BJT11] F. Bogomolov, B. Hassett, Y. Tschinkel *Constructing rational curves on K3 surfaces.* Duke Math. J. 157 (2011), 535–550.

[BuRa75] D. Burns, M. Rapoport *On the Torelli problem for kählerian K3 surfaces.* Ann. Sci. École Norm. Sup. (4), 8(2) (1975), 235–273.

[Bu19] N. Buskin *Every rational Hodge isometry between two K3 surfaces is algebraic.* Crelle J. Reine Angew. Math. 755 (2019), 127–150.

[BL00] G. Buzzard, S. Lu *Algebraic surfaces holomorphically dominable by* $\mathbb{C}^2$. Invent. Math. 139 (2000), 617–659.

[CM13] F. Charles, E. Markman *The Standard Conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of K3 surfaces.* Comp. Math. 149 (2013), 481–494.

[Ch13] F. Charles *The Tate conjecture for K3 surfaces over finite fields.* Invent. Math. 194 (2013), 119–145.

[CDH14] M. Cheng, J. Duncan, J. Harvey *Umbral moonshine and the Niemeier lattices.* Res. Math. Sc. 1, (2014)

[CGL19] X. Chen, F. Gounelas, C. Liedtke *Curves on K3 surfaces.* arXiv:1907.01207.

[CG72] H. Clemens, P. Griffiths *The intermediate Jacobian of the cubic threefold.* Ann. Math. 95 (1972), 281–356.

[De72] P. Deligne *La conjecture de Weil pour les surfaces K3.* Invent. Math. 15 (1972), 206–226.

[FV19] L. Fu, C. Vial *A motivic global Torelli theorem for isogenous K3 surfaces.* arXiv:1907.10868.

[GV12] M. Gaberdiel, S. Hohenegger, R. Volpato *Symmetries of K3 sigma models.* Commun. Num. Theor. Phys. 6 (2012), 1–50.

[Ha00] B. Hassett *Special cubic fourfolds.* Comp. Math. 120 (2000), 1–23.

[HK13] D. Huybrechts, M. Kemeny *Stable maps and Chow groups.* Doc. math. 18 (2013), 507–517.

[Hu99] D. Huybrechts *Compact hyperkähler manifolds: Basic results.* Invent. Math. 135 (1999), 63–113.

[Hu03] D. Huybrechts *Finitenes results for hyperkähler manifolds.* J. reine angew. Math. 558 (2003), 15–22.

[Hu11a] D. Huybrechts *A Global Torelli theorem for hyperkähler manifolds (after Verbitsky).* Séminaire Bourbaki Exp. No. 1040 Juin 2011 Astérisque No. 348 (2012), 375–403.

[Hu11b] D. Huybrechts *A note on the Bloch–Beilinson conjecture for K3 surfaces and spherical objects.* Pure Appl. Math. Q. 7 (2011), 1395–1405.

[Hu12] D. Huybrechts *Symplectic automorphisms of K3 surfaces of arbitrary order.* Math. Res. Lett. 19 (2012), 947–951.

[Hu13] D. Huybrechts *On derived categories of K3 surfaces and Mathieu groups.* Advanced Studies in Pure Math. 69 (Math. Soc. Japan) (2016), 387–405.

[Hu16] D. Huybrechts *Lectures on K3 surfaces.* Cambridge University Press, 2016.

[Hu18] D. Huybrechts *Motives of derived equivalent K3 surfaces.* Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 88 (2018), 201–207.

[Hu19] D. Huybrechts *Motives of isogenous K3 surfaces.* Comm. Math. Helv. 94 (2019), 445–458.

[Hu20] D. Huybrechts *The geometry of cubic hypersurfaces.* Notes `http://www.math.uni-bonn.de/people/huybrech/Notes.pdf`

[Ki05] S. Kimura *Chow groups are finite dimensional, in some sense.* Math. Ann. 331 (2005), 173–201.

[Ko06] S. Kondo *Maximal subgroups of the Mathieu group M23 and symplectic automorphisms of supersingular K3 surfaces.* IMRN 9 (2006), pages Art. ID 71517.

[Ko20] S. Kondo K3 *surfaces.* EMS Tracts in Mathematics Vol. 32, 2020.

[KS67] M. Kuga, I. Satake *Abelian varieties attached to polarized K3 surfaces.* Math. Ann. 169 (1967), 239–242.

[Ku10] A. Kuznetsov *Derived categories of cubic fourfolds.* in *Cohomological and Geometric Approaches to Rationality Problems*, Progr. Math. 282, Birkhäuser, Boston, 2010, 219–243.

[Labs] O. Labs *Mathe-in-Glas.* `https://oliverlabs.net`

[LL11] J. Li, C. Liedtke *Rational curves on K3 surfaces.* Invent. math. 188 (1911), 713–727.

[Lo19] E. Looijenga *Teichmüller spaces and Torelli theorems for hyperkähler manifolds.* arXiv:1912.02550

[Ma15] K. Madapusi Pera *The Tate conjecture for K3 surfaces in odd characteristic.* Invent. Math. 201 (2015), 625–668.

[Ma11] E. Markman *A survey of Torelli and monodromy results for holomorphic-symplectic varieties.* Complex and differential geometry, 257–322, Springer Proc. Math., 8, 2011.

[Ma20] E. Markman *The Beauville–Bogomolov class as a characteristic class.* JAG 29 (2020), 199–245.

[Ma14] D. Maulik *Supersingular K3 surfaces for large primes.* With an appendix by Andrew Snowden. Duke Math. J. 163 (2014), 2357–2425.

[Mo17] B. Moonen *On the Tate and Mumford–Tate conjectures in codimension one for varieties with $h^{2,0} = 1$.* Duke Math J. 166 (2017), 739–799.

[Mu84] S. Mukai *On the moduli space of bundles on K3 surfaces. I.* In *Vector bundles on algebraic varieties* (Bombay, 1984), Tata Inst. Fund. Res. Stud. Math. 11 341–413.

[Mu88] S. Mukai *Finite groups of automorphisms of K3 surfaces and the Mathieu group.* Invent. Math. 94 (1988), 183–221.

[Mu68] D. Mumford *Rational equivalence of 0-cycles on surfaces.* J. Math. Kyoto Univ. 9 (1968), 195–204.

[MNP13] J. Murre, J. Nagel, C. Peters *Lectures on the theory of pure motives.* AMS University Lecture Series 61, Providence (2013).

[O'G99] K. O'Grady *Desingularized moduli spaces of sheaves on a K3.* J. Reine Angew. Math. 512 (1999), 49–117.

[O'G03] K. O'Grady *A new six-dimensional irreducible symplectic manifold.* JAG 12(3) (2003), 435–505.

[Or97] D. Orlov *Equivalences of derived categories and K3 surfaces.* J. Math. Sci. (New York), 84(5) (1997), 1361–1381.

[Pa88] K. Paranjape *Abelian varieties associated to certain K3 surfaces.* Comp. Math. 68 (1988), 11–22.

[PSS75] I. Piatetskij-Shapiro, I. Shafarevich *The arithmetic of K3 surfaces.* Proc. Steklov Inst. Math. 121 (1975), 45–57.

[PSS71] I. Piatetskij-Shapiro, I. Shafarevich *Torelli's theorem for algebraic surfaces of type K3.* Izv. Akad. Nauk SSSR Ser. Mat., 35 (1971), 530–572.

[Ra72] M. Rapoport *Complément à l'article de P. Deligne "La conjecture de Weil pour les surfaces K3".* Invent. Math. 15 (1972), 227–236.

[Ri16] U. Rieß *On Beauville's conjectural weak splitting property.* IMRN 20 (2016), 6133–6150.

[Ro80] A. Roitman *The torsion of the group of 0-cycles modulo rational equivalence.* Ann. Math. 111 (1980), 553–569.

[TW13] A. Taormina, K. Wendland *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}.* JHEP 1308:152 (2013).

[To80] A. Todorov *Applications of the Kähler–Einstein–Calabi–Yau metric to moduli of K3 surfaces.* Invent. Math. 61(1980), 251–265.

[vG00] B. van Geemen *Kuga–Satake varieties and the Hodge conjecture.* In *The arithmetic and geometry of algebraic cycles* (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci. 548 (2000), 51–82.

[Ve13] M. Verbitsky *Mapping class group and a global Torelli theorem for hyperkähler manifolds.* Duke Math. J. 162 (2013), 2929–2986. Erratum: Duke Math. J. 169 (2020), 1037–1038. `https://arxiv.org/abs/1908.11772`

[Vo02] C. Voisin *Théorie de Hodge et géométrie algébrique complexe.* Cours Spécialisés 10. SMF, Paris, 2002.

[Vo08] C. Voisin *On the Chow ring of certain algebraic hyper–Kähler manifolds.* Pure Appl. Math. Q. 4 (2008), 613–649.

[Vo12] C. Voisin *Symplectic involutions of K3 surfaces act trivially on* CH_0. Doc. math. 17 (2012) 851–860

[We79] A. Weil *Scientific works.* Collected papers. Vol. II (1951Ð1964). Springer-Verlag, New York, 1979.

[Yi15] Q. Yin *Finite-dimensionality and cycles on powers of K3 surfaces.Ê*Comment. Math. Helv. 90 (2015), 503–511.

[Za80] Y. Zarhin *Hodge groups of K3 surfaces.* Crelle J. Reine Angew. Math. 341 (1983), 193–220.

Mathematisches Institut and Hausdorff Center for Mathematics, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Email address: huybrech@math.uni-bonn.de