Lagrangian intersections as d-critical loci and their perverse sheaves

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Born 15th Mai 1992 in Bonn

13th April 2017

Master's Thesis Mathematics

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Contents

1. Introduction

We are interested in intersections of Lagrangian submanifolds inside a holomorphic symplectic manifold. Our particular goal is to understand the construction of a perverse sheaf $\mathcal{P}^\bullet_{L,M}$ on such an intersection as defined in [Bu14], [BBDJS] and [KL16]. It is a special case of the construction of a perverse sheaf on a d-critical locus. Therefore, the paper is divided in two parts: Part 1 can be viewed as an introduction to the theory of d-critical loci from [Joy13]. It covers the definition of a d-critical locus (X, s), symmetric obstruction theories as defined in [BF05], and the construction of a perverse sheaf $\mathcal{P}^\bullet_{X,s}$ on (X, s) from [BBDJS]. Part 2 applies the theory to Lagrangian intersections. The starting point is that any such intersection admits the structure of a d-critical locus.

We will now describe the main results in more detail:

Part 1. ***d*-critical loci.** First of all, the local setting we are interested in is as follows. Let S be a smooth scheme and $f \in H^0(S, \mathcal{O}_S)$. We set $X := \mathrm{crit}(f)$ and generally care about the following constructions on X:

(i) *Symmetric obstruction theories*, i.e. a triple

$$(\mathbb{E} \in \mathrm{D}(X),\ \phi\colon \mathbb{E} \to \mathbb{L}_X,\ \theta\colon \mathbb{E}^\vee[1] \xrightarrow{\sim} \mathbb{E})$$

such that $\mathbb{E}$ is locally isomorphic to a complex $[\mathcal{E}^{-1} \to \mathcal{E}^0]$ of locally free sheaves, $\mathcal{H}^{-1}(\phi)$ is surjective and $\mathcal{H}^0(\phi)$ is an isomorphism. Here, $\mathbb{L}_X$ denotes the (truncated) cotangent complex of X. For instance, a symmetric obstruction theory on X is given by

$$\begin{tikzcd} \mathbb{E}_{\partial^2 f} \arrow[d, "\psi"] & = & {[\mathcal{T}_S|_X} \arrow[r, "\partial^2 f"] \arrow[d, two heads, "df"] & \Omega_S|_X] \arrow[d, equal] \\ \mathbb{L}_X & = & {[\mathscr{I}_{XS}/\mathscr{I}_{XS}^2} \arrow[r, "d"] & \Omega_S|_X] \end{tikzcd} \tag{1.1}$$

(Lemma 3.10).

(ii) The *perverse sheaf of vanishing cycles* defined by

$$\mathcal{PV}^\bullet_{S,f} := \phi_f\, \underline{\mathbb{C}}_S[\dim S - 1] \in \mathbf{Perv}(X). \tag{1.2}$$

In general, we deal with schemes that are locally of the form $\mathrm{crit}(f)$. These are d-critical loci.

According to [Joy13, Definition 2.5], a d-critical locus is either a scheme or an analytic space X together with a section s of a certain sheaf $\mathcal{S}_X$. This section satisfies the property that for every point $x \in X$ there exists an open neighborhood $U \subset X$, a smooth scheme S together with a closed embedding $U \subset S$ and $f \in H^0(S, \mathcal{O}_S)$ such that $U = \mathrm{crit}(f)$ and f defines $s|_U$. The triple (U, S, f) is called a chart for (X, s). In Section 2.1 we will give the precise definition of the sheaf $\mathcal{S}_X$ involving what we call the *thickened cotangent complex* $\mathbb{M}_X$. This is a non-coherent version of the truncated cotangent complex $\mathbb{L}_X$. More precisely, for any scheme X we construct a morphism

$$D_X\colon \mathcal{O}_X \to \mathbb{L}_X \text{ in } \mathrm{D}(\mathbf{Sh}(X))$$

over the natural differential $d_X\colon \mathcal{O}_X \to \Omega_X$. The thickened cotangent complex is the cone of D_X and we set

$$\mathcal{S}_X := \mathcal{H}^{-1}(\mathbb{M}_X).$$

Since the differential in $\mathbb{M}_X$ is not $\mathcal{O}_X$-linear, $\mathcal{S}_X$ is in general not a coherent sheaf but only a sheaf of $\mathbb{C}$-algebras.

On any d-critical locus (X, s) we can locally define a symmetric obstruction theory as in (1.1). Unfortunately, these obstruction theories need not glue to a global obstruction theory on X. We discuss this in Section 3.

Proposition 1.1 ([Joy13]). *There exists a line bundle $\mathcal{K}_{X,s}$ on X_{red} such that*

$$\mathcal{K}_{X,s}|_{U_{\mathrm{red}}} \xrightarrow{\sim} \det \mathbb{E}_{\partial^2 f}|_{U_{\mathrm{red}}} = \omega_S^{\otimes 2}|_{U_{\mathrm{red}}}$$

for every critical chart (U, S, f).

For the precise statement see Proposition 3.12. The line bundle $\mathcal{K}_{X,s}$ is called *canonical bundle*.

In Section 4 we address the question whether the perverse sheaves of vanishing cycles as in (1.2) can be glued to a global perverse sheaf $\mathcal{P}^\bullet_{X,s}$ on (X,s).

Proposition 1.2 ([BBDJS]). *Assume that there is a square root $\mathcal{K}^{1/2}_{X,s}$ of the canonical bundle $\mathcal{K}_{X,s}$. Then there exists a perverse sheaf $\mathcal{P}^\bullet_{X,s}$ on X such that*

$$\mathcal{P}^\bullet_{X,s}|_U \xrightarrow{\sim} \mathcal{PV}^\bullet_{S,f} \otimes \mathfrak{L}^{\mathrm{or}}_{U,S}$$

for every critical chart (U,S,f). Here, $\mathfrak{L}^{\mathrm{or}}_{U,S}$ is a local system on U_{red} such that $\mathfrak{L}^{\mathrm{or}}_{U,S} \otimes_{\mathbb{C}} \mathcal{O}_{U_{\mathrm{red}}} \cong (\mathcal{K}^{1/2}_{X,s})^\vee|_{U_{\mathrm{red}}} \otimes \omega_S|_{U_{\mathrm{red}}}$.

For the precise statement see Proposition 4.20.

Part 2. Lagrangian intersections. Let (S,σ) be a symplectic manifold of dimension $2n$ and let $X = L \cap M$ be the intersection of two Lagrangian submanifolds. For instance, let M be a complex manifold and $S = |\Omega_M|$ be the total space of its cotangent bundle with the canonical symplectic structure. Then, for any $f \in H^0(M,\mathcal{O}_M)$ we define a Lagrangian submanifold by $L := \Gamma_{df}$. Here, Γ_{df} denotes the image of the embedding $df\colon M \hookrightarrow |\Omega_M|$, which is given by the section $df \in H^0(M,\Omega_M)$. The intersection $X = L \cap M$ is the locus where df vanishes, i.e. $X = \mathrm{crit}(f) \subset M$. It turns out that any Lagrangian intersection is locally of this form.

Proposition 1.3 ([Bu14]). *The intersection of two Lagrangian submanifolds $X = L \cap M$ admits the structure of a d-critical locus with canonical bundle*

$$\mathcal{K}_{X,s} \cong \omega_L|_{X_{\mathrm{red}}} \otimes \omega_M|_{X_{\mathrm{red}}}.$$

In particular, we can once again locally define a symmetric obstruction theory on X. On the other hand, we prove in Lemma 6.1 that X carries a global symmetric obstruction theory with $\mathbb{E}$ given by

$$\mathbb{E}_{LM} := \left[\Omega_S|_X \xrightarrow{(-\mathrm{res}_L,\mathrm{res}_M)} \Omega_L|_X \oplus \Omega_M|_X\right].$$

From here we can conclude (Lemma 6.2) that

$$\omega_X \otimes \omega_X \cong \omega_L|_X \otimes \omega_M|_X$$

if X is smooth. Moreover, we compare these two obstruction theories in the above situation.

Proposition 1.4. *Let M be a complex manifold considered as Lagrangian submanifold inside its cotangent bundle $S = |\Omega_M|$. Let $f \in H^0(M,\mathcal{O}_M)$ and $L := \Gamma_{df} \subset S$. There is an isomorphism of symmetric obstruction theories*

$$\Phi\colon \mathbb{E}_{\partial^2 f} \xrightarrow{\sim} \mathbb{E}_{LM},$$

where $\mathbb{E}_{\partial^2 f}$ is defined as in (1.1) and $\mathbb{E}_{LM}$ as above.

The precise definition of Φ can be found in Proposition 6.4.

In the case that a square root of $\omega_L|_{X_{\mathrm{red}}} \otimes \omega_M|_{X_{\mathrm{red}}}$ exists the d-critical structure on X also yields the existence of a perverse sheaf $\mathcal{P}^\bullet_{L,M}$. In Section 7.2 we will describe $\mathcal{P}^\bullet_{L,M}$ explicitly in some special situations. For instance, we have (Lemma 7.6)

Lemma 1.5. *Assume that the intersection $X = L \cap M$ is smooth and let $\mathcal{K}^{1/2}_{X,s} = \omega_X$. Then*

$$\mathcal{P}^\bullet_{L,M} \cong \underline{\mathbb{C}}_X[\dim X].$$

The following Section 8 is dedicated to the study of two concrete examples of Lagrangian intersections in dimension 4. Here, the Hilbert scheme $\mathrm{Hilb}^2(Z)$ of a K3 surface serves as a symplectic manifold.

Finally, in the last Section 9 we aim at setting up a connection between $\dim \mathbb{H}^\bullet(\mathcal{P}^\bullet_{L,M})$ and $\dim \mathrm{Ext}^\bullet(\omega_L^{1/2}, \omega_M^{1/2})$. More precisely, under the assumption that

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\omega_L^{1/2}, \omega_M^{1/2})) \Rightarrow \mathrm{Ext}^{p+q}(\omega_L^{1/2}, \omega_M^{1/2})$$

degenerates, we prove

Proposition 1.6. *Assume that $X = L \cap M$ is smooth and compact. Let $\omega_L^{1/2}$ and $\omega_M^{1/2}$ be fixed square roots of the canonical bundles of L and M, respectively. Then*

$$\dim \mathrm{Ext}^{i-n}_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2}) = \dim \mathbb{H}^i(\mathcal{P}^\bullet_{L,M}).$$

Here, $\mathcal{K}^{1/2}_{X,s} = \omega_L^{1/2}|_X \otimes \omega_M^{1/2}|_X$ serves for the computation of $\mathcal{P}^\bullet_{L,M}$.

Acknowledgements. I want to heartily thank my advisor Daniel Huybrechts for all his time and motivation. Many thanks to Johannes Anschütz for continuously answering all my questions and also to Andrey Soldatenkov for the computation of Example 2.12.

Notation. All schemes are of finite type over $\mathbb{C}$. Let X be a scheme. We abbreviate $D(\mathbf{Qcoh}(X))$ by $\mathrm{D}(X)$ and call this the derived category of X. We use the notation $\mathrm{D}^+(X), \mathrm{D}^-(X), \mathrm{D}^b(X)$ for the derived category of complexes that are bounded from below, from above or from both sides respectivly. Moreover, $\mathrm{D}^b_c(X)$ is the derived category of bounded complexes of constructible sheaves. We denote the category of sheaves of $\mathbb{C}$ vector spaces by $\mathbf{Sh}(X)$. We do not especially indicate when tensor products are derived tensor products.

The theory of d-critical loci works in the analytic and in the Zariski topology. We will not mention this explicitly. All our perverse sheaves are elements of $\mathrm{D}^b_c(\mathbf{Sh}(X))$, where constructibility is understood with respect to the analytic topology. In Part 2 we are only interested in the analytic topology.

If $\mathcal{E}^\bullet$ is a complex of sheaves, then $\mathcal{E}^\bullet[1]$ is the complex with $(\mathcal{E}^\bullet[1])^i = \mathcal{E}^{i+1}$ and differential $d^i_{\mathcal{E}^\bullet[1]} = -d^{i+1}_{\mathcal{E}^\bullet}$. If $\phi\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a morphism of complexes, then $\phi[1]$ is given by $\phi[1]^i = \phi^{i+1}$.

Let X be a scheme. We write $\mathcal{N}il_X$ for the Nilradical of $\mathcal{O}_X$. If $X \subset S$ is a closed embedding with ideal sheaf $\mathscr{I}$, then $\mathcal{N}il_X = \sqrt{\mathscr{I}}/\mathscr{I}$, where $\sqrt{\mathscr{I}}$ is the radical ideal of $\mathscr{I}$ defining X_{red}.

If X is a scheme and $\mathcal{E}$ a vector bundle on X, we denote the total space of $\mathcal{E}$ by $|\mathcal{E}| = \mathrm{Spec}_X(\mathrm{Sym}^\bullet(\mathcal{E}^\vee))$. For any section $s \in H^0(X, \mathcal{E})$ let $\Gamma_s \subset |\mathcal{E}|$ be the image of the embedding $s\colon X \hookrightarrow |\mathcal{E}|$. If no confusion arises, we write $X \subset |\mathcal{E}|$ for the zero section Γ_0. For projective bundles, we use the convention $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_X(\mathrm{Sym}^\bullet(\mathcal{E}^\vee))$.

Let M be a scheme and $f\colon M \to \mathbb{A}^1$ a morphism, or equivalently, $f \in \Gamma(M, \mathcal{O}_M)$. Then $\mathrm{crit}(f) \subset M$ is the vanishing locus of the section $df \in \Gamma(M, \Omega_M)$. We could also write $\mathrm{crit}(f) = M \cap \Gamma_{df}$.

Part 1. d-critical loci

2. Definition

In this section we will establish the notion of a d-critical locus (X, s). This is a scheme or a compelex analytic space X that is locally of the form $\mathrm{crit}(f)$. Any two local reprensentations of the form $\mathrm{crit}(f)$ satisfy a certain compatibility condition, which is encoded by a section $s \in H^0(X, \mathcal{S}_X)$. The sheaf $\mathcal{S}_X$ is a cohomology sheaf of the thickened cotangent complex that we define now.

2.1. The thickened cotangent complex.

In the following, we let X be any scheme or complex analytic space. We assume that there is a closed embedding $X \subset S$ with ideal sheaf $\mathscr{I}$ into some smooth scheme S. Consider the natural differential $d_S\colon \mathcal{O}_S \to \Omega_S$. It induces an $\mathcal{O}_X$-linear map

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega_S|_X$$

whose cokernel is Ω_X and whose kernel carries information on the smoothness of X. The complex

$$\mathbb{L}_X := [\mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega_S|_X] \tag{2.1}$$

concentrated in degrees -1 and 0 is called the *truncated cotangent complex* of X (in S). By construction, $\mathbb{L}_X$ comes with a natural morphism $\mathbb{L}_X \to \Omega_X$. In fact, $\mathbb{L}_X$ is quasi-isomorphic to the truncation $\tau^{\geq -1}\mathbf{L}^\bullet_X$ of Illusie's full cotangent complex and therefore only depends on X. Following [HT10], we can also give a direct proof that $\mathbb{L}_X$ does (up to quasi-isomorphism) not depend on the embedding $X \subset S$. Assume that $X \subset S_i$ with ideal $\mathscr{I}_i$ for some smooth scheme S_i and $i \in \{1, 2\}$. Then the diagonal embedding gives an embedding $X \subset S_1 \times S_2$, whose ideal sheaf we denote by $\mathscr{I}_{12}$. The composition $X \subset S_1 \times S_2 \to S_1$ induces a short exact sequence of two term complexes

$$\begin{tikzcd}
{[\mathscr{I}_1/\mathscr{I}_1^2} \arrow[r] \arrow[d] & \Omega_{S_1}|_X] \arrow[d] \\
{[\mathscr{I}_{12}/\mathscr{I}_{12}^2} \arrow[r] \arrow[d] & \Omega_{S_1}|_X \oplus \Omega_{S_2}|_X] \arrow[d] \\
{[\Omega_{S_2}|_X} \arrow[r, equal] & \Omega_{S_2}|_X].
\end{tikzcd}$$

This proves that $\mathbb{L}_X$ defined via $X \subset S_1$ and $X \subset S_2$ are quasi-isomorphic.

Another property of the truncated cotangent complex is that for any open subset $U \subset X$ we have

$$\mathbb{L}_U \cong \mathbb{L}_X|_U \text{ in } \mathrm{D}(U). \tag{2.2}$$

If we do not insist on dealing exclusively with $\mathcal{O}_X$-linear morphisms, we can also consider the following $\mathbb{C}$-linear map

$$\mathcal{O}_S/\mathscr{I}^2 \longrightarrow \Omega_S|_X$$

that is likewise induced by d_S. It is well defined because d_S being a derivation implies $d_S(\mathscr{I}^2) \subset \mathscr{I}\Omega_S$. This map will allow us to lift the classical differential $d_X : \mathcal{O}_X \to \Omega_X$ to a morphism $D_X : \mathcal{O}_X \to \mathbb{L}_X$ defined in $\mathrm{D}(\mathbf{Sh}(X))$ but not in $\mathrm{D}(X)$.

Consider the inclusion $i : \mathscr{I}/\mathscr{I}^2 \to \mathcal{O}_S/\mathscr{I}^2$, yielding a short exact sequence of two-term complexes

$$[\mathscr{I}/\mathscr{I}^2 \to \Omega_S|_X] \xrightarrow{u} [\mathcal{O}_S/\mathscr{I}^2 \to \Omega_S|_X] \longrightarrow [\mathcal{O}_X \to 0], \tag{2.3}$$

which we consider as a distinguished triangle in $\mathrm{D}(\mathbf{Sh}(X))$. Therefore, after rotation, we obtain a morphism

$$D_X : \mathcal{O}_X \longrightarrow \mathbb{L}_X \text{ in } \mathrm{D}(\mathbf{Sh}(X)). \tag{2.4}$$

More explicitly, (2.3) gives an isomorphism

$$\mathcal{O}_X[1] \cong \mathrm{Cone}(u) = \left[\mathscr{I}/\mathscr{I}^2 \xrightarrow{(i,-d)} \mathcal{O}_S/\mathscr{I}^2 \oplus \Omega_S|_X \xrightarrow{d_S+\mathrm{id}} \Omega_S|_X\right]$$

in $\mathrm{D}(\mathbf{Sh}(X))$. Here, $\mathcal{O}_X[1]$ is the complex with $\mathcal{O}_X$ in degree -1, so that the projection to $\mathbb{L}_X[1]$ defines (2.4). Given two embeddings $X \subset S_i$ for $i = 1, 2$ we constructed above a quasi-isomorphism between $\mathbb{L}_X$ defined via S_1 and $\mathbb{L}_X$ defined via S_2. The construction of D_X is compatible with this quasi-isomorphism as we can apply the above proof to (2.3). We conclude that the definition of D_X is independent of the embedding $X \subset S$.

Finally, for later use, we will consider the composition

$$\mathcal{N}il_X \hookrightarrow \mathcal{O}_X \xrightarrow{D_X} \mathbb{L}_X.$$

Lemma 2.1 ([Joy13, Proposition 2.3]). *The construction of D_X is functorial in the following way: For every morphism $\Phi : X \to Y$ there is a commutative* *diagram*

$$\begin{array}{ccc} \Phi^{-1}\mathcal{O}_Y & \xrightarrow{\Phi^{-1}D_Y} & \Phi^{-1}\mathbb{L}_Y \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \xrightarrow{D_X} & \mathbb{L}_X. \end{array}$$

Proof. This follows essentially from functoriality of $\mathbb{L}_X$ and exactness of Φ^{-1}. □

Remark 2.2. Let $f \in H^0(X, \mathcal{O}_X)$. We can give a more explicit description of $D_X(f) \in \mathbb{H}^0(X, \mathbb{L}_X)$ as follows: Consider f as a morphism $f\colon X \to \mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$. Then functoriality of D_X gives the commutative diagram

$$\begin{array}{ccc} H^0(X, f^{-1}\mathcal{O}_{\mathbb{A}^1}) & \longrightarrow & \mathbb{H}^0(X, f^{-1}\mathbb{L}_{\mathbb{A}^1}) \cong H^0(X, f^{-1}\mathcal{O}_{\mathbb{A}^1}\langle dt\rangle) \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X) & \longrightarrow & \mathbb{H}^0(X, \mathbb{L}_X), \end{array}$$

where $t \in H^0(X, f^{-1}\mathcal{O}_{\mathbb{A}^1})$ maps to $f \in H^0(X, \mathcal{O}_X)$. Hence, $D_X(f)$ is the image of the generator dt under the right-hand vertical arrow

$$\mathbb{H}^0(X, f^{-1}\mathbb{L}_{\mathbb{A}^1}) \cong H^0(X, f^{-1}\mathcal{O}_{\mathbb{A}^1}\langle dt\rangle) \longrightarrow \mathbb{H}^0(X, \mathbb{L}_X).$$

Definition 2.3. The *thickened cotangent complex* is

$$\mathbb{M}_X := \operatorname{Cone}(\mathcal{O}_X \xrightarrow{D_X} \mathbb{L}_X) \in \mathrm{D}(\mathbf{Sh}(X))$$

and the *reduced thickened contangent complex* is

$$\mathbb{M}_X^0 := \operatorname{Cone}(\mathcal{N}il_X \to \mathbb{L}_X) \in \mathrm{D}(\mathbf{Sh}(X)).$$

We define two sheaves of $\mathbb{C}$-vector spaces on X by

$$\mathcal{S}_X := \mathcal{H}^{-1}(\mathbb{M}_X) \quad \text{and} \quad \mathcal{S}_X^0 := \mathcal{H}^{-1}(\mathbb{M}_X^0).$$

Remark 2.4. This definition of $\mathcal{S}_X$ is suggested in [Joy13, Remark 2.2(b)].

Remark 2.5. It follows from the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathbb{M}_X)) \Rightarrow \mathbb{H}^{p+q}(X, \mathbb{M}_X)$$

that $H^0(X, \mathcal{S}_X) \cong \mathbb{H}^{-1}(X, \mathbb{M}_X)$.

Lemma 2.6. *With the above notations we have the following first properties of the thickened cotangent complex.*

(i) *For any closed embedding $X \subset S$ into a smooth scheme S with ideal sheaf $\mathscr{I}$. The thickened cotangent complex can be represented by*

$$\mathbb{M}_X \cong \left[\mathcal{O}_S/\mathscr{I}^2 \longrightarrow \Omega_S\,|_X\right]$$

and its reduced version by

$$\mathbb{M}_X^0 \cong \left[\sqrt{\mathscr{I}}/\mathscr{I}^2 \longrightarrow \Omega_S\,|_X\right] \cong \left[\mathcal{O}_S/\mathscr{I}^2 \longrightarrow \Omega_S\,|_X \oplus \mathcal{O}_{X_{\mathrm{red}}}\right].$$

(ii) *$\mathcal{S}_X$ and $\mathcal{S}_X^0$ are sheaves of $\mathbb{C}$-algebras.*

(iii) *Let $U \subset X$ be an open subset. Then*

$$\mathbb{M}_U \cong \mathbb{M}_X|_U \text{ in } \mathrm{D}(\mathbf{Sh}(U)).$$

In particular, we have $\mathcal{S}_U = \mathcal{S}_X|_U$.

(iv) *For all morphisms $\Phi\colon X \to Y$ there is an induced map $\Phi^{-1}\mathbb{M}_Y \to \mathbb{M}_X$ and thus $\Phi^{-1}\mathcal{S}_Y \to \mathcal{S}_X$.*

Proof. (i) follows directly from the definition, (ii) from the Leibniz rule, (iii) from (2.2) and (iv) from the functoriality of D_X (cf. Lemma 2.1). $\square$

Lemma 2.7. *The connecting homomorphism δ in (2.5) is the differential $d_X\colon \mathcal{O}_X \to \Omega_X$. Therefore, we have a commuting diagram in* $\mathrm{D}(\mathbf{Sh}(X))$

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "D_X"] \arrow[d, equal] & \mathbb{L}_X \arrow[d] \\
\mathcal{O}_X \arrow[r, "d_X"] & \Omega_X .
\end{tikzcd}$$

Proof. We embed $X \subset S$ as above. The Lemma follows from the definitions and the commutative diagram

$$\begin{tikzcd}
 & \mathcal{H}^{-1}(\mathbb{L}_X) \arrow[r] \arrow[d] & \mathcal{S}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d, "\wr"'] & \\
0 \arrow[r] & \mathscr{I}/\mathscr{I}^2 \arrow[r] \arrow[d] & \mathcal{O}_S/\mathscr{I}^2 \arrow[r] \arrow[d, "d"'] & \mathcal{O}_S/\mathscr{I} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega_S|_X \arrow[r, equal] \arrow[d] & \Omega_S|_X \arrow[r] & 0 \arrow[r] & 0 \\
 & \mathcal{H}^0(\mathbb{L}_X) \cong \Omega_X . & & &
\end{tikzcd}$$

$\square$

By definition, there is a commutative diagram in $\mathrm{D}(\mathbf{Sh}(X))$ with exact rows and columns

$$\begin{tikzcd}
\mathcal{N}il_X \arrow[r] \arrow[d] & \mathbb{L}_X \arrow[r] \arrow[d, equal] & \mathbb{M}^0_X \arrow[d] \\
\mathcal{O}_X \arrow[r] \arrow[d] & \mathbb{L}_X \arrow[r] \arrow[d] & \mathbb{M}_X \arrow[d] \\
\mathcal{O}_{X_{\mathrm{red}}} \arrow[r] & 0 \arrow[r] & \mathcal{O}_{X_{\mathrm{red}}}[1].
\end{tikzcd}$$

This induces a commutative diagram of long exact sequences (cf. [Joy13, Theorem 2.1(b)])

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^{-1}(\mathbb{L}_X) \arrow[r] \arrow[d, equal] & \mathcal{S}_X^0 \arrow[r] \arrow[d] & \mathcal{N}il_X \arrow[r] \arrow[d] & \Omega_X \arrow[r] \arrow[d, equal] & \mathcal{H}^0(\mathbb{M}_X^0) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{H}^{-1}(\mathbb{L}_X) \arrow[r] & \mathcal{S}_X \arrow[r] & \mathcal{O}_X \arrow[r, "\delta"] & \Omega_X \arrow[r] & \mathcal{H}^0(\mathbb{M}_X) \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

and a long exact sequence

$$0 \longrightarrow \mathcal{S}_X^0 \longrightarrow \mathcal{S}_X \xrightarrow{\alpha} \mathcal{O}_{X_{\mathrm{red}}} \longrightarrow \mathcal{H}^0(\mathbb{M}_X^0) \longrightarrow \mathcal{H}^0(\mathbb{M}_X) \longrightarrow 0. \tag{2.6}$$

Form these sequences we now deduce several properties of the thickened cotangent complex.

Corollary 2.8. *The zero-th cohomology sheaves of the thickened cotangent complexes are given by*

$$\mathcal{H}^0(\mathbb{M}_X) \cong \Omega_X/d_X(\mathcal{O}_X) \quad \textit{and} \quad \mathcal{H}^0(\mathbb{M}_X^0) \cong \Omega_X/d_X(\mathcal{N}il_X).$$

□

This allows us to give a precise formulation of [Joy13, Corollary 2.14].

Corollary 2.9. *We have*

$$\mathcal{H}^{-1}(\mathbb{L}_X) \cong \mathcal{S}_X^0 \iff \ker(\mathcal{O}_X \xrightarrow{d_X} \Omega_X) \cap \mathcal{N}il_X = 0.$$

In particular, $\mathcal{S}_X^0$ is a coherent sheaf on X in this case.

□

Proposition 2.10 (cf. [Joy13, Theorem 2.1(a),(c)]). (i) *There is a natural decomposition*

$$\mathcal{S}_X = \underline{\mathbb{C}}_X \oplus \mathcal{S}_X^0,$$

where the constant sheaf $\underline{\mathbb{C}}_X$ is embedded via the isomorphim $\underline{\mathbb{C}}_X \cong (\underline{\mathbb{C}}_S + \mathscr{I}^2)/\mathscr{I}^2$.

(ii) *The sheaf $\mathcal{S}_X^0$ is isomorphic to the middle cohomology sheaf of the complex*

$$\mathbb{E} := \left[\mathscr{I}^2 \xrightarrow{d} \mathscr{I} \cdot \Omega_S \xrightarrow{d} \Omega_S^2\right].$$

Proof. (i) We claim that $\operatorname{im}(\alpha) = \underline{\mathbb{C}}_X = i_* \underline{\mathbb{C}}_{X_{\mathrm{red}}}$ in (2.6) with $i\colon X_{\mathrm{red}} \hookrightarrow X$ being the inclusion. Obviously we have $\operatorname{im}(\alpha) \supset \underline{\mathbb{C}}_X$. For the converse inclusion we choose an embedding of X into a smooth scheme S. Then α is given by the upper row in the following commutative diagram

$$\begin{tikzcd} \mathcal{S}_X \arrow[r, hook] \arrow[dr, "0"'] & \mathcal{O}_S/\mathscr{I}^2 \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & i_*\mathcal{O}_{X_{\mathrm{red}}} \arrow[d] \\ & \Omega_S|_X \arrow[r] & \Omega_X \arrow[r] & i_*\Omega_{X_{\mathrm{red}}} \end{tikzcd}$$

and thus $\operatorname{im}(\alpha) \subset \ker(i_*\mathcal{O}_{X_{\mathrm{red}}} \to i_*\Omega_{X_{\mathrm{red}}}) = i_* \underline{\mathbb{C}}_{X_{\mathrm{red}}}$. Together, this gives

$$0 \to \mathcal{S}_X^0 \to \mathcal{S}_X \to \underline{\mathbb{C}}_X \to 0$$

and the inclusion $\underline{\mathbb{C}}_X \hookrightarrow \mathcal{S}_X$ defines a split.
(ii) This time, we denote by i the inclusion $X \hookrightarrow S$. Consider the commutative diagram with short exact columns

$$\begin{tikzcd}
\mathscr{I}^2 \arrow[r] \arrow[d] & \mathscr{I}\Omega_S \arrow[r] \arrow[d] & \Omega_S^2 \arrow[d, equal] \\
\mathcal{O}_S \arrow[r] \arrow[d] & \Omega_S \arrow[r] \arrow[d] & \Omega_S^2 \arrow[d] \\
i_*(\mathcal{O}_S/\mathscr{I}^2) \arrow[r] & i_*(\Omega_S\,|_X) \arrow[r] & 0.
\end{tikzcd}$$

We consider the rows as complexes concentrated in the interval $[-1,1]$. Taking cohomology sheaves yields an exact sequence

$$\mathcal{H}^{-1}(\mathbb{E}) = j_!\,\underline{\mathbb{C}}_{S\setminus X} \to \underline{\mathbb{C}}_S \to i_*\mathcal{S}_X \to \mathcal{H}^0(\mathbb{E}) \to 0,$$

where $j\colon X \setminus S \hookrightarrow S$ is the inclusion. As $\mathcal{H}^0(\mathbb{E})$ is actually supported on X, applying i^{-1} gives the short exact sequence

$$0 \to \underline{\mathbb{C}}_X \longrightarrow \mathcal{S}_X \longrightarrow \mathcal{H}^0(\mathbb{E}) \to 0.$$

By construction, $\underline{\mathbb{C}}_X \to \mathcal{S}_X$ is the embedding from part (i), so that applying part (i) finishes the proof. $\square$

The following example is central to our application of the thickened cotangent complex.

Example 2.11. Let $f \in H^0(S, \mathcal{O}_S)$ and consider $\operatorname{crit}(f) = Z(df) \subset S$ with ideal $\mathscr{I} = \operatorname{im}(\mathcal{T}_S \xrightarrow{df} \mathcal{O}_S)$. Then, by definition, $df \in \mathscr{I}\Omega_S$ and thus

$$\bar{f} \in H^0(X, \mathcal{S}_X) \subset H^0(X, \mathcal{O}_S/\mathscr{I}^2).$$

Moreover, $\bar{f} \in H^0(X, \mathcal{S}_X^0)$ if and only if $f \in \sqrt{\mathscr{I}}$. In Example 2.12 below, we see that f is not necessarily locally constant on X as one might expect thinking of manifolds.

Example 2.12 (cf. [Joy13, Example 2.13]). In general, we can not expect that $\mathcal{H}^{-1}(\mathbb{L}_X) \cong \mathcal{S}_X^0$. The following example shows that this is not even true when $X = \operatorname{crit}(f)$ for some $f \in H^0(S, \mathcal{O}_S)$.

Let $S = \operatorname{Spec}\mathbb{C}[x,y]$ and $f = x^5 + x^2y^2 + y^5$. Set $X = \operatorname{crit}(f)$. Then, $X = V(I)$ with $I = (d_xf = 5x^4 + 2xy^2, d_yf = 2x^2y + 5y^4)$. In Example 2.11 we already noticed that $df\,|_X = 0$. Write X as the intersection of two reducible curves

$$X = (L_1 \cup C_1) \cap (L_2 \cup C_2),$$

where L_i are lines and C_i are cuspidal cubics, $i = 1, 2$. Hence, the intersection number is 16. Looking more closely, we see that X consists of 5 distinct reduced intersection points of C_1 and C_2 and the point $(0,0)$ occuring consequently with multiplicity 11. We claim that the local ring at $(0,0)$ is given by $\operatorname{Spec}\mathbb{C}[x,y]/J$, where $J = (d_xf, d_yf, xy^3, y^3x)$. This follows, since $2yd_xf - 5x^2d_yf = xy^3(4-25xy)$ and $4-25xy$ is invertible in the local ring at $(0,0)$. Similarly for x^3y. Note that $\mathbb{C}[x,y]/J$ really has a basis consisting of 11 elements, namely $\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^2y^2\}$.

Now, $5f - x^2y^2 = d_xf + d_yf$ shows that $0 \neq f \in \mathbb{C}[x,y]/J$ meaning that f is not locally constant near $(0,0)$. Moreover, we see $f \in \sqrt{J}$. Finally, after altering f by its value on the other points of X we find

$$0 \neq f \in \mathcal{N}il_X \cap \ker(\mathcal{O}_X \to \Omega_X)$$

or, put differently, $f \in H^0(X, \mathcal{S}_X^0)$ but $f \notin H^0(X, \mathcal{H}^{-1}(\mathbb{L}_X))$.
We also see that, in this example $\mathcal{S}_X^0$ is not an natural $\mathcal{O}_X$-module. Otherwise we would have $x \cdot x^2y^2 \in H^0(X, \mathcal{S}_X^0)$. However,

$$d(x^3y^2) = xd(x^2y^2) + x^2y^2dx = x^2y^2dx \neq 0 \in \Omega_{\mathbb{A}^2}|_X .$$

2.2. **Definition and first examples.** We come now to the definition of a d-critical locus from [Joy13]. A d-critical locus is a scheme that is locally of the form $\mathrm{crit}(f)$ with a compatibility condition given by a section of the sheaf $\mathcal{S}_X^0$. Depending on the situation, we apply the theory using the analytic or the Zariski topology, so we let X be a scheme or a complex analytic space.

Definition 2.13 ([Joy13, Definition 2.5]). A structure of a *d-critical locus* on X is defined by a section $s \in H^0(X, \mathcal{S}_X^0)$ satisfying the condition that for each $x \in X$ there is an open neighborhood $U \subset X$ of x and a closed embedding $U \subset S$ into a smooth scheme S such that there is $f \in H^0(S, \mathcal{O}_S)$ with the properties

$$s|_U = \bar{f} \in H^0(U, \mathcal{O}_S/\mathscr{I}^2) \qquad \text{and} \qquad U = \mathrm{crit}(f) \subset S.$$

We call (U, S, f) a *critical chart* around x for the d-critical locus (X, s).

An *embedding of critical charts* $\iota\colon (U,S,f) \hookrightarrow (V,T,g)$ is a locally closed embedding $\iota\colon S \hookrightarrow T$ such that $g \circ \iota = f$ and $\iota|_U : U \hookrightarrow V \subset X$ is the inclusion in X. A *subchart* is an open embedding of critical charts. If (U,S,f) and (V,T,g) are critical charts, then any étale morphism $\iota\colon S \to T$ such that $g \circ \iota = f$ restricts to a morphism $\iota|_U : U \to V$. If $\iota|_U$ is the inclusion in X we write $\iota\colon (U,S,f) \to (V,T,g)$ and call ι an *étale morphism of critical charts*.

A *morphism of d-critical loci* $\Phi\colon (X,s) \to (Y,t)$ is a morphism $\Phi\colon X \to Y$ such that the induced map (cf. Lemma 2.6(iv)) $\Phi^{-1}\mathcal{S}_Y \longrightarrow \mathcal{S}_X$ maps t to s. With this notion, d-critical loci form a category.

Remark 2.14. The theory also works with $s \in H^0(X, \mathcal{S}_X)$. However, choosing $s \in H^0(X, \mathcal{S}_X^0)$ has the practical advantage that, with the above notations, it enforces $f|_{U_{\mathrm{red}}} = 0 \in H^0(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$.

Example 2.15. Let us apply the definition to the zero-dimensional example $X = \operatorname{Spec} \mathbb{C}[x]/(x^n) \subset \mathbb{A}^1$. For this, consider the exact sequence of $\mathbb{C}$-vector spaces

$$\begin{array}{ccccc} 0 \longrightarrow \mathcal{S}_X \longrightarrow & \mathbb{C}[x]/(x^{2n}) & \xrightarrow{d} & \mathbb{C}[x]/(x^n)\langle dx\rangle \\ & x^i & \longmapsto & ix^{i-1}. \end{array}$$

We find $\mathcal{S}_X^0 = (x^{n+1}) \subset \mathbb{C}[x]/(x^{2n})$. Note that this is an example of Corollary 2.9 and indeed $\mathbb{L}_X$ is given by

$$\Big[(x^n)/(x^{2n}) \xrightarrow{d} \mathbb{C}[x]/(x^n)\langle dx\rangle\Big]$$

hence $\mathcal{S}_X^0 \cong \mathcal{H}^{-1}(\mathbb{L}_X)$.
Next, we wonder which $f \in \mathbb{C}[x]$ actually define a d-critical structure on X. This is the case if and only if there is an open subset $U \subset \mathbb{A}^1$ such that $X = V(df) \cap U$. Hence, we need that $f = a_{n+1}x^{n+1} + \dots$ with $a_{n+1} \neq 0$ and this gives $s \in (x^{n+1}) \setminus (x^{n+2})$.

Example 2.16. Assume that X is smooth. Then $\mathbb{M}_X^0 \cong [0 \to \Omega_X]$ and thus $\mathcal{S}_X^0 = 0$. Therefore, X has a unique structure of a d-critical locus given by $(X, 0)$.

2.3. **Critical charts.** Let (X, s) be a d-critical locus. In [Joy13, 2.3] Joyce studies how to manipulate critical charts on (X, s). His major results are that any two critical charts can locally be embedded into a third one and that any embedding of critical charts can locally be modified into an étale morphism of a certain standard form.

Proposition 2.17 ([Joy13, Theorem 2.20]). *Let (U, S, f) and (V, T, g) be two critical charts on (X, s). For every $x \in U \cap V$ there exist subcharts $(U', S', f') \subset (U, S, f)$ and $(V', T', g') \subset (V, T, g)$ around x and another critical chart (W, R, h) together with embeddings of critical charts $(U', S', f') \hookrightarrow (W, R, h)$ and $(V', T', g') \hookrightarrow (W, R, h)$.*

□

Proposition 2.18 ([Joy13, Proposition 2.22]). *Let (X, s) be an analytic critical locus and let $(U, S, f) \hookrightarrow (V, T, g)$ be an embedding of critical charts on (X, s). For every $x \in U$ there exists an open neighborhood $T' \subset T$ and holomorphic maps $\alpha \colon T' \to S, \beta \colon T' \to \mathbb{C}^n$, where $n = \dim T - \dim S$ such that $\alpha \times \beta \colon (V \cap T', T', g|_{V'}) \to (U, S \times \mathbb{C}^n, f \boxplus z_1^2 + \dots + z_n^2)$ is an étale morphism of critical charts, i.e. $g|_{T'} = f \circ \alpha + (z_1^2 + \dots + z_n^2) \circ \beta$. Furthermore, $\alpha \times \beta|_{S'} : S' := S \cap T' \hookrightarrow S \times \mathbb{C}^n$ is the inclusion of $S' \times \{0\}$.*

□

Remark 2.19. There is an algebraic analogue of this Proposition ([Joy13, Proposition 2.24]).

These propositions are the key for working with d-critical loci. Roughly speaking, when constructing a geometric object on (X, s) we only have to care about

(1) étale morphisms of critical charts and
(2) the embedding $(U, S, f) \hookrightarrow (U, S \times \mathbb{C}^n, f \boxplus z_1^2 + \dots + z_n^2)$.

In [KL16] Kiem and Li present another approach to (analytic) d-critical loci. They make use of the above Propositions in order to fix a well-behaved set of critical charts from the beginning on. More precisely, they prove the following equivalent definition of d-critical loci.

Lemma 2.20 ([KL16, Proposition 1.44]). *Let X be a complex analytic space. Then X has a d-critical structure if and only if there exists an open covering $X = \bigcup_\alpha X_\alpha$ and complex manifolds V_α together with a closed embedding $X_\alpha \subset V_\alpha$ and holomorphic functions $f_\alpha \colon V_\alpha \to \mathbb{C}$ satisfying*

(1) $X_\alpha = \mathrm{crit}(f_\alpha) \subset V_\alpha$*;*
(2) *for each pair of indices* (α, β) *and* $X_{\alpha\beta} = X_\alpha \cap X_\beta$ *there is an open neighborhood* $V_{\alpha\beta}$ *(resp.* $V_{\beta\alpha}$*) of* $X_{\alpha\beta}$ *in* V_α *(resp.* V_β*) and a biholomorphic map* $\varphi_{\alpha\beta} \colon V_{\alpha\beta} \to V_{\beta\alpha}$ *making the following diagram commute*

$$\begin{tikzcd}
& X_{\alpha\beta} \arrow[dl, hook'] \arrow[dr, hook] & \\
V_{\alpha\beta} \arrow[rr, "\varphi_{\alpha\beta}"] \arrow[dr, "f_\alpha\big|_{V_{\alpha\beta}}"'] & & V_{\beta\alpha} \arrow[dl, "f_\beta\big|_{V_{\beta\alpha}}"] \\
& \mathbb{C}; &
\end{tikzcd}$$

(3) $\varphi_{\alpha\beta}^{-1} = \varphi_{\beta\alpha}$ *and* $\varphi_{\alpha\alpha} = \mathrm{id}_{V_\alpha}$*.*

In this situation, we find for all $X_{\alpha\beta\gamma} := X_\alpha \cap X_\beta \cap X_\gamma$ an open neighborhood $V_{\alpha\beta\gamma}$ in V_α such that

$$\varphi_{\alpha\beta\gamma} := \varphi_{\alpha\beta} \circ \varphi_{\gamma\alpha} \circ \varphi_{\beta\gamma} \colon V_{\alpha\beta\gamma} \to V_\alpha \tag{2.7}$$

is biholomorphic onto its image ([KL16, Remark 1.7]). Then $\varphi_{\alpha\beta\gamma}$ is the identity on $X_{\alpha\beta\gamma}$, yet in general not on all of $V_{\alpha\beta\gamma}$. This is the reason why the gluing of objects (as for example the symmetric obstruction theories from the consecutive chapter) over the intersection of critical charts is troublesome.

3. Symmetric obstruction theories on d-critical loci

We are going to explain the concept of symmetric obstructions theories as introduced in [BF05]. We will see that a symmetric obstruction theory is naturally defined in the situation of Lagrangian intersections and locally defined on any d-critical locus. For a discussion of the relation between the category of d-critical loci and the category of schemes with (symmetric) obstruction theory we refer to [Joy13, Examples 2.16 and 2.17].

3.1. **Symmetric obstruction theories.** Let X be a scheme. We denote by $\mathbb{L}_X$ the truncated cotangent complex as in (2.1). If $\mathbb{E} \in \mathrm{D}^b(X)$ is a bounded complex, we can define

$$\mathbb{E}^\vee := R\mathcal{H}om(\mathbb{E}, \mathcal{O}_X) \in \mathrm{D}^b(X)$$

as well as $\mathbb{E}^{\vee\vee} = R\mathcal{H}om(\mathbb{E}^\vee, \mathcal{O}_X) \in \mathrm{D}^b(X)$. There is a natural map $\mathbb{E} \to \mathbb{E}^{\vee\vee}$. If $\mathbb{E}$ is given by a complex $[\mathcal{E}^{-1} \xrightarrow{\alpha} \mathcal{E}^0]$ of locally free sheaves, then, following our sign conventions, $\mathbb{E}^\vee = [(\mathcal{E}^0)^\vee \xrightarrow{-\alpha^\vee} (\mathcal{E}^{-1})^\vee]$ in degree 0 and 1, such that we get $\mathbb{E}^\vee[1] = [(\mathcal{E}^0)^\vee \xrightarrow{\alpha^\vee} (\mathcal{E}^{-1})^\vee]$ again in degree -1 and 0. We start with a technical result.

Lemma 3.1. *Let* $\mathbb{E} \in \mathrm{D}(X)$ *be such that* $\mathcal{H}^i(\mathbb{E}) = 0$ *if* $i \notin \{0, 1\}$*. Then there is an isomorphism*

$$\mathcal{H}^{-1}(\mathbb{E}^\vee) \xrightarrow{\sim} \mathcal{H}^1(\mathbb{E})^\vee = \mathcal{H}om(\mathcal{H}^1(\mathbb{E}), \mathcal{O}_X).$$

Proof. We are looking for an isomorphism

$$\mathcal{H}^{-1}(R\mathcal{H}om(\mathbb{E},\mathcal{O}_X)) = \mathcal{E}xt^{-1}(\mathbb{E},\mathcal{O}_X) \longrightarrow \mathcal{H}^1(\mathbb{E})^\vee = \mathcal{E}xt^0(\mathcal{H}^1(\mathbb{E}),\mathcal{O}_X).$$

Consider the spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p(\mathcal{H}^{-q}(\mathbb{E}),\mathcal{O}_X) \Rightarrow E^{p+q} = \mathcal{E}xt^{p+q}(\mathbb{E},\mathcal{O}_X).$$

By assumption, we have $E_2^{p,q} = 0$ if $p < 0$ or $-q \notin \{0,1\}$ and hence the only non-zero entry on the diagonal $p+q=-1$ occurs when $p=0$. Thus the spectral sequence degenerates at the E_2 page and yields the desired isomorphism

$$E^{-1} = \mathcal{E}xt^{-1}(\mathbb{E},\mathcal{O}_X) \xrightarrow{\sim} E_2^{0,-1} = \mathcal{E}xt^0(\mathcal{H}^{-1}(\mathbb{E}),\mathcal{O}_X).$$

□

Corollary 3.2. *Let $\mathbb{E} \in \mathrm{D}(X)$ be a perfect complex such that $\mathcal{H}^i(\mathbb{E}) = 0$ if $i \notin \{-1,0\}$. Then there is an isomorphism*

$$\mathcal{H}^{-1}(\mathbb{E}) \xrightarrow{\sim} \mathcal{H}^1(\mathbb{E}^\vee)^\vee.$$

Proof. Since $\mathbb{E}$ is perfect the natural map $\mathbb{E} \to \mathbb{E}^{\vee\vee}$ is an isomorphism (cf. [St17, Tag 07VI]) and thus $\mathcal{H}^{-1}(\mathbb{E}) \xrightarrow{\sim} \mathcal{H}^{-1}(\mathbb{E}^{\vee\vee})$. The corollary follows after postcomposing with the isomorphism $\mathcal{H}^{-1}(\mathbb{E}^{\vee\vee}) \longrightarrow \mathcal{H}^1(\mathbb{E}^\vee)^\vee$ from Lemma 3.1. □

This result applies in particular to the following type of complexes.

Definition 3.3. We say that $\mathbb{E} \in \mathrm{D}(X)$ is *perfect of amplitude in* $[-1,0]$ if $\mathbb{E}$ is locally isomorphic to a complex $[\mathcal{E}^{-1} \to \mathcal{E}^0]$ of locally free sheaves.

Definition 3.4 ([BF05, Section 1.3])**.** A *perfect obstruction theory* for X is tuple $(\mathbb{E},\phi)$, where $\mathbb{E}$ is perfect of amplitude in $[-1,0]$ and $\phi\colon \mathbb{E} \to \mathbb{L}_X$ is a morphism in $\mathrm{D}(X)$ such that

(1) $\mathcal{H}^{-1}(\phi)\colon \mathcal{H}^{-1}(\mathbb{E}) \to \mathcal{H}^{-1}(\mathbb{L}_X)$ is surjective and
(2) $\mathcal{H}^0(\phi)\colon \mathcal{H}^0(\mathbb{E}) \to \mathcal{H}^0(\mathbb{L}_X) \cong \Omega_X$ is an isomorphism.

A *symmetric obstruction theory* is a triple $(\mathbb{E},\phi,\theta)$, where $(\mathbb{E},\phi)$ is a perfect obstruction theory and $\theta\colon \mathbb{E}^\vee[1] \to \mathbb{E}$ is a non-degenerate symmetric bilinear form, i.e. an isomorphism in $\mathrm{D}(X)$ such that $\theta^\vee[1] = \theta$.

An *isomorphism of perfect obstruction theories* $\Phi\colon (\mathbb{E}_1,\phi_1) \to (\mathbb{E}_2,\phi_2)$ is given by an isomorphism $\Phi\colon \mathbb{E}_1 \to \mathbb{E}_2$ over $\mathbb{L}_X$ in $\mathrm{D}(X)$. Moreover, Φ is an *isomorphism of symmetric obstruction theories* $(\mathbb{E}_1,\phi_1,\theta_1) \to (\mathbb{E}_2,\phi_2,\theta_2)$ if it is an isometry $(\mathbb{E}_1,\theta_1) \to (\mathbb{E}_2,\theta_2)$, i.e. if

$$\begin{tikzcd} \mathbb{E}_1^\vee[1] \arrow[d, "\theta_1"'] & \mathbb{E}_2^\vee[1] \arrow[l, "\Phi^\vee[1]"'] \arrow[d, "\theta_2"] \\ \mathbb{E}_1 \arrow[r, "\Phi"] & \mathbb{E}_2 \end{tikzcd}$$

commutes.

Example 3.5. Let S be any scheme containing two smooth and closed subschemes Y, Z with ideal sheaves $\mathscr{I}_{YS}, \mathscr{I}_{ZS}$. Then the intersection $X = Y \cap Z$ carries a perfect obstruction theory

$$\mathbb{E} := \left[\Omega_S\,|_X \xrightarrow{(-\mathrm{res}_Y, \mathrm{res}_Z)} \Omega_Y\,|_X \oplus \Omega_Z\,|_X\right] \tag{3.1}$$

with $\mathbb{E} \xrightarrow{\phi} \mathbb{L}_X$ represented by

$$\begin{tikzcd}
\mathbb{E} & = & {[\Omega_S\,|_X} \arrow[r, "(-\mathrm{res},\mathrm{res})"] & {\Omega_Y\,|_X \oplus \Omega_Z\,|_X]} \\
\mathbb{F} \arrow[u, "\varphi_1"] \arrow[d, "\varphi_2"'] & := & {[\mathscr{I}_{YS}/\mathscr{I}_{YS}^2\,|_X} \arrow[u] \arrow[d, two heads] \arrow[r] & {\Omega_Z\,|_X]} \arrow[u, hook, "(0,1)"'] \arrow[d, equal] \\
\mathbb{L}_X & = & {[\mathscr{I}_{XZ}/\mathscr{I}_{XZ}^2} \arrow[r] & {\Omega_Z\,|_X].}
\end{tikzcd}$$

Here, the differential of $\mathbb{F}$ is the composition

$$\mathscr{I}_{YS}/\mathscr{I}_{YS}^2\,|_X \twoheadrightarrow \mathscr{I}_{XZ}/\mathscr{I}_{XZ}^2 \xrightarrow{d} \Omega_Z\,|_X\;.$$

This can be seen at follows. First of all the exact triangle

$$\mathbb{F} \xrightarrow{\varphi_1} \mathbb{E} \longrightarrow [\Omega_Y\,|_X = \Omega_Y\,|_X]$$

implies that $\varphi_1 \colon \mathbb{F} \to \mathbb{E}$ is a quasi-isomorphism. Moreover, we see immediately that $\mathcal{H}^0(\varphi_2)$ is an isomorphism and $\mathcal{H}^{-1}(\varphi_2)$ is surjective.

Let us remark that interchanging the roles of Y and Z yields isomorphic obstruction theories.

Remark 3.6. In Proposition 6.1 we will see that if S is a symplectic manifold and $Y, Z \subset S$ are Lagrangian submanifolds, then the above Example 3.5 can be enhanced by an isometry and thus turned into a symmetric obstruction theory.

Lemma 3.7 ([BF05, Corollary 1.15])**.** *Let* $(\mathbb{E}, \phi, \theta)$ *be a symmetric obstruction theory. Then*

$$\mathcal{H}^{-1}(\mathbb{E}) \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) =: \mathcal{T}_X$$

Proof. In Corollary 3.2 we showed that $\mathcal{H}^{-1}(\mathbb{E}) = \mathcal{H}^1(\mathbb{E}^\vee)^\vee$. However, this is all we need to show, since

$$\mathcal{H}^1(\mathbb{E}^\vee) = \mathcal{H}^0(\mathbb{E}^\vee[1]) \cong \mathcal{H}^0(\mathbb{E}) \cong \Omega_X$$

by symmetry of the obstruction theory. $\square$

Remark 3.8. Let $\mathbb{E} = [\mathcal{E}^{-1} \to \mathcal{E}^0] \to \mathbb{L}_X$ be a symmetric obstruction theory on X. Then, if $\mathcal{H}^{-1}(\mathbb{E}) \cong \Omega_X^\vee$ and $\mathcal{H}^0(\mathbb{E}) \cong \Omega_X$ admit a finite resolution by vector bundles their determinants are well-defined and we have

$$\begin{aligned}
\det \mathbb{E} &= (\det \mathcal{E}^{-1})^\vee \otimes \det \mathcal{E}^0 \cong (\det \mathcal{H}^{-1}(\mathbb{E}))^\vee \otimes \det \mathcal{H}^0(\mathbb{E}) \\
&\overset{3.7}{=} \det(\Omega_X^\vee)^\vee \otimes \det \Omega_X.
\end{aligned}$$

In particular, in this case $\det \mathbb{E}$ is independent of the obstruction theory. For example, if X is smooth we find

$$\det \mathbb{E} = \omega_X^{\otimes 2}. \tag{3.2}$$

The same is true if X is normal and Gorenstein. In this situation ω_X is a line bundle. Moreover, any line bundle on X is uniquely determined by its restriction to X_{reg}, i.e. for every line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ the natural morphism

$$\mathcal{L} \longrightarrow j_* j^* \mathcal{L}$$

is an isomorphism, where $j\colon X_{\mathrm{reg}} \hookrightarrow X$ is the inclusion. This can be seen as follows. As X is normal, we have $\mathrm{codim}(X_{\mathrm{sing}} \subset X) \geq 2$ and therefore the local cohomology sheaves $\underline{H}^k_{X_{\mathrm{sing}}}(\mathcal{L})$ are trivial for $k = 0, 1$ by [SGA2, VII, Corollaire 1.4]. Here, we use that any line bundle is a coherent Cohen–Macauley module. Now, there is an exact sequence

$$0 \to \underline{H}^0_{X_{\mathrm{sing}}}(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow j_* j^* \mathcal{L} \longrightarrow \underline{H}^1_{X_{\mathrm{sing}}}(\mathcal{L}) \to 0$$

(see [SGA2, I, Corollaire 2.11]), which proves the claim. In particular, we have $\omega_X \cong j_* \omega_{X_{\mathrm{reg}}}$ and also

$$\det \mathbb{E} \cong j_* j^* \det \mathbb{E} \cong j_* \det(\mathbb{E}\,\big|_{X_{\mathrm{reg}}}) \cong j_* \omega^{\otimes 2}_{X_{\mathrm{reg}}} \cong \omega_X^{\otimes 2}$$

for every symmetric obstruction theory on $\mathbb{E}$ on X. We conclude this discussion with

Question 3.9. Does $\det \mathbb{E}$ only depend on X? In other words, if $\mathbb{E}$ and $\mathbb{E}'$ are two symmetric obstruction theories on X. Is there a (canonical) isomorphism

$$\det \mathbb{E} \cong \det \mathbb{E}'?$$

3.2. **Symmetric obstruction theories on d-critical loci.** The reason why we are interested in symmetric obstruction theories is that every d-critical locus carries locally a symmetric obstruction theory.

Let M be a smooth scheme and $f\colon M \to \mathbb{A}^1$ a regular function with critical locus X. We consider $df \in H^0(M, \Omega_M)$ as a morphism $\mathcal{T}_M \to \mathcal{O}_M$. By definition, its image is the ideal sheaf $\mathscr{I}_{XM}$ defining X. Therefore, we obtain $df\colon \mathcal{T}_M\,|_X \twoheadrightarrow \mathscr{I}_{XM}/\mathscr{I}^2_{XM}$. A local computation shows that the composition

$$\mathcal{T}_M\,|_X \xrightarrow{df} \mathscr{I}_{XM}/\mathscr{I}^2_{XM} \xrightarrow{d} \Omega_M\,|_X$$

is just the Hessian $\partial^2 f$.

Lemma 3.10. *With the above notations $X = \mathrm{crit}(f)$ carries a symmetric obstruction theory $(\mathbb{E}_{\partial^2 f}, \psi, \theta)$. It is given by*

$$\mathbb{E}_{\partial^2 f} := \left[\mathcal{T}_M\,|_X \xrightarrow{\partial^2 f} \Omega_M\,|_X\right],$$

the morphism of complexes

$$\begin{array}{ccc} \mathbb{E}_{\partial^2 f} & = & [\mathcal{T}_M\,|_X \xrightarrow{\partial^2 f} \Omega_M\,|_X] \\ \downarrow{\scriptstyle \psi} & & \quad\downarrow{\scriptstyle df} \qquad\qquad \| \\ \mathbb{L}_X & = & [\mathscr{I}_{XM}/\mathscr{I}^2_{XM} \xrightarrow{d} \Omega_M\,|_X] \end{array}$$

and the isometry $\theta := \mathrm{id}_{\mathbb{E}_{\partial^2 f}} : \mathbb{E}^\vee_{\partial^2 f}[1] \to \mathbb{E}_{\partial^2 f}$.

Proof. Again, we see immediately that $(\mathbb{E}_{\partial^2 f}, \psi)$ is a perfect obstruction theory. Moreover, the symmetry of the Hessian yields $(\partial^2 f)^\vee[1] = \partial^2 f$ so that $\mathbb{E}^\vee_{\partial^2 f}[1] \cong \mathbb{E}_{\partial^2 f}$ and after this identification the identity serves indeed as a non-degenerate bilinear form. □

It turns out that these local obstruction theories are in general not compatible and thus do not glue to a global symmetric obstruction theory on (X, s). For instance, in [Joy13, Example 2.17] one finds an example of a smooth scheme X admitting two global critical charts $(X, X, 0)$ and (X, U, f). However, in this example $\mathbb{E}_0$ is not isomorphic to $\mathbb{E}_{\partial^2 f}$. Nevertheless, because X is smooth we have $\mathrm{Ext}^2(\Omega_U, \mathcal{T}_U) = 0$ for any affine open subscheme $U \subset X$ and therefore, $\mathbb{E}_{\partial^2 f}|_U \cong \mathbb{E}_0|_U$. We do not know, whether it is in general (locally) possible to find a distinguished symmetric obstruction theory on (X, s). At least, if we restrict ourselves to a set of charts as in Lemma 2.20 this can not happen.

Remark 3.11. Let $(X_\alpha, V_\alpha, f_\alpha)$ as in Lemma 2.20. Then the biholomorphism $\varphi_{\alpha\beta} \colon V_{\alpha\beta} \to V_{\beta\alpha}$ is an isomorphism. Moreover it induces

$$d\varphi_{\alpha\beta}\big|_{X_{\alpha\beta}} : \Omega_{V_\alpha}\big|_{X_{\alpha\beta}} = \Omega_{V_{\alpha\beta}}\big|_{X_{\alpha\beta}} \xrightarrow{\sim} \Omega_{V_{\beta\alpha}}\big|_{X_{\alpha\beta}} = \Omega_{V_\beta}\big|_{X_{\alpha\beta}}$$

so that we can define an isomorphism of symmetric obstruction theories

$$\begin{tikzcd}
\mathbb{E}_{\partial^2 f_\alpha} \arrow[d] & = & {[\mathcal{T}_{V_\alpha}\big|_{X_{\alpha\beta}}} \arrow[r, "\partial^2 f_\alpha"] \arrow[d] & {\Omega_{V_\alpha}\big|_{X_{\alpha\beta}}]} \arrow[d] \\
\mathbb{E}_{\partial^2 f_\beta} & = & {[\mathcal{T}_{V_\beta}\big|_{X_{\alpha\beta}}} \arrow[r, "\partial^2 f_\beta"] & {\Omega_{V_\beta}\big|_{X_{\alpha\beta}}].}
\end{tikzcd} \tag{3.3}$$

Here, the vertical arrows are given by $(d\varphi^{-1}_{\alpha\beta}\big|_{X_{\alpha\beta}})^\vee$ and $d\varphi_{\alpha\beta}\big|_{X_{\alpha\beta}}$, respectively. Commutativity of (3.3) means

$$d\varphi_{\alpha\beta}\big|_{X_{\alpha\beta}} \circ \partial^2 f_\alpha \circ (d\varphi_{\alpha\beta}\big|_{X_{\alpha\beta}})^\vee = \partial^2 f_\beta$$

which follows from $f_\alpha = f_\beta \circ \varphi_{\alpha\beta}$. In particular, we see that fixing an atlas as in Lemma 2.20 provides us with a distinguished symmetric obstruction theory in the neighborhood of any point $x \in X$.

3.3. **Canonical bundle.** Let (X, s) be a d-critical locus. For each critical chart (U, S, f) we consider the line bundle $\omega_S|_U$. One could hope, that it is possible to glue these in order to obtain a 'canonical bundle' associated to (X, s). However, this is not the case (see Example 3.18). What is possible instead, is to glue the square $\omega_S^{\otimes 2}|_{U_{\mathrm{red}}} \cong \det \mathbb{E}_{\partial^2 f}|_{U_{\mathrm{red}}}$ to a line bundle $\mathcal{K}_{X,s}$ on X_{red}. As discussed in Section 2.3 we have to define gluing isomorphisms for every étale morphism of critical charts and for the embedding $(U, S, f) \hookrightarrow (U, S \times \mathbb{C}^n, f \boxplus z_1^2 + \ldots + z_n^2)$. In the latter situation we will identify $\omega_{S \times \mathbb{C}^n}|_U$ with $\omega_S|_U$ without mentioning it explicitly.

Let $\iota \colon (U, S, f) \to (V, T, g)$ be an étale morphism of critical charts (Definition 2.13). Recall that this signifies among other things that $\iota|_U : U \hookrightarrow V$

is just the inclusion in X. The natural morphism $d\iota\colon \iota^*\Omega_T \xrightarrow{\sim} \Omega_S$ induces an isomorphism

$$\xi_\iota^{\otimes 2} = \det(d\iota)^{\otimes 2}\colon\ \det \mathbb{E}_{\partial^2 g}\,|_U = \omega_T^{\otimes 2}\,|_U \longrightarrow \det \mathbb{E}_{\partial^2 f}\,|_U = \omega_S^{\otimes 2}\,|_U\,.$$

This shall be the gluing datum for $\mathcal{K}_{X,s}$.

Proposition 3.12 ([Joy13, Theorem 2.28] or [KL16, Remark 1.18])**.** *There exists a unique line bundle $\mathcal{K}_{X,s}$ on X_{red} such that for each critical chart (U,S,f) there is an isomorphism*

$$\lambda_{(U,S,f)}\colon \mathcal{K}_{X,s}\,|_{U_{\mathrm{red}}} \xrightarrow{\sim} \det \mathbb{E}_{\partial^2 f}\,|_{U_{\mathrm{red}}}$$

satisfying for each étale morphism of critical charts $\iota\colon (U,S,f) \to (V,T,g)$ the equality

$$\lambda_{(U,S,f)} = \xi_\iota^{\otimes 2}\,|_{U_{\mathrm{red}}} \circ \lambda_{(V,T,g)}\,|_{U_{\mathrm{red}}}\,.$$

□

We will not give the proof here, but at least point out its most important step. One has to show that $\xi_\iota^{\otimes 2}$ defines a gluing datum. This holds only true on the reduced critical locus X_{red} and relies on the following

Lemma 3.13 ([KL16, Lemma 1.16] or [BBDJS, Corollary 3.2])**.** *Let S be a smooth scheme and $f\colon S \to \mathbb{C}$ a regular function. Set $X = \mathrm{crit}(f)$ and assume that there is an étale morphism $\iota\colon S \to S$ such that the following diagram commutes*

$$\begin{tikzcd} & X \arrow[dl, hook'] \arrow[dr, hook] & \\ S \arrow[rr, "\iota"] \arrow[dr, "f"'] & & S \arrow[dl, "f"] \\ & \mathbb{C}. & \end{tikzcd}$$

Then

$$\det(d\iota)\,|_{X_{\mathrm{red}}}\ \colon \iota^*\omega_S\,|_{X_{\mathrm{red}}} = \omega_S\,|_{X_{\mathrm{red}}} \longrightarrow \omega_S\,|_{X_{\mathrm{red}}}$$

is locally constant with values in $\{\pm 1\}$.

Proof. Consider the commutative diagram with exact lines

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T}_X \arrow[r] \arrow[d, equal] & \mathcal{T}_S\,|_X \arrow[r, "\partial^2 f"] & \Omega_S\,|_X \arrow[r] \arrow[d, "d\iota"] & \Omega_X \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{T}_X \arrow[r] & \mathcal{T}_S\,|_X \arrow[r, "\partial^2 f"] \arrow[u, "(d\iota)^\vee"] & \Omega_S\,|_X \arrow[r] & \Omega_X \arrow[r] & 0. \end{tikzcd} \tag{3.4}$$

If $\mathcal{T}_X$ and Ω_X admit a finite resolution by locally free sheaves such that their determinants satisfy $\det(\mathcal{T}_X)^\vee \cong \det(\Omega_X)$, the Lemma will follow from functoriality of the determinant. For the general proof let $x \in X$. We will show that $(\det \iota)^2(x) = 1$ in the residue field $\kappa(x)$. For this aim, we choose a decomposition $\Omega_S|_X(x) \cong \Omega_X(x) \oplus W$ so that the commutative diagram of vector spaces

$$\begin{tikzcd}
\mathcal{T}_X(x) \arrow[r] \arrow[d, equal] & \mathcal{T}_S|_X(x) \arrow[r, "\partial^2 f(x)"] & \mathcal{T}_S|_X(x) \arrow[r] \arrow[d, "d\iota(x)"] & \Omega_X(x) \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{T}_X(x) \arrow[r] & \Omega_S|_X(x) \arrow[r, "\partial^2 f(x)"] \arrow[u, "(d\iota(x))^\vee"] & \Omega_S|_X(x) \arrow[r] & \Omega_X(x) \arrow[r] & 0
\end{tikzcd}$$

takes the following form

$$\begin{tikzcd}
\mathcal{T}_X(x) \arrow[r] \arrow[d, equal] & \Omega_X(x)^\vee \oplus W^\vee \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}"] & \Omega_X(x) \oplus W \arrow[r] \arrow[d, "\begin{pmatrix} 1 & * \\ 0 & A \end{pmatrix}"] & \Omega_X(x) \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{T}_X(x) \arrow[r] & \Omega_X(x)^\vee \oplus W^\vee \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix}"] \arrow[u, "\begin{pmatrix} 1 & 0 \\ * & {}^tA \end{pmatrix}"] & \Omega_X(x) \oplus W \arrow[r] & \Omega_X(x) \arrow[r] & 0,
\end{tikzcd}$$

where H is some invertible matrix. It follows that $H =^t AHA$ and thus $\det A \in \{\pm 1\}$, which implies the claim. □

Remark 3.14 ([KL16, Proof of Proposition 1.15]). If one fixes charts $(X_\alpha, V_\alpha, f_\alpha)$, appropriate neighborhoods $V_{\alpha\beta}$ and biholomorphisms $\varphi_{\alpha\beta}\colon V_{\alpha\beta} \to V_{\beta\alpha}$ as in Lemma 2.20, the existence of $\mathcal{K}_{X,s}$ follows directly from Lemma 3.13. Namely, in this case we have to show that the isomorhisms

$$\xi_{\alpha\beta} := \det(d\varphi_{\alpha\beta})^{\otimes 2}\big|_{X_{\alpha\beta}} : \det(\mathbb{E}_{\partial^2 f_\alpha})\big|_{X_{\alpha\beta}} \to \det(\mathbb{E}_{\partial^2 f_\beta})\big|_{X_{\alpha\beta}}$$

from Remark 3.11 satisfy

$$\xi_{\beta\gamma} \circ \xi_{\alpha\beta} = \xi_{\alpha\gamma} \text{ on } (X_{\alpha\beta\gamma})_{\mathrm{red}}$$

which is equivalent to

$$\xi_{\gamma\alpha} \circ \xi_{\beta\gamma} \circ \xi_{\alpha\beta} = \det(d\varphi_{\gamma\alpha\beta})^2 = 1 \text{ on } (X_{\alpha\beta\gamma})_{\mathrm{red}},$$

where $\varphi_{\gamma\alpha\beta}$ is defined as in (2.7). And this follows after applying Lemma 3.13 with $S = \varphi_{\gamma\alpha\beta}(V_{\gamma\alpha\beta}) \cap V_{\gamma\alpha\beta}, \phi = \varphi_{\gamma\alpha\beta}$ and $f = f_\gamma$.

Definition 3.15. An *orientation* of (X, s) is the choice of a square root line bundle $\mathcal{L} = \mathcal{K}_{X,s}^{1/2}$ on X_{red} together with an isomorphism $\mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{K}_{X,s}$. We will suppress the choice of the latter in our notation.

Question 3.16. When is it possible to define $\mathcal{K}_{X,s}$ on X and not only on X_{red}? Of course, this is possible, if we are provided with a global symmetric obstruction theory, together with isomorphisms $\mathbb{E}|_U \xrightarrow{\sim} \mathbb{E}_{\partial^2 f}$ for every critical chart (U, S, f) satisfying the correct functoriality. On the other hand, as mentioned in the proof of Lemma 3.13, the question also has a positive answer, if $\det(\Omega_X)$ and $\det(\mathcal{T}_X)$ are well-defined and satisfy $\det(\mathcal{T}_X)^\vee \cong \det(\Omega_X)$. In this case, we saw in Remark 3.8 that $\det \mathbb{E} = \det(\Omega_X)^{\otimes 2}$ for every symmetric obstruction theory $\mathbb{E}$ on X. Hence, if we apply this to a covering of critical charts, we find $\mathcal{K}_{X,s} \cong \det(\Omega_X)^{\otimes 2}$.

Example 3.17. Assume that X is smooth. Then we have

$$\mathcal{K}_{X,0} \cong \omega_X^{\otimes 2}.$$

In particular, the d-critical locus $(X, 0)$ is canonically oriented.

Example 3.18. An example of a non-orientable d-critical locus X can be found in [Joy13, Example 2.39] or [KL16, Example 1.19]. In these examples, X is a projective line with a non-reduced point.

4. Perverse sheaves

We want to explain the existence of a perverse sheaf $\mathcal{P}^\bullet_{X,s}$ on any d-critical locus (X, s). First, let us review some basics about the perverse sheaf of vanishing cycles following [Di04].

4.1. **Perverse sheaves of vanishing cycles.** Let X be a topological space and $\mathrm{D}^b_c(\mathbf{Sh}(X))$ the essential image of the category of bounded complexes of constructible sheaves in $\mathrm{D}(\mathbf{Sh}(X))$. We denote by $\mathbf{Perv}(X)$ the subcategory of perverse sheaves, i.e. (cf. [BBD, Définition 2.1.2]) for any $\mathcal{F}^\bullet \in \mathrm{D}^b_c(\mathbf{Sh}(X))$ we have $\mathcal{F}^\bullet \in \mathbf{Perv}(X)$ if and only if for each stratification $\{X_\alpha\}$

(1) $\mathcal{H}^k(i_\alpha^* \mathcal{F}^\bullet) = 0$ for all $k > -\dim_{\mathbb{C}} X_\alpha$ and

(2) $\mathcal{H}^k(i_\alpha^! \mathcal{F}^\bullet) = 0$ for all $k < -\dim_{\mathbb{C}} X_\alpha$

for all α, where $i_\alpha \colon X_\alpha \hookrightarrow X$ denotes the inclusion. By a stratification we mean a finite partition of X into (non-empty) locally closed subsets, called strata. We assume that the closure of each stratum is the union of strata. As an immediate consequence we record

Remark 4.1. Let $\mathcal{F}^\bullet \in \mathbf{Perv}(X)$ then $\mathcal{H}^k(\mathcal{F}^\bullet) = 0$ if $k \notin [-\dim X, 0]$.

Proposition 4.2 ([Di04, Theorem 5.1.20]). *Let X be a complex analytic space of pure dimension n which is locally a complete intersection. Then the shifted constant sheaf $\underline{\mathbb{C}}_X[n]$ is a perverse sheaf.*

□

Remark 4.3. Let $\mathcal{P}^\bullet \in \mathbf{Perv}(X)$ and $\mathfrak{L}$ a local system on X. Then also $\mathcal{P}^\bullet \otimes \mathfrak{L} \in \mathbf{Perv}(X)$, because tensoring with a local system is exact. In particular, in the situation of the previous proposition, it follows that $\mathfrak{L}[n]$ is a perverse sheaf.

Next, we define the perverse sheaf of vanishing cycles as in [Di04, Section 4.2]. Let S be a complex analytic variety and $f \colon S \to \mathbb{C}$ an analytic map. We set $S_0 := f^{-1}(0)$ with inclusion $i \colon S_0 \hookrightarrow S$ and consider the commutative diagram

$$\begin{tikzcd}
\tilde{S} := S \times_\rho \widetilde{\mathbb{C}^*} \arrow[r] \arrow[d] \arrow[rr, bend left, "p"] & S \setminus S_0 \arrow[r, hook] \arrow[d] & S \arrow[d, "f"] \\
\widetilde{\mathbb{C}^*} \arrow[r] \arrow[rr, bend right, "\rho"'] & \mathbb{C}^* \arrow[r, hook] & \mathbb{C},
\end{tikzcd} \tag{4.1}$$

where $\widetilde{\mathbb{C}^*}$ is the universal cover of $\mathbb{C}^*$. More especially, $\widetilde{\mathbb{C}^*} = \mathbb{C}$ and $\rho(z) = \exp(2\pi i z)$. The *nearby cycle functor* $\psi_f \colon \mathrm{D}^b_c(S) \to \mathrm{D}^b_c(S_0)$ is defined by sending $\mathcal{F}^\bullet \in \mathrm{D}^b_c(S)$ to

$$\psi_f \mathcal{F}^\bullet := i^* Rp_* p^* \mathcal{F}^\bullet \in \mathrm{D}^b_c(S_0)$$

and the *vanishing cycle functor* $\phi_f \colon \mathrm{D}^b_c(S) \to \mathrm{D}^b_c(S_0)$ is defined such that there is an exact triangle in $\mathrm{D}^b_c(S_0)$

$$i^*\mathcal{F}^\bullet \longrightarrow \psi_f \mathcal{F}^\bullet \longrightarrow \phi_f \mathcal{F}^\bullet \longrightarrow i^*\mathcal{F}^\bullet[1],$$

where the first arrow comes from the adjunction map $\mathcal{F}^\bullet \to Rp_*p^*\mathcal{F}^\bullet$. We denote by ψ^p_f and ϕ^p_f the shifted functors $\psi_f[-1]$ and $\phi_f[-1]$, respectively. We need

Proposition 4.4 ([Di04, Theorem 5.2.21]). *The functors $\psi^p_f, \phi^p_f \colon \mathrm{D}^b_c(S) \to \mathrm{D}^b_c(S_0)$ preserve perversity. In particular, we have induced functors*

$$\psi^p_f, \phi^p_f \colon \mathbf{Perv}(S) \to \mathbf{Perv}(S_0).$$

$\square$

Proposition 4.5. *Let $\Phi \colon T \to S$ be a morphism between two analytic spaces that is smooth of relative dimension d and $f \colon S \to \mathbb{C}$. Write $g = f \circ \Phi$ and $\Phi_0 = \Phi|_{T_0} : T_0 \to S_0$. Then we have natural isomorphisms*

$$\Phi_0^* \circ \psi_f \xrightarrow{\sim} \psi_g \circ \Phi^* \quad \textit{and} \quad \Phi_0^* \circ \phi_f \xrightarrow{\sim} \phi_g \circ \Phi^*.$$

Proof. It suffices to prove the statement for the nearby cycle functor. Plugging in the definitions, we get

$$\begin{aligned}\Phi_0^* \circ \psi_f &= \Phi_0^* i_S^* Rp_{S*} p_S^* = i_T^* \Phi^* Rp_{S*} p_S^* \xrightarrow{\sim} i_T^* Rp_{T*} \tilde{\Phi}^* p_S^* = i_T^* Rp_{T*} p_T^* \Phi^* \\ &= \psi_g \circ \Phi^*.\end{aligned}$$

Here, we used $i_S \circ \Phi_0 = \Phi \circ i_T$ for the first, $\Phi^* = \Phi^![2d]$ and the base change isomorphism for the second and $\Phi \circ p_T = p_S \circ \tilde{\Phi}$ for the third equality, where $\tilde{\Phi} = id_{\widetilde{\mathbb{C}^*}} \times_\rho \Phi$. $\square$

Lemma 4.6. *Let S and $f \colon S \to \mathbb{C}$ as above. We denote by $f^n \colon S \to \mathbb{C}$ the morphism given by $f^n(s) = f(s)^n$. Then there is a natural isomorphism*

$$\psi_{f^n} \mathcal{F}^\bullet \cong \bigoplus_{k=1}^n \psi_f \mathcal{F}^\bullet$$

for every $\mathcal{F}^\bullet \in \mathrm{D}^b_c(S)$.

Proof. By definition, ψ_{f^n} is computed by means of the fibre product

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] \arrow[rr, bend left, "p"] & S \setminus S_0 \arrow[r] \arrow[d] & S \arrow[d, "f^n"] \\ \mathbb{C} \arrow[r, "\rho"] & \mathbb{C}^* \arrow[r, hook] & \mathbb{C}, \end{tikzcd}$$

where $\rho(z) = \exp(2\pi i z)$. We define $\mu_n, \nu_n \colon \mathbb{C} \to \mathbb{C}$ by $\mu_n(z) = nz$ and $\nu_n(z) = z^n$ and replace $\tilde{S}, p$ by $\tilde{S} \times_{\mu_n} \mathbb{C}, \mu_n^* p$, respectively. As μ_n is an isomorphism this does not affect the vanishing cycle but now, using $\rho \circ \mu_n = \nu_n \circ \rho$, we can express S as the double fibre product

$$\begin{tikzcd} \tilde{S} \arrow[r, "q"] \arrow[rr, bend left, "p"] \arrow[d] & \bigsqcup_{k=1}^n S \arrow[r, "\pi"] \arrow[d, "\sqcup \zeta^k f"] & S \arrow[d, "f^n"] \\ \mathbb{C} \arrow[r, "\rho"] & \mathbb{C} \arrow[r, "\nu_n"] & \mathbb{C}, \end{tikzcd}$$

where $\pi = \sqcup\, \mathrm{id}$ and ζ is a n-th root of unity. Let $i\colon S_0\colon \hookrightarrow S$ be the inclusion and $j = \sqcup i$ and $\pi_0 = \pi\,|_{\sqcup S_0}$. We find

$$\begin{aligned} \psi_{f^n} &\cong i^* Rp_* p^* = i^* R\pi_* Rq_* q^* \pi^* \cong R\pi_{0*} j^* Rq_* q^* \pi^* \\ &= R\pi_{0*} \psi_{\sqcup_{\zeta^k f}} \pi^* \cong \bigoplus_{k=1}^n \psi_f. \end{aligned}$$

using $i^* R\pi_* \cong R\pi_{0*} j^*$ by proper base change. □

We can compute the stalk cohomology of ψ_f or ϕ_f via the Milnor fibre (cf. [Di04, Proposition 4.2.2 and Example 4.2.6]). Recall that if f vanishes in $s \in S$, then this is the fibre of the Milnor-Lê fibration (cf. [Mi68, Theorem 4.8]) which is given by $MF_s = B_\delta(s) \cap f^{-1}(t)$ for some small $t, \delta > 0$ and $B_\delta(s)$ is an open ball of radius δ around s in S defined via embedding the germ (S, s) in an affine space. If $S = \mathbb{C}^{n+1}$ with $n \geq 1$ and f has an isolated singularity in s, then Milnor proved that MF_s has the homotopy type of a bouqet of n-dimensional spheres (cf. [Mi68, Theorem 6.5]). The number of these spheres is given by the Milnor number $\mu = \dim_{\mathbb{C}} \mathbb{C}[x_0, \dots, x_n]/J(f)$, where $J(f) = \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)$ is the ideal defining $\mathrm{crit}(f)$. Finally, we have for all $s \in S_0$ a natural isomorphism

$$\mathcal{H}^k(\psi_f\, \underline{\mathbb{C}}_S)_s \cong H^k(MF_s, \mathbb{C}) \tag{4.2}$$

and if S is smooth

$$\mathcal{H}^k(\phi_f\, \underline{\mathbb{C}}_S)_s \cong \tilde{H}^k(MF_s, \mathbb{C}), \tag{4.3}$$

where $\tilde{H}^k(MF_s, \mathbb{C})$ is the reduced cohomology of the Milnor fibre at s.

Now, we let $X = \mathrm{crit}(f)$, then $f\,|_X$ is locally constant as a map of topological spaces $f\colon X \to \mathbb{C}$ with finite image. For simplicity, we assume that $f\,|_X\, \equiv 0$ such that $X = \mathrm{Sing}(S_0)$ is the singular locus of the fibre S_0. We define the *perverse sheaf of vanishing cycles*

$$\mathcal{PV}^\bullet_{S,f} := \phi_f^p(\underline{\mathbb{C}}_S[\dim S]) = \phi_f\, \underline{\mathbb{C}}_S[\dim S - 1]. \tag{4.4}$$

It follows from Propositions 4.2 and 4.4 that $\mathcal{PV}^\bullet_{S,f} \in \mathrm{Perv}(S_0)$. Moreover, by (4.3) we have $\mathcal{H}^k(\mathcal{PV}^\bullet_{S,f})_s = \tilde{H}^{k+\dim S-1}(MF_s, \mathbb{C})$ and therefore:

Corollary 4.7. *The perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{S,f}$ is supported on $X = \mathrm{crit}(f)$. In particular, we can consider $\mathcal{PV}^\bullet_{S,f}$ as an element of* $\mathbf{Perv}(X)$.

□

Remark 4.8. More precisely, if $i\colon X \hookrightarrow S$ is a closed subspace, then there is an equivalence between the category of perverse sheaves on S supported on X and $\mathbf{Perv}(X)$ established by i^* with inverse Ri_*.

Remark 4.9 ([Di04, Cor. 6.1.18]). If S is a connected complex manifold and f non-constant, then $\operatorname{supp}(\phi_f \underline{\mathbb{C}}_S) = \operatorname{Sing}(S_0)$.

Remark 4.10. If f has several critical values, we set

$$\mathcal{PV}^\bullet_{S,f} := \bigoplus_{c \in f(X)} \phi^p_{f-c}(\underline{\mathbb{C}}_S[\dim S]).$$

This defines a pervese sheaf on X (cf. [Bu14, Definition 1.7]).

We will now compute some examples.

Example 4.11. Let S be any smooth sheme and $f = 0$. Applying the definition we find $\psi_f = 0$ and thus $\phi^p_f = \mathrm{id}_{\mathrm{D}^b_c(S)}$ yielding

$$\mathcal{PV}^\bullet_{S,f} \cong \underline{\mathbb{C}}_S[\dim S].$$

Example 4.12. Let $S = \operatorname{Spec} \mathbb{C}[x]$ and $f = x^{n+1}$. Combining Remark 4.1 and Corollary 4.7 we find that $\mathcal{PV}^\bullet_{S,f}$ is a complex concentrated in degree zero and supported on a point. Therefore, $\mathcal{PV}^\bullet_{S,f}$ is determined by

$$\mathcal{H}^0(\mathcal{PV}^\bullet_{S,f})_0 = \tilde{H}^0(MF_0, \mathbb{C}).$$

However, MF_0 is given by $n+1$ points. Hence,

$$\mathcal{PV}^\bullet_{S,f} \cong \underline{\mathbb{C}}^n_0\,.$$

Example 4.13. Let $S = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ and $f = \sum_{i=1}^n x_i^2$. This time, we have to compute $\tilde{H}^{n-1}(MF_0)$ and MF_0 has the homotopy type of a $(n-1)$-sphere. Hence,

$$\mathcal{PV}^\bullet_{S,f} \cong \underline{\mathbb{C}}_0\,.$$

Example 4.14. Let $S = \operatorname{Spec} \mathbb{C}[x, y]$ and $f = x^n y^n$. Then

$$\psi_f \, \underline{\mathbb{C}}_S \cong \bigoplus_{i=1}^n \psi_{xy} \, \underline{\mathbb{C}}_S$$

by Lemma 4.6. We can compute $\psi_{xy} \, \underline{\mathbb{C}}_S$ as follows. Using the same arguments as above we find $\phi_{xy} \, \underline{\mathbb{C}}_S \cong \underline{\mathbb{C}}_0[-1]$ and thus $\psi_{xy} \, \underline{\mathbb{C}}_S$ fits into the exact triangle

$$\underline{\mathbb{C}}_{X_0} \longrightarrow \psi_{xy} \, \underline{\mathbb{C}}_S \longrightarrow \underline{\mathbb{C}}_0[-1],$$

where $X_0 = Z(xy) \subset S$. This allows to conclude that $\psi_{xy} \, \underline{\mathbb{C}}_S \cong [\underline{\mathbb{C}}_{X_0} \xrightarrow{d} \underline{\mathbb{C}}_0]$ in degrees 0 and 1. Moreover, using (4.2), i.e. $\mathcal{H}^1(\psi_{xy} \, \underline{\mathbb{C}}_S)_s \cong H^1(MF_s, \mathbb{C})$ we see that $d = 0$. Hence,

$$\psi_{xy} \, \underline{\mathbb{C}}_S \cong \underline{\mathbb{C}}_{X_0} \oplus \underline{\mathbb{C}}_0[-1]$$

yielding

$$\mathcal{PV}^\bullet_{S,f} \cong \underline{\mathbb{C}}^{n-1}_{X_0}[1] \oplus \underline{\mathbb{C}}^n_0\,.$$

4.2. **Local system corresponding to a torsion line bundle.** Before stating the construction of the perverse sheaf $\mathcal{P}^\bullet_{X,s}$ we need yet another technical paranthesis.

Let X be a scheme and $\mathcal{L}$ a line bundle on X such that $\alpha\colon \mathcal{L}\otimes\mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$. Then $\mathcal{O}_X\oplus\mathcal{L}$ is an $\mathcal{O}_X$-algebra via $(a,b)(a',b') = (aa'+\alpha(bb'), ab'+a'b)$ and we define $Y := \mathrm{Spec}_X(\mathcal{O}_X\oplus\mathcal{L})$. The structure map $f\colon Y\to X$ defines an étale covering of degree two such that $f_*\mathcal{O}_Y \cong \mathcal{O}_X\oplus\mathcal{L}$ and, more importantly for our purposes,

$$f_*\,\underline{\mathbb{C}}_Y \cong \underline{\mathbb{C}}_X \oplus \mathfrak{L},$$

for some rank one local system $\mathfrak{L}$ on X. This can be seen as follows. First of all, $f_*\,\underline{\mathbb{C}}_Y$ is a local system of rank 2 because f is finite and flat with 2-dimensional fibres. Now, we consider the natural morphism $\gamma\colon\,\underline{\mathbb{C}}_X \to f_*\,\underline{\mathbb{C}}_Y$, which is obviously injective. We claim that

$$\mathfrak{L} := \mathrm{coker}(\underline{\mathbb{C}}_X \xrightarrow{\gamma} f_*\,\underline{\mathbb{C}}_Y)$$

is a local system and satisfies $f_*\,\underline{\mathbb{C}}_Y \cong \underline{\mathbb{C}}_X\oplus\mathfrak{L}$. Indeed, a retract of γ is given explicitly given by mapping a section $s\in H^0(f^{-1}(U),\underline{\mathbb{C}}_Y)$ to the locally constant map $X\ni x\mapsto \frac{1}{2}(s(x_1)+s(x_2))$, where x_1 and x_2 are the two preimages of x under f. We call $\mathfrak{L}$ the *local system corresponding to* $\mathcal{L}$. If X is an analytic space Y, can either be constructed by gluing disjoint copies of X or as a closed subspace of the total space of the line bundle $\mathcal{L}$. By construction, $\mathfrak{L}$ satisfies

$$\mathfrak{L}\otimes\mathfrak{L}\cong\underline{\mathbb{C}}_X \quad\text{and}\quad \mathcal{L} = \mathfrak{L}\otimes_{\underline{\mathbb{C}}_X}\mathcal{O}_X.$$

To be more precise, we should say that $\mathfrak{L}$ is the local system corresponding to $(\mathcal{L},\alpha)$. However, if X is proper and reduced this makes no difference.

Lemma 4.15. *Let X be proper and reduced. Then the local system $\mathfrak{L}$ depends only on $\mathcal{L}$ and not on the isomorphism $\mathcal{L}\otimes\mathcal{L}\cong\mathcal{O}_X$.*

Proof. Consider the short exact sequence of groups

$$1\to\mathbb{Z}/2\mathbb{Z}\longrightarrow\mathbb{C}^*\xrightarrow{\nu_2}\mathbb{C}^*\to 1, \tag{4.5}$$

where ν_2 is given by $\nu_2(z)=z^2$. We denote the two-torsion part of $H^1(X,\mathcal{O}_X^*)$ by $H^1(X,\mathcal{O}_X^*)[2]$ and similarly for $H^1(X,\underline{\mathbb{C}}_X^*)$. Then (4.5) induces a commutative diagram with exact rows

$$\begin{tikzcd}
H^0(X,\underline{\mathbb{C}}_X^*) \arrow[r, "\nu_2"] \arrow[d] & H^0(X,\underline{\mathbb{C}}_X^*) \arrow[r] \arrow[d] & H^1(X,\mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(X,\underline{\mathbb{C}}_X^*)[2] \arrow[r] \arrow[d, two heads, "\alpha"] & 0 \\
H^0(X,\mathcal{O}_X^*) \arrow[r, "\nu_2"] & H^0(X,\mathcal{O}_X^*) \arrow[r] & H^1(X,\mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X^*)[2] \arrow[r] & 0.
\end{tikzcd}$$

Here, α is given by $\alpha(\mathfrak{L}) = \mathfrak{L}\otimes\mathcal{O}_X$. We want to see that α is an isomoprhism. First of all, we know from the above construction that α is surjective. The morphism $\nu_2\colon H^0(X,\underline{\mathbb{C}}_X^*)\to H^0(X,\underline{\mathbb{C}}_X^*)$ is surjective for any analytic space so that $H^1(X,\mathbb{Z}/2\mathbb{Z})\to H^1(X,\underline{\mathbb{C}}_X^*)[2]$ is an isomorphism. This implies that the surjectivity of $\nu_2\colon H^0(X,\mathcal{O}_X^*)\to H^0(X,\mathcal{O}_X^*)$ is equivalent to the injectivity of α and thus holds true if X is proper and reduced. $\square$

Remark 4.16. The same construction works for line bundles with n-torsion.

Remark 4.17. The construction is functorial in the following sense. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles with two-torsion given by the isomorphisms $\alpha_i\colon \mathcal{L}_i \otimes \mathcal{L}_i \xrightarrow{\sim} \mathcal{O}_X$. Any morphism $\varphi\colon \mathcal{L}_1 \to \mathcal{L}_2$ such that $\alpha_2 \circ (\varphi \otimes \varphi) = \alpha_1$ induces a morphism $\mathfrak{L}_1 \to \mathfrak{L}_2$ between the corresponding local systems.

If X is smooth, the Riemann–Hilbert correspondence ([De70, Théorème 2.17]) provides a second description of the local system $\mathfrak{L}$. Namely, there is a unique flat connection $\nabla\colon \mathcal{L} \to \mathcal{L} \otimes \Omega^1_X$ such that $\mathfrak{L} = \ker \nabla$. If X is moreover compact, Lemma 4.15 says that ∇ is the unique flat connection on $\mathcal{L}$ such that $\ker \nabla \otimes \ker \nabla \cong \underline{\mathbb{C}}_X$.

Lemma 4.18. *Let X be a compact Kähler manifold and $\mathcal{L}, \mathfrak{L}$ as above. Then there exists a hermitian metric h on X with corresponding Chern connection $\nabla_h = \nabla^{1,0} + \bar{\partial}$ such that*
$$\mathfrak{L} = \ker \nabla^{1,0}.$$

Proof. By Lemma 4.15 is suffices to show that there is a hermitian metric h such that
$$\ker \nabla^{1,0} \otimes \ker \nabla^{1,0} \cong \underline{\mathbb{C}}_X\,.$$
Under the Riemann–Hilbert correspondence, $\ker \nabla^{0,1} \otimes \ker \nabla^{0,1}$ corresponds to the pair $(\mathcal{L} \otimes \mathcal{L}, \nabla^{0,1} \otimes 1 + 1 \otimes \nabla^{0,1})$ and $\nabla_h \otimes 1 + 1 \otimes \nabla_h$ is the Chern connection to $h \otimes h$ on $\mathcal{L} \otimes \mathcal{L}$. Therefore, we have to show that there is a hermitian metric h on $\mathcal{L}$ such that $h \otimes h$ is the constant metric h_0 on $\mathcal{O}_X$ via the isomorphism $\alpha\colon \mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$. Let g be any hermitian metric on $\mathcal{L}$. Then $g \otimes g = e^f h_0$ via α for some real function f on X and $h := e^{-1/2f} g$ satisfies $h \otimes h = e^{-f} g \otimes g = h_0$ as desired. $\square$

4.3. **Perverse sheaves on d-critical loci.** If $X \subset S$ is of the form $\operatorname{crit}(f)$ for some $f \in H^0(S, \mathcal{O}_S)$, we defined in (4.4) the perverse sheaf of vanishing cycles $\mathcal{PV}^\bullet_{S,f}$ on X. Let (X, s) be a d-critical locus. Our goal is to define a perverse sheaf $\mathcal{P}^\bullet_{X,s}$ on (X, s) that is locally isomorphic to $\mathcal{PV}_{U,f}$. The following example shows that this will be more complicated than naively gluing the perverse sheaves $\mathcal{PV}_{U,f}$ over critical charts.

Example 4.19 ([BBDJS, Example 5.5]). Let $X = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$. We will define two different global critical charts (X, S, f) and (X, V, g) on X such that $\mathcal{PV}_{S,f}$ and $\mathcal{PV}_{T,g}$ are not isomorphic.

Let $S := X$ with $f = 0$ and $T := \operatorname{Spec}[x, x^{-1}, y]$ with $g\colon T \to \operatorname{Spec} \mathbb{C}[t]$ given by $g \mapsto x^k y^2$ for some $k \geq 0$. Then
$$\operatorname{crit}(g) = \operatorname{Spec} \mathbb{C}[x, x^{-1}, y]/(kx^{k-1}y^2, 2x^k y) \cong \operatorname{Spec} \mathbb{C}[x, x^{-1}] = X.$$
We find $\mathcal{PV}_{S,f} = \underline{\mathbb{C}}_X[1]$ by Example 4.11. However, we claim that
$$\mathcal{PV}_{T,g} \cong \begin{cases} \underline{\mathbb{C}}_X[1] & \text{if } k \equiv 0 \bmod 2 \\ \mathfrak{L}_X[1] & \text{if } k \equiv 1 \bmod 2. \end{cases}$$
Here, $\mathfrak{L}_X$ is the unique two-torsion local system on X, i.e. it corresponds to the representation $\pi_1(X) \cong \mathbb{Z} \to \mathbb{C}^*$ that is given by -1. This can be seen as follows. First of all, let us consider X and T as topological spaces, i.e. $X = \mathbb{C} \setminus \{0\}$ and $T = \mathbb{C} \setminus \{0\} \times \mathbb{C}$. If $k = 2n$ for some $n \in \mathbb{N}$, then $\Phi\colon T \to T$

given by $\Phi(x,y) = (x, x^n y)$ defines an isomorphism such that $g = \Phi \circ h$, where $h(x,y) = y^2$. Moreover, $\Phi|_X = \mathrm{id}_X$ and thus

$$\mathcal{PV}_{T,g} \cong \mathcal{PV}_{T,h} \cong \underline{\mathbb{C}}_X[1].$$

If $k = 2n+1$ we cover $T = (U_0 \times \mathbb{C}) \cup (U_1 \times \mathbb{C})$, where $U_0 = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $U_1 = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. For $i = 0, 1$ the isomorphism

$$\begin{array}{rccc} \Phi_i\colon & U_i \times \mathbb{C} & \xrightarrow{\sim} & U_i \times \mathbb{C} \\ & (x,y) & \longmapsto & (x, \sqrt{x^k}, y) \end{array}$$

allows to compute

$$\mathcal{PV}_{T,g}|_{U_i} \cong \underline{\mathbb{C}}_{U_i}[1].$$

Finally, these constant sheaves will be glued as specified by the chart transition of the square root. This proves the claim.

In [BBDJS, Theorem 5.4] it is proven that the above example represents the general situation. For any embedding $(U,S,f) \hookrightarrow (V,T,g)$ of critical charts the perverse sheaves $\mathcal{PV}_{S,f}$ and $\mathcal{PV}_{T,g}$ differ by a two-torsion local system of rank one. Given the datum of a square root $\mathcal{K}_{X,s}^{1/2}$ of the canonical bundle $\mathcal{K}_{X,s}$ one can control the local systems arising this way and in this case it is actually possible to glue the perverse sheaves of vanishing cycles. We will state this result below. First, let us define the gluing maps. In Section 2.3 we discussed that it suffices to consider étale morphisms of critical charts and the embedding $(U,S,f) \hookrightarrow (U, S \times \mathbb{C}^n, f \boxplus z_1^2 + \ldots + z_n^2)$. In the latter case the Thom-Sebastini Theorem (cf. [BBDJS, Theorem 2.13]) provides a natural isomoprhism

$$\mathcal{PV}_{S\times\mathbb{C}^n, f\boxplus z_1^2+\ldots+z_n^2} \xrightarrow{\sim} \mathcal{PV}_{S,f} \boxtimes \mathcal{PV}_{\mathbb{C}^n, z_1^2+\ldots+z_n^2} \cong \mathcal{PV}_{S,f}. \tag{4.6}$$

Let $\iota\colon (U,S,f) \to (V,T,g)$ be an étale morphism of critical charts. Then Proposition 4.5 gives an isomorphims of functors

$$\iota_0^* \circ \psi_g \xrightarrow{\sim} \psi_f \circ \iota^*$$

and consequently, because $\dim S = \dim T$, we obtain an isomorphism

$$\rho_\iota\colon \mathcal{PV}_{T,g}|_U \xrightarrow{\sim} \mathcal{PV}_{S,f} \tag{4.7}$$

in $\mathbf{Perv}(U) = \mathbf{Perv}(U_{\mathrm{red}})$. Let $\xi_\iota\colon \iota^*\omega_T \to \omega_S$ be the natural isomorphism coming from $\iota^*\Omega_T \to \Omega_S$. Then

$$id \otimes \xi_\iota\colon (\mathcal{K}_{X,s}^{1/2})^\vee|_{U_{\mathrm{red}}} \otimes \omega_T|_{U_{\mathrm{red}}} \to (\mathcal{K}_{X,s}^{1/2})^\vee|_{U_{\mathrm{red}}} \otimes \omega_S|_{U_{\mathrm{red}}}$$

is per definition compatible with the trivializations of the squares and therefore induces a morphism

$$\tilde{\xi}_\iota\colon \mathfrak{L}_{V,T}^{\mathrm{or}}|_U \to \mathfrak{L}_{U,S}^{\mathrm{or}} \tag{4.8}$$

(cf. Remark 4.17).

Proposition 4.20 ([BBDJS, Theorem 6.9]). *Let (X,s) be a d-critical locus with orientation $\mathcal{K}_{X,s}^{1/2}$. Then there exists a perverse sheaf $\mathcal{P}_{X,s}^\bullet$ in* $\mathrm{Perv}(X)$ *such that for each critical chart (U,S,f) on (X,s) there is a natural isomorphism*

$$\kappa_{(U,S,f)}\colon\ \mathcal{P}_{X,s}^\bullet|_U \xrightarrow{\sim} \mathcal{PV}_{S,f}^\bullet \otimes_{\underline{\mathbb{C}}_U} \mathfrak{L}_{U,S}^{\mathrm{or}}$$

in $\mathbf{Perv}(U)$*. Here,* $\mathfrak{L}^{\mathrm{or}}_{U,S}$ *is the local system on* U *that corresponds to*

$$(\mathcal{K}^{1/2}_{X,s})^{\vee}\,|_{U_{\mathrm{red}}} \otimes \omega_S\,|_{U_{\mathrm{red}}}\ .$$

Moreover, $\mathcal{P}^{\bullet}_{X,s}$ *is uniquely determined by the conditions*

(1) *For every étale morphism* $\iota\colon (U,S,f) \to (V,T,g)$ *of critical charts*

$$\kappa_{(U,S,f)} = (\rho_\iota \otimes \tilde{\xi}_\iota) \circ \kappa_{(V,T,g)}\,|_{U_{\mathrm{red}}}\ ,$$

where ρ_ι *is as in* (4.7) *and* $\tilde{\xi}_\iota$ *as in* (4.8)*;*

(2) *After identification via the Thom-Sebastiani isomorphism from* (4.6)*, we have*

$$\kappa_{(U,S,f)} = \kappa_{(U,S\times\mathbb{C}^n,f\boxplus z_1^2+\ldots+z_n^2)}.$$

$\square$

Remark 4.21. Note that $\mathcal{PV}^{\bullet}_{S,f} \otimes \mathfrak{L}^{\mathrm{or}}_{U,S}$ is a perverse sheaf by Remark 4.3.

Remark 4.22. We omit the indication of the reduced structure, when this is irrelevant, i.e. in the case of local systems and perverse sheaves.

Once again, the key of the proof is to consider the situation from Lemma 3.13. In this case ρ_ι is the identity multiplied with a sign.

Proposition 4.23 ([BBDJS, Theorem 3.1] or [KL16, Theorem 2.12])**.** *Let* S *be a smooth scheme,* $f\colon S \to \mathbb{C}$ *a holomorphic function and set* $X = \mathrm{crit}(f)$*. For every étale morphism* $\iota\colon (X,S,f) \to (X,S,f)$ *we have*

$$\rho_\iota = \pm\,\mathrm{id}\colon \mathcal{PV}_{S,f} \longrightarrow \mathcal{PV}_{S,f}$$

and the sign is given by $\det(d\iota)\,|_{X_{\mathrm{red}}}$*.*

$\square$

Recall that we saw in Lemma 3.13 that $\det(d\iota)\,|_{X_{\mathrm{red}}}$ is locally constant with values in $\{\pm 1\}$ and, with the approporiate choices for S and ι, it defined a gluing datum for $\mathcal{K}^{1/2}_{X,s}$.

Remark 4.24 (Proof of [KL16, Theorem 2.15]). Again, we can immediately deduce the existence of $\mathcal{P}^{\bullet}_{X,s}$ in the situation of Lemma 2.20, i.e. after fixing charts $(X_\alpha, V_\alpha, f_\alpha)$, appropriate neighborhoods $V_{\alpha\beta}$ and biholomorphisms $\varphi_{\alpha\beta}\colon V_{\alpha\beta} \to V_{\beta\alpha}$. In this situation, we have to show that the isomorphisms

$$\rho_{\alpha\beta}\colon \mathcal{PV}_{V_{\alpha\beta},f_\alpha} \longrightarrow \mathcal{PV}_{V_{\alpha\beta},f_\beta}$$

satisfy

$$\rho_{\beta\gamma} \circ \rho_{\alpha\beta} = \rho_{\alpha\gamma} \text{ on } (X_{\alpha\beta\gamma})_{\mathrm{red}}.$$

Applying Proposition 4.23 in the analogous manner we did in Remark 3.14, we conclude that this is equivalent to $\det(\varphi_{\alpha\beta\gamma}) = 1$ for all (α,β,γ), which in turn was equivalent to the existence of $\mathcal{K}^{1/2}_{X,s}$.

Part 2. **Lagrangian intersections**

5. Basics on Symplectic Geometry

Let S be a smooth $\mathbb{C}$-scheme. A *symplectic structure* on S is a closed, non-degenerate two-form $\sigma \in H^0(S, \Omega_S^2)$ this is the same as an alternating isomorphism, we shall likewise denote by σ,

$$\sigma \colon \mathcal{T}_S \xrightarrow{\sim} \Omega_S.$$

We call (S, σ) a *symplectic manifold*. The existence of the symplectic form σ implies that the dimension of S is even. Let $\dim S = 2n$. Then $\sigma^n \in H^0(S, \omega_S)$ vanishes nowhere and therefore defines a trivialization $\omega_S \cong \mathcal{O}_S$.

Let (S, σ) be a symplectic manifold of dimension $2n$ and let $L \subset S$ be a smooth subscheme. We say that L is *Lagrangian* if $\dim L = n$ and the symplectic form vanishes on L. This means that the image of $\sigma\,|_L$ in $H^0(L, \Omega_L^2)$ is zero or, equivalenty, that the compostion

$$\mathcal{T}_L \longrightarrow \mathcal{T}_S\,|_L \xrightarrow{\sigma} \Omega_S\,|_L \longrightarrow \Omega_L$$

is trivial. Thus, the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_L \arrow[r] \arrow[d, "\wr"] & \mathcal{T}_S\,|_L \arrow[r] \arrow[d, "\wr"] & \mathcal{N}_{LS} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathscr{I}_{LS}/\mathscr{I}_{LS}^2 \arrow[r] & \Omega_S\,|_L \arrow[r] & \Omega_L \arrow[r] & 0.
\end{tikzcd} \tag{5.1}$$

Here, $\mathcal{N}_{LS}$ denotes the normal bundle of L in S.

Given any smooth scheme M, the total space of its cotangent bundle $|\Omega_M|$ is naturally endowed with a symplectic structure. If we choose standard coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ for $|\Omega_M|$, i.e. coordinates $(x_1, \ldots, x_n)$ for M and $y_i = dx_i$, the symplectic form on $|\Omega_M|$ is given by

$$\sigma = \sum dx_i \wedge dy_i \tag{5.2}$$

or, when fixing the first variable, we find $\sigma \colon \mathcal{T}_S \xrightarrow{\sim} \Omega_S$ given by

$$\sigma(\tfrac{\partial}{\partial x_i}) = dy_i \quad \text{and} \quad \sigma(\tfrac{\partial}{\partial y_i}) = -dx_i.$$

We can also write $\sigma = -d\eta$ with $\eta = \sum y_i dx_i \in H^0(S, \Omega_S)$. For a section $s \in H^0(M, \Omega_M)$ we denote by Γ_s the image of the embedding $s \colon M \hookrightarrow |\Omega_M|$. Then, by definition, Γ_s is Lagrangian if and only if $s^*\sigma = 0$ and this is the case if and only if s is a closed form. This follows from the equality $s^*\eta = s$ (cf. [Le03, Proposition 22.12]) and thus

$$s^*\sigma = -s^*(d\eta) = -d(s^*\eta) = -ds.$$

If (S, σ) is a symplectic manifold, coordinates such that σ is of the form (5.2) are called *Darboux coordinates*.

6. Symmetric obstruction theories on Lagrangian intersections

Let (S, σ) be a complex symplectic manifold containing two Lagrangian submanifolds $L, M \subset S$ with intersection $X := L \cap M$. We define the two-term complex

$$\mathbb{E}_{LM} := \left[\Omega_S|_X \xrightarrow{(-\mathrm{res}_L, \mathrm{res}_M)} \Omega_L|_X \oplus \Omega_M|_X \right]. \tag{6.1}$$

This is the same complex that we encountered in Example 3.5, where we defined a perfect obstruction theory $\mathbb{E}_{LM} \xrightarrow{\phi} \mathbb{L}_X$. Now, given that S is symplectic and L, M are Lagrangians it turns out that this can be made into a symmetric obstruction theory. We set $\mathcal{T}_X := \mathcal{H}om(\Omega_X, \mathcal{O}_X)$, regardless of X being smooth or not.

Lemma 6.1. *The intersection $X = L \cap M$ carries a symmetric obstruction theory $(\mathbb{E}_{LM}, \phi, \theta)$ with $\mathbb{E}_{LM} \xrightarrow{\phi} \mathbb{L}_X$ as defined in Example 3.5 and $\theta\colon \mathbb{E}_{LM}^\vee[1] \to \mathbb{E}_{LM}$ given by*

$$\begin{tikzcd}
\mathcal{T}_L|_X \oplus \mathcal{T}_M|_X \arrow[r, "-\mathrm{incl}_L + \mathrm{incl}_M"] \arrow[d, "\frac{1}{2}\mathrm{incl}_L + \frac{1}{2}\mathrm{incl}_M"] \arrow[dd, bend right=60, "\theta_{-1} := \theta_0^\vee"'] & \mathcal{T}_S|_X \arrow[d, "\sigma"'] \arrow[dd, bend left=60, "\theta_0"] \\
\mathcal{T}_S|_X \arrow[d, "-\sigma"] & \Omega_S|_X \arrow[d, "(\frac{1}{2}\mathrm{res}_L, \frac{1}{2}\mathrm{res}_M)"'] \\
\Omega_S|_X \arrow[r, "(-\mathrm{res}_L, \mathrm{res}_M)"] & \Omega_L|_X \oplus \Omega_M|_X \,,
\end{tikzcd} \tag{6.2}$$

where the indicated isomorphisms are induced by the symplectic form.

Proof. First of all, the diagram (6.2) commutes and hence θ is well-defined. It follows from (5.1) that the cokernel of the upper row identifies with the cokernel of

$$\mathscr{I}_{LS}/\mathscr{I}_{LS}^2|_X \oplus \mathscr{I}_{MS}/\mathscr{I}_{MS}^2|_X \xrightarrow{d_L + d_M} \Omega_S|_X \,,$$

which equals $\mathrm{coker}\left(\mathscr{I}_{XS}/\mathscr{I}_{XS}^2 \to \Omega_S|_X\right) \cong \Omega_X$. Hence, via this identification, we have $\mathcal{H}^0(\theta) = \mathrm{id}_{\Omega_X}$ and after dualizing $\mathcal{H}^{-1}(\theta) = -\mathrm{id}_{\mathcal{T}_X}$. Here, the minus sign occurs since we always identify $\mathcal{T}_X$ with $\mathcal{H}^{-1}(\mathbb{E}_{LM}) \subset \Omega_S|_X$ using σ and not $\sigma^\vee$. Therefore, θ is an isomorphism in $\mathrm{D}(X)$ and by definition it satisfies $\theta^\vee[1] = \theta$. This proves that $(\mathbb{E}_{LM}, \phi, \theta)$ is a symmetric obstruction theory. $\square$

Lemma 6.2. *Assume that $X = L \cap M$ is smooth. Then*

$$\omega_X \otimes \omega_X \cong \omega_L|_X \otimes \omega_M|_X \,. \tag{6.3}$$

Proof. As S is symplectic, it follows that $\omega_S \cong \mathcal{O}_S$ and thus

$$\det \mathbb{E}_{LM} = (\det \Omega_S|_X)^\vee \otimes \det(\Omega_L|_X \oplus \Omega_M|_X) \cong \omega_L|_X \otimes \omega_M|_X \,.$$

On the other hand, as in (3.2), we have

$$\det \mathbb{E}_{LM} = (\det \mathcal{H}^{-1}(\mathbb{E}_{LM}))^\vee \otimes \det \mathcal{H}^0(\mathbb{E}_{LM}) = \omega_X^{\otimes 2}.$$

$\square$

Corollary 6.3. *In the situation of Lemma 6.2, we have*

$$\det \mathcal{N}_{XL} \otimes \det \mathcal{N}_{XM} \cong \mathcal{O}_X.$$

Proof. Since X is smooth, the adjunction formula yields

$$\omega_X \cong \omega_M|_X \otimes \det \mathcal{N}_{XM} \quad \text{and} \quad \omega_X \cong \omega_L|_X \otimes \det \mathcal{N}_{XL}.$$

Therefore,

$$\omega_M|_X \otimes \omega_L|_X \cong \omega_X^{\otimes 2} \otimes \det \mathcal{N}_{XM}^\vee \otimes \det \mathcal{N}_{XL}^\vee$$

and the result follows with Lemma 6.2. □

Now, let M be a smooth scheme and consider its cotangent bundle $|\Omega_M|$ as a symplectic manifold. Let $L := \Gamma_{df}$ for $f \in H^0(M, \mathcal{O}_M)$. Recall that this was the image of the embedding $df\colon M \hookrightarrow |\Omega_M|$ and note that the restriction of the projection $|\Omega_M| \xrightarrow{\pi} M$ to Γ_{df} defines an isomorphism $\alpha\colon \Gamma_{df} \xrightarrow{\sim} M$. Then $X = L \cap M$ is the intersection of two Lagrangian submanifolds and at the same time we have $X = \mathrm{crit}(f)$. This gives two symmetric obstruction theories on X. It is natural to ask, whether these are isomorphic.

Proposition 6.4. *There is an isomorphism*

$$\Phi\colon (\mathbb{E}_2 := \mathbb{E}_{\partial^2 f}, \psi, \theta_2) \xrightarrow{\sim} (\mathbb{E}_1 := \mathbb{E}_{LM}, \phi, \theta_1)$$

between the symmetric obstructions theories defined in Lemma 6.1 and in Lemma 3.10. It is given by

$$\begin{tikzcd}
\mathbb{E}_2 \arrow[dd, "\Phi"] & = & {[}\mathcal{T}_M|_X \arrow[rr, "\partial^2 f"] \arrow[d, "\mathrm{incl}_M"] \arrow[dd, bend right, "\Phi_{-1}"'] & & \Omega_M|_X{]} \arrow[dd, "\Phi_0=(\alpha^*{,}0)"] \\
& & \mathcal{T}_S|_X \arrow[d, "-\sigma"] & & \\
\mathbb{E}_1 & = & {[}\Omega_S|_X \arrow[rr, "(-\mathrm{res}_L{,}\mathrm{res}_M)"] & & \Omega_L|_X \oplus \Omega_M|_X{]}.
\end{tikzcd} \tag{6.4}$$

Proof. We start by checking that (6.4) commutes. Therefore, choose coordinates $x_1, \ldots, x_n$ on M and $x_1, \ldots, x_n, y_1, \ldots, y_n$ on S. Then

$$\Phi_0 \circ \partial^2 f(\tfrac{\partial}{\partial x_i}) = \Phi_0 \left(\sum\nolimits_{j=1}^n \tfrac{\partial^2 f}{\partial x_i \partial x_j} dx_j\right) = \left(\sum\nolimits_{j=1}^n \tfrac{\partial^2 f}{\partial x_i \partial x_j} dx_j, 0\right).$$

On the other hand $\sigma \circ \mathrm{incl}_M(\frac{\partial}{\partial x_i}) = dy_i$ and thus

$$(-\mathrm{res}_L, \mathrm{res}_M) \circ \Phi_{-1}(\tfrac{\partial}{\partial x_i}) = \left(\sum\nolimits_{j=1}^n \tfrac{\partial^2 f}{\partial x_i \partial x_j} dx_j, 0\right).$$

Next, we show that Φ is defined over $\mathbb{L}_X$. For this aim, we factor $\mathbb{E}_2 \xrightarrow{\psi} \mathbb{L}_X$ as follows

$$\begin{tikzcd} \mathbb{E}_2 \arrow[dd, "\tau"] & = & {[\mathcal{T}_M|_X} \arrow[d, "(\alpha^{-1})_*"'] \arrow[rr, "\partial^2 f"] & & {\Omega_M|_X]} \arrow[dd, equal] \\ & & \mathcal{T}_L|_X \arrow[d, "-\sigma'"'] & & \\ \mathbb{F} \arrow[d, "\varphi_2"] & = & {[\mathscr{I}_{LS}/\mathscr{I}_{LS}^2|_X} \arrow[rr] \arrow[d, two heads] & & {\Omega_M|_X].} \arrow[d, equal] \\ \mathbb{L}_X & = & {[\mathscr{I}_{XM}/\mathscr{I}_{XM}^2} \arrow[rr] & & {\Omega_M|_X].} \end{tikzcd} \tag{6.5}$$

with $\mathbb{F}$ and φ_2 as in Example 3.5 and $\sigma' \colon \mathcal{T}_L|_X \xrightarrow{\sim} \mathscr{I}_{LS}/\mathscr{I}_{LS}^2|_X$ being the isomorphism induced by σ as in (5.1). Using that

$$(\alpha^{-1})_*(\tfrac{\partial}{\partial x_i}) = dx_i + \sum\nolimits_{j=1}^n \tfrac{\partial^2 f}{\partial x_i \partial x_j} dy_j$$

we see that τ is an isomorphism of complexes and since ϕ was given by $\mathbb{E}_1 \xleftarrow{\varphi_1} \mathbb{F} \xrightarrow{\varphi_2} \mathbb{L}_X$, this means that have to show that

$$\begin{tikzcd} \mathbb{E}_2 \arrow[rr, "\Phi"] \arrow[dr, "\tau"'] & & \mathbb{E}_1 \\ & \mathbb{F} \arrow[ur, "\varphi_1"'] & \end{tikzcd}$$

commutes in $\mathrm{D}(X)$. We claim that $\pi^* \colon \Omega_M|_X \to \Omega_S|_X$ defines a homotopy between Φ and $\varphi_1 \circ \tau$. This means that

$$(\varphi_1 \circ \tau)_{-1} - \Phi_{-1} = \pi^* \circ \partial^2 f \tag{6.6}$$

and

$$(\varphi_1 \circ \tau)_0 - \Phi_0 = (-\mathrm{res}_L, \mathrm{res}_M) \circ \pi^*. \tag{6.7}$$

Using (5.1) we find that $(\varphi_1 \circ \tau)_{-1}$ is given by the composition

$$\mathcal{T}_M|_X \xrightarrow{(\alpha^{-1})_*} \mathcal{T}_L|_X \xrightarrow{\mathrm{incl}_L} \mathcal{T}_S|_X \xrightarrow{-\sigma} \Omega_S|_X$$

which allows to see that (6.6) holds. Finally, it follows immediately from the definitions that both sides in (6.7) are given by

$$(-\alpha^*, \mathrm{id}) \colon \Omega_M|_X \longrightarrow \Omega_L|_X \oplus \Omega_M|_X .$$

Hence, Φ defines an isomorphism of perfect obstruction theories and it is left to see that it is an isometry. We set $\eta := \Phi \circ \theta_2 \circ \Phi^\vee[1]$ then we have to show that $\eta = \theta_1$ in $\mathrm{D}(X)$. Let $\epsilon \colon S \to S$ be induced by the multiplication with -1 on Ω_M and set $h := -\frac{1}{2} d\epsilon \circ \sigma \colon \mathcal{T}_S|_X \to \Omega_S|_X$. In local coordinates we have

$$h(\tfrac{\partial}{\partial x_i}) = \tfrac{1}{2} dy_i \quad \text{and} \quad h(\tfrac{\partial}{\partial y_i}) = \tfrac{1}{2} dx_i.$$

We claim that h defines a homotopy between η and θ_1. Indeed, we compute

$$\begin{array}{rccl} (\eta-\theta_1)_0\colon & \mathcal{T}_S\,|_X & \longrightarrow & \Omega_L\,|_X \oplus \Omega_M\,|_X \\ & \frac{\partial}{\partial x_i} & \longmapsto & \left(-\frac{1}{2}\sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j, 0\right) \\ & \frac{\partial}{\partial y_i} & \longmapsto & \left(-\frac{1}{2}dx_i, \frac{1}{2}dx_i\right) \end{array}$$

and thus

$$(\eta-\theta_1)_0 = (-\mathrm{res}_L, \mathrm{res}_M)\circ h.$$

Finally, we conclude using $\eta^\vee[1]=\eta$ as well as $h^\vee = h$ that

$$(\eta-\theta_1)_{-1} = (\eta-\theta_1)_0{}^\vee = h\circ(-\mathrm{incl}_L + \mathrm{incl}_M),$$

which finishes the proof. □

7. Lagrangian intersections and perverse sheaves

In this chapter, we explain how Bussi (cf. [Bu14]) defines a perverse sheaf on the oriented intersection of two Lagrangian submanifolds. It is a special case of the construction of a perverse sheaf $\mathcal{P}^\bullet_{X,s}$ on an oriented d-critical locus (X,s) from [BBDJS] that we explained in Section 4.3.

7.1. Lagrangian intersections are d-critical.

From now on, let (S,σ) be a symplectic manifold containing two Lagrangian submanifolds L, M with intersection $X := L\cap M$.
In [Bu14, Theorem 3.1] Bussi shows that X has a d-critical structure. Let us explain the procedure. The Lagrangian neighborhood theorem states that for any $x\in M$ there are open neighborhoods U in S and $\tilde U$ in $|\Omega_{M\cap U}|$ together with an isomorphism $\Phi\colon U\to\tilde U$ satisfying the following properties

(1) Φ is compatible with the symplectic structures;
(2) Φ identifies $M\cap U$ with the zero section in $|\Omega_{M\cap U}|$.

Moreover, we can arrange that

(3) $\tilde L := \Phi(L\cap U)$ intersects each fiber F of the projection $|\Omega_{M\cap U}|\to M\cap U$ transversally in exactly one point x, i.e.

$$\mathcal{T}_{\tilde L}(x)\oplus\mathcal{T}_F(x)\cong\mathcal{T}_{|\Omega_{M\cap U}|}(x).$$

Remark 7.1. In contrast to the real version of this theorem ([We71, Theorem 6.1]), we can in general not assume that $M\subset U$. Unfortunately, we could not find a reference for the complex version.

Now, it follows from property (3) that $\Phi(L\cap U)=\Gamma_s\cap\tilde U$ for some section $s\in H^0(M\cap U,\Omega_{M\cap U})$. On the other hand, $\Phi(L\cap U)$ is Lagrangian in $|\Omega_{M\cap U}|$ by property (1) and therefore, s is a closed form (cf. Section 5) so that after shrinking U we can assume that $s = df$ for some $f\in H^0(M\cap U,\mathcal{O}_{M\cap U})$. Thus, $\Phi(X\cap U)=\Gamma_{df}\cap\Gamma_0$ or just

$$X\cap U = \mathrm{crit}(f)\subset M\cap U.$$

In other words, $(X \cap U, M \cap U, f)$ is a d-critical chart. This is what Bussi calls an *M-chart* for X. Summing up, it is the datum of (U, Φ, f) fitting in the commutatives diagrams

$$\begin{tikzcd} U \arrow[r, "\Phi"] & \tilde{U} \subset |\Omega_{M\cap U}| \\ M \cap U \arrow[u, hook] \arrow[ur, hook, "0"] & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\Phi"] & \tilde{U} \subset |\Omega_{M\cap U}| \\ L \cap U \arrow[u, hook] \arrow[r, "\sim"] & \Gamma_{df}. \arrow[u, hook] \end{tikzcd}$$

Bussi also applies the procedure with the roles of L and M interchanged yielding so called L-charts and writing $X = \Delta_S \cap (L \times M) \subset S \times S$ she obtains so called LM-charts. All these charts are compatible, as stated by the following result.

Proposition 7.2 ([Bu14, Theorem 3.1])**.** *The intersection $X = L \cap M$ of two Lagrangians admits a unique structure of a d-critical locus (X, s) such that all of the above charts, i.e. L-charts, M-charts and LM-charts are d-critical charts for (X, s). The canonical bundle $\mathcal{K}_{X,s}$ is isomorphic to $\omega_L\,|_{X_{\mathrm{red}}} \otimes \omega_M\,|_{X_{\mathrm{red}}}$.*

□

Remark 7.3. Proposition 7.2 says in particular that the d-critical structure is independent of the order of L and M.

7.2. **Lagrangian intersections and perverse sheaves.** The structure of a d-critical locus together with a square root of $\mathcal{K}_{X,s}$ allows one to define a perverse sheaf on X as done in Section 4.3.

Corollary 7.4. *Consider X as a d-critical locus as in Proposition 7.2 and assume that there exists a square root of $\omega_L\,|_{X_{\mathrm{red}}} \otimes \omega_M\,|_{X_{\mathrm{red}}}$, i.e. that X is oriented. Then there exists a perverse sheaf*

$$\mathcal{P}^{\bullet}_{L,M} \cong \mathcal{P}^{\bullet}_{M,L}$$

on X that is uniquely determined by the properties of Proposition 4.20.

□

Remark 7.5. A special case of the above orientation assumption is provided by the existence of square roots $\omega_L^{1/2}$ and $\omega_M^{1/2}$, which we call *orientation of the Lagrangians* L and M, respectively. This is the only case that is considered in [Bu14].

We will compute $\mathcal{P}^{\bullet}_{L,M}$ in some special situations.

Lemma 7.6 (Smooth intersection)**.** *Assume that $X = L \cap M$ is smooth and fix a square root $\mathcal{K}^{1/2}_{X,s}$ of $\omega_L\,|_X \otimes \omega_M\,|_X$. Then*

$$\mathcal{P}^{\bullet}_{L,M} \xrightarrow{\sim} \mathfrak{L}_{\mathrm{or}}[\dim X],$$

where $\mathfrak{L}_{\mathrm{or}}$ is the local system corresponding to $(\mathcal{K}^{1/2}_{X,s})^{\vee} \otimes \omega_X$.

Remark 7.7. Note that by Lemma 6.2 we have $\omega_X^{\otimes 2} \cong \omega_L|_X \otimes \omega_M|_X$ for smooth X. Therefore, $\mathcal{K}_{X,s}^{1/2}$ always exists and $\mathfrak{L}_{\mathrm{or}}$ is well defined. The perverse sheaf is independent of the orientation if X is moreover compact and $\mathrm{Pic}(X)$ has no two-torsion.

Proof. As X is smooth, we have $\mathcal{S}_X^0 = 0$. Hence, $(X, 0)$ with canonical bundle $\omega_X^{\otimes 2}$ is the unique structure of a d-critical locus on X. Therefore by Example 4.11 we have

$$\mathcal{P}^\bullet_{L,M} \xrightarrow{\sim} \mathcal{PV}^\bullet_{X,0} \otimes \mathfrak{L}_{\mathrm{or}} = \underline{\mathbb{C}}_X[\dim X] \otimes \mathfrak{L}_{\mathrm{or}} = \mathfrak{L}_{\mathrm{or}}[\dim X]$$

with $\mathfrak{L}_{\mathrm{or}}$ as stated. □

Remark 7.8 (Self intersection). A special case of the above Lemma 7.6 is the case that $X = L = M$. If one chooses the same orientation for L and M, then

$$\mathcal{P}^\bullet_{L,L} \xrightarrow{\sim} \underline{\mathbb{C}}_L[n].$$

Remark 7.9 (Transversal intersection). Another special case of Lemma 7.6 is the case that L and M intersect transversally. Then $\dim X = 0$ and any local system on X is trivial. Hence,

$$\mathcal{P}^\bullet_{L,M} \xrightarrow{\sim} \underline{\mathbb{C}}_X .$$

One can also use the holomorphic Morse Lemma to produce this result, i.e. locally around a point in X we can assume that $X = Z(df)$ is given by $f = x_1^2 + \cdots + x_n^2 \colon \mathbb{C}^n \to \mathbb{C}$ and thus $\mathcal{PV}^\bullet_{\mathbb{C}^n,f} \cong \underline{\mathbb{C}}_0$ by Example 4.13.

Example 7.10 (The case of surfaces). We wish to describe $\mathcal{P}^\bullet_{L,M}$ in the case of a two-dimensional symplectic manifold S, for example, a K3 surface. In this situation any smooth curves L, M in S are automatically Lagrangian since trivially $\Omega_L^2 = 0$ and $\Omega_M^2 = 0$.
If $X = L = M$ we saw in Lemma 7.6 that $\mathcal{P}^\bullet_{L,M} \cong \underline{\mathbb{C}}_L[1]$ modulo a possible twist by a two-torsion local system of rank one. Therefore assume that L and M are connected and do not coincide. In this case we have $\dim X = 0$. Let $x \in X$ and choose an analytic neighborhood U such that there is an étale morphism $U \to \mathbb{A}^2 = \mathrm{Spec}\,\mathbb{C}[x, y]$ identifying M with $V(y) \subset \mathbb{A}^2$ and L with $V(y - h(x))$ for some holomorphic function $h(x)$ on M. After changing coordinates and shrinking U, we can assume that $h = x^m$, where m is the multiplicity of the intersection of L and M in x and therefore, we are reduced to $X = L \cap M = \mathrm{crit}(f) \subset \mathrm{Spec}\,\mathbb{C}[x]$ with $f = x^{m+1}$. Finally, we know from Example 4.12 that

$$\mathcal{PV}^\bullet_{\mathbb{C},x^{m+1}} = \underline{\mathbb{C}}_0^m .$$

Hence, if $X = \{x_1, \ldots, x_n\}$ and the multiplicity of the intersection at the point x_i is m_i, we find

$$\mathcal{P}^\bullet_{L,M} \cong \underline{\mathbb{C}}_{x_1}^{m_1} \oplus \ldots \oplus \underline{\mathbb{C}}_{x_n}^{m_n} .$$

8. Examples of Lagrangian intersections in dimension 4

In this chapter, we study two examples of two-dimensional Lagrangians with one-dimensional intersection.

8.1. **Symplectic structure on the Hilbert scheme.** Let Z be a compact complex surface which has trivial canonical bundle ω_Z, for example, take a K3 surface. Then the Hilbert scheme $\mathrm{Hilb}^r(Z)$ that parametrizes subschemes of length r in Z is a complex symplectic variety of dimension $2r$ (see [Be83, Proposition 5]). In the following, we consider $S = \mathrm{Hilb}^2(Z)$. A point in S corresponds to two points in Z or one point in Z together with a tangential direction and we have the description as in [Be83, Chapter 6]

$$S \cong \mathrm{Bl}_\Delta(Z \times Z/S_2) \cong \mathrm{Bl}_\Delta(Z \times Z)/S_2,$$

where the S_2-action permutes the factors and Δ denotes the diagonal in $Z \times Z$ and in $S^2 Z = (Z \times Z)/S_2$. We fix the following notation

$$\begin{tikzcd} \mathrm{Bl}_\Delta(Z \times Z) \arrow[r, "\eta"] \arrow[d, "\rho"] & Z \times Z \arrow[d, "\pi"] \\ S = \mathrm{Hilb}^2 Z \arrow[r, "\varepsilon"] & S^2 Z. \end{tikzcd} \tag{8.1}$$

Let $pr_i \colon Z \times Z \to Z$ be the projections and consider

$$pr_1^*\omega_Z \oplus pr_2^*\omega_Z \subset \bigwedge^2 (pr_1^*\Omega_Z \oplus pr_2^*\Omega_Z) = \Omega^2_{Z\times Z}.$$

The symplectic form $\sigma_S \in H^0(S, \Omega_S^2)$ is constructed such that

$$\rho^*\sigma_S = \eta^*(pr_1^*\sigma_Z + pr_2^*\sigma_Z)$$

for some non-zero $\sigma_Z \in H^0(Z, \omega_Z) = H^0(Z, \mathcal{O}_Z)$.
If we choose coordinates x_1, y_1, x_2, y_2 for $Z \times Z$, we can describe σ_S as follows. Set $x = x_1 - x_2, x' = x_1 + x_2, y = y_1 - y_2$ and $y' = y_1 + y_2$ such that $S^2 Z = \mathrm{Spec}\,\mathbb{C}[x^2, xy, y^2, x', y']$ and since $\Delta = V(x, y)$ the blow up

$$\mathrm{Bl}_\Delta(Z \times Z) = V(xt - ys) \subset Z \times Z \times \mathbb{P}^1$$

is covered by the two S_2-invariants charts $\{t \neq 0\}$ and $\{s \neq 0\}$, both isomorphic to $\mathbb{A}^4$. This gives two charts for S namely $U_0 = \mathrm{Spec}\,\mathbb{C}[y^2, x', y', s]$ and $U_1 = \mathrm{Spec}\,\mathbb{C}[x^2, x', y', t]$. We claim that the symplectic form on S is given by $2\sigma_S\,|_{U_0} = ds \wedge d(y^2) + 2dx' \wedge dy'$. Indeed, we have

$$2(dx_1 \wedge dy_1 + dx_2 \wedge dy_2) = dx \wedge dy + dx' \wedge dy'$$

and

$$\begin{aligned} \eta^*(dx \wedge dy + dx' \wedge dy')\,\big|_{\{t\neq 0\}} &= d(ys) \wedge dy + dx' \wedge dy' \\ &= yds \wedge dy + dx' + dy' \\ &= \tfrac{1}{2}\rho^*(ds \wedge d(y^2) + 2dx' \wedge dy')\,\big|_{\{t\neq 0\}} \end{aligned}$$

Analogously, we find $2\sigma_S\,|_{U_1} = d(x^2) \wedge dt + 2dx' \wedge dy'$.

8.2. **Two examples.** We give a smooth and a non-smooth example of Lagrangian intersections in $\mathrm{Hilb}^2(Z)$ for a K3 surface Z. We use the notations from the previous section.

Example 8.1. Let Z be a K3 surface containing a smooth rational curve $\mathbb{P}^1 \cong C \subset Z$. For example, the Fermat quartic in $\mathbb{P}^3$ defined by $x_0^4 + \ldots + x_3^4$ containing the line $V(x_0 - \zeta x_1) \cap V(x_2 - \zeta x_3)$, where $\zeta^4 = -1$.
We consider the exceptional divisor E of $\pi\colon \mathrm{Bl}_\Delta(Z \times Z) \to Z \times Z$, then $E \cong \mathbb{P}(\mathcal{T}_Z)$. Since S_2 acts trivially on E, this is also the exceptional divisor

of $\mathrm{Bl}_\Delta(Z\times Z/S_2)\to Z\times Z/S_2$ and therefore we write $\mathbb{P}(\mathcal{T}_Z)\subset S$. We define two submanifolds in $S=\mathrm{Hilb}^2 Z$ by

$$M:=\mathbb{P}(\mathcal{T}_Z|_{\mathbb{P}^1})\quad\text{and}\quad L:=\mathrm{Hilb}^2(\mathbb{P}^1)\cong\mathbb{P}^2.$$

Both L and M are smooth and their canonical bundles do not have global sections. This is clear in the case of L and also true for M, since M is a $\mathbb{P}^1$-bundle. Therefore, we have defined two Lagrangian submanifolds. Their intersection $X:=L\cap M\hookrightarrow L$ arises as the fibre product

$$\begin{tikzcd} X=\mathbb{P}(\mathcal{T}_{\mathbb{P}^1}) \arrow[r] \arrow[d, "\wr"'] & L=\mathrm{Bl}_{\Delta_{\mathbb{P}^1}}(S^2\mathbb{P}^1) \arrow[d, "\wr"'] \\ \mathbb{P}^1 \arrow[r, hook, "\Delta"] & S^2\mathbb{P}^1. \end{tikzcd}$$

The isomorphism $X\cong\mathbb{P}^1$ can be understood geometrically. A point in the intersection corresponds to a point in $\mathbb{P}^1\subset Z$ and a tangential direction in $\mathbb{P}^1$ at this point, which is unique.

Now, we compute $\omega_L|_X$ and $\omega_M|_X$, respectively. We consider X as a divisor in L with corresponding line bundle $\mathcal{O}_L(X)\cong\mathcal{O}_{S^2\mathbb{P}^1}(\Delta)\cong\mathcal{O}_{\mathbb{P}^2}(2)$ and normal sheaf $\mathcal{N}_{XL}=\mathcal{O}_L(X)|_X\cong\mathcal{O}_{\mathbb{P}^1}(4)$. Thus,

$$\omega_L|_X\cong\omega_X\otimes\mathcal{N}_{XL}^\vee\cong\mathcal{O}_{\mathbb{P}^1}(-6)$$

and by Lemma 6.2 it follows that

$$\omega_M|_X\cong\mathcal{O}_{\mathbb{P}^1}(2)\quad\text{and}\quad\mathcal{N}_{XM}\cong\mathcal{O}_{\mathbb{P}^1}(-4).$$

In particular, we see that $\omega_L|_X\ncong\omega_X\ncong\omega_M|_X$ and $\mathcal{N}_{XL}\ncong\mathcal{O}_X\ncong\mathcal{N}_{XM}$. Moreover, there is a square root of $\omega_M\cong\mathcal{O}_M(-2)\otimes(\pi|_M)^*\mathcal{O}_{\mathbb{P}^1}(-2)$ and of $\omega_L|_X$, yet not of $\omega_L\cong\omega_{\mathbb{P}^2}(-3)$. However, as there are no non-trivial local systems on X we always find

$$\mathcal{P}^\bullet_{L,M}\cong\underline{\mathbb{C}}_X[1],$$

independent of matters of orientation.

Example 8.2. Let Z be a K3 surface containing three smooth curves C_1,C_2 and C_3 that intersect in one point $z\in Z$. Assume that C_1 intersects C_2 and C_3 transversally. We now define two submanifolds in $\mathrm{Bl}_\Delta(Z\times Z)$ by first letting

$$S_{12}:=\mathrm{Bl}_{(z,z)}(C_1\times C_2)\quad\text{and}\quad S_{13}:=\mathrm{Bl}_{(z,z)}(C_1\times C_3)$$

and then

$$M:=\rho(S_{12})\quad\text{and}\quad L:=\rho(S_{13}).$$

Actually, it follows from a local calculation that $M\cong S_{12}$. Here, we need that the intersection of C_1 and C_2 is transversal. We conclude that M is smooth. Alternatively we could write $M\cong\mathrm{Bl}_{\{z,z\}}(\pi(C_1\times C_2))$ and $\pi(C_1\times C_2)\cong C_1\times C_2$. Analogously, it follows that L is smooth. Moreover,

$$\sigma_S|_L=\eta^*(pr_1^*\sigma_Z+pr_2^*\sigma_Z)\Big|_{\mathrm{Bl}_{(z,z)}(C_1\times C_2)}=(pr_1^*\sigma_Z+pr_2^*\sigma_Z)|_{C_1\times C_2}=0.$$

Hence, L and M are Lagrangian submanifolds. Their set-theoretical intersection $X=L\cap M$ is going to be the union of the exceptional divisor E' of $\mathrm{Bl}_{\{z,z\}}(\pi(C_1\times C_2))$ and the strict transform $\widetilde{C_1}$ of C_1 that intersect in a point $\tilde{z}$. In order to understand the non-reduced structure we will describe X étale locally near $\tilde{z}$. Therefore, choose coordinates x and y on Z and assume

$$C_1 = V(x), C_2 = V(y) \text{ and } C_3 = V(y - x^n) \text{ for some } n \in \mathbb{N}.$$

Then, on $Z \times Z$ with coordinates x_1, y_1, x_2, y_2 or x, y, x', y' as introduced above we have

$$C_1 \times C_2 = V(x_1, y_2) \cong V(x' + x, y' - y)$$

and thus

$$M \cap U_0 = \rho(V(x' + ys, y' - y)) = V(x' + y's, y'^2 - y^2) \subset U_0.$$

We note that $ds \wedge dy^2 + 2dx' \wedge dy' = dsdy'^2 + 2d(y's) \wedge dy' = 0$ signifying that M is Lagrangian. Similarly, $M \cap U_1 = V(y' + x't, x^2 - x'^2)$. On the other hand, we have

$$\begin{aligned} C_1 \times C_3 &= V(x_1, y_2 - (x_2)^n) \\ &\cong V(x' + x, 2^{n-1}(y' - y) - (x' - x)^n) = V(x' + x, y' - y - 2x'^n) \end{aligned}$$

and this gives

$$L \cap U_0 = V(x' + (y' - 2x'^n)s, (y' - 2x'^n)^2 - y^2).$$

Finally, using $M \cap U_0 \cong \operatorname{Spec} \mathbb{C}[y', s]$ via the identifications $y^2 = y'^2$ and $x' = -y's$ we find

$$\begin{aligned} X \cap U_0 &\cong V(-y's + (y' - 2(-y's)^n)s, (y' - 2(-y's)^n)^2 - y'^2) \\ &= V((y's)^n s, (y's)^n y') \subset \operatorname{Spec} \mathbb{C}[y', s]. \end{aligned}$$

In other words, $X \cap U_0 \cong \operatorname{crit}((y's)^{n+1})$.

The canonical bundle of L is given by

$$\omega_L \cong \varepsilon^* \omega_{\pi(C_1 \times C_2)} \otimes \mathcal{O}(E').$$

Similarly, we compute ω_M. Unfortunately, there are no square roots $\omega_L^{1/2}, \omega_M^{1/2}$.

We will now specify a situation, in which $\mathcal{P}^\bullet_{L,M}$ is defined and given by

$$\mathcal{P}^\bullet_{L,M} \cong \underline{\mathbb{C}}^n_X[1] \oplus \underline{\mathbb{C}}^{n+1}_{\tilde{z}}.$$

Assume that there is an étale morphism $\phi \colon L \to M$ such that $\phi|_X = \mathrm{id}_X$. For instance, ϕ could be induced by an isomorphism $C_2 \xrightarrow{\sim} C_3$ that fixes the point z. In this situation, we obtain an isomorphism $\omega_M|_X \cong \omega_L|_X$ and therefore $\mathcal{K}^{1/2}_{X,s} := \omega_L|_{X_{\mathrm{red}}}$ defines a square root of the canonical bundle. As a consequence, we have $\mathcal{K}^{1/2}_{X,s}|_{U_{\mathrm{red}}} \otimes \omega_S|_{U_{\mathrm{red}}} \cong \mathcal{O}_{U_{\mathrm{red}}}$ for any L-chart (U, S, f). Since X is compact, this yields that the corresponding local system $\mathfrak{L}^{\mathrm{or}}_{U,S}$ is trivial and therefore

$$\mathcal{P}^\bullet_{X,s}|_U \cong \mathcal{PV}_{U,f} \text{ for every } L\text{-chart } (U, S, f).$$

Now, around any point $x \in X$ such that $x \neq \tilde{z}$ we find a neighborhood $U \subset X$ such that $U \cong \operatorname{crit}(x^{n+1}) \subset D$, where $D \subset \mathbb{C}^2$ is the open unit disk. This gives

$$\mathcal{P}^\bullet_{X,s}|_U \cong \underline{\mathbb{C}}^n_U[1].$$

As discussed above, around $\tilde{z}$ there is a neighborhood $V \subset X$ such that $V \cong \operatorname{crit}((yz)^{n+1}) \subset D$ and thus

$$\mathcal{P}^{\bullet}_{X,s}|_V \cong \underline{\mathbb{C}}^n_V[1] \oplus \underline{\mathbb{C}}^{n+1}_{\tilde{z}}$$

by Example 4.14. If for instance, $\pi_1(C_1) = 0$ and thus also $\pi_1(X) = 0$ by the Seifert-van Kampen Theorem (using that $E' \cong \mathbb{P}^1$ is simply connected), these constant sheaves will be trivially glued, i.e.

$$\mathcal{P}^{\bullet}_{L,M} \cong \underline{\mathbb{C}}^n_X[1] \oplus \underline{\mathbb{C}}^{n+1}_{\tilde{z}}\,.$$

Remark 8.3. One could also combine these examples intersecting $\mathbb{P}(\mathcal{T}_Z|_{\mathbb{P}^1})$ from Example 8.1 and $\rho(\mathrm{Bl}_{(z,z)}(C_1 \times C_2))$ from Example 8.2 with $C_1 = \mathbb{P}^1$. Then $X = \mathbb{P}(\mathcal{T}_Z(z)) \cong \mathbb{P}^1$.

9. Comparison with $\mathcal{E}xt^i(\mathcal{O}_L, \mathcal{O}_M)$

In [BF09] Behrend and Fantechi claim to construct a $\mathbb{C}$-linear differential

$$d\colon \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \to \mathcal{E}xt^{i+1}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$$

such that $d^2 = 0$ and $(\mathcal{E}^\bullet := \mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), d)$ is a constructible complex. Moreover, they conjecture ([BF09, Conjecture 5.8]) that

$$\mathbb{H}^p(X, (\mathcal{E}^\bullet, d)) = \mathrm{Ext}^p_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$$

and ([BF09, Conjecture 5.16]) that there is a perverse sheaf on X, that is locally modelled on the perverse sheaf of vanishing cycles and related to $(\mathcal{E}^\bullet, d)$ by a spectral sequence. Unfortunately, the construction of the differential turned out to be false, yet Brav et al. claim ([BBDJS, Remark 6.15]) that this mistake can be fixed by working with $\mathcal{E}xt^i_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2})$ instead, where $\omega_L^{1/2}$ and $\omega_M^{1/2}$ are square roots of the canonical bundles.

In the following, we will point out a connection between $\dim \mathbb{H}^p(X, \mathcal{P}^\bullet_{L,M})$ and $\dim \mathrm{Ext}^p_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2})$ under the assumption that X is smooth and that the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\omega_L^{1/2}, \omega_M^{1/2})) \Rightarrow \mathrm{Ext}^{p+q}(\omega_L^{1/2}, \omega_M^{1/2}) \tag{9.1}$$

degenerates. For instance, this assumption is satisfied if $\dim X = 1$. We also elaborate on the question, why one should consider $\omega_L^{1/2}, \omega_M^{1/2}$ rather than $\mathcal{O}_L, \mathcal{O}_M$. Let us start by studying the case of a symplectic surface.

Example 9.1. Let S be a symplectic manifold of dimension two. In Example 7.10 we saw that we have to understand the following situation. Assume that $S = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$, $L = V(y)$ and $M = V(y - x^n)$. Then

$$X = L \cap M = \operatorname{Spec} \mathbb{C}[x]/(x^n) \subset L.$$

We compute $\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ using the following resolution of $\mathcal{O}_L$

$$0 \to \mathbb{C}[x, y] \xrightarrow{\cdot y} \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y]/(y) \to 0.$$

After applying $\mathrm{Hom}(\_, \mathbb{C}[x, y]/(y - x^n))$ we find

$$\mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) = \begin{cases} 0 & \text{if } i \neq 1 \\ \mathcal{O}_X & \text{if } i = 1 \end{cases}$$

and thus

$$\dim \mathrm{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) = \left\{ \begin{array}{ll} 0 & \text{if } i \neq 1 \\ m & \text{if } i = 1 \end{array} \right\} = \dim \mathbb{H}^{i-1}(\mathcal{P}^\bullet_{L,M})$$

by Example 7.10. This solves the case of surfaces.

Unfortunately, the example is misleading as we will see later. Before, we need some technical results in order to compute Ext groups and sheaves.

9.1. **Computing Ext sheaves.** In the following, we let S be a smooth, quasi-projective scheme, containing two smooth closed subschemes $i : Y \hookrightarrow S$ and $j \colon Z \hookrightarrow S$. We set $X = Y \cap Z$

Lemma 9.2. *Let $\mathcal{E}, \mathcal{F}$ be locally free sheaves on Y, Z, respectively, and $\mathcal{M}$ any coherent sheaf on S. Then we have isomorphisms*

(i) $\mathcal{E}xt^p_{\mathcal{O}_S}(i_*\mathcal{E}, \mathcal{M}) \cong \mathcal{E}xt^p_{\mathcal{O}_S}(i_*\mathcal{O}_Y, \mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{E}^\vee$
(ii) $\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{M}, j_*\mathcal{F}) \cong \mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{M}, j_*\mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \mathcal{F}.$

In particular,

$$\mathcal{E}xt^p_{\mathcal{O}_S}(i_*\mathcal{E}, j_*\mathcal{F}) \cong \mathcal{E}xt^p(i_*\mathcal{O}_Y, j_*\mathcal{O}_Z) \otimes_{\mathcal{O}_X} \mathcal{E}^\vee|_X \otimes_{\mathcal{O}_X} \mathcal{F}|_X \,.$$

Remark 9.3. For any two sheaves $\mathcal{E}, \mathcal{F}$ on S we have for all p

$$\operatorname{supp} \mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{E}, \mathcal{F}) \subset \operatorname{supp} \mathcal{E} \cap \operatorname{supp} \mathcal{F}.$$

Therefore, we suppress respective pullbacks in our notation. In the following, we will also cease from writing down pushforwards by embeddings.

Proof. (i) For $p = 0$ composition defines a natural map

$$\begin{array}{ccccc} \mathcal{H}om_{\mathcal{O}_S}(i_*\mathcal{O}_Y, \mathcal{M}) & \otimes_{\mathcal{O}_Y} \mathcal{E}^\vee & \longrightarrow & \mathcal{H}om_{\mathcal{O}_S}(i_*\mathcal{E}, \mathcal{M}) \\ \phi & \otimes \psi & \longmapsto & \phi \circ i_*\psi. \end{array}$$

We check locally on Y that this is an isomorphism. Therefore, assume that $S = \operatorname{Spec} A, Y = \operatorname{Spec} B$ and that $\mathcal{E}, \mathcal{F}$ correspond to B^n, M, where M is any A-module. We find

$$\mathrm{Hom}_A(B, M) \otimes_B \mathrm{Hom}_B(B^n, B) \longrightarrow \mathrm{Hom}_A(B^n, M) \otimes B.$$

This is evidently an isomorpism. Now, both sides are effaceable δ-functors in $\mathcal{M}$ and we get the required isomorphism for each $p > 0$.
(ii) Follows analogously. Here, we need the assumption that S is quasi-projectice and $\mathcal{M}$ coherent in order to guarantee effaceability. □

From now on, we assume that X is smooth.

Proposition 9.4. *Let $Y, Z \subset S$ smooth such that $X = Y \cap Z$ is smooth. Then*

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z) \cong \bigwedge^{p-c} \tilde{\mathcal{N}} \otimes_{\mathcal{O}_X} \det \mathcal{N}_{XZ},$$

where $c = \mathrm{rk} \mathcal{N}_{XZ}$ and $\tilde{\mathcal{N}} = \mathcal{T}_S|_X / (\mathcal{T}_Y|_X + \mathcal{T}_Z|_X)$.

The proof of Proposition 9.4 will be achieved in several steps translating from [CKS03, Proof of Proposition A.5], where one finds analogous statements involving $\mathcal{T}or$ instead of $\mathcal{E}xt$.

To begin with, we will assume that Y is the zero locus of a regular section $s \in H^0(S, \mathcal{G})$ of a locally free sheaf $\mathcal{G}$ of rank c. In this situation we have $\mathcal{G}|_Y \cong \mathcal{N}_{YS}$ and $\mathcal{O}_Y$ can be resolved by the Koszul complex

$$0 \to \bigwedge^c \mathcal{G}^\vee \to \ldots \to \mathcal{G}^\vee \xrightarrow{s^\vee} \mathcal{O}_S \to \mathcal{O}_Y \to 0.$$

Hence, $\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z)$ is given by the cohomology sheaves of

$$0 \to \mathcal{H}om_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z) \to \mathcal{O}_Z \xrightarrow{s|_Z} \mathcal{G}|_Z \to \ldots \to \bigwedge^c \mathcal{G}|_Z \to 0. \tag{9.2}$$

Lemma 9.5 (cf. [CKS03, Proposition A.3]). *Assume that $Z \subset Y$. Then*

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z) \cong \bigwedge^q \mathcal{N}_{ZS}|_Y .$$

Proof. As $Z \subset Y$ all differentials in (9.2) are zero, hence the result. □

We say that Y and Z *intersect properly* if

$$\dim X + \dim S = \dim Y + \dim Z.$$

Lemma 9.6 (cf. [CKS03, Proposition A.4]). *Assume that Y and Z intersect properly. Then*

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z) \cong \begin{cases} 0 & \text{if } p \neq c \\ \det \mathcal{N}_{XZ} & \text{if } p = c \end{cases},$$

where $c = \operatorname{codim}(X \subset Z) = \operatorname{codim}(Y \subset S)$.

Proof. By assumption, $s|_Z$ remains regular (cf. [St17, TAG 00N6]) and we have $X = Z(s|_Z)$. Thus (9.2) is everywhere exact besides at the right end. There we find

$$\operatorname{coker}(\bigwedge^{c-1} \mathcal{G}|_Z \xrightarrow{-\wedge s|_Z} \bigwedge^c \mathcal{G}|_Z) \cong \bigwedge^c \mathcal{G}|_X$$

so that the Lemma follows from the isomorphisms

$$\mathcal{G}|_X \cong \mathcal{N}_{YS}|_X \cong \mathcal{N}_{XZ}.$$

□

Remark 9.7. Lemmas 9.5 and 9.6 remain true without the assumption $Y = Z(s)$. We will give the proof in the context of Lemma 9.5 copying the argument from [Hu06, Proof of Proposition 11.8]. Let $\mathcal{E}^\bullet \to \mathcal{O}_Y$ be any global locally free resolution. Then $\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z) = \mathcal{H}^p(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}^\bullet, \mathcal{O}_Z))$. Let $x \in X$ and consider the free resolution $\mathcal{E}^\bullet_x \to \mathcal{O}_{Y,x}$ of $\mathcal{O}_{S,x}$-modules. Locally around x we also have the Koszul resolution $\bigwedge^\bullet \mathcal{G}^\vee_x \to \mathcal{O}_{Y,x}$. Using projectivity of free modules, we obtain a morphism of complexes $\varphi \colon \mathcal{E}^\bullet_x \to \bigwedge^\bullet \mathcal{G}^\vee_x$ inducing an isomorphism

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z)_x \cong H^p(\mathcal{H}om_{\mathcal{O}_{S,x}}(\mathcal{E}^\bullet_x, \mathcal{O}_{Z,x})) \xrightarrow{\sim} H^p(\mathcal{H}om_{\mathcal{O}_{S,x}}(\bigwedge^\bullet \mathcal{G}^\vee_x, \mathcal{O}_{Z,x})) \cong \bigwedge^q \mathcal{N}_{ZS,x}.$$

The point is that this isomorphism does not depend on any choices and therefore leads to a global isomorphism

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z) \cong \bigwedge^q \mathcal{N}_{ZS}\,|_Y\ .$$

This is true because first of all any other choice of φ is homotopic to the original one and hence induces the very same map on cohomology. Secondly, the latter isomorphism is independet of the chosen Koszul resolution. If we choose a second Koszul resolution, defined by a section $\tilde{s} \in \tilde{\mathcal{G}}$, then any isomorphism $\mathcal{G} \xrightarrow{\sim} \tilde{\mathcal{G}}$ mapping s to $\tilde{s}$ induces the identity on $\mathcal{N}_{ZS}\,|_Y$. The general proof of Lemma 9.6 works analogously. We also conclude that the isomorphism in Lemma 9.6 depends only on the intersection $Y \cap Z$ and not on Y itself.

Lemma 9.8. *For any $x \in X$ there is an open neighborhood $U \subset S$ and a smooth subvariety $W \subset U$ with $Y \subset W$ such that*

(1) *$X \cap U = W \cap Z$ as schemes and*
(2) *W and Z intersect properly.*

Proof. See [CKS03, Proposition A.2]. □

Proof of Proposition 9.4. We prove the result locally on S first. Therefore, using the above Lemma we can assume that there is a smooth subscheme $W \subset S$ such that $Y \subset W$ and $X = W \cap Z$, where W and Z intersect properly. We have a spectral sequence

$$E_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_W}(\mathcal{O}_Y, \mathcal{E}xt^q_{\mathcal{O}_S}(\mathcal{O}_W, \mathcal{O}_Z)) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z).$$

and by Lemma 9.6 the only non-zero entries occur for $q \neq c$, where $c = \operatorname{codim}(X \subset Z)$ and are given by

$$\begin{aligned} E_2^{p,c} &= \mathcal{E}xt^p_{\mathcal{O}_W}(\mathcal{O}_Y, \det \mathcal{N}_{XZ}) \cong \mathcal{E}xt^p_{\mathcal{O}_W}(\mathcal{O}_Y, \mathcal{O}_X) \otimes \det \mathcal{N}_{XZ} \\ &\cong \bigwedge^p \mathcal{N}_{YW}\,|_X \otimes \det \mathcal{N}_{XZ} \end{aligned}$$

Here, we used Lemma 9.2 and Lemma 9.5. Next, the natural map

$$\mathcal{N}_{YW}\,|_X \cong \mathcal{T}_W\,|_X\,/\mathcal{T}_Y\,|_X \hookrightarrow \mathcal{T}_S\,|_X\,/\mathcal{T}_Y\,|_X \twoheadrightarrow \mathcal{T}_S\,|_X\,/(\mathcal{T}_Y\,|_X + \mathcal{T}_Z\,|_X)$$

is an isomorphism. Firstly, it is injective because $\mathcal{T}_Z\,|_X \cap \mathcal{T}_W\,|_X \subset \mathcal{T}_Y\,|_X$ and secondly, because X is reduced, we can check surjectivity fibrewise. Hence it suffices to compare dimensions. On the left hand side we find $\dim W - \dim Y$ which is equal to the right-hand side $\dim S - \dim Y - \dim Z + \dim X$ because W and Z intersect properly. Putting together, we have proved

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_Z) \cong \bigwedge^{p-c} \tilde{\mathcal{N}} \otimes \det \mathcal{N}_{XZ}. \tag{9.3}$$

Actually, this isomorphism does not depend on the choice of W (cf. [CKS03, Proof of Proposition A.5]) and along the lines of Remark 9.7 we conclude that the isomorphism (9.3) exists globally. □

9.2. **Application to Lagrangian intersections.** From now on we let (S,σ) be a symplectic manifold of dimension $2n$ containing two Lagrangian submanifold L, M with smooth intersection $X = L \cap M$.

Corollary 9.9. *We have*

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \cong \bigwedge^{p-c} \Omega_X \otimes \det \mathcal{N}_{XM},$$

where $c = n - \dim X$.

Proof. We saw in Lemma 6.1 that there is an exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_L|_X \oplus \mathcal{T}_M|_X \to \mathcal{T}_S|_X \to \Omega_X \to 0.$$

Hence, $\tilde{\mathcal{N}} \cong \Omega_X$. □

Corollary 9.10. *Assume that there exist square roots* $\omega_L^{1/2}$ *and* $\omega_M^{1/2}$. *Then*

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2}) \cong \bigwedge^{p-c} \Omega_X \otimes \mathcal{L}_{\mathrm{or}},$$

where $\mathcal{L}_{\mathrm{or}} = (\omega_L^{1/2}|_X \otimes \omega_M^{1/2}|_X)^\vee \otimes \omega_X$ *and* $c = n - \dim X$.

Proof. We have

$$\begin{aligned} \mathcal{E}xt^p_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2}) &\cong \mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \otimes \omega_L^{-1/2}|_X \otimes \omega_M^{1/2}|_X \\ &\cong \bigwedge^{p-c} \Omega_X \otimes \det \mathcal{N}_{XM} \otimes \omega_L^{-1/2}|_X \otimes \omega_M^{1/2}|_X \\ &= \bigwedge^{p-c} \Omega_X \otimes \mathcal{L}_{\mathrm{or}}, \end{aligned}$$

where we used $\det \mathcal{N}_{XM} \cong \omega_M^\vee|_X \otimes \omega_X$. □

Question 9.11. In particular, it follows from Corollary 9.2 that $\mathcal{E}xt^i(\omega_L^{1/2}, \omega_M^{1/2})$ is already defined if there are square roots of $\omega_L|_X$ and $\omega_M|_X$. On the other hand, for the defintion of the perverse sheaf $\mathcal{P}^\bullet_{L,M}$ it is even sufficient that a square root of $\omega_L|_X \otimes \omega_M|_X$ exists. In this case, we could set

$$\mathcal{E}xt^i(\omega_L^{1/2}, \omega_M^{1/2}) := (\omega_L|_X \otimes \omega_M|_X)^{-1/2} \otimes \omega_X,$$

which is well-defined if $\mathrm{Pic}(X)$ has no two-torsion. We wonder, whether we can also give a meaning to $\mathrm{Ext}^i(\omega_L^{1/2}, \omega_M^{1/2})$ in these cases?

Corollary 9.12. *We have*

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \otimes \omega_M|_X \cong \mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_L) \otimes \omega_L|_X$$

and

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2}) \cong \mathcal{E}xt^p_{\mathcal{O}_S}(\omega_M^{1/2}, \omega_L^{1/2}).$$

□

Recall that $\mathcal{P}^\bullet_{LM} \cong \mathcal{P}^\bullet_{ML}$ so it seems that $\dim \mathrm{Ext}^p_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2})$ is a better candidate than $\dim \mathrm{Ext}^p_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ for the comparison with $\dim \mathbb{H}^p(\mathcal{P}^\bullet_{L,M})$.

Example 9.13. In Example 8.1 we constructed an intersection $X \cong \mathbb{P}^1$ with $\mathcal{N}_{XM} \cong \mathcal{O}(-4)$ and $\mathcal{N}_{XL} \cong \mathcal{O}(4)$. We find

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \cong \begin{cases} \mathcal{O}(-4) \\ \mathcal{O}(-6) \\ 0 \end{cases}, \quad \mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_L) \cong \begin{cases} \mathcal{O}(4) & \text{if } p=1 \\ \mathcal{O}(2) & \text{if } p=2 \\ 0 & \text{else.} \end{cases}$$

Hence, the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{O}_L, \mathcal{O}_M)) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{O}_L, \mathcal{O}_M)$$

degenerates at the E_2-page yielding

$$\dim \operatorname{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \cong \begin{cases} 0 \\ 3 \\ 5 \end{cases}, \quad \dim \operatorname{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_L) \cong \begin{cases} 3 & \text{if } i=1 \\ 5 & \text{if } i=2 \\ 0 & \text{if } i=3. \end{cases}$$

In particular, there is no hope to compare this to

$$\dim \mathbb{H}^i(\mathcal{P}^\bullet_{LM}) = \dim H^{i-1}(\mathbb{P}^1, \mathbb{C}).$$

Proposition 9.14. *Assume that $X = L \cap M$ is smooth and compact. Let $\omega_L^{1/2}$ and $\omega_M^{1/2}$ be fixed orientations of L and M. Then, provided that the spectral sequence*

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\omega_L^{1/2}, \omega_M^{1/2})) \Rightarrow \operatorname{Ext}^{p+q}(\omega_L^{1/2}, \omega_M^{1/2}) \tag{9.4}$$

degenerates, we have

$$\dim \operatorname{Ext}^i_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2}) = \dim \mathbb{H}^{i-n}(\mathcal{P}^\bullet_{L,M}).$$

Here, the orientation $\mathcal{L}_{\mathrm{or}} = \omega_L^{-1/2}|_{X_{\mathrm{red}}} \otimes \omega_M^{-1/2}|_{X_{\mathrm{red}}} \otimes \omega_X|_{X_{\mathrm{red}}}$ serves for the computation of $\mathcal{P}^\bullet_{L,M}$.

Proof. We know from Lemma 7.6 that $\mathbb{H}^i(\mathcal{P}^\bullet_{L,M}) = H^{i+\dim X}(X, \mathfrak{L}_{\mathrm{or}})$, where $\mathfrak{L}_{\mathrm{or}}$ is the local system corresponding to $\mathcal{L}$. Moreover, we saw in Lemma 4.18 that there is a hermitian metric on $\mathcal{L}_{\mathrm{or}}$ with corresponding Chern connection ∇ such that $\mathfrak{L}_{\mathrm{or}} = \ker(\mathcal{L}_{\mathrm{or}} \xrightarrow{\nabla^{1,0}} \Omega^1_X \otimes \mathcal{L}_{\mathrm{or}})$. This yields a Hodge–de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_X \otimes \mathcal{L}_{\mathrm{or}}) \Rightarrow H^{p+q}(X, \mathfrak{L}_{\mathrm{or}})$$

degenerating at the E_1 page (cf. [Ba06, Theorem 5.32]). Therefore,

$$\dim H^i(X, \mathfrak{L}_{\mathrm{or}}) = \sum_{p+q=i} H^q(X, \Omega^p_X \otimes \mathcal{L}_{\mathrm{or}}).$$

On the other hand, we have by Corollary 9.10

$$\mathcal{E}xt^i(\omega_L^{1/2}, \omega_M^{1/2}) \cong \Omega_X^{i-c} \otimes \mathcal{L}_{\mathrm{or}},$$

where $c = n - \dim X$. Finally, if (9.4) degenerates, putting everything togehter gives

$$\begin{aligned} \dim \operatorname{Ext}^i(\omega_L^{1/2}, \omega_M^{1/2}) &= \textstyle\sum_{p+q=i} H^p(X, \mathcal{E}xt^q(\omega_L^{1/2}, \omega_M^{1/2})) \\ &= \textstyle\sum_{p+q=i} H^p(X, \Omega_X^{q-c} \otimes \mathcal{L}_{\mathrm{or}}) \\ &= \dim H^{i-c}(X, \mathfrak{L}_{\mathrm{or}}) \\ &= \dim \mathbb{H}^{i-n}(\mathcal{P}^\bullet_{L,M}). \end{aligned}$$

□

Remark 9.15. Independent of degeneration we have the equalities of the Euler characteristics

$$\begin{aligned} \textstyle\sum_i (-1)^i \operatorname{Ext}^i(\omega_L^{1/2}, \omega_M^{1/2}) &= \textstyle\sum_{i,j} (-1)^{i+j} \dim H^i(X, \mathcal{E}xt_{\mathcal{O}_S}(\omega_L^{1/2}, \omega_M^{1/2})) \\ &= \textstyle\sum_{i,j} (-1)^{i+j} \dim H^i(X, \Omega_X^{j-c} \otimes \mathcal{L}_{\mathrm{or}}) \\ &= (-1)^c \textstyle\sum_i (-1)^i \dim \mathbb{H}^i(\mathcal{P}^\bullet_{L,M}). \end{aligned}$$

References

[Ba06] W. Ballmann: *Lectures on Kähler manifolds.* ESI Lectures in Mathematics and Physics (2006).

[Be83] A. Beauville: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geometry **18** (1983) 755–782.

[BF05] K. Behrend, B. Fantechi: Symmetric obstruction theories and Hilbert schemes of points on threefolds. Algebra Number Theory 2 (2008), no. **3**, 313–345.

[BF09] K. Behrend, B. Fantechi: Gerstenhaber and Batalin-Vilkovisky structures on Lagrangian intersections. Algebra, arithmetic, and geometry: In honor of Yu. I. Manin. Vol. I, 1–47, Progr. Math. **269**, Birkhäuser Boston (2009).

[BBD] A. A. Beilinson, J. Bernstein, P. Deligne: Faisceaux pervers. Astérisque **100** (1983).

[BBDJS] C. Brav, V. Bussi, D. Dupont, D. Joyce, B. Szendrői: Symmetries and stabilization for sheaves of vanishing cycles. J. Singul. **11** (2015) 85–151.

[Bu14] V. Bussi: Categorification of Lagrangian intersections on complex symplectic manifolds using perverse sheaves of vanishing cycles arXiv:1404.1329v1 (2014).

[CKS03] A. Căldăraru, S. Katz, E. Sharpe: D-branes, B fields and Ext groups. Adv. Theor. Math. Phys. **7** (2003) 381–405.

[De70] P. Deligne: *Equations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics **163**. Springer (1970).

[Di04] A. Dimca: *Sheaves in topology.* Universitext. Springer (2004).

[SGA2] A. Grothendieck, M. Raynaud: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux. Documents Mathématiques **4**. Société Mathématique de France (1986).

[Hu06] D. Huybrechts: *Fourier–Mukai Transforms in Algebraic Geometry.* Oxford Mathematical Monographs (2006).

[HT10] D. Huybrechts, R. Thomas: Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes. Math. Ann. **346** (2010), 545–569.

[Joy13] D. Joyce: A classical model for derived critical loci. J. Differential Geom. **101** (2015), no. 2, 289–367.

[KL16] Y. Kiem, J. Li: Categorification of Donaldson–Thomas invariants via perverse sheaves. arXiv:1212.6444v5 (2016).

[Le03] J. M. Lee: *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics **218**. Springer (2003).

[Mi68] J. Milnor: *Singular Points of Complex Hypersurfaces.* Ann. of Math. Studies, **61**, Princeton Univ. Press (1968).

[St17] The Stacks Project authors. stacks.math.columbia.edu.

[We71] A. Weinstein: Symplectic manifolds and their Lagrangian submanifolds. Advances in Math. **6** (1971) 329–346.