Geometry of K3 surfaces and hyperkähler manifolds

Open problems and new perspectives

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For references see: http://www.math.uni-bonn.de/people/huybrech/K3HKDMV.pdf

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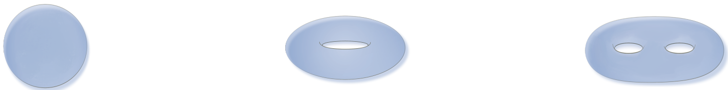
§1 From elliptic curves to K3 surfaces and beyond

Topology of Riemann surfaces ($\sim$ algebraic curves)



$g = 0$ $\qquad$ $g = 1$ $\qquad$ $g > 1$

Geometry: Add conformal / complex / algebraic structure

Elliptic curves:

- complex: As torus $\mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$, $\tau \in$
- algebraic: As zero set of $y^2 \overset{(*)}{=} x^3 + ax + b$, e.g. $x^3 + y^3 = 1$
- arithmetic: Solutions of $(*)$ but with $x, y \in \mathbb{Z},\ \mathbb{Q},\ \mathbb{F}_p$

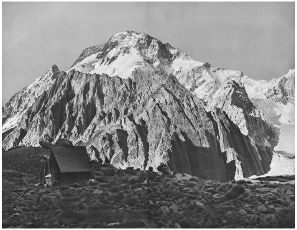
Choices in higher dimensions

Complex: $\mathbb{C}^2/\Gamma$, $\mathbb{Z}^4 \cong \Gamma \subset \mathbb{C}^2$ . . . *equations?*

Algebraic: $x^4 + y^4 + z^4 = 1$ . . . *uniformization?*

$\rightsquigarrow$ K3 surfaces: **K**ähler, **K**odaira, and **K**ummer

'. . . et de la belle montagne K2 au Cachemir' (A. Weil)



Think of K3 surfaces (hyperkähler) as alternative
to replacing elliptic curves $\mathbb{C}/\Gamma$ by complex tori $\mathbb{C}^n/\Gamma$.

K3 surface: simply connected & $\exists$ holomorphic symplectic form σ.

(locally $\sigma = f\, dz_1 \wedge dz_2$ with f holomorphic without zeros)

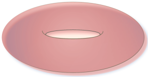
Hyperkähler: same definition in $\dim > 2$ & σ unique & $\exists$ Kähler metric

§2 What do we want to know

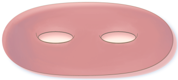
How many geometric structures?

Riemann surfaces

unique



1-dim family



3g-3-dim family: $\mathcal{M}_g$

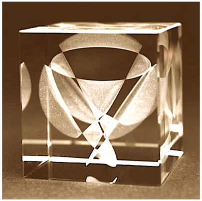
K3 surfaces:
- complex 20-dim: wild & ergodic
- algebraically 19-dim: M_d, $d = 2, 4, 6, \dots$

Topological description

Elliptic curves, abelian varieties, complex tori:

$$\text{topologically} \approx \mathbb{C}^n/\Gamma \approx \underbrace{S^1 \times \cdots \times S^1}_{\text{2n copies}}$$

K3 surfaces: topologically all the same $\approx$ Fermat quartic



Hyperkähler manifolds $\dim > 2$: More interesting!

- At least two topological types in each dimension!
- Only few topological restrictions known. Classification largely open.

Some typical questions

Uniformization

- $\exists\ \mathbb{C}^2 \xrightarrow{f} S$ with $\det(df) \not\equiv 0$?
- $\exists\ \mathbb{C} \xrightarrow{f} S$ with $f(\mathbb{C}) \subset S$ Zariski dense ?

Points and curves

- $x \in S \overset{?}{\Longrightarrow} \exists\ \mathbb{P}^1 \xrightarrow{f} S$ with $x \in f(\mathbb{P}^1)$
- $x_1, x_2 \in S \overset{?}{\Longrightarrow} \exists\ C \xrightarrow{f} S$ with $C \cong \mathbb{P}^1$ or $C \cong \mathbb{C}/\Gamma$ and $x_1, x_2 \in f(C)$

Bundles

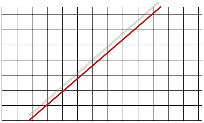
- $E \longrightarrow S$: parametrization & moduli spaces
- Existence & Chern classes

§3 Linear algebra: the most efficient tool

Abstract Hodge theory Fix $\Lambda = \mathbb{Z}^n$ or $\Lambda = \mathbb{Q}^n$ $\rightsquigarrow$ $\Lambda_{\mathbb{C}} = \mathbb{C}^n$

Hodge structure of weight 1: $\Lambda_{\mathbb{C}} = \Lambda^{1,0} \oplus \Lambda^{0,1}$

- $\mathbb{C}$-vector spaces
- $v \in \Lambda^{1,0} \Leftrightarrow \bar{v} \in \Lambda^{0,1}$



Hodge structure of weight 2: $\Lambda_{\mathbb{C}} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$

- $\mathbb{C}$-vector spaces
- $v \in \Lambda^{2,0} \Leftrightarrow \bar{v} \in \Lambda^{0,2}$
- $v \in \Lambda^{1,1} \Leftrightarrow \bar{v} \in \Lambda^{1,1}$

$\Lambda \mapsto \bigwedge^2 \Lambda$

{ weight one } $\rightleftarrows$ { weight two }

Kuga–Satake

Hodge theory from geometry (weight one)

- Torus $T = \mathbb{C}^2/\Gamma \quad \rightsquigarrow \quad \Lambda := H^1(T,\mathbb{Z}) \cong \Gamma^*$

$$\Lambda_{\mathbb{C}} = H^1(T,\mathbb{C}) \overset{(*)}{=} \underbrace{H^{1,0}(T)}_{=\langle dz_1, dz_2\rangle} \oplus H^{0,1}(T)$$

- Riemann surface / algebraic curve $C \quad \rightsquigarrow \quad \Lambda := H^1(C,\mathbb{Z})$

$$\Lambda_{\mathbb{C}} = H^1(C,\mathbb{C}) \overset{(*)}{=} \underbrace{H^{1,0}(C)}_{g\text{-dim}} \oplus H^{0,1}(C)$$

Torelli theorem

- Tori: $T_1 \cong T_2 \quad \Leftrightarrow \quad H^1(T_1,\mathbb{Z}) \cong H^1(T_2,\mathbb{Z})$ compatible with $(*)$
- Curves: $C_1 \cong C_2 \quad \Leftrightarrow \quad H^1(C_1,\mathbb{Z}) \cong H^1(C_2,\mathbb{Z})$ compatible with $(*)$ & $(\,.\,)$

Open: What happens if $(\,.\,)$ is dropped?

Hodge theory from geometry (weight two)

- Torus $T = \mathbb{C}^2/\Gamma \quad \rightsquigarrow \quad \Lambda := H^2(T,\mathbb{Z}) \cong \bigwedge^2 H^1(T,\mathbb{Z})$

$$\Lambda_{\mathbb{C}} = H^2(T,\mathbb{C}) \stackrel{(*)}{=} \underbrace{H^{2,0}(T)}_{= \langle dz_1 \wedge dz_2 \rangle} \oplus H^{1,1}(T) \oplus H^{0,2}(T)$$

- K3 / hyperkähler $S \quad \rightsquigarrow \quad \Lambda := H^2(S,\mathbb{Z})$ & (.)

$$\Lambda_{\mathbb{C}} = H^2(S,\mathbb{C}) \stackrel{(*)}{=} \underbrace{H^{2,0}(S)}_{= \langle \sigma = f\, dz_1 \wedge dz_2 \text{ loc.} \rangle} \oplus H^{1,1}(S) \oplus H^{0,2}(S) \text{ orthogonal wrt. (.)}$$

Kuga–Satake: weight **two** $\rightsquigarrow$ weight **one**

K3 surface $\rightsquigarrow$ complex torus.

Open: Geometric description.

Lattices $\Lambda := H^2(S,\mathbb{Z})$ with $\mathrm{rk} = 22$ & $\mathrm{sign} = (3,19)$

$\widetilde{\Lambda} := H^*(S,\mathbb{Z})$ with $\mathrm{rk} = 24$ & $\mathrm{sign} = (4,20)$

$\overset{??}{\leftrightsquigarrow}$ Leech / Niemeier lattices $N_1,\ldots,N_{24}$ with $\mathrm{rk} = 24$ & $\mathrm{sign} = (0,24)$

Theorem Finite group G and Mathieu group $M_{23} \subset M_{24} \subset \mathrm{O}(N_2)$:

- $G \subset \mathrm{Aut}_s(S) \;\Leftrightarrow\; G \subset M_{23}$ with ≥ 5 orbits
- $\mathrm{Aut}(\mathrm{D}(S))$ & Conway groups
- Mathieu moonshine & CFT

Higher dimensions: $\Lambda = H^2(S,\mathbb{Z})$ with $\mathrm{rk} = r$ & $\mathrm{sign} = (3, r-3)$

Open: Which lattices can arise ($\approx$ topological classification)?

Known: $\mathrm{rk}(\Lambda) = 7, 8, 22, 23, 24$. Can $\mathrm{rk}(\Lambda) = 3$ be excluded?

Complex geometry S_0 K3 / hyperkähler

$$\rightsquigarrow \quad H^2(S_0,\mathbb{C}) \overset{(*)}{=} H^{2,0}(S_0) \oplus H^{1,1}(S_0) \oplus H^{0,2}(S_0)$$

Surjectivity of period map

- Every Hodge structure on $\Lambda = H^2(S_0,\mathbb{Z})$ comes from some S

Algebraic description of S, e.g. as zero set of $x^4 + y^4 + z^4 = 1$

Linear description of S by Hodge structure $H^2(S,\mathbb{Z})$

Example

- Not all $S =$ K3 / hyperkähler are described as zero sets (algebraic)
- S is algebraic $\Leftrightarrow$ $\exists\, \alpha \in H^2(S,\mathbb{Z}) \cap H^{1,1}(S) : (\alpha.\alpha) > 0$

Torelli theorem S_1, S_2 K3 surface / hyperkähler manifolds

- $S_1 \cong S_2 \;\Leftrightarrow\; H^2(S_1,\mathbb{Z}) \cong H^2(S_2,\mathbb{Z})$ & $(*)$ & (.) & $(**)$ in dim > 2

Open: What happens if (.) is dropped or $\mathbb{Z} \rightsquigarrow \mathbb{Q}$?

$\rightsquigarrow$ **Hodge conjecture**

K3 surface S

- $\gamma \in H^{1,1}(S) \cap H^2(S,\mathbb{Z}) \stackrel{\checkmark}{\Longrightarrow} \exists\, C \subset S\ \forall\, \alpha \in H^2(S)\colon (\gamma.\alpha) = \int_C \alpha.$

Products of K3 surfaces $S_1 \times S_2$

- $\varphi\colon H^2(S_1,\mathbb{Q}) \longrightarrow H^2(S_2,\mathbb{Q})$ with $\varphi(H^{p,q}(S_1)) \subset H^{p,q}(S_2)$

$$\stackrel{??}{\Longrightarrow} \exists\, Z \subset S_1 \times S_2\colon (\varphi(\alpha_i).\alpha_j) = \int_Z (\alpha_i \times \alpha_j)$$

Theorem

- OK for φ orthogonal (⇒ OK for S with CM)

§4 New ways in geometry

Points and curves

$\rightsquigarrow$ Chow groups and Chow motives (linearizing geometry)

Linearizing points: $x \in S \rightsquigarrow \mathrm{CH}_0(S) := \{ \sum n_i x_i \}/\sim$, where

$x_1 \sim x_2 \in S$ equivalence relation generated by:

$\exists \ f\colon \mathbb{P}^1 \longrightarrow S : x_1, x_2 \in f(\mathbb{P}^1)$ & natural generalization

Theorem If $g\colon S \xrightarrow{\sim} S$, then:

- $g = \mathrm{id}$ on $H^{2,0}(S) \Leftrightarrow g = \mathrm{id}$ on $\mathrm{CH}_0(S)$

Linearizing S: Grothendieck's philosophy of motives . . .

...Linearizing S: Replace geometry by linear category Mot (pseudo-abelian)

$$S \quad \rightsquigarrow \quad \mathfrak{h}_S \in \mathrm{Mot}$$

$$\mathrm{Mor}(S_1, S_2) \ \text{set} \ \rightsquigarrow \ \mathrm{Mor}(\mathfrak{h}_{S_1}, \mathfrak{h}_{S_2}) \ \mathbb{Q}\text{-vector space}$$
$$= \{ S_1 \longrightarrow S_2 \} \qquad\qquad = \mathrm{CH}_2(S_1 \times S_2)$$

Open: Are all $\mathfrak{h}_S$ ‘finite-dimensional’ ?

Motivic Hodge / **Torelli** S_1, S_2 K3 surfaces

- $\mathfrak{h}_{S_1} \cong \mathfrak{h}_{S_2}$ multiplicative $\Leftrightarrow$ $H^2(S_1, \mathbb{Q}) \cong H^2(S_2, \mathbb{Q})$ compatible with $(*)$ & $(\,.\,)$

Open: What happens if $(\,.\,)$ is dropped?

Bundles $\rightsquigarrow$ linear category

Instead of one bundle $\rightsquigarrow$ consider complexes

$$E \longrightarrow S \qquad \cdots \xrightarrow{d} E_1 \xrightarrow{d} E_2 \xrightarrow{d} \cdots \quad \text{with } d^2 = 0$$

$\rightsquigarrow$ $\mathrm{D}(S)$ triangulated (dg) category (or 'twisted' version $D(S, \alpha)$)

Categorical Hodge / Torelli S_1, S_2 K3 surfaces

• $\mathrm{D}(S_1, \alpha_1) \cong \cdots \cong \mathrm{D}(S_2, \alpha_2) \Leftrightarrow H^2(S_1, \mathbb{Q}) \cong H^2(S_2, \mathbb{Q})$ compatible with $(*)$ & (.)

Open: What happens if (.) is dropped?

Further questions: Relation to other motivic and categorical invariants:

• $K_0(\mathrm{Var})$, $K_0(\mathrm{trcat})$ • $\mathrm{CH}(S)$ & $\mathrm{D}(S)$ for $S/\bar{\mathbb{Q}}$ • ...

§5 Unexpected findings

Cubic hypersurfaces $X \subset \mathbb{P}^5$, e.g. zero set of $x_0^3 + \cdots + x_4^3 = 1$

Geometric: $\rightsquigarrow$ $F(X) = \{ L \subset X \mid \text{line} \}$ (Fano variety of lines)
is hyperkähler of $\dim = 4$

Hodge theory: $\rightsquigarrow$ $H^4(X, \mathbb{C}) = H^{3,1}(X) \oplus H^{2,2}(X) \oplus H^{1,3}(X)$
frequently $\approx H^2(S_X, \mathbb{C}) = H^{2,0}(S_X) \oplus H^{1,1}(S_X) \oplus H^{0,2}(S_X)$

Open: What is the geometric relation:

$$F(X) \overset{?}{\longleftrightarrow} S_X \overset{?}{\longleftrightarrow} X$$

Clearer picture: Using motives and categories

$$\mathfrak{h}_{F(X)} \approx S^2 \mathfrak{h}_{S_X} \text{ and } \mathfrak{h}^{\mathrm{tr}}_{S_X} \approx \mathfrak{h}^{\mathrm{tr}}_X$$

$$\mathrm{D}(F(X)) \leftrightsquigarrow S^2 \mathrm{D}(S_X) \text{ and } \mathrm{D}(S_X) \hookrightarrow \mathrm{D}(X)$$

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