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Hyperkähler manifolds and sheaves

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Abstract. Moduli spaces of hyperkähler manifolds or of sheaves on them are often non-separated. We will discuss results where this phenomenon reflects interesting geometric aspects, e.g. deformation equivalence of birational hyperkähler manifolds or cohomological properties of derived autoequivalences. In these considerations the Ricci-flat structure often plays a crucial role via the associated twistor space providing global deformations of manifolds and bundles.

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K3 surfaces and over the last ten years or so also their higher dimensional analogues, compact hyperkähler manifolds, have been studied intensively from various angles. As for abelian varieties, the interplay between algebraic, arithmetic, and complex geometric techniques makes the study of this particular class of varieties interesting and rewarding. In many respects, K3 surfaces and hyperkähler manifolds behave very much like abelian varieties, one can even pass from one to the other via the Kuga–Satake construction. There are however two features that are new: Non-separation (of various moduli spaces) and twistor spaces (associated to Ricci-flat metrics).

In a way, it is the group structure that prevents both issues from playing any role for abelian varieties. E.g. Ricci-flat metrics on complex tori are actually flat and hence without much geometric significance. As for the non-separation, we will discuss birational hyperkähler manifolds giving rise to non-separated points in the moduli space of varieties, whereas the group structure allows one to extend any birational correspondence between abelian varieties to an isomorphism right away.

The first is the more intriguing of the two features. Usually, non-Hausdorff phenomena, e.g. for an algebraic geometer non-separated schemes, are considered unpleasant and better avoided. As it turns out, the occurrence of non-separated points, e.g. in the moduli space of manifolds or of (complexes of) sheaves, can be turned into a useful technique applicable to various problems. This general idea seems to work best when combined with the existence of twistor spaces. The latter also allows one to go back and forth between algebraic and non-algebraic complex geometry. For purists this technique might be a weakness of the theory, but we shall try to convince the reader that it is indeed very powerful.

The aim of this note is to review a few scattered results for which non-separation phenomena and twistor spaces play a decisive role. We will touch upon questions concerning the birational geometry of hyperkähler manifolds, derived categories of coherent sheaves on K3 surfaces and their autoequivalences, Brauer classes, hyperholomorphic bundles, Chow groups, etc. There is no attempt at completeness and I apologize for not covering the material in a more concise form. I believe that some of the techniques can be pushed further to treat other interesting open problems in the area, some of which will be mentioned at the end.

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1. Introduction

To get an idea what kind of non-Hausdorff phenomena we have in mind let us recall the following two classical examples.

– The bundles E_t on the projective line $\mathbb{P}^1$ (say over a field k) parametrized by classes $t \in \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) \cong H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong k$ are isomorphic to $\mathcal{O} \oplus \mathcal{O}$ for $t \neq 0$ and to $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ for $t = 0$. In other words, there exists a vector bundle E on $\mathbb{P}^1 \times \mathbb{A}^1$ such that on all fibres of the projection $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ with the exception of the fibre over the origin the bundle is the trivial bundle of rank two. Equivalently, there exist two bundles E and E' on $\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ which are isomorphic on the open set $\mathbb{P}^1 \times \mathbb{A}^1 \setminus \{0\}$ but with different restrictions $E_0 \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$ respectively $E'_0 \cong \mathcal{O} \oplus \mathcal{O}$ to the special fibre. This classical observation can easily be translated into a more geometric non-separation phenomenon for Hirzebruch surfaces: $\mathbb{F}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ and $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ define non-separated points in the moduli space of varieties

– The Atiyah flop describes two crepant resolutions $Z \to Z_0 \leftarrow Z'$ of the three-dimensional rational double point $Z : xy - zw = 0$ both replacing the singular point by a $\mathbb{P}^1$. Equivalently, the blow-up $\tilde{Z} \to Z_0$ of the singular point admits two projections $Z \leftarrow \tilde{Z} \to Z'$ extending the two projections of the exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$. Put in a more global context this observation can be used to construct two non-isomorphic families of K3 surfaces $\mathcal{X} \to D \leftarrow \mathcal{X}'$ over a disk D isomorphic over the punctured disk D^*, i.e. $\mathcal{X}|_{D^*} \cong \mathcal{X}'|_{D^*}$. In particular, all fibres $\mathcal{X}_t, \mathcal{X}'_t$, $t \neq 0$, are isomorphic in a way compatible with the projection to D, but these isomorphisms do not converge to an isomorphism of the special fibres $\mathcal{X}_0$ and $\mathcal{X}'_0$. In fact, the graph Γ_t of the fibrewise isomorphism for $t \neq 0$ degenerates to a cycle $\Gamma_0 + \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathcal{X}_0 \times \mathcal{X}'_0$ with Γ_0 itself being, somewhat accidentally due to the small dimension, the graph of an isomorphism.

A compact *hyperkähler manifold* or irreducible holomorphic *symplectic manifold* is, by the definition we adopt here, a simply-connected compact complex Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by a nowhere degenerate two-form σ. Since all our manifolds will be compact, we simply call them hyperkähler. The definition can be adapted to projective varieties over other fields, but most of the existing theory is concerned with complex manifolds. K3 surfaces are the two-dimensional hyperkähler manifolds and for them there is also a rich theory over number fields and in finite characteristic.

What makes the complex case special is the Calabi–Yau theorem proving the existence of a unique Ricci-flat Kähler metric in each Kähler class on X. In fact, Ricci-flat Kähler metrics exist on the larger class of Calabi–Yau manifolds, but for hyperkähler manifolds they lead to a global complex geometric structure, the *twistor space*.

To be more precise, let $\mathcal{K}_X \subset H^2(X,\mathbb{R}) \cap H^{1,1}(X)$ denote the open cone of Kähler classes (among them all ample classes if X is projective). With any $\alpha \in \mathcal{K}_X$ there is associated a complex manifold $\mathcal{X}(\alpha)$ together with a smooth proper holomorphic map $\pi : \mathcal{X}(\alpha) \to \mathbb{P}^1$. One of the fibres, say $\mathcal{X}_0$ is actually isomorphic to X, but most of the other fibres are not.

Note that by construction the twistor space as a differentiable manifold is simply $X \times \mathbb{P}^1$ and π is the second projection. Moreover, the natural (twistor) sections $\{x\} \times \mathbb{P}^1$ of π are holomorphic with normal bundle $\mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(1)$.

2. Non-separation for hyperkähler manifolds

Birational K3 surfaces are always isomorphic, e.g. because the minimal model of a surface of non-negative Kodaira dimension is unique. In fact, any birational correspondence extends to an isomorphism. (Note that by abuse of language we will speak about birational maps etc. even when the manifolds are not algebraic and one should more accurately say bimeromorphic.)

In higher dimension the situation changes. The easiest example of a non-trivial birational correspondence between, in general non-isomorphic, hyperkähler manifolds has been constructed already in [20] and is called the Mukai flop. Any hyperkähler manifold containing a half-dimensional projective space can be flopped replacing the projective space $\mathbb{P}$ by its dual $\mathbb{P}^*$. The new manifold is holomorphic symplectic, but not always Kähler and hence not hyperkähler (see [28] for an example that starts with a projective moduli space of sheaves). In general and in particular in dim > 4, birational correspondences between hyperkähler manifolds will be more complicated than simple Mukai flops. But as it turns out, any birational correspondence between hyperkähler manifolds can be obtained as the limit of isomorphisms (see [9, 10]):

2.1. *Any two birational hyperkähler manifolds X and X' define non-separated points in the moduli space of varieties. Equivalently, there exist two smooth proper families $\mathcal{X} \to D \leftarrow \mathcal{X}'$ over a disk D with central fibres $\mathcal{X}_0 \cong X$ respectively $\mathcal{X}'_0 \cong X'$ and such that the two families are isomorphic over the punctured disk D^*, i.e. $\mathcal{X}|_{D^*} \cong \mathcal{X}'|_{D^*}$.*

This result had first been proved for projective hyperkähler manifolds and under an additional assumption on the codimension of the exceptional locus by projective techniques which are valid over arbitrary fields. Later, twistor spaces have been used instead to prove the result in the above form. Note that even for X and X' projective, the nearby fibres in the families in (2.1) are usually non-projective.

The result is intimately related to the description of the Kähler cone and its birational variant. For K3 surfaces the Kähler cone is determined by smooth rational curves and a less explicit version of this holds true also in higher dimensions. In particular, for generic hyperkähler manifolds, which do not admit any curves, the Kähler cone is maximal, i.e. coincides with the positive cone. For the general theory see the survey [6] and references therein. A detailed investigation of the shape of the ample cone in the known examples has been initiated by Hassett and Tschinkel, see e.g. [7].

Let us state explicitly the following immediate consequence of (2.1):

2.2. *Two birational hyperkähler manifolds are deformation equivalent. In particular, their Hodge, Betti, and Chern numbers coincide.*

The result was used to show that most of the known examples, with the exception of O'Grady's exceptional examples in dimension six and ten, are deformations of the two standard series provided by Hilbert schemes of points on K3 surfaces and generalized Kummer varieties.

Note that deformation equivalence does not hold for birational Calabi–Yau manifolds in general, which need not even be homeomorphic and might even have different Chern numbers (see [2, 25]). For general Calabi–Yau manifolds the result that comes close to (2.1) is due to Batyrev and Kontsevich and proves equality of Hogde and Betti numbers. Motivic integration originated by Kontsevich for this purpose has been developed to a beautiful general theory by Denef and Loeser (see [19]). Applied to birational Calabi–Yau manifolds X and X' it shows that the (infinite-dimensional) spaces of formal arcs $J(X)$ respectively $J(X')$ differ only by insignificant bits. For birational hyperkähler manifolds and the non-separating families $\mathcal{X}, \mathcal{X}'$ as in (2.1) one can consider the spaces $J_0(\mathcal{X})$ and $J_0(\mathcal{X}')$ of formal arcs with support in the central fibre. The twistor sections provide a canonical section of the projection $J_0(\mathcal{X}) \to X$ which should lead to a stratified isomorphism of X and X' (non-holomorphic on the exceptional locus). It would be interesting to incorporate the Ricci-flat metric in a stronger way into birational geometry of hyperkähler manifolds and also to extend some of it to general Calabi–Yau manifolds.

The graph Γ_t of the isomorphism of the general fibres $\mathcal{X}_t \cong \mathcal{X}'_t$ in (2.1) degenerates to a cycle $Z + \sum Y_i \subset X \times X'$ where Z is the original birational correspondence. This is reminiscent of the Atiyah flop. The additional components Y_i do not dominate the factors but are in general difficult to describe explicitly. E.g. in the case of a Mukai flop there is only one additional component which is simply $\mathbb{P} \times \mathbb{P}^*$. So, more in the spirit of our philosophy here, (2.1) says that up to adding non-dominating components any birational correspondence $X \leftarrow Z \to X'$ between hyperkähler manifolds can be deformed to an isomorphism of generic deformations of X respectively X'. Derived versions will be discussed later, see (4.2) and (5.1).

3. Twistor spaces

Deformation theory is a technical but well developed subject. The standard techniques deal with finite order or formal deformations. Convergence or algebraicity is usually more difficult. Global deformations of a variety X, i.e. a flat family $\mathcal{X} \to B$ with $\mathcal{X}_0 \cong X$ over a proper base B of positive dimension are hardly ever constructed explicitly. This makes twistor spaces stand apart. The twistor space $\mathcal{X} = \mathcal{X}(\alpha) \to \mathbb{P}^1$ associated with a Kähler class α on a hyperkähler manifold X connects X with other, possibly far away, hyperkähler manifolds $\mathcal{X}_t$. The price one has to pay is the loss of algebraicity. In fact, the total space $\mathcal{X}$ is not even Kähler and only countable many fibres $\mathcal{X}_t$ are projective. Nevertheless, it seems that essential information about the geometry of a projective hyperkähler manifold X is preserved along the twistor space deformation to other projective fibres.

Twistor spaces or almost equivalently hyperkähler metrics play a central role already in the standard theory of K3 surfaces and, partially due to the absence of a proper analogue of the Global Torelli theorem, even more so in higher dimensions (see [6, 10]). We will not go into the details of the general theory of hyperkähler manifolds, but let us mention that twistor spaces are crucial e.g. for the proof of the surjectivity of the period map and the description of the (birational) Kähler cone.

To underpin the global nature of twistor spaces let us just mention that for any polarized K3 surface (X, L), e.g. $X \subset \mathbb{P}^3$ a quartic and L the restriction of $\mathcal{O}(1)$, and any Kähler class on X, e.g. the one given by $\mathrm{c}_1(L)$, the associated twistor space will also parametrize polarized K3 surfaces (X', L') of other degrees, e.g. a double cover of the plane. In dimension four the twistor space can be used to connect e.g. the Hilbert scheme $\mathrm{Hilb}^2(S)$ of a K3 surface S with the Fano variety of lines on a cubic fourfold. The reason behind this observation is that the base of the twistor space yields a curve in the moduli space of marked hyperkähler manifolds whereas the other loci are of codimension one, which therefore are expected to intersect.

By construction, twistor spaces are associated to hyperkähler metrics. A similar relation exists, due to the work of Donaldson, Hitchin and others, between stable vector bundles and Hermite–Einstein metrics. A combination of both leads to the following result of Verbitsky [26] which applies to stable vector bundles with trivial first Chern class on K3 surfaces.

3.1. *Let X be a hyperkähler manifold and E a holomorphic bundle on X which is stable with respect to a Kähler class α. If the first and second Chern classes of E stay algebraic (i.e. of type $(1,1)$ resp. $(2,2)$) on the fibres of the associated twistor space $\mathcal{X}(\alpha) \to \mathbb{P}^1$, then E is hyperholomorphic, i.e. extends naturally to a holomorphic vector bundle on $\mathcal{X}(\alpha)$.*

The idea of the proof is to show that the curvature of the Hermite–Einstein connection on E is of type $(1,1)$ with respect to all complex structures associated to the Ricci-flat structure given by α. That this is controlled by the first two Chern classes is reminiscent of Simpson's observation that the vanishing of the second Chern character of a stable bundle implies its (projective) flatness. Then on each fibre $\mathcal{X}_t$ the $(0,1)$-part of the natural Hermite–Einstein connection defines the $\bar{\partial}$-operator for E on this fibre.

The result can be applied to cases where the first Chern class is not trivial or not even orthogonal to the Kähler class α, but then it is only $\mathbb{P}(E)$ that deforms and not the bundle E itself. In [11] this was used to prove that cohomological and geometric Brauer group coincide for K3 surfaces, a result well known for algebraic surfaces. Roughly, the idea is to follow a given cohomological Brauer class along a twistor space and show that it becomes trivial somewhere. (Picard and hence Brauer group jump in a countable and dense subset.) When the class is trivial one represents it by a stable vector bundle which deforms back to the original K3 surface as a projective bundle that represents the chosen Brauer class.

Verbitsky used his result to deduce that very general (and hence non-projective) K3 surfaces have equivalent abelian categories $\mathrm{Coh}(X)$. This is in contrast to Gabriel's result that the abelian category $\mathrm{Coh}(X)$ of an algebraic variety, or more generally any scheme, determines X, but confirms the believe that for non-algebraic manifolds the abelian category of coherent sheaves is too small. Note however that $\mathrm{Coh}(X)$ even for very general K3 surfaces is a very rich category due to the many stable bundles that continue to exist.

Another point of view on Verbitsky's result, already studied by Itoh and others, is that the moduli space of stable vector bundles on a K3 surface inherits a natural hyperkähler structure. Equivalently, the relative moduli space of stable bundles on the fibres of $\mathcal{X}(\alpha) \to \mathbb{P}^1$ is nothing but the twistor space of the moduli space on one fibre. Note however that this does not extend to the boundary, i.e. to the moduli space of (semi-)stable sheaves and hence does not allow one to construct the hyperkähler structure on the Hilbert scheme or on the moduli space of stable sheaves.

4. Non-separation for sheaves and complexes

That there are bundles that define, like $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ and $\mathcal{O} \oplus \mathcal{O}$ on $\mathbb{P}^1$, non-separated points in the moduli space of bundles is a common feature and not special to $\mathbb{P}^1$. Also the existence of non-trivial homomorphisms between the bundles (in both directions) is frequently observed even for simple bundles (see [23]). The moduli space of simple bundles on a variety, an algebraic space, is in general not expected to be separated. Only stability prevents sheaves of the same slope (or normalized Hilbert polynomial, or phase, etc.) to have non-trivial homomorphisms between each other and this leads to separated and in fact quasi-projective moduli spaces.

The situation seems easier for simple sheaves not allowing any deformation, they do define isolated and hence separated points in their moduli space. Recall that a sheaf F has no infinitesimal deformations if and only if $\mathrm{Ext}^1(F,F) = 0$. Simple sheaves with this property on a K3 surface X are called *spherical*, i.e. they satisfy $\mathrm{Ext}^*(F,F) = H^*(S^2,k)$. So in particular, two spherical sheaves F and F' on X will always define separated points in the moduli space of sheaves on X, but this changes if also deformations of X are allowed. For the rest of this section X will be a projective K3 surface.

4.1. *Suppose F and F' are spherical sheaves with the same numerical invariants on a K3 surface X. Then there exists a deformation $\mathcal{X} \to D$ of X over a disk D and two D-flat sheaves $\mathcal{F}$ and $\mathcal{F}'$ on $\mathcal{X}$ with isomorphic restrictions to $\mathcal{X}^* := \mathcal{X}|_{D^*}$ and special fibres F respectively F'.*

In fact, X can be deformed together with F and F' such that simple implies stable with respect to any Kähler class or polarization. A beautiful observation going back to Mukai says that moduli spaces of stable sheaves with fixed numerical invariants are irreducible (see e.g. [8] or the original [21]). This allows one to conclude that in particular the generic deformations of F and F' are isomorphic.

A rather straightforward consequence of this is that numerically equivalent spherical bundles can also not be distinguished by any other continuous invariants, e.g. they are also rationally equivalent, i.e. their Chern characters in $\mathrm{CH}^*(X)$ coincide. The result also holds for spherical objects in the derived category, see below.

The result can be generalized to sheaves on products of K3 surfaces. This is central for the proof of a conjecture of Szendrői [24] as we shall explain shortly. Let X be an algebraic K3 surface and let $\Phi := \Phi_{\mathcal{E}_0} : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$ be a linear exact autoequivalence of the derived category $\mathrm{D^b}(X) := \mathrm{D^b}(\mathrm{Coh}(X))$ given as a Fourier–Mukai transform $F \mapsto \mathrm{pr}_{2*}(\mathcal{E}_0 \otimes \mathrm{pr}_1^* F)$ with $\mathcal{E}_0 \in \mathrm{D^b}(X \times X)$. Then $\mathcal{E}_0$ is rigid, i.e. $\mathrm{Ext}^1(\mathcal{E}_0, \mathcal{E}_0) = 0$, but $\mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0)$ is of dimension 22. In [14] it was proved that X and Φ (or rather $\mathcal{E}_0$) can be deformed together to a very general K3 surface and an equivalence that can be written as a product of explicitly described autoequivalences (shifts and spherical twists $T_{\mathcal{O}}$) whenever the action of Φ on the cohomology of X allows it. Informally, it can be rephrased by saying that any autoequivalence acting trivially on cohomology is a degeneration of the identity. This should be compared to (2.1). As we will mention later, the group of cohomologically trivial autoequivalences is a very rich group. More in the spirit of this review we state the result as (see [14]):

4.2. *If $\Phi, \Phi' : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$ are two linear exact autoequivalences inducing the same action on $H^*(X, \mathbb{Z})$, then there exist formal deformations $\mathcal{X} \to \mathrm{Spf}(\mathbb{C}[[t]])$ of X and $(\tilde{\Phi}, \tilde{\Phi}')$ of (Φ, Φ') whose restrictions to the generic fibre $\mathcal{X}_K$ of $\mathcal{X}$ over $K := \mathbb{C}((t))$ are isomorphic Fourier–Mukai transforms up to shift and a power of the simple spherical twist $T_{\mathcal{O}}$.*

The assumption in (4.1) that the two sheaves have the same Chern characters in $H^*(X)$ is here replaced by Φ, Φ' inducing the same action on $H^*(X)$. In fact, any spherical sheaf F induces an autoequivalence T_F, the spherical twist, which on cohomology acts by reflection. In this sense, (4.2) is a generalization of (4.1), but due to the deformation theory involved its proof is rather more technical.

Note also that the deformation $\mathcal{X} \to \mathrm{Spf}(\mathbb{C}[[t]])$ used in [13] is the formal neighbourhood of a very generic twistor space $\mathcal{X}(\alpha) \to \mathbb{P}^1$ and thus highly non-algebraic. In particular, the generic fibre $\mathcal{X}_K$ does not exist as a projective variety.

Instead of working with rigid analytic varieties, [13] makes only use of $\mathrm{Coh}(\mathcal{X}_K)$ and its derived category which can both be constructed directly as quotients of $\mathrm{Coh}(\mathcal{X})$ respectively $\mathrm{D^b}(\mathcal{X})$ without ever defining $\mathcal{X}_K$.

The result (4.2) has interesting consequences. Firstly, in [14] it was proved that autoequivalences of K3 surfaces, thought of as mirrors of symplectomorphisms, behave as predicted by mirror symmetry (see [24]):

4.3. *If $\Phi : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ is a linear exact autoequivalence, then the induced action on $H^*(X,\mathbb{Z})$ preserves the natural orientation of any positive four-space.*

The orthogonal group $\mathrm{O}(H^*(X,\mathbb{R}))$ has four connected components and the result says that derived autoequivalences avoid two of them. The result completes earlier work of Mukai, Orlov, and others and allows one to describe the image of $\mathrm{Aut}(\mathrm{D^b}(X)) \to \mathrm{O}(H^*(X,\mathbb{Z}))$ explicitly as the group of orientation preserving Hodge isometries. This can be seen as a derived version of the Global Torelli theorem for automorphisms of K3 surfaces.

Secondly, since the Fourier–Mukai kernels $\mathcal{E}_0$ and $\mathcal{E}'_0$ of Φ respectively Φ' as in (4.2) cannot be separated in the larger moduli space of complexes on deformations of $X \times X'$, all their usual invariants will be the same. E.g. the action on cohomology determines the action on the much larger (at least over $\mathbb{C}$) Chow groups $\mathrm{CH}^*(X)$. Combined with Lazarsfeld's result that indecomposable curves on K3 surfaces are Brill–Noether general this leads to (see [15]):

4.4. *For $\rho(X) \geq 2$ all spherical complexes $F \in \mathrm{D^b}(X)$ take Chern classes in the Beauville–Voisin ring $R(X) \subset \mathrm{CH}^*(X)$.*

Recall that the Beauville–Voisin ring naturally splits the cycle map $\mathrm{CH}^*(X) \to H^*(X,\mathbb{Z})$ (see [1]) and for X defined over $\bar{\mathbb{Q}}$ it is conjectured (Bloch–Beilinson) to be the Chow ring of X over $\bar{\mathbb{Q}}$ (see [15]). The assumption on the Picard number $\rho(X)$ in (4.4) should be superfluous.

5. Open problems

5.1. It is generally believed that birational Calabi–Yau varieties are derived equivalent. The conjecture seems more accessible for hyperkähler manifolds. One approach could be to use (2.1) and put the two birational hyperkähler manifolds X and X' as special fibres of the same family and then construct the Fourier–Mukai kernel as a degeneration of the diagonal. How to degenerate the diagonal explicitly is not clear. This has been worked out in a few cases (e.g. [16, 22]) and progress in the non-compact situation has been made in [5]. One could wonder whether the autoequivalence can be produced without actually explicitly giving the Fourier–Mukai kernel $\mathcal{E}$. Again, the degeneration argument could be helpful, but since not a single Fourier–Mukai equivalence has ever been described without also giving its Fourier–Mukai kernel, this seems not obvious.

5.2. As explained, all equivalences in the kernel of the natural representation

$$\rho : \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \to \mathrm{O}(H^*(X, \mathbb{Z}))$$

can be obtained as degenerations of the diagonal on deformations of X (up to shift and twist $T_{\mathcal{O}}$). Conjecturally, $\ker(\rho)$ is described by Bridgeland [3] as the fundamental group of a certain period domain depending only on the Hodge structure of $H^2(X, \mathbb{Z})$. In particular, $\ker(\rho)$ is usually a non-residually finite group. Also, it should be viewed as the group of deck-transformations of the space of stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$. How exactly the spaces of stability conditions $\mathrm{Stab}(\mathcal{X}_t)$ on the generic deformation $\mathcal{X}_t$ of $X = \mathcal{X}_0$, which has been shown to be simply connected in [12], fit together and 'degenerate' to $\mathrm{Stab}(X)$ is unclear.

5.3. Can non-separation be avoided for hyperkähler manifolds? I believe it cannot and this should be seen as a good thing. E.g. hyperkähler manifolds giving rise to non-separated points are birational and non-isomorphic birational correspondences produce rational curves. The existence and the counting of rational curves on K3 surfaces is a highly interesting subject, see e.g. [4] and [17]. Clearly, if a hyperkähler manifold contains a rational curve, it cannot be hyperbolic as predicted by the Kobayashi conjecture. In fact, non-separated points should be dense in the moduli space of hyperkähler manifolds which could eventually prove non-hyperbolicity for all hyperkähler manifolds. Note that non-separation would also imply topological restrictions, e.g. $b_2 > 3$ which is widely expected but proved only in small dimensions.

5.4. Another interesting subject concerns the arithmetic of hyperkähler manifolds and whether certain arithmetic properties, e.g. to be defined over particular fields or to admit (many) rational points, is transferred along the twistor space from one algebraic fibre to another. (Compare the work of Hausel and Rodriguez-Villegas on moduli spaces of bundles on curves and the character variety.)

5.5. In analogy to (3.1) it would be interesting to define hyperholomorphic complexes, i.e. complexes of sheaves that naturally deform to the whole twistor space. The stability condition should be phrased in terms of Bridgeland stability. However, since (3.1) works only for bundles, one would also need to find a derived version for locally freeness. How exactly the Hermite–Einstein metric should come in seems unclear.

5.6. The general fibre of a twistor space is a rigid analytic variety. In [13] its category of sheaves was studied, and somehow identified with the variety. As a geometric object or as a category, it should be viewed as naturally associated to the Ricci-flat metric. However, in the construction only the formal neighbourhod of one twistor fibre was used and it would be interesting to see whether this leads to equivalent notions for all fibres. For an algebraic family it would just be the fibre over the generic point.

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