Hyperholomorphic bundles

① Connection + curvature

M diffb mfd + $\mathbb{C}$-vb E $\rightsquigarrow$ Consider 3 additional structures

i) hermitian structure h: $E \times E \to \mathbb{C}$, i.e. $\forall x \in M$: h_x hermitian metric on E_x + h_x C^∞ fct of x — always exists!

ii) connection: $\nabla: \mathcal{A}^0(E) \to \mathcal{A}^1(E) = \mathcal{A}^0(E \otimes T^*_{\mathbb{C}} M)$ sheaf homom. ($\mathcal{A}^0$ = sheaf of C^∞-sections) st

Leibniz rule holds: $\nabla(f \cdot s) = df \otimes s + f \nabla(s)$ $\quad f \in \mathcal{A}^0$, $s \in \mathcal{A}^0(E)$ local

always exists!

iii) holomorphic structure $\bar\partial_E$: Assume $X = (M, I)$ cpl mfd

$\rightsquigarrow$ $T^*_{\mathbb{C}} M = T^* M^{1,0} \oplus T^* M^{0,1} \rightsquigarrow \mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$

$\bar\partial_E: \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$ st

$\bar\partial$-Leibniz rule: $\bar\partial_E(f \cdot s) = \bar\partial(f) \otimes s + f \bar\partial_E(s)$ & $\bar\partial_E^2 = 0$

$\rightsquigarrow$ $\mathcal{E} := \ker(\bar\partial_E)$ locally free coherent sheaf $\hat{=}$ holomorphic vb

(This uses Frobenius integrability etc, not easy)

Does not always exist!

Compatibility conditions

i) + ii): Hermitian connection: $dh(s_1, s_2) \overset{\circledast}{=} h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$, $\forall s_1, s_2$ local

always exists!

ii) + iii): $\bar\partial_E \leftrightarrow \nabla$: $\nabla^{0,1} = \bar\partial_E$

~~Chern connec~~

i) + ii) + iii): $h, \nabla, \bar\partial_E$: $\circledast$ + $\nabla^{0,1} = \bar\partial_E$ — Chern connection

Proposition (easy): $\bar\partial_E$, h given $\Rightarrow \exists^1$ Chern connection

Proof: Local calculation $\nabla = d + A$ $\quad \circledast$ $dH = A^t H + H \bar A$, A $(1,0)$

$\rightsquigarrow A = \bar H^{-1} \partial H$ unique $\square$

Curvature: $D: \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ connection $\rightsquigarrow$ $D: \mathcal{A}^1(E) \to \mathcal{A}^2(E)$

by $D(\alpha \otimes s) = d(\alpha) \wedge s - \alpha \wedge D(s)$

$\rightsquigarrow$ $F_D := D \circ D: \mathcal{A}^0(E) \to \mathcal{A}^2(E)$

Easy facts (computation):

- F_D is $\mathcal{A}^0$-linear, ie $F_D \in \mathcal{A}^2(\mathrm{End}\, E)$
- $D_{\mathrm{End}(E)}(F_D) = 0 \in \mathcal{A}^3(\mathrm{End}(E))$ Bianchi identity

Chern-Weil theory $c_1(E) = \frac{i}{2\pi} \mathrm{tr}(F_D) \in \mathcal{A}^2(M)$ closed $\rightsquigarrow c_1(E) \in H^2(M)$

$ch_2(E) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}(F_D \wedge F_D) \in \mathcal{A}^4(M)$ closed $\rightsquigarrow ch_2(E) \in H^4$

Curvature properties of special connections

- (E, h) + hermitian D $\Rightarrow$ $h(F_D s_1, s_2) + h(s_1, F_D s_2) = 0$

  Locally $F_D + \bar{F}_D^t = 0$, $F_D \in \mathcal{A}^2_{\mathbb{R}}(\underline{\mathrm{End}(E,h)})$ $\mathbb{R}$-bdl

  ($rk = 1 \Rightarrow \mathrm{End}(E,h) = i \cdot \mathbb{R}$)

- $(E, \bar{\partial}_E)$ with $D^{0,1} = \bar{\partial}_E$ $\Rightarrow$ $F_D \in (\mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1})(\mathrm{End}(E))$
- (E, D) s.t. $F_D^{0,2} = 0$ $\Rightarrow$ $\bar{\partial}_E := D^{0,1}$ defines holomorphic structure
- $(E, \bar{\partial}_E, h)$ + Chern connection $\Rightarrow$ $F_D \in \mathcal{A}^{1,1}_{\mathbb{R}}(\mathrm{End}(E,h))$

② Hermite-Einstein conditions

(X,ω) = cpt Kähler manifold & $(E, \bar\partial_E, h, \nabla = \text{Chern connection})$

$$\rightsquigarrow F_\nabla \in \mathcal{A}^{1,1}_{\mathbb{R}}(\mathrm{End}(E,h))$$

locally F_∇ matrix (α_{ij}) $\qquad \alpha_{ij} \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$

$$\rightsquigarrow \Lambda_\omega \alpha_{ij} \in \mathcal{A}^0_{\mathbb{R}}(X)$$

(Alternatively i.e. $\omega^{n-1} \wedge \alpha_{ij} = \omega^n \cdot (\Lambda_\omega \alpha_{ij})$ up to universal factor

$\cdot \ \alpha_{ij} = f \cdot \omega + \alpha_{ij}'$ primitive)

$(E, \bar\partial_E)$ h is HE metric if $\boxed{i \Lambda_\omega F_\nabla = \lambda \cdot id_E}$ for ∇ = Chern connection
($\rightsquigarrow \nabla$ is HE connection) $\qquad \lambda \in \mathbb{C}$

($\Leftrightarrow$ $i F_\nabla = (\lambda/n)\, \omega \cdot id_E + F_\nabla'$ locally = matrix of primitive (1,1)-forms

Computation using $c_1 = tr(F) \frac{i}{2\pi}$ $\Rightarrow$ $\lambda \cdot \int \omega^n = (2\pi) \cdot n\, \mu(E)$

$$\mu(E) := \frac{\int c_1(E)\, \omega^{n-1}}{rk E}$$

Examples:

i) ∇ flat connection, i.e. $F_\nabla = 0$. $\Rightarrow$ HE

ii) $rk E = 1$ $\Rightarrow$ $\exists$ HE metric

Pf $\cdot$ $i F_{\nabla_0} \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ closed $\forall$ Chern connection ∇_0 of h_0

$h := e^f h_0$ $\Rightarrow$ $F_\nabla = F_{\nabla_0} + \partial\bar\partial f$

$\cdot$ $\alpha \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$ closed $\Rightarrow$ $\alpha = \underbrace{\alpha_0}_{\text{harmonic}} + i\partial\bar\partial f \qquad f \in \mathcal{A}^0_{\mathbb{R}}(X)$

$\rightsquigarrow \exists h :$ $i F_\nabla$ harmonic (1,1) $\Rightarrow$ $= \lambda\omega +$ primitive

$\Rightarrow \Lambda_\omega i F_\nabla = \lambda$

Bogomolov–Lübke inequality: (X,ω) cpt Kähler, $(E,\bar\partial_E)$ + HE metric h. If $c_1(E)=0$ (primitive is enough), then

$$\int ch_2(E)\wedge\omega^{n-2}\le 0$$

and "$=0$" $\iff$ $F_\nabla=0$ (i.e. HE connection is flat)

Proof. $\int\omega^{n-1}\wedge c_1(E)=0 \Rightarrow F_\nabla$ = matrix (α_{ij}) α_{ij} primitive (1,1) forms, $\overline{F_\nabla}^t=-F_\nabla$

$$\Rightarrow \left(\frac{i}{2\pi}\right)^2 tr(F_\nabla\wedge F_\nabla)=\frac{1}{2\pi}tr(F_\nabla\wedge\overline{F_\nabla}^t)=\sum_{i,j}\alpha_{ij}\overline{\alpha_{ij}}$$

$0\neq\alpha$ primitive (1,1) form $\Rightarrow \int\alpha\bar\alpha\wedge\omega^{n-2}<0$ positive
HR pairing

$\square$

Remark: If $c_1(E)$ not primitive, consider $End(E)$ instead
$\Rightarrow \int ch_2(End(E))\wedge\omega^{n-2}\le 0$ | $End(E)$ flat $\iff$ $\mathbb{P}(E)$ flat

$$\left(\iff \int(2rk(E)c_2(E)-(rkE-1)c_1^2(E))\wedge\omega^{n-2}\ge 0\right)$$

Theorem (Kobayashi–Hitchin, Donaldson, Uhlenbeck–Yau)
Let $(E,\bar\partial_E)$ be a holomorphic VB on cpt Kähler mfd (X,ω)
E is σ-polystable $\iff$ E admits HE connection

Recall: E is σ-polystable $\iff$ · $E\cong\oplus E_i$ $\quad \mu(E_i)=\mu(E)$
· E_i σ-stable, i.e. $\forall F\subsetneq E$ $\mu(F)<\mu(E)$
$0<rkF<rkE$

Pf "$\Leftarrow$" idea

③ Hyperholomorphic Bundles

(X,ω) compact HK (eg K3) $X=(M,I)$ + g metric

$\omega=\omega_I := g(-,I-)$; g HK metric if $\exists\, J, K=IJ=-JI$

s.t. $\omega_J=g(-,J-)$, $\omega_K=g(-,K-)$ closed, $\sigma_J=\omega_K+i\omega_I$ hol. 2-f

$\rightsquigarrow$ $\mathcal{X}=M\times\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ holomorphic $\sigma_I=\omega_J+i\omega_K$

$\uparrow$

$\lambda=(aI+bJ+cK)$ $a^2+b^2+c^2=1$

$\mathcal{X}_\lambda=(M,\lambda)$

Def. $(E,\bar\partial_E)$ on (X,ω) is hyperholomorphic if $\exists\ \mathcal{E}$ hol. vb on $\mathcal{X}$, $\mathcal{E}|_{\mathcal{X}_I=X}\cong E$

Define $\mathbb{C}$-vb $\hat E$ on $\mathcal{X}=M\times\mathbb{P}^1$ as pull-back of E

if ∇ = connection on E $\rightsquigarrow$ $\hat\nabla:\mathcal{A}^0(\hat E)\to\mathcal{A}^1(\hat E)$ pull-back connection

Theorem (Verbitsky) E ω-polystable vb, $c_1(E)\in H^{1,1}((M,\lambda))$, $c_2(E)\in H^{2,2}(M,\lambda)$ for all λ $\Rightarrow$ E is hyperholomorphic

Proof "Easy case" X K3 $\Rightarrow$ only $c_1(E)\in H^{1,1}(X)_\lambda$ as condition

E ω-polystable $\Rightarrow$ $\exists h$. HE = metric $\rightsquigarrow$ ∇ Chern connection

$\rightsquigarrow$ $\hat\nabla$ $\rightsquigarrow$ $F_{\hat\nabla}=p_1^*F_\nabla\in\mathcal{A}^0(\Lambda^2T^*_{M\mathbb{C}}\otimes End(\hat E))$

$\bar\partial_{\hat E}:=\hat\nabla^{0,1}$ defines holomorphic structure if $F_{\hat\nabla}^{0,2}=0$

$\Leftrightarrow$ $F_{\hat\nabla}^{0,2}|_{\mathcal{X}_\lambda}=0$

$\rightsquigarrow$ Enough $(0,2)_\lambda$-part of F_∇ is trivial $\forall\lambda$ ($\Leftrightarrow$ $(0,2)_J$-part $=0$)

~~Recall then $\bar\partial_\lambda:=(0,1)_\lambda$-part of ∇ defines hol. str. on~~

$F=F_\nabla=(\alpha_{ij})$ α_{ij} has trivial $(0,2)_J$ part

$\Leftrightarrow$ $\sigma_J\wedge\alpha_{ij}=0$

$$\left.\begin{array}{l}\sigma_J=\omega_K+i\omega_I,\ \Lambda_{\omega_I}F\underset{HE}{=}0 \Rightarrow \omega_I\wedge\alpha_{ij}=0\\ \sigma_I=\omega_J+i\omega_K + F\in\mathcal{A}^{1,1}_{\mathbb{R}}(End(E,h)) \Rightarrow \omega_K\wedge F=0\end{array}\right\}\Rightarrow \sigma_J\wedge F=0$$

$\dim X > 2$. Combination of HB case + Bogomolov–Lübke

<u>Preparation</u>: $\dim = 2$ $\quad \Lambda^{2} = \Lambda^{1,1}_{pr} \oplus \underbrace{\mathbb{C}\omega_I \oplus \mathbb{C}\sigma_I \oplus \mathbb{C}\bar\sigma_I}_{= \mathbb{C}\omega_J \oplus \mathbb{C}\sigma_J \oplus \mathbb{C}\bar\sigma_J}$ pointwise

$\dim > 2$ More explicitly

$$\Lambda^2 = \Lambda^{1,1}_{pr} \oplus \boxed{\quad\quad} \oplus \underbrace{\mathbb{C}\omega_I \oplus \mathbb{C}\sigma_I \oplus \mathbb{C}\bar\sigma_I}_{= \mathbb{C}\omega_J \oplus \mathbb{C}\sigma_J \oplus \mathbb{C}\bar\sigma_J}$$

other $(2,0)_I + (0,2)_I$ stuff

wrt I $\quad\quad\quad\quad$ wrt J

$$\underset{\perp\omega_I}{\Lambda^{1,1}_{pr}} \oplus \underset{\perp\sigma_I}{(\Lambda^{2,0}\oplus\Lambda^{0,2})_{pr}} = \underset{\perp\omega_J}{\Lambda^{1,1}_{pr}} \oplus \underset{\perp\sigma_J}{(\Lambda^{2,0}\oplus\Lambda^{0,2})_{pr}}$$

$$F_\nabla = (A_I, 0) = (A_J, B_J)$$

$ch_2 \in H^{2,2}(X, J) \Rightarrow \sigma_J^{n-2}\bar\sigma_J^{n}\, ch_2 = 0$

$\Rightarrow \int tr(F_\nabla \wedge F_\nabla)\,\sigma_J^{n-2}\bar\sigma_J^{n} = 0$

$tr(F_\nabla \wedge F_\nabla) = -tr(F_\nabla \wedge \bar F_\nabla^{t}) = -tr((A_J + B_J) \wedge \overline{(A_J+B_J)}^{t})$

$\Rightarrow tr(F_\nabla \wedge F_\nabla)\,\sigma_J^{n-2}\bar\sigma_J^{n} = -tr(B_J \wedge \bar B_J^{t})\,\sigma_J^{n-2}\bar\sigma_J^{n}$

$B_J = (b_{ij}) \Rightarrow B_J \wedge \bar B_J^{t} = \sum b_{ij}\bar b_{ij}$ $\quad\quad b_{ij} \in (\Lambda^{2,0}\oplus\Lambda^{0,2})_{pr} \leftarrow \sigma_J$

Use "Hodge–Riemann relation": $\pm\, b\bar b\,\sigma_J^{n-2}\bar\sigma_J^{n} \cancel{\neq} > 0$ for $b \neq 0$

$\Rightarrow B_J = 0 \Rightarrow F_\nabla$ stays $(1,1)$ wrt J (and then all λ)

$\Rightarrow \bar\partial_J := (0,1)_J$-part of ∇ defines holo. str.

(Need: On $(\Lambda^{2,0}\oplus\Lambda^{0,2})_{pr} \cap \Lambda^{1,1}_{\cancel{pr}\, I}$ have HR rel.)