Derived categories of smooth projective varieties. Survey

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Basic questions

X = smooth projective variety $/\mathbb{C}$
$\rightsquigarrow \mathrm{D^b}(X) := \mathrm{D^b}(\mathrm{Coh}(X)) = \mathbb{C}$-linear triangulated category.

Basic questions:

- $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Leftrightarrow$??
- $\mathrm{Aut}(\mathrm{D^b}(X)) =$?

?? could be *geometric*, e.g. '$X \simeq X'$' or 'X, X' birational',
or *cohomological*, e.g. '$H^*(X) \simeq H^*(X')$'.

Variant: $\mathrm{D^b}(X, \alpha) := \mathrm{D^b}(\mathrm{Coh}(X, \alpha))$ with $\alpha \in \mathrm{Br}(X)$.

Ample $K_X^{\pm}$

Invariants from classification theory: $K_X := \Omega_X^{\dim(X)}$, $\mathrm{kod}(X, K_X) := \mathrm{trdeg} \bigoplus_{n \geq 0} H^0(X, K_X^n) - 1$.

Bondal, Orlov: Suppose $K_X^{\pm}$ is ample. Then

- $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Leftrightarrow X \simeq X'$
- $\mathrm{Aut}(\mathrm{D^b}(X)) \simeq \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$

Orlov: $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Rightarrow \mathrm{kod}(X) = \mathrm{kod}(X')$.

Kawamata: Suppose $\mathrm{kod}(X, K_X^{\pm}) = \dim(X)$. Then $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Rightarrow X, X'$ birational (K-equivalent).

$K_X \equiv 0$

If $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$ and $K_X^n \simeq \mathcal{O}_X$, then $K_{X'}^n \simeq \mathcal{O}_{X'}$.

Decomposition theorem: Suppose $0 = \mathrm{c}_1(X) \in H^2(X, \mathbb{R})$.
Then $\exists\ \widetilde{X} \longrightarrow X$ finite, étale (minimal) such that

$$\widetilde{X} \simeq \mathrm{A} \times \prod X_i \times \prod Y_i$$

with $\mathrm{A} =$ abelian variety; $X_i =$ hyperkähler manifold (HK);
$Y_i =$ Calabi-Yau manifold (CY).

- $X =$ **HK** if X is projective, $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$ with $\sigma : \mathcal{T}_X \xrightarrow{\sim} \Omega_X$ and $\pi_1(X) = \{1\}$.
- $Y =$ **CY** if $K_Y \simeq \mathcal{O}_Y$, $H^0(Y, \Omega_Y^i) = 0$ for $0 < i < \dim(Y) \geq 3$ and $\pi_1(Y) = \{1\}$.

Examples of HKs:

i) dim = 2: K3 surfaces.

ii) Hilbert schemes: $\mathrm{Hilb}^n(S)$ for $S =$ K3 surface.

iii) Generalized Kummer varieties: $\mathrm{K}^n(\mathrm{A})$ for $\mathrm{A} =$ abelian surface.
Fibre of $\sum : \mathrm{Hilb}^n(\mathrm{A}) \longrightarrow \mathrm{A}$.

iv) Certain moduli spaces of stable sheaves on K3 surfaces.
(Deformations of ii).)

iv) O'Grady's sporadic examples in dimension 6 and 10.
(Resolutions of singular moduli spaces.)

Question: Suppose $\mathrm{c}_1(X) = 0$ and $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$. Let

$$\mathrm{A} \times \prod X_i \times \prod Y_j \longrightarrow X \quad \text{and} \quad \mathrm{A}' \times \prod X_i' \times \prod Y_j' \longrightarrow X'$$

be the minimal covers. Is then

$$\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(\mathrm{A}'),$$

$$\mathrm{D^b}(X_i) \simeq \mathrm{D^b}(X'_{\sigma(i)}), \quad \text{and} \quad \mathrm{D^b}(Y_j) \simeq \mathrm{D^b}(Y'_{\tau(j)}) \ ?$$

A = abelian variety

Mukai: The Poincaré line bundle $\mathcal{P}$ on $\mathrm{A} \times \hat{\mathrm{A}}$ induces $\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(\hat{\mathrm{A}})$.

Easy: If $\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(X)$, then X is an abelian variety.

Orlov, Polishchuk:

- $\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(\mathrm{B}) \Leftrightarrow \mathrm{A} \times \hat{\mathrm{A}} \simeq \mathrm{B} \times \hat{\mathrm{B}}$ isometry.
- $\mathrm{Aut}(\mathrm{D^b}(\mathrm{A}))/(\mathbb{Z} \times \mathrm{A}(\mathbb{C}) \times \hat{\mathrm{A}}(\mathbb{C})) = \mathrm{Aut}(\mathrm{A} \times \hat{\mathrm{A}}, q_{\mathrm{A}})$

$X = \mathrm{HK}$

H., Nieper-Wißkirchen: If $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$, then $X' = \mathrm{HK}$.

- $(HH^*, HH_*) \simeq (HT^*, H^{*,*})$ (for $K \simeq \mathcal{O}$).
- $\mathrm{D^b}(X) \simeq \mathrm{D^b}(Y) \Rightarrow HH_*^*(X) \simeq HH_*^*(Y)$.
- $X = \mathrm{HK}$: $HH^2(X) \simeq H^2(\mathcal{O}_X) \oplus H^1(\mathcal{T}_X) \oplus H^0(\bigwedge^2 \mathcal{T}_X)$.
  Points $k(x) \in \mathrm{D^b}(X)$ do not deform in $H^0(\bigwedge^2 \mathcal{T}_X)$-direction.
- $Y = \mathrm{CY}$: $HH^2(Y) \simeq H^1(\mathcal{T}_Y)$.
  Points $k(y) \in \mathrm{D^b}(Y)$ deform in all directions.

$X = \mathrm{HK}$

H., Nieper-Wißkirchen: If $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$, then $X' = \mathrm{HK}$.

Known: If $\mathrm{D^b}(S) \simeq \mathrm{D^b}(S')$ with S, S' K3 surfaces, then $\mathrm{D^b}(\mathrm{Hilb}^n(S)) \simeq \mathrm{D^b}(\mathrm{Hilb}^n(S'))$ (cf. Ploog).

Conjecture (Bondal, Orlov): $X \sim X'$ birational HKs $\Rightarrow$ $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$.

Kawamata, Namikawa: OK for Mukai flops.

H.: Birational HKs are non-separated, i.e. $\exists$ deformations $\mathcal{X}, \mathcal{X}' \longrightarrow D = \text{disk}$, $\mathcal{X}_0 \simeq X$, $\mathcal{X}'_0 \simeq X'$ with $\mathcal{X}|_{D^*} \simeq \mathcal{X}'|_{D^*}$. In particular $H^{*,*}(X) \simeq H^{*,*}(X')$.

Open:

- $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Leftrightarrow$??
- $\mathrm{D^b}(\mathrm{K}^n(\mathrm{A})) \simeq \mathrm{D^b}(\mathrm{K}^n(\hat{\mathrm{A}}))$??

$Y = \mathrm{CY}$

Question: $\mathrm{D^b}(Y) \simeq \mathrm{D^b}(Y') \Rightarrow Y'$ CY ?

Conjecture (B/O): $Y \sim Y'$ birational CYs $\Rightarrow \mathrm{D^b}(Y) \simeq \mathrm{D^b}(Y')$.

Bondal, Orlov: OK for standard flops.

Bridgeland: OK in dimension three.

Question: $\mathrm{D^b}(Y) \simeq \mathrm{D^b}(Y') \Rightarrow h^{p,q}(Y) = h^{p,q}(Y')$?

Batyrev, Kontsevich: $Y \sim Y'$, $Y' = \mathrm{CY} \Rightarrow h^{p,q}(Y) = h^{p,q}(Y')$.

$\mathrm{Aut}(\mathrm{D^b}(X))$ for $K_X = \mathcal{O}_X$

- $E \in \mathrm{D^b}(X)$ is *spherical* if $\mathrm{Ext}^*(E,E) \simeq H^*(S^n, \mathbb{C})$.

Seidel, Thomas: $\mathrm{C}(E^\vee \boxtimes E \longrightarrow \mathcal{O}_\Delta)$ defines the *spherical twist*

$$T_E : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X), \quad F \longmapsto \mathrm{C}(\mathrm{Hom}^*(E,F) \otimes E \longrightarrow F).$$

- $E \in \mathrm{D^b}(X)$ is a $\mathbb{P}$-*object* if $\mathrm{Ext}^*(E,E) \simeq H^*(\mathbb{P}^n, \mathbb{C})$.

H., Thomas: $\mathrm{C}(\mathrm{C}(E^\vee \boxtimes E[-2] \longrightarrow E^\vee \boxtimes E) \longrightarrow \mathcal{O}_\Delta)$ defines the $\mathbb{P}$-*twist*

$$P_E : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X).$$

Examples: Line bundles on CYs and HKs are spherical respectively $\mathbb{P}$-objects. There is no spherical object E on a HK with $\mathrm{rk}(E) \neq 0$.

$X, X' =$ K3 surfaces

Mukai, Orlov: $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$
i) $\Leftrightarrow$ $T(X) \simeq T(X')$ Hodge isometry. ($T=$ transcendental lattice)
ii) $\Leftrightarrow$ $\widetilde{H}(X,\mathbb{Z}) \simeq \widetilde{H}(X',\mathbb{Z})$ Hodge isometry.
iii) $\Leftrightarrow$ $X' \simeq X$ or $\simeq$ moduli space of slope stable vector bundles.
iv) $\Leftrightarrow$ $\mathcal{A}_X(B+i\omega) \simeq \mathcal{A}_{X'}(B'+i\omega')$ (abelian categories).

H., Stellari: $\mathrm{D^b}(X,\alpha) \simeq \mathrm{D^b}(X',\alpha') \Leftrightarrow \widetilde{H}(X,\alpha,\mathbb{Z}) \simeq \widetilde{H}(X',\alpha',\mathbb{Z})$ Hodge isometry.

Hodge conjecture: Every isomorphism of Hodge structures $T(X)_{\mathbb{Q}} \simeq T(X')_{\mathbb{Q}}$ is algebraic.

$X =$ K3 surface

H., Macri, Stellari: The image of

$$\rho : \mathrm{Aut}(\mathrm{D^b}(X)) \longrightarrow \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))$$

is $\mathrm{O}_+(\widetilde{H}(X, \mathbb{Z}))$ (= group of Hodge isometries preserving the orientation of the positive directions).

For '$\supset$' use Mukai, Orlov, Hosono et al., Ploog.

Conjecture (Bridgeland): $\mathrm{Ker}(\rho) = \pi_1(\mathcal{P}_0^+(X))$.

$\mathcal{P}_0^+(X) =$ period domain for the space of stability conditions.

Stability

$\mathcal{T} = \mathbb{C}$-linear triangulated category (e.g. $\mathcal{T} = \mathrm{D^b}(X)$)
Fix $K(\mathcal{T}) \twoheadrightarrow K$ $(\mathrm{rk}(K) < \infty)$ (e.g. algebraic classes in $H^*(X)$).

Stability condition $\sigma \in \mathrm{Stab}(\mathcal{T})$: $=$ bounded t-structure with heart $\mathcal{A}$ and additive $Z : \mathcal{A} \longrightarrow K \longrightarrow \mathbb{C}$ such that:

- $Z(E) = r(E) \exp(i\pi\phi(E))$ with $\phi(E) \in (0, 1]$ and $r(E) > 0$.
- Every $E \in \mathcal{A}$ has a HN filtration with respect to ϕ.

Bridgeland: The projection $\pi : \mathrm{Stab}(\mathcal{T}) \longrightarrow K_{\mathbb{C}}^*$ is a local homeomorphism from each connected component Σ to a linear subspace $V_\Sigma \subset K_{\mathbb{C}}^*$.

Why?

Examples

i) Curves (Bridgeland, Macri, Okada)
ii) Enriques surfaces (Macri, Mehrotra, Stellari)
iii) del Pezzo surfaces, $\mathbb{P}^n$ (Macri)
iv) abelian surfaces (Bridgeland), generic tori (Meinhardt)
v) K3 surfaces (Bridgeland)

Open: Examples of stability conditions on abelian varieties (eg. product of elliptic curves), HKs, CYs.

Bridgeland: For K3 surface X there is a distinguished component $\Sigma \subset \mathrm{Stab}(X)$ such that:

- $\mathcal{P}_0^+(X) := \pi(\Sigma)$ admits explicit description.
- If $\Sigma = \mathrm{Stab}(X)$ and simply connected, then $\mathrm{Ker}(\rho) = \pi_1(\mathcal{P}_0^+(X))$.

$C =$ curve, $g(C) \geq 1$

$K(C) = \mathbb{Z} \oplus \mathrm{Pic}(C) \twoheadrightarrow K := \mathbb{Z} \oplus \mathrm{NS}(C) = \mathbb{Z} \oplus \mathbb{Z}.$

Example: Define stability function on $\mathrm{Coh}(C)$ by

$$Z(E) := -\deg(E) + i \cdot \mathrm{rk}(E)$$

Semi-stable objects of phase $\phi \in (0,1)$ are the semi-stable vector bundles E with $\cot(\pi\phi) = -\mu(E) = -d/r$.

Bridgeland, Macri: $\mathrm{Stab}(\mathrm{D^b}(C)) \simeq \widetilde{\mathrm{Gl}}_+(2,\mathbb{R})$ and $\mathrm{Stab}(C) \longrightarrow K^*_{\mathbb{C}}$ is universal covering $\widetilde{\mathrm{Gl}}_+(2,\mathbb{R}) \longrightarrow \mathrm{Gl}_+(2,\mathbb{R})$.

Generic K3

Bridgeland: Polarization $B + i\omega \rightsquigarrow$ torsion theory:

$$\mathcal{F} := \{E \mid \text{torsion free, } \mu_{\max} \leq (B.\omega),$$
$$\mathcal{T} := \{E \mid \mu_{\min}(E/T(E)) > (B.\omega)\}$$

$\rightsquigarrow$ Tilt is heart $\mathcal{A}(B + i\omega)$.
Then

$$Z(E) := \langle v(E), \exp(B + i\omega)\rangle$$

defines stability condition $\sigma_{B+i\omega}$.

H., Macri, Stellari: Suppose $\mathrm{Pic}(X) = 0$. Then

- $\mathrm{Stab}(X) = \bigcup T_{\mathcal{O}}^n \cdot \{\sigma_{B+i\omega}\} \cdot \widetilde{\mathrm{Gl}}_+(2, \mathbb{R})$.
- $\mathrm{Stab}(X)$ is connected and simply-connected.
- $\mathrm{Aut}(\mathrm{D^b}(X)) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Aut}(X)$.

X modulo $\mathrm{codim} \geq 2$

Let $\mathrm{Coh}(X) \twoheadrightarrow \mathrm{Coh}'(X)$ be the quotient by the subcategory of sheaves concentrated in $\mathrm{codim} \geq 2$ and

$$\mathcal{T} := \mathrm{D^b}(\mathrm{Coh}'(X)).$$

Fix $K(\mathcal{T}) = \mathbb{Z} \oplus \mathrm{Pic}(X) \twoheadrightarrow \mathbb{Z} \oplus \mathrm{NS}(X)$.

Example: For ω an ample class one defines a stability function with HN-property on $\mathrm{Coh}'(X)$ by

$$Z(E) := -(\mathrm{c}_1(E).\omega) + i \cdot \mathrm{rk}(E).$$

So $\mathrm{Stab}(\mathcal{T}) \neq \emptyset$.

Meinhardt, Partsch:

- $\mathrm{Coh}'(X)$ has homological dimension one.
- Each $\widetilde{\mathrm{Gl}}_+(2, \mathbb{R})$-orbit is a connected component in $\mathrm{Stab}(\mathcal{T})$.
- For $\rho(X) > 1$ there are infinitely many connected components.