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EFFECTIVITY OF SEMI-POSITIVE LINE BUNDLES

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Abstract. We review work by Campana–Oguiso–Peternell [COP10] and Verbitsky [Ver10] showing that a semi-positive line bundle on a hyperkähler manifold admits at least one non-trivial section. This is modest but tangible evidence towards the SYZ conjecture for hyperkähler manifolds.

1. Main theorem and motivation

1.1. **Main theorem.** The following result was first proved in the non-algebraic setting by Campana–Oguiso–Peternell [COP10] and later, applying similar techniques, by Verbitsky [Ver10].

Theorem 1.1. *Any semi-positive line bundle L on a compact hyperkähler manifold is $\mathbb{Q}$-effective, i.e. $H^0(X, L^m) \neq 0$ for some $m > 0$.*

A line bundle L, say on a compact Kähler manifold, is *semi-positive* if it admits a smooth hermitian metric with semi-positive curvature. *Warning:* The term semi-positive is used with different meaning in other contexts.

Clearly, any ample line bundle is semi-positive, as due to Kodaira's theorem being ample is equivalent to admitting a hermitian metric with positive curvature. Also, semi-positive line bundles are nef. However, the converse is not true. There exist line bundles on projective manifolds which are nef but not semi-positive, e.g. one finds in [Har70, Thm. I.10.5] Mumford's example of a nef line bundle that is not semi-ample and in [DPS01, Sec. 2.5] an example of nef line bundles that is not semi-positive. However, the situation is expected to be better on compact hyperkähler manifolds (or, more generally, on Calabi–Yau manifolds).

Conjecture 1.2. *Any nef line bundle on a compact hyperkähler manifold is semi-ample, i.e. some positive power L^m is globally generated, and, in particular, semi-positive.*

There are two cases to be considered here: For a nef line bundle on a compact hyperkähler manifold either $q(L) > 0$ or $q(L) = 0$, where q is the Beauville–Bogomolov quadratic pairing on $H^2(X, \mathbb{Z})$. In the first case, L is nef and big, X is projective [Huy99, Thm. 3.11], and, therefore, L is semi-ample by the base-point free theorem [CKM88]. Hence, only the case of a nef line bundle L with $q(L) = 0$ needs to be dealt with and we shall restrict to this case in what follows.

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Remark 1.3. The naive idea to approach Theorem 1.1 and Conjecture 1.2 is of course to apply the hyperkähler Riemann–Roch formula $\chi(L) = \sum a_i q(L)^i$, see [Huy99, Huy03a], which for $q(L) = 0$ reduces to $\chi(L) = n + 1$. The problem now is that we have a priori no control over the higher cohomology groups $H^q(X, L)$ as the usual vanishing results do not apply. In fact, by applying a result of Matsushita [Mat05, Thm. 1.3] showing that $R^j f_* \mathcal{O}_X \simeq \Omega^j_{\mathbb{P}^n}$ for a Lagrangian fibration $f\colon X \longrightarrow \mathbb{P}^n$, we know that in this case $H^q(X, f^*\mathcal{O}(k)) \simeq H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k))$ for $k > 0$ by Bott vanishing and Leray spectral sequence. Hence, $H^q(X, f^*\mathcal{O}(k)) \neq 0$ for $k > q \leqslant n$.

1.2. **SYZ conjecture.** Assume that L is a non-trivial semi-ample line bundle with $q(L) = 0$ on a compact hyperkähler manifold X of dimension $2n$. Then the linear system $|L^m|$, $m \gg 0$, defines a Lagrangian fibration $X \twoheadrightarrow B$ over a normal base B of dimension n.

As any compact hyperkähler manifold X with $b_2(X) \geqslant 5$ deforms to a compact hyperkähler manifold X' that admits a nef line bundle of square zero, Conjecture 1.2 would thus confirm the following version of the Stromminger–Yau–Zaslov (SYZ) conjecture for Calabi–Yau threefolds.

Conjecture 1.4. *Every compact hyperkähler manifold is deformation equivalent to a compact hyperkähler manifold with a Lagrangian fibration.*

The conjecture has been verified for all known deformation types of compact hyperkähler manifolds. This is obvious for those deformation equivalent to Hilbert schemes of K3 surfaces or to generalized Kummer varieties. See [Rap07, Cor. 1.1.10], for those deformation equivalent to the examples of O'Grady in dimension six and ten.

1.3. **Notation.** Typically X will denote a compact hyperkähler manifold X of dimension $2n$, i.e. X is a simply-connected, compact Kähler manifold with $H^0(X, \Omega_X^2)$ spanned by an everywhere non-degenerate holomorphic two-form σ. The second cohomology $H^2(X, \mathbb{Z})$ is endowed with the Beauville–Bogomolov form q which is of signature $(3, b_2(X) - 3)$ and which satisfies $q(\alpha)^n = c_X \cdot \int \alpha^{2n}$ for all classes $\alpha \in H^2(X, \mathbb{Z})$ and some positive rational number $c_X \in \mathbb{Q}$, the Fujiki constant. The square of a class $\alpha \in H^{1,1}(X)$ can alternatively be computed as $q(\alpha) = \int \alpha^2 (\sigma\bar{\sigma})^{n-1}$ (up to a positive scaling factor not depending on α).

2. Preparation

We shall prepare the ground for the actual proof by recalling the main results and techniques that go into it.

2.1. **Hard Lefschetz theorem.** The following result is due to Mourougane [Mou99, Thm. 2.6].[1]

[1] In [Mou99] one finds the dual statement, namely that $H^q(X, \Omega_X^p \otimes F) \hookrightarrow H^{d-p}(X, \Omega_X^{d-q} \otimes F)$ is injective. Is there a typo in his result? Should he not assume F to be semi-negative?

Proposition 2.1. *Let L be a semi-positive line bundle on a compact Kähler manifold X of dimension d with Kähler class ω. Then the product with ω^q defines surjective maps*

$$H^0(X, \Omega_X^{d-q} \otimes L) \twoheadrightarrow H^q(X, \Omega_X^d \otimes L).$$

For $L \simeq \mathcal{O}_X$, this is the content of the Hard Lefschetz theorem which, in fact, asserts the bijectivity of the map. The techniques to prove the more general statement are similar.

If X is a hyperkähler manifold of dimension $d = 2n$ and so $\Omega_X^{2n} \simeq \mathcal{O}_X$, one obtains surjections

$$H^0(X, \Omega^{2n-q} \otimes L) \twoheadrightarrow H^q(X, L).$$

This allows one to turn the non-vanishing of higher cohomology groups of L into the existence of global sections of powers of L.

Remark 2.2. (i) The result fails if L is only assumed to be nef without having a semi-positive metric, see [DPS01, Sec. 2.5] for an example. In this case, there is a variant of the above due to Takegoshi [Tak97, Thm. 1] for nef line bundles and to Demailly–Peternell–Schneider [DPS01, Thm. 2.1] for pseudo-effective line bundles: For a line bundle L on a compact Kähler manifold X of dimension d with a singular hermitian metric h with semi-positive curvature current the product with ω^q defines a surjection

$$H^0(X, \Omega_X^{d-q} \otimes L \otimes \mathcal{I}(h)) \twoheadrightarrow H^q(X, \Omega_X^d \otimes L \otimes \mathcal{I}(h)), \tag{2.1}$$

where $\mathcal{I}(h)$ denotes the multiplier ideal sheaf.

(ii) Due to [Tak97, Thm. 2], for $q > n$ the morphism $H^q(X, L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, L)$ induced by the inclusion $L \otimes \mathcal{I}(h) \subset L$ is the zero map for any nef line bundle L on a compact hyperkähler manifold X of dimension $2n$. In fact, according to another result of Verbitsky [Ver07, Thm. 1.6], one has $H^q(X, L) = 0$, $q > n$, for any nef and, more generally, for any pseudo-effective line bundle L.

2.2. **Finiteness of non-polar hypersurfaces.** An integral hypersurface $Y \subset X$ of a compact complex manifold is called *polar* if there exists a meromorphic function $f \in K(X)$ that has a pole along Y, i.e. Y is contained in the pole divisor $(f)_\infty$ of f. On a projective manifold, every integral hypersurface is polar. However, for general non-projective manifolds this fails, but the following result was proved by Fischer–Forster [FF79] and in the case that $K(X) = \mathbb{C}$ by Krasnov [Kra75].

Proposition 2.3. *A compact connected complex manifold X contains at most finitely many integral hypersurfaces $Y \subset X$ that are not polar. More precisely, the number of non-polar hypersurfaces is bounded by $h^{1,1}(X) + \dim(X) - h^{1,0}(X)$.*

For the proof one needs the following elementary but useful observation, see [Kra75, Prop. 1].

Lemma 2.4. *Let E be a vector bundle of rank r on X. Then the space of meromorphic sections of E is of dimension at most r, i.e.*

$$\dim_{K(X)} H^0(X, E \otimes \mathcal{K}_X) \leqslant \mathrm{rk}(E),$$

where $\mathcal{K}_X$ is the sheaf or rational (meromorphic) functions and $K(X) = H^0(X, \mathcal{K}_X)$ is the function field of X. In particular, if $K(X) = \mathbb{C}$, then for any vector bundle E one has $h^0(X, E) \leqslant \mathrm{rk}(E)$.

Proof. Suppose there exist sections $s_1, \ldots, s_{r+1} \in H^0(X, E \otimes \mathcal{K}_X)$ linearly independent over $K(X)$. Then there is a proper closed analytic subset such that all sections s_i are holomorphic on its open complement $U \subset X$ and such that (after renumbering) the sections $s_1, \ldots, s_{r'}$ span the subspace $\langle s_1(x), \ldots, s_r(x) \rangle \subset E(x)$ of constant (maximal) dimension r' at every point $x \in U$. In particular, on U we can write $(*)$ $s_r = \sum_{i=1}^{r'} a_i \cdot s_i$ for certain holomorphic functions $a_i \in \mathcal{O}_X(U)$.

It suffices to check that the a_i are meromorphic functions which is a local question. Thus, we may think of the s_i as vectors $s_i = (s_j^i)_{j=1,\ldots,r}$ of meromorphic functions and view (a_i) as a solution of the system of linear equations $(*)$. Expressing (a_i) in terms of the adjoint matrix and the vector (s_j^r) proves that all a_i are indeed meromorphic. $\square$

Proof of proposition. We shall follow [Kra75] and assume $K(X) = \mathbb{C}$. This is the only case that will be needed for Corollary 2.5 and its application later on. For the general case we refer to [FF79].

Applying $d\log$, the sheaf of complexified Cartier divisors $\mathcal{K}_X^*/\mathcal{O}_X^* \otimes_{\mathbb{Z}} \mathbb{C}$ is identified with the the quotient of $\Omega_X^1 \subset \Omega_{X,\log}^1$, where the latter sheaf is by definition locally generated by all holomorphic one-forms and logarithmic one-forms $d\log f$ with f a local section of $\mathcal{K}_X^*$. Taking cohomology yields a long exact sequence

$$0 \longrightarrow H^0(X, \Omega_X^1) \longrightarrow H^0(X, \Omega_{X,\log}^1) \longrightarrow \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^1(X, \Omega_X) \longrightarrow \cdots .$$

Since $H^0(X, \Omega_{X,\log}^1) \subset H^0(X, \Omega_X^1 \otimes \mathcal{K}_X)$ and since we assume $K(X) = \mathbb{C}$, the lemma implies $\dim_{\mathbb{C}}(\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{C}) \leqslant h^1(X, \Omega_X^1) + \dim(X) - h^0(X, \Omega_X^1)$. $\square$

Corollary 2.5. *If a compact complex manifold X contains infinitely many integral hypersurfaces, then its algebraic dimension satisfies $a(X) > 0$, i.e. $K(X) \neq \mathbb{C}$.* $\square$

2.3. **Sections of twists of vector bundles.** Proposition 2.1 does not directly produce sections of powers of L. For this one needs the following result due to Demailly–Peternell–Schneider [DPS01, Prop. 2.15]

Proposition 2.6. *Let L be a line bundle and E a vector bundle (or, more generally, a torsion free sheaf) on a compact complex manifold X. Assume m_i is an unbounded sequence of positive integers such that $H^0(X, E \otimes L^{m_i}) \neq 0$.*

(i) *Then there exists a line bundle M on X and an unbounded sequence m_i' of positive integers such that $H^0(X, M \otimes L^{m_i'}) \neq 0$.*

(ii) *There exist infinitely many integral hypersurfaces $Y \subset X$, and in particular $K(X) \neq \mathbb{C}$, or L is $\mathbb{Q}$-effective.*

Proof. Let $F \subset E$ be the coherent subsheaf of E generated by all homomorphisms $L^{-m_i} \longrightarrow E$. By assumption, F is non-trivial, so F is a torsion free sheaf of positive rank $r := \mathrm{rk}(F) > 0$. Furthermore, we may assume that there exist integers $n_1, \ldots, n_{r-1}$ (among the m_i) and an unbounded subsequence m_j of m_i for which there exist injections $L^{-n_1} \oplus \cdots \oplus L^{-n_{r-1}} \oplus L^{-m_j} \hookrightarrow F$.

Taking determinants yields non-trivial global sections $s_j \in H^0(X, M \otimes L^{m_j'})$, where $M := \det(F)$ and $m_j' := \sum n_i + m_j$. This proves (i).

Let us turn to (ii). There is nothing to prove in the case that X is projective or weaker that $K(X) \neq \mathbb{C}$. So we assume that X contains only finitely many integral hypersurfaces $Y_1, \ldots, Y_N \subset X$. Then the zero loci $Z(s_j) \subset X$ as Weil divisors are in the convex hull of the finitely many Y_i, i.e. $M \otimes L^{m_j'} \in \sum \mathbb{Z}_{\geqslant 0} \cdot \mathcal{O}(Y_i)$. As the sequence m_j' is unbounded, also $L^{m_j''} \in \sum \mathbb{Z}_{\geqslant 0} \cdot \mathcal{O}(Y_i)$ for an unbounded sequence m_j'' and, in particular, L is $\mathbb{Q}$-effective. $\square$

A priori, X could contain only one integral hypersurface $Y \subset X$ and all sections of powers of L^m are of the form s^m for some $s \in H^0(X, L)$ with $Z(s) = Y$. In other words, the above result only ensures the existence of one non-trivial section of the line bundles L^m up to passing to powers, which is not enough to make progress on Conjecture 1.2.

2.4. **Cones on hyperkähler manifolds.** We recall some basic notations and facts concerning the various cones relevant for the arguments below.

The *positive cone* $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ of a compact hyperkähler manifold X is the connected component of the open set of all classes $\alpha \in H^{1,1}(X, \mathbb{R})$ with $q(\alpha) > 0$. It contains the *Kähler cone* $\mathcal{K}_X \subset \mathcal{C}_X$ of all Kähler classes as an open subcone. The closure of the Kähler cone $\overline{\mathcal{K}}_X \subset \overline{\mathcal{C}}_X$, the *nef cone*, is the set of all classes $\alpha \in \overline{\mathcal{C}}_X$ with $\int_C \alpha \geqslant 0$ for all rational curves $C \subset X$, cf. [Huy03b, Prop. 3.2], and the open Kähler cone $\mathcal{K}_X \subset \mathcal{C}_X$ is the set of all classes $\alpha \in \mathcal{C}_X$ with $\int_C \alpha > 0$ for all rational curves $C \subset X$, cf. [Bou01, Thm. 1.2].

The *birational Kähler cone* $\mathcal{BK}_X$ is by definition the union $\bigcup \mathcal{K}_{X'}$ of all Kähler cones of birational compact hyperkähler manifolds X'. Here, we use that any birational correspondence $X \sim X'$ induces a natural Hodge isometry $H^2(X, \mathbb{Z}) \simeq H^2(X', \mathbb{Z})$, cf. [Huy03a, Prop. 25.14]. Clearly, $\mathcal{BK}_X \subset \mathcal{C}_X$ and according to [Huy03a, Prop. 28.7] its closure $\overline{\mathcal{BK}}_X \subset \overline{\mathcal{C}}_X$, the *modified nef cone*, is the set of all classes $\alpha \in \overline{\mathcal{C}}_X$ with $q(\alpha, D) \geqslant 0$ for all uniruled divisors. Of course, it suffices to test this for *prime exceptional divisors*, i.e. irreducible divisors $D \subset X$ with $q(D) < 0$. Alternatively, $\overline{\mathcal{BK}}_X$ can be described as the dual of the pseudo-effective cone $\mathcal{E}_X$ of all classes $\alpha \in H^{1,1}(X, \mathbb{R})$ that can be represented by a positive current, see [Huy03b, Cor. 4.6]. In particular, all effective divisors $D \subset X$ define classes in $\mathcal{E}_X$. Note that in particular $q(L, D) \geqslant 0$ for any nef line bundle L and any effective divisor $D \subset X$.

According to Boucksom [Bou04, Thm. 4.8], any pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ admits a *Zariski decomposition* $\alpha = P(\alpha) + N(\alpha)$, where $P(\alpha) \in \overline{\mathcal{BK}}_X$ and $N(\alpha)$ is the class of an exceptional $\mathbb{R}$-divisor, i.e. $N(\alpha) = \sum a_i D_i$ with $D_i \subset X$ irreducible divisors such that the matrix $(q(D_i, D_j))$ is negative definite. Furthermore, $P(\alpha)$ and $N(\alpha)$ are orthogonal, i.e. $q(P(\alpha), N(\alpha)) = 0$. We shall need the Zariski decomposition only for divisor classes $\alpha \in H^{1,1}(X, \mathbb{Z})$ and in this case $P(\alpha)$ and $N(\alpha)$ are in fact rational.

2.5. **Stability of the tangent bundle.** Due to existence of a Kähler–Einstein metric in each Kähler class, the tangent bundle $\mathcal{T}_X$ of a compact hyperkähler manifold X is μ-stable with respect to any Kähler class $\omega \in \mathcal{K}_X$. In fact, stability holds with respect to all ω in the interior of $\overline{\mathcal{BK}}_X$, see also Section 5.1.

Proposition 2.7. *Let X be a compact hyperkähler manifold and $M \subset \Omega_X^{\otimes N}$ a line bundle in some tensor power of its cotangent bundle. Then the dual M^* is pseudo-effective.*

Proof. In the projective case, the assertion is a consequence of a general result due to Campana–Peternell [CP11, Thm. 0.1] showing that any torsion free quotient $(\Omega_X^1)^{\otimes q} \twoheadrightarrow \mathcal{F}$ has a pseudo-effective determinant $\det(\mathcal{F})$ unless X is uniruled.

Verbitsky [Ver10] gives an alternative argument relying on the observation that all tensor powers $\Omega_X^{\otimes N}$ of the cotangent bundle are μ-semistable with respect to any class in the birational Kähler cone. More precisely, let $\alpha \in \mathcal{BK}_X$ be a class corresponding to Kähler class $\omega' \in \mathcal{K}_{X'}$ on some birational model X' of X. Since X and X' are isomorphic in codimension one, the inclusion $M \subset \Omega_X^{\otimes N}$ carries over to an inclusion $M' \subset \Omega_{X'}^{\otimes N}$. To conclude use the stability of $\Omega_{X'}$, which proves $q(\alpha, M) = q(\omega', M') \leqslant 0$. Hence, $q(\alpha, M^*) \geqslant 0$ for all $\alpha \in \overline{\mathcal{BK}}_X$, i.e. M^* is pseudo-effective. □

3. Proofs

In this section we present two proofs of the main theorem. The original of Campana–Oguiso–Peternell [COP10] applies only to the case that the hyperkähler manifold is non-projective. Verbitsky [Ver10] showed how to combine the original approach with Boucksom's Zariski decomposition to also cover the algebraic case. In the next section we will sketch a different approach that reduces the projective case to the non-projective one.

3.1. **Non-algebraic case.** We follow the arguments in [COP10].

Proof. Assume L is a non-trivial nef line bundle on a non-projective compact hyperkähler manifold X of dimension $2n$ and assume $q(L) = 0$. Suppose $H^0(X, L^m) = 0$ for all $m > 0$. The Riemann–Roch formula [Huy99, Huy03a] simply states $\chi(X, L^m) = n + 1$. Thus, there exists an even number $q > 0$ and an unbounded sequence m_i of positive integers such that $H^q(X, L^{m_i}) \neq 0$. By virtue of Proposition 2.1 this implies $H^0(X, \Omega_X^{2n-q} \otimes L^{m_i}) \neq 0$.

Combining Corollay 2.5 and Proposition 2.6, we conclude that L is $\mathbb{Q}$-effective, in which case we are done, or $K(X) \neq \mathbb{C}$. For example, the former case holds if $\rho(X) = 1$. In the latter case, the algebraic reduction [Uen75, Ch. 3] provides us with a diagram

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[rr, dashed, "f"'] & & B. \end{tikzcd}$$

Here, $\tilde{X}$ is a compact complex manifold, B is smooth and projective of dimension at least one, and π is birational. The pull-back $\tilde{f}^*H$ of a very ample line bundle H on B can be written as $\tilde{f}^*H \simeq \pi^*M \otimes \mathcal{O}(-E)$ with $E \subset \tilde{X}$ effective, in fact π-exceptional but possibly trivial, and $M \in \mathrm{Pic}(X)$. This yields inclusions

$$H^0(B, H) \hookrightarrow H^0(\tilde{X}, \tilde{f}^*H) \hookrightarrow H^0(\tilde{X}, \pi^*M) \simeq H^0(X, M).$$

Since $\dim(B) \geqslant 1$, this shows that M is non-trivial and effective. In fact, as H is very ample, we may assume that M admits two linearly independent sections with distinct zero divisors $D_1, D_2 \subset X$ without common irreducible components.

According to [Bou04, Prop. 4.2], for any two such divisors $D_1, D_2 \subset X$ we have $q(D_1, D_2) \geqslant 0$. Indeed, up to a positive scalar $q(D_1, D_2) = \int_{D_1 \cap D_2} (\sigma\bar{\sigma})^{n-1} \geqslant 0$, since $(\sigma\bar{\sigma})^{n-1}$ is a positive form. Applied to our situation this yields $q(M) \geqslant 0$. On the other hand, since X is assumed to be non-projective, the projectivity criterion [Huy99, Thm. 3.11] implies $q(M) \leqslant 0$. Therefore, $q(M) = 0$. However, as the form q restricted to $H^{1,1}(X, \mathbb{R})$ satisfies the Hodge index theorem, every line bundle M on X with $q(M) = 0$ is a rational multiple of L. As L was assumed semi-positive (hence, nef) and M is effective, M is a positive rational multiple of L. Therefore, L is $\mathbb{Q}$-effective. $\square$

3.2. **Algebraic case.** In fact, the following arguments taken from [Ver10] apply also to non-algebraic hyperkähler manifolds and thus subsume the original proof in [COP10]

Proof. The first part of the proof is identical to the one in the non-algebraic case. From Proposition 2.6 we deduce the existence of a line bundle M with $H^0(X, M \otimes L^{m_i}) \neq 0$ for an unbounded sequence of positive integers. Since L is nef with $q(L) = 0$, this implies $q(M \otimes L^{m_i}, L) \geqslant 0$ and $q(M, L) \geqslant 0$. On the other hand, the line bundle M in the proof of Proposition 2.6 was constructed as the determinant $M = \det(F)$ of a subsheaf $F \subset E = \Omega_X^{2n-q}$ and, therefore, $M \subset \Omega_X^{\otimes N}$ for some N. Then, Proposition 2.7 shows that M^* is pseudo-effective and, hence, $q(M, L) \leqslant 0$, see Section 2.4. Therefore, $q(M, L) = 0$. In the case that $\rho(X) = 2$, we can conclude already that M is a rational multiple of L and that, therefore, L is $\mathbb{Q}$-effective.

For $\rho(X) > 2$, we consider the Zariski decomposition of the pseudo-effective line bundle M^* as $P + N$ with P contained in the closure of the birational Kähler cone and N exceptional effective. In particular, $q(P) \geqslant 0$ with $q(P, L) > 0$ unless P is a rational multiple of L and $q(N) < 0$ unless $N = 0$. Then, $0 = q(L, M) = q(L, P) + q(L, N)$ with both summands non-negative and, therefore, both zero. Thus, the Zariski decomposition of M^* is of the form $\lambda L + N$, with $\lambda \in \mathbb{Q}_{\geqslant 0}$. On the other hand, $M \otimes L^{m_i}$ is effective for an unbounded sequence of positive integers m_i. Hence, $(m_i - \lambda)L$ can be written as the sum of the two effective divisors $M \otimes L^{m_i}$ and N. Therefore, L itself is $\mathbb{Q}$-effective. $\square$

4. Semi-positivity under deformations

We will now show that alternatively the proof in the algebraic case can be reduced via deformation to the non-algebraic case. The techniques are potentially relevant to make progress on Conjecture 1.2.

First recall that for a smooth proper family $\mathcal{X} \longrightarrow \Delta$ is of complex manifolds with central fibre $X = \mathcal{X}_0$ of Kähler type, all nearby fibres $\mathcal{X}_t$ are of Kähler type as well, i.e. for all t after shrinking Δ to an open neighbourhood of $0 \in \Delta$. More precisely, if the Kähler class on X stays of type $(1,1)$ on the nearby fibres, then it is Kähler there as well. This classical result is due to Kodaira and Spencer [KS60]. Note that since the Kähler property is a combination of the open condition that a real $(1,1)$-form ω is positive and the closed condition $d\omega = 0$, this is a priori not clear. In the case of closed semi-positive forms ω, the corresponding statement fails. Similarly, if $\alpha \in H^{1,1}(X, \mathbb{R})$ is a nef class that stays of type $(1,1)$ on all the fibres $\mathcal{X}_t$, as a class on $\mathcal{X}_t$ it need not be nef, see [Mor92] for an example.

4.1. **Degenerate twistor lines.** We shall describe a one-parameter deformation of a compact hyperkähler manifold endowed with a semi-positive isotropic $(1,1)$-form, see [Ver15]. In the following, we let X be a compact hyperkähler manifold of dimension $2n$ with a fixed holomorphic two-form σ and a Ricci-flat Kähler form ω. Furthermore, we assume that η is a semi-positive closed real $(1,1)$-form with isotropic cohomology class $[\eta] \in H^{1,1}(X, \mathbb{R})$, i.e. $q([\eta]) = 0$. Note that we also know

$$q([\sigma]) = 0, \; q([\sigma], [\eta]) = 0 \text{ and } q([\sigma] + [\eta]) = 0.$$

Now, by virtue of Verbitsky's description of the cohomology generated by $H^2(X, \mathbb{R})$, see [Bog96, Ver96], these equalities imply

$$[\eta]^{n+1} = 0 \text{ and } ([\sigma] + [\eta])^{n+1} = 0$$

in $H^{2n+2}(X, \mathbb{R})$. The semi-positivity of η implies that the same equalities hold on the level of forms, see [Ver15, Sec. 3].

Lemma 4.1. *Under the above assumptions, the following assertions hold true:*

(i) $\eta^{n+1} = 0$.

(ii) $(\sigma+\eta)^{n+1}=0$.

(iii) $(\sigma+\eta)^n \wedge (\bar\sigma+\eta)^n$ *is a volume form.*

Proof. We skip the proof. This is a point-wise statement which boils down to linear algebra. $\square$

Of course, the same results hold for all positive multiples $t\eta$ which will be used to define a family of complex structures on X.

First recall that the kernel of the map $\sigma\colon T_{\mathbb{C}}X \longrightarrow T_{\mathbb{C}}X^*$ naturally induced by the complex two-form σ is the bundle of $(0,1)$-vector fields $T^{0,1}\subset T_{\mathbb{C}}X$. Copying this, one defines for the closed two-forms

$$\sigma_t := \sigma + t\cdot\eta$$

the bundle $T_t^{0,1}\subset T_{\mathbb{C}}X$ as the kernel of the map induced by the complex two-form σ_t. Lemma 4.1, (iii) implies that $T_t^{0,1}$ really is a complex vector bundle of dimension $2n$ and that $T_{\mathbb{C}}X = T_t^{1,0}\oplus T_t^{0,1}$, where $T_t^{1,0}$ is the complex conjugate of $T_t^{0,1}$. This direct sum decomposition describes an almost complex structure I_t on the differentiable manifold M underlying X.

Lemma 4.2. *The almost complex structure I_t is integrable and σ_t is a holomorphic symplectic form on (M,I_t). Furthermore, for all t the form η is of type $(1,1)$ with respect to I_t and η is semi-positive for small t.*

Proof. Due to the Newlander–Nirenberg theorem, it suffices to show that $T_t^{0,1}$ is preserved under the Lie bracket, i.e. $[T_t^{0,1},T_t^{0,1}]\subset T_t^{0,1}$, cf. [Huy05, Sec. 2.6]. The standard formula for the derivative of differential forms applied to $v_1,v_2\in T_t^{0,1}$ and arbitrary $v\in T_{\mathbb{C}}X$ shows

$$\begin{aligned} 0=(d\sigma_t)(v_1,v_2,v) &= v_1(\sigma_t(v_2,v)) - v_2(\sigma_t(v_1,v)) + v(\sigma_t(v_1,v_2)) \\ &\quad -\sigma_t([v_1,v_2],v) + \sigma_t([v_1,v],v_2) - \sigma_t([v_2,v],v), \end{aligned}$$

which by definition of $T_t^{0,1}$ yields $\sigma([v_1,v_2],v)=0$ for all $v\in T_{\mathbb{C}}X$ and, therefore, $[v_1,v_2]\in T_t^{0,1}$.

By construction, σ_t is of type $(2,0)$ on (M,I_t), closed since σ and η are closed, and non-degenerate by Lemma 4.1, (iii). Hence, σ_t is a holomorphic symplectic form on (M,I_t). The form η is of type $(1,1)$ with respect to I_t, for $\sigma_t^n\wedge\eta = (\sigma+t\cdot\eta)^n\wedge\eta = 0$ by Lemma 4.1, (ii).

The semipositivity of a smooth form can be checked point-wise and it is suffices to verify it at a general point $x\in M$, where we can assume η to be of maximal rank n.[2] Now, choose a family of forms $\alpha_i(t)$, $i=1,\ldots,2n$, that are of type $(1,0)$ with respect to I_t, vary smoothly with t, and form a basis of $(T_t^{1,0})^*$ at x with respect to which η is diagonal, i.e. $\eta(x) = i\sum a_i(t)\cdot(\alpha_i(t)\wedge\overline{\alpha}_i(t))(x)$. By Lemma 4.1, (i) and the semipositivity of η with respect to I_0, the coefficients satisfy $a_i(0)\geqslant 0$ and exactly n of them, say $a_1(0),\ldots,a_n(0)$, are strictly

[2] This is oversimplifying things a little: There is such an open subset, but it may be open only in the analytic topology. To make this rigorous one has to work with an analytically dense union of open subsets $\bigcup U_i\subset M$ such that on each U_i the rank n_i of η is constant. Then $\eta^{n_i+1}|_{U_i}\neq 0$ but $\eta^{n_i}|_{U_i}=0$. The rest of the argument remains unchanged.

positive. By continuity, $a_1(t), \dots, a_n(t) > 0$ for all small t and $\eta^{n+1} = 0$ then implies that we still must have $a_{n+1}(t) = \cdots = a_{2n}(t) = 0$ for those t. Hence, η is semi-positive for small t. $\square$

Remark 4.3. It is possible to show that in fact η is semi-positive with respect to all I_t, $t \in \mathbb{C}$, but we will not need this stronger statement.

Altogether the above describes a smooth family $\mathcal{X} \longrightarrow \mathbb{C}$ of compact hyperkähler manifolds $\mathcal{X}_t := (M, I_t)$, called *degenerate twistor family*, together with a constant closed real $(1,1)$-form η that is semi-positive for small $t \in \Delta \subset \mathbb{C}$.[3] The central fibre $\mathcal{X}_0$ is the original compact hyperkähler manifold X.

4.2. **Semi-continuity.** We apply the above construction to the case of a non-trivial semi-positive line bundle L with $q(L) = 0$. By assumption, $\mathrm{c}_1(L) = [\eta]$ for some semi-positive closed real $(1,1)$-form η. For example, if $f \colon X \longrightarrow \mathbb{P}^n$ is a Lagrangian fibration and $L \simeq f^*\mathcal{O}(1)$, then any positive form η_0 on $\mathbb{P}^n$ representing the hyperplane class $\mathrm{c}_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^n, \mathbb{Z})$ induces a semi-positive form $\eta = f^*\eta_0$ that represents $\mathrm{c}_1(L)$. The properties in Lemma 4.1, for example $\eta^{n+1} = 0$, are obvious in this case. Note that in contrast to the uniqueness of Ricci-flat Kähler forms in any given Kähler class, the form η satisfying the degenerate Monge–Ampère equation $\eta^{2n} = 0$, or even $\eta^{n+1} = 0$, is certainly not unique.

Lemma 4.4. *The degenerate twistor line $\mathcal{X} \longrightarrow \mathbb{C}$ associated to a semi-positive form η representing $\mathrm{c}_1(L)$ has the property that for very general $t \in \mathbb{C}$, the fibre $\mathcal{X}_t = (M, I_t)$ is non-projective. Furthermore, the complex line bundle L is holomorphic with respect to all $t \in \mathbb{C}$ and semi-positive for small $t \in \Delta$.*

Proof. As the first Chern class of the complex line bundle L satisfies $q(\mathrm{c}_1(L), \sigma_t) = q([\eta], [\sigma]) + t \cdot q([\eta], [\eta]) = 0$, the line bundle L is holomorphic on all fibres $\mathcal{X}_T$. Since the non-trivial nef class $\mathrm{c}_1(L)$ is not orthogonal to any class in the positive cone $\mathcal{C}_X$, the very general fibre is not projective.

On every fibre the class $\mathrm{c}_1(L)$ is represented by the closed real $(1,1)$-form η and any such form is the curvature of a uniquely determined hermitian structure on L. Since for small t the form η is still semi-positive, one finds that L is a semi-positive holomorphic line bundle on all fibres $\mathcal{X}_t$ for small t. $\square$

This allows one to show that the original result of Campana–Oguiso–Peternell [COP10] for non-projective hyperkähler manifolds is enough to conclude the result for all hyperkähler manifolds, which gives an alternative proof of Verbitsky's result [Ver10].

Corollary 4.5. *Assume any semi-positive line bundle on a non-projective hyperkähler manifold X is $\mathbb{Q}$-effective. Then the same also holds for projective hyperkähler manifolds.*

[3]There is a minor technical issue here. The parameter t above was assumed to be real and positive. Either, Lemma 4.1 has to be adapted in (iii) to say that $(\sigma + t\eta)^n \wedge (\bar{\sigma} + \bar{t}\eta)^n$ is a volume form or the family is first constructed just over $\mathbb{R}_{>0} \cap \Delta \subset \mathbb{C}$ and then extended from there.

Proof. Indeed, as explained above, any compact hyperkähler manifold X with a semi-positive line bundle L with $q(L) = 0$ can be realized as the central fibre of a degenerate twistor family $\mathcal{X} \twoheadrightarrow \mathbb{C}$. The very general fibre $\mathcal{X}_t$ is non-projective and [COP10] thus applies to L considered as a semi-positive line bundle on $\mathcal{X}_t$. Therefore, L is $\mathbb{Q}$-effective on the very general fibre $\mathcal{X}_t$, $t \in \Delta$, and, by semi-continuity, the same holds for the central fibre. $\square$

5. Open questions

Besides the two conjectures stated in the introduction, there are a number of related questions that seem approachable.

5.1. **Stability of the tangent bundle.**

Due to the existence of a hyperkähler (and hence Kähler–Einstein) metric on a hyperkähler manifold X, the tangent bundle $\mathcal{T}_X$ is μ-stable. In fact, $\mathcal{T}_X$ is μ-stable with respect to every Kähler class and, as explained in Section 2.5, with respect to the generic class in the birational Kähler cone.

Question 5.1. *Is the tangent bundle $\mathcal{T}_X$ of a hyperkähler manifold μ-stable with respect to any class in the positive cone?*

This would subsume Proposition 2.7 and would allow one to conclude the stronger statement that the line bundle M constructed in the proof of Proposition 2.6 and used in the two proofs in Section 3 is contained in the closure of the positive cone.

5.2. **Elliptic and parabolic hyperkähler manifolds.**

The paper [COP10] discusses the possibilities for the algebraic dimension $a(X) = \mathrm{trdeg} K(X)$ of a compact hyperkähler manifold and how the algebraic dimension is related to the intersection form on the Néron–Severi group. We only touch upon one aspect here.

Question 5.2. *Assume X is a compact hyperkähler manifold of algebraic dimension zero, i.e. $K(X) = \mathbb{C}$. Is the Beauville–Bogomolov form q on $\mathrm{NS}(X) \simeq H^{1,1}(X, \mathbb{Z})$ negative definite?*

Following [COP10], X is called *elliptic* if q is negative definite on $\mathrm{NS}(X)$. It is known that elliptic hyperkähler manifolds satisfy $K(X) = \mathbb{C}$. The above question is the converse.

Similarly, X is called *parabolic* if q on $\mathrm{NS}(X)$ is semi-negative definite with one isotropic direction and *hyperbolic* if it has signature $(1, \rho(X) - 1)$. By the Hodge index theorem and the projectivity criterion for hyperkähler manifolds, the latter is equivalent to X being projective. The analogue of Question 5.2 in the parabolic case is the conjecture that X is parabolic if and only if $a(X) = n$. According to [COP10, Thm. 3.6], any non-algebraic compact hyperkähler manifold satisfies $a(X) \leqslant n$, so that the cases $0 < a(X) < n$ would need to be excluded.

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THE LOOIJENGA–LUNTS–VERBITSKY ALGEBRA AND VERBITSKY'S THEOREM

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Abstract. In these notes we review some basic facts about the LLV Lie algebra. It is a rational Lie algebra, introduced by Looijenga–Lunts and Verbitsky, acting on the rational cohomology of a compact Kähler manifold. We study its structure and describe one irreducible component of the rational cohomology in the case of a compact hyperkähler manifold.

1. Introduction

1.1. Let $V = \bigoplus_{k\in\mathbb{Z}} V_k$ be a finite dimensional graded vector space over a field $\mathbb{F}$ of characteristic 0, and denote by h the operator:

$$h|_{V_k} = k \cdot \mathrm{id}.$$

Definition 1.1. Let $e\colon V \longrightarrow V$ be a degree 2 endomorphism. We say e has the *Lefschetz property* if

$$e^k\colon V_{-k} \longrightarrow V_k$$

is an isomorphism.

Remark 1.2. The degree two operators with the Lefschetz property form a Zariski open subset of $\mathrm{End}_2(V)$.

Theorem 1.3 (Jacobson–Morozov, [?, Theorem 3])**.** *An operator e has the Lefschetz property if and only if there exists a unique degree -2 endomorphism $f\colon V \longrightarrow V$ such that*

$$[e, f] = h.$$

Moreover, if $L \subset \mathrm{End}(V)$ is a semisimple Lie subalgebra and $e, h \in L$, then $f \in L$.

We say that the triple (e, h, f) is an $\mathfrak{sl}_2$-triple, the reason is that we can define a representation $\mathfrak{sl}_2(\mathbb{F}) \longrightarrow \mathrm{End}(V)$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ on the vector space V as follows

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto f.$$

In the rest of these notes, we will mostly be interested in the graded rational vector space $V = H^*(X, \mathbb{Q})[N]$, where X is a compact Kähler manifold of dimension N. Here $[m]$ indicates

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the shift by m, so that $V_0 = H^N(X,\mathbb{Q})$. To any class $a \in H^2(X,\mathbb{Q})$ we can associate the operator in cohomology obtained by taking cup product

$$e_a \colon H^*(X,\mathbb{Q}) \longrightarrow H^*(X,\mathbb{Q}), \quad \omega \longmapsto a.\omega.$$

The operator h becomes

$$h|_{H^k(X,\mathbb{Q})} := (k-N)\mathrm{id}.$$

From Theorem 1.3 we see that if e_a has the Lefschetz property (for example if a is a Kähler class), there is an operator f_a of degree -2 that makes (e_a, h, f_a) an $\mathfrak{sl}_2$-triple. Moreover, the map

$$f \colon H^2(X,\mathbb{Q}) \dashrightarrow \mathrm{End}_{-2}(H^*(X,\mathbb{Q})),$$

that sends a to the operator f_a is defined on a Zariski open subset and rational.

Remark 1.4. If $a \in H^{1,1}(X,\mathbb{Q})$ is Kähler, it follows from standard Hodge theory that everything can be defined at the level of forms. The dual operator is $f_a = *^{-1}e_a*$, where $*$ is the Hodge star operator. The $\mathfrak{sl}_2$-action preserves the harmonic forms, so it induces an action on cohomology.

Definition 1.5 ([?, ?])**.** Let X be a compact Kähler manifold. The *total Lie algebra* $\mathfrak{g}_{\mathrm{tot}}(X)$ of X is the Lie algebra generated by the $\mathfrak{sl}_2$-triples

$$(e_a, h, f_a),$$

where $a \in H^2(X,\mathbb{Q})$ is a class with the Lefschetz property.

The following is a general result about this Lie algebra for compact Kähler manifolds. Denote by ϕ the pairing on $H^*(X,\mathbb{C})$ given by

$$\phi(\alpha,\beta) = (-1)^q \int_X \alpha.\beta,$$

if α has degree $N+2q$ or $N+2q+1$.

Proposition 1.6 ([?, Proposition 1.6])**.** *The Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ is semisimple and preserves ϕ infinitesimally. Moreover, the degree-*0 *part $\mathfrak{g}_{\mathrm{tot}}(X)_0$ is reductive.*

1.2. Now let X be a compact hyperkähler manifold of complex dimension $2n$. In this case, the Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ is also called the Looijenga-Lunts-Verbitsky Lie algebra. It is well known that for each hyperkähler metric g on X we get an action of the quaternion algebra $\mathbb{H}$ on the real tangent bundle TX. This means that we have three complex structures I, J, K such that

$$IJ = -JI = K. \tag{1.1}$$

To each of these complex structures we can associate a Kähler form $\omega_I := g(I(-),-), \omega_J := (J(-),-), \omega_K := g(K(-),-)$ and a holomorphic symplectic form $\sigma_I = \omega_J + i\omega_K, \sigma_J = \omega_K + i\omega_I, \sigma_K = \omega_I + i\omega_J$.

Definition 1.7. The *charateristic* 3*-plane* $F(g)$ of the metric g is

$$F(g) := \langle [\omega_I], [\omega_J], [\omega_K] \rangle = \langle [\omega_I], [\Re\sigma_I], [\Im\sigma_I] \rangle \subset H^2(X, \mathbb{R}).$$

Definition 1.8 ([**?**])**.** Denote by $\mathfrak{g}_g \subset \mathrm{End}(H^*(X, \mathbb{R}))$ the Lie algebra generated by the $\mathfrak{sl}_2$-triples (e_a, h, f_a) where $a \in F(g)$.

Remark 1.9. This Lie algebra is generated by the three $\mathfrak{sl}_2$-triples associated to the classes $[\omega_I], [\omega_J], [\omega_K]$. Indeed, from the discussion in the following section we will see that the subalgebra generated by these three $\mathfrak{sl}_2$-triples is semisimple. From the Jacobson-Morozov Theorem and the linearity of $e\colon H^2(X, \mathbb{R}) \longrightarrow \mathrm{End}(H^*(X, \mathbb{R}))$ we conclude that it contains every $\mathfrak{sl}_2$-triple (e_a, h, f_a) with $a \in F(g)$.

2. The algebra associated to a metric

2.1. In this section we study the smaller algebra $\mathfrak{g}_g$ and its action on cohomology. These results are due to Verbitsky [**?**], see also [**?**].

We start with a general algebraic construction. Let $\mathbb{H}$ be the quaternion algebra. As a real vector space it is generated by $1, I, J, K$, where I, J, K satisfy the relations (1.1). We denote by $\mathbb{H}_0$ the pure quaternions, i.e. the linear combinations of I, J, K.

Let V be a left $\mathbb{H}$-module, equipped with an inner product

$$\langle -, - \rangle \colon V \times V \longrightarrow \mathbb{R},$$

and assume that I, J, K act on V via isometries. The $\mathbb{H}$-action gives three complex structures I, J, K on V, satisfying the relations (1.1). Consider the forms

$$\omega_I = \langle I(-), - \rangle,$$

$$\omega_J = \langle J(-), - \rangle,$$

$$\omega_K = \langle K(-), - \rangle$$

and the holomorphic symplectic forms $\sigma_I = \omega_J + i\omega_K, \sigma_J = \omega_K + i\omega_I, \sigma_K = \omega_I + i\omega_J$.

Remark 2.1. Note that the operators e_λ for $\lambda = \omega_I, \omega_J, \omega_K$ have the Lefschetz property; the dual operator is given by $f_\lambda = *^{-1} e_\lambda *$, where $*$ is the Hodge star operator on $\Lambda^\bullet V^*$ induced by the inner product.

Definition 2.2. Let $\mathfrak{g}(V) \subset \mathrm{End}(\bigwedge^\bullet V^*)$ be the Lie algebra generated by the $\mathfrak{sl}_2$-triples

$$(e_\lambda, h, f_\lambda)_{\lambda = \omega_I, \omega_J, \omega_K},$$

where h is the shifted degree operator.

In particular, this definition makes sense for the rank one module $\mathbb{H}$ equipped with the standard inner product. This gives a Lie algebra $\mathfrak{g}(\mathbb{H}) \subset \mathrm{End}(\bigwedge^\bullet \mathbb{H}^*)$. We denote by $\mathfrak{g}(\mathbb{H})_0$ the degree-0 component of $\mathfrak{g}(\mathbb{H})$ (here the degree is viewed as an endomorphism of the graded vector space). It is a Lie subalgebra, and we denote it by $\mathfrak{g}(\mathbb{H})_0' := [\mathfrak{g}(\mathbb{H})_0, \mathfrak{g}(\mathbb{H})_0]$ its derived Lie algebra.

Theorem 2.3. *With the above notation we have the following.*

(1) *There is a natural isomorphism* $\mathfrak{g}(V) \simeq \mathfrak{g}(\mathbb{H})$.
(2) *There is an isomorphism* $\mathfrak{g}(\mathbb{H}) \simeq \mathfrak{so}(4,1)$.
(3) *The algebra decomposes with respect to the degree as*

$$\mathfrak{g}(\mathbb{H}) = \mathfrak{g}(\mathbb{H})_{-2} \oplus \mathfrak{g}(\mathbb{H})_0 \oplus \mathfrak{g}(\mathbb{H})_2.$$

Furthermore, $\mathfrak{g}(\mathbb{H})_{\pm 2} \simeq \mathbb{H}_0$ *as Lie algebras, and* $\mathfrak{g}(\mathbb{H})_0 = \mathfrak{g}(\mathbb{H})_0' \oplus \mathbb{R}h$ *with* $\mathfrak{g}(\mathbb{H})_0' \simeq \mathbb{H}_0$*; this last isomorphism is compatible with the actions on* $\bigwedge^\bullet V^*$.

Proof. (1) Since $\langle -, - \rangle$ is $\mathbb{H}$-invariant, we can find an orthogonal decomposition

$$V = \mathbb{H} \oplus \cdots \oplus \mathbb{H}.$$

Taking exterior powers we get $\bigwedge^\bullet V^* = \bigwedge^\bullet \mathbb{H}^* \otimes \cdots \otimes \bigwedge^\bullet \mathbb{H}^*$. This gives an injective map $\mathfrak{g}(\mathbb{H}) \longrightarrow \mathrm{End}(\bigwedge^\bullet V^*)$, given by the natural tensor product representation. It is a direct check that the image of this morphism is exactly the algebra $\mathfrak{g}(V)$.

(2) Consider the subrepresentation $W \subset \bigwedge^\bullet \mathbb{H}^*$ given by

$$W = \bigwedge^0 \mathbb{H}^* \oplus \langle \omega_I, \omega_J, \omega_K \rangle \oplus \bigwedge^4 \mathbb{H}^*.$$

We equip it with the quadratic form given by setting $\bigwedge^0 \mathbb{H}^* \oplus \bigwedge^4 \mathbb{H}^*$ to be a hyperbolic plane, orthogonal to the 3-plane, and $\{\omega_I, \omega_J, \omega_K\}$ to be an orthonormal basis of the 3-plane. By a direct computation we can see that the action of $\mathfrak{g}(\mathbb{H})$ respects infinitesimally this quadratic form. This gives a map

$$\mathfrak{g}(\mathbb{H}) \longrightarrow \mathfrak{so}(W) \simeq \mathfrak{so}(4,1), \tag{2.1}$$

that we next show to be an isomorphism.

Since W has dimension 5 the Lie algebra $\mathfrak{so}(W)$ has dimension 10. Now consider the following 10 elements of $\mathfrak{g}(\mathbb{H})$:

$$h, e_I, e_J, e_K, f_I, f_J, f_K, K_{IJ}, K_{IK}, K_{JK},$$

where $K_{IJ} := [e_I, f_J], K_{IK} = [e_I, f_K]$ and $K_{JK} = [e_J, f_K]$. Verbitsky [?] showed that K_{IJ} acts like the Weil operator associated with the Hodge structure on $\bigwedge^\bullet \mathbb{H}^*$ given by K, and similarly K_{JK} and K_{IK}. This means that on a (p,q) form with respect to K it acts as multiplication by $i(p-q)$. It follows that the ten operators above are linearly independent over W, hence the map is surjective. Moreover they generate $\mathfrak{g}(\mathbb{H})$ as a vector space. Indeed, they generate $\mathfrak{g}(\mathbb{H})$ as a Lie algebra, and one has the following relations (see [?]):

$$\begin{aligned}[K_{\lambda,\mu}, K_{\mu,\nu}] &= K_{\lambda,\nu}, \quad [K_{\lambda,\mu}, h] = 0,\\ [K_{\lambda,\mu}, e_\mu] &= 2e_\lambda, \quad [K_{\lambda,\mu}, f_\mu] = 2f_\lambda,\\ [K_{\lambda,\mu}, e_\nu] &= 0, \quad [K_{\lambda,\mu}, f_\nu] = 0,\end{aligned}$$

where $\lambda, \mu, \nu \in \{I, J, K\}$ and $\nu \neq \lambda, \nu \neq \mu$. This implies that they are a basis of $\mathfrak{g}(\mathbb{H})$, hence the map (2.1) is an isomorphism.

Point (3) follows using this explicit basis. Indeed we have

$$\begin{aligned}&\mathfrak{g}(\mathbb{H})_{-2} = \langle f_I, f_J, f_K\rangle, \quad \mathfrak{g}(\mathbb{H})_2 = \langle e_I, e_J, e_K\rangle, \text{ and}\\ &\mathfrak{g}(\mathbb{H})_0 = \langle K_{IJ}, K_{JK}, K_{IK}\rangle \oplus \mathbb{R}h.\end{aligned}$$

In particular we have

$$\begin{aligned}\mathfrak{g}(\mathbb{H})'_0 &\xrightarrow{\sim} \mathbb{H}_0,\\ K_{IJ} &\longmapsto K,\\ K_{JK} &\longmapsto I,\\ K_{IK} &\longmapsto J.\end{aligned}$$

Since $I, J, K \in \mathbb{H}_0$ act on $\bigwedge^\bullet \mathbb{H}^*$ as Weil operators for the corresponding complex structures on $\mathbb{H}$, the isomorphism is compatible with the actions. □

Now we can compute the Lie algebra $\mathfrak{g}_g$. As above we denote by $(\mathfrak{g}_g)_0$ the degree-0 part, and by $(\mathfrak{g}_g)'_0 := [(\mathfrak{g}_g)_0, (\mathfrak{g}_g)_0]$ its derived Lie algebra.

Proposition 2.4. *Let (X, g) be a hyperkähler manifold with a fixed hyperkähler metric.*

(1) *There is a natural isomorphism of graded Lie algebras $\mathfrak{g}_g \simeq \mathfrak{g}(\mathbb{H})$. In particular $\mathfrak{g}_g \simeq \mathfrak{so}(4, 1)$.*
(2) *The semisimple part $(\mathfrak{g}_g)'_0$ acts on $H^*(X, \mathbb{R})$ via derivations.*

Proof. (1). Consider the Lie subalgebra $\hat{\mathfrak{g}}_g \subset \mathrm{End}(\Omega^\bullet_X)$, generated by the $\mathfrak{sl}_2$-triples (e_a, h, f_a) with $a \in F(g)$, at the level of forms (in particular $f_a = *^{-1}e_a*$). From the previous proposition, we see that for every point $x \in X$ there is an inclusion $\mathfrak{g}(\mathbb{H}) \hookrightarrow \mathrm{End}(\Omega^\bullet_{X,x})$. This gives an inclusion $\mathfrak{g}(\mathbb{H}) \hookrightarrow \prod_{x\in X} \mathrm{End}(\Omega^\bullet_{X,x})$. It follows from the definitions that the two algebras of $\mathfrak{g}(\mathbb{H})$ and $\hat{\mathfrak{g}}_g$ are equal as subalgebras of $\prod_{x\in X} \mathrm{End}(\Omega^\bullet_{X,x})$.

Since the metric g is fixed, the $\mathfrak{sl}_2$-triples (e_a, h, f_a) preserve the harmonic forms $\mathcal{H}^*(X)$, and so does $\hat{\mathfrak{g}}_g$. Since $\mathcal{H}^*(X) \simeq H^*(X, \mathbb{R})$ we get a morphism

$$\mathfrak{g}(\mathbb{H}) \simeq \hat{\mathfrak{g}}_g \longrightarrow \mathfrak{g}_g.$$

This map is surjective, because the image contains the $\mathfrak{sl}_2$-triples that generate $\mathfrak{g}_g$. Moreover, by explicit computations similar to the proof of the previous proposition, we can see that $\dim \mathfrak{g}_g \geqslant 10$. Hence the map is an isomorphism.

(2). From the previous proposition we have an isomorphism compatible with the actions on cohomology

$$(\mathfrak{g}_g)'_0 \simeq \mathfrak{g}(\mathbb{H})'_0 \simeq \mathbb{H}_0.$$

Hence, it suffices to prove the statement for the action of I, J, K. Each of them gives a complex structure, and acts as the Weil operator on the associated Hodge decomposition. So, the action on (p, q) forms is given by multiplication by $i(p - q)$, which is a derivation. □

3. The total Lie algebra

The goal of this section is to prove the following result due to Looijenga and Lunts [?, Proposition 4.5] and Verbitsky [?, Theorem 1.6].

Theorem 3.1. *Let X be a hyperkähler manifold. With the above notation we have the following.*

(1) *The total Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ lives only in degrees $-2, 0, 2$, so it decomposes as:*

$$\mathfrak{g}_{\mathrm{tot}}(X) = \mathfrak{g}_{\mathrm{tot}}(X)_{-2} \oplus \mathfrak{g}_{\mathrm{tot}}(X)_0 \oplus \mathfrak{g}_{\mathrm{tot}}(X)_2.$$

(2) *There are canonical isomorphisms $\mathfrak{g}_{\mathrm{tot}}(X)_{\pm 2} \simeq H^2(X, \mathbb{Q})$.*

(3) *There is a decomposition $\mathfrak{g}_{\mathrm{tot}}(X)_0 = \mathfrak{g}_{\mathrm{tot}}(X)'_0 \oplus \mathbb{Q}h$ with $\mathfrak{g}_{\mathrm{tot}}(X)'_0 \simeq \mathfrak{so}(H^2(X, \mathbb{Q}), q)$, where q is the Beauville–Bolomov–Fujiki quadratic form [?]. Furthermore $\mathfrak{g}_{\mathrm{tot}}(X)'_0$ acts on $H^*(X, \mathbb{Q})$ by derivations.*

The main geometric input in the proof is the following lemma.

Lemma 3.2. *If X is a compact hyperkähler manifold, then $[f_a, f_b] = 0$ for every $a, b \in H^2(X, \mathbb{R})$ for which f is defined.*

The proof relies on the following fact.

Proposition 3.3. *The set of charateristic 3-planes is open in the Grassmannian of 3-planes in $H^2(X, \mathbb{R})$.*

In turn, this follows from a celebrated Theorem of Yau.

Theorem 3.4 (Yau)**.** *Let X be a hyperkähler manifold, and let I be a complex structure on X. If ω is a Kähler class, then there is a unique hyperkähler metric g such that $[\omega_I] = \omega$.*

Proof of Lemma 3.2. If we fix a hyperkähler metric g on X, then for every $a, b \in F(g)$ we have $[f_a, f_b] = 0$. This holds already at the level of forms, using the definition $f_a = *^{-1} e_a *$ and the fact that $*$ depends only on the metric. Let $a \in H^2(X, \mathbb{R})$ be a class for which f_a is defined.

Since f is rational, the condition $[f_a, f_b] = 0$ is Zariski closed with respect to $b \in H^2(X, \mathbb{R})$. From Proposition 3.3 it follows that the set

$$\{b \in H^2(X, \mathbb{R}) \mid a, b \in F(g) \text{ for some metric } g\}$$

is open. Since $[f_a, f_b] = 0$ for every b in this open set, we get $[f_a, f_b] = 0$ for every b where f_b is defined. □

While the statement of Theorem 3.1 is over $\mathbb{Q}$, we will give the proof over $\mathbb{R}$ following [**?**].

Proof of Theorem 3.1. Consider the subspace

$$V := V_{-2} \oplus V_0 \oplus V_2 \subset \mathfrak{g}_{\text{tot}}(X),$$

where V_2 is the abelian Lie subalgebra generated by e_a with $a \in H^2(X, \mathbb{R})$, V_{-2} is the abelian Lie subalgebra generated by the f_a with $a \in H^2(X, \mathbb{R})$ where f_a is defined, and V_0 is the Lie subalgebra generated by $[e_a, e_b]$. To prove (1) and (2), it is enough to show that V is a Lie subalgebra of $\mathfrak{g}_{\text{tot}}(X)$. Indeed, since $\mathfrak{g}_{\text{tot}}(X)$ is generated by elements contained in V this would imply $V = \mathfrak{g}_{\text{tot}}(X)$. Since V_2 and V_{-2} are abelian, it suffices to show that $[V_0, V_2] \subset V_2$ and $[V_0, V_{-2}] \subset V_{-2}$.

Claim. Define $V_0' := [V_0, V_0]$. We have $V_0 = V_0' \oplus \mathbb{R}h$ where V_0' acts on cohomology via derivations.

Proof of the claim. Proposition 3.3 implies that the set $\{(a, b) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \mid a, b \in F(g) \text{ for some metric } g\}$ is open. Arguing as in the proof of Lemma 3.2 we see that V_0 is generated by the elements $[e_a, f_b]$ with $a, b \in F(g)$. If we fix the hyperkähler metric g, the elements $[e_a, f_b]$ with $a, b \in F(g)$ generate the Lie algebra $(\mathfrak{g}_g)_0$ and their brackets the Lie subalgebra $(\mathfrak{g}_g)_0'$. Thus, V_0' is generated by the Lie algebras $(\mathfrak{g}_g)_0'$ and their brackets. Since the Lie algebras $(\mathfrak{g}_g)_0'$ act on cohomology via derivations, the same is true for their brackets, hence V_0' acts via derivations. Moreover, from point (3) of Theorem 2.3 we get the decomposition $V_0 = V_0' + \mathbb{R}h$. Since $\mathfrak{g}_{\text{tot}}(X)_0$ is reductive (Proposition 1.6) and h is in the center, we get $h \notin V_0' \subset \mathfrak{g}_{\text{tot}}(X)_0'$, so the sum is direct. □

Now we show that $[V_0, V_2] \subset V_2$. Since the adjoint action of h gives the grading, it is enough to show that $[V_0', V_2] \subset V_2$. Let $u \in V_0'$ and $e_a \in V_2$. For every $x \in H^2(X, \mathbb{R})$ we have

$$[u, e_a](x) = u(a.x) - a.u(x) = u(a).x = e_a(x), \tag{3.1}$$

because u is a derivation.

The inclusion $[V_0, V_{-2}] \subset V_{-2}$ is more difficult. Let $G_0' \subset GL(H^*(X, \mathbb{R}))$ be the closed Lie subgroup with Lie algebra V_0'. For every $t \in G_0'$ we have $te_at^{-1} = e_{t(a)}$ and $tht^{-1} = h$, by integrating the analogous relations at the level of Lie algebras. Since the third element of an $\mathfrak{sl}_2$-triple is unique, we get that $tf_at^{-1} = f_{t(a)}$. This implies that the adjoint action of G_0' leaves V_{-2} invariant, hence so does the Lie algebra V_0'.

To summarize, at this point we showed (1) and (2), and also that $\mathfrak{g}_{\mathrm{tot}}(X)'_0$ acts via derivations. It remains to show that $\mathfrak{g}_{\mathrm{tot}}(X)'_0 \simeq \mathfrak{so}(H^2(X,\mathbb{R}),q)$.

We begin by defining the map $\mathfrak{g}_{\mathrm{tot}}(X)'_0 \longrightarrow \mathfrak{so}(H^2(X,\mathbb{R}),q)$. For this, we consider the restriction of the action of $\mathfrak{g}_{\mathrm{tot}}(X)'_0$ to $H^2(X,\mathbb{R})$, and show that it preserves infinitesimally the Beauville–Bogomolov–Fujiki form q. We can fix a hyperkähler metric g and check this for $(\mathfrak{g}_g)'_0$, because these Lie subalgebras generate $\mathfrak{g}_{\mathrm{tot}}(X)'_0$. From Theorem 2.3 it is enough to check it for the Weil operators associated to the three complex structures I, J, K induced from g. Fix one of them, say I; we have to verify that

$$q(I\alpha,\beta)+q(\alpha,I\beta)=0,$$

for every $\alpha,\beta \in H^2(X,\mathbb{R})$. This follows from a direct verification using the q-orthogonal decomposition

$$H^2(X,\mathbb{R}) = (H^{2,0}(X)\oplus H^{0,2}(X))_{\mathbb{R}})\oplus H^{1,1}(X,\mathbb{R}),$$

induced by the Hodge decomposition with respect to the complex structure I.

To conclude the proof it remains to show that this map is bijective; we begin with the surjectivity. Fix a hyperkähler metric g, the image of the Lie algebra $(\mathfrak{g}_g)'_0$ in $\mathfrak{so}(H^2(X,\mathbb{R}),q)$ is generated (as a vector space) by the Weil operators associated to I, J, K. Using this, it is easy to see that $(\mathfrak{g}_g)'_0$ kills the q-orthogonal complement to the characteristic 3-plane $F(g)$, and it maps onto $\mathfrak{so}(F(g),q|_{F(g)})$. One can check that varying the metric g the Lie subalgebras $\mathfrak{so}(F(g),q|_{F(g)})$ generate $\mathfrak{so}(H^2(X,\mathbb{R}))$, hence the surjectivity.

For the injectivity we proceed as follows. Let $SH^2(X,\mathbb{R})\subset H^*(X,\mathbb{R})$ be the graded subalgebra generated by $H^2(X,\mathbb{R})$; it is a $\mathfrak{g}_{\mathrm{tot}}(X)$ representation for Corollary 4.5. By Lemma 4.6, the map $\mathfrak{g}_{\mathrm{tot}}(X)\longrightarrow \mathfrak{gl}(SH^2(X,\mathbb{R}))$ is injective. Since $\mathfrak{g}_{\mathrm{tot}}(X)'_0$ acts via derivations, the map must be injective already at the level of $H^2(X,\mathbb{R})$. □

Definition 3.5. We define the *Mukai completion* of the quadratic vector space $(H^2(X,\mathbb{Q}),q)$ as the quadratic vector space

$$(\tilde{H}(X,\mathbb{Q}),\tilde{q}) \coloneqq (H^2(X,\mathbb{Q}),q)\oplus U$$

where U is a two dimensional vector space with quadratic form given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Corollary 3.6. *There is a natural isomorphism*

$$\mathfrak{g}_{\mathrm{tot}}(X)\simeq \mathfrak{so}(\tilde{H}(X,\mathbb{Q}),\tilde{q}).$$

Proof. Recall that for a rational quadratic space (V,q) there is an isomorphism

$$\begin{aligned} \bigwedge^2 V &\xrightarrow{\simeq} \mathfrak{so}(V,q),\\ x\wedge y &\longmapsto \frac{1}{2}(q(x,-)y-q(y,-)x) \end{aligned}$$

The desired isomorphism follows from this, at least at the level of vector spaces. The computations to show that it is in fact an isomorphism of Lie algebras are carried out in [**?**, Proposition 2.7]. $\square$

Example 3.7. If X is a $K3$ surface, then the Mukai completion $\tilde{H}(X, \mathbb{Q})$ is the rational cohomology $H^*(X, \mathbb{Q})$ with the usual Mukai pairing. This identification is compatible with the action of $\mathfrak{g}_{\text{tot}}(X)$.

Corollary 3.8. *The Hodge structure on $H^*(X, \mathbb{R})$ is determined by the Hodge structure on $H^2(X, \mathbb{R})$ and by the action of $\mathfrak{g}_{\text{tot}}(X)_{2,\mathbb{R}} \simeq H^2(X, \mathbb{R})$ on $H^*(X, \mathbb{R})$.*

Proof. Let I, J, K be the three complex structures associated to a hyperkähler metric g, and assume I is the given one. As recalled before, the commutator $K_{JK} = [e_J, f_K]$ acts like the Weil operator for I; hence it recovers the Hodge structure. By definition, it depends only on the classes $[\omega_I], [\omega_K]$ and their action on $H^*(X, \mathbb{R})$. Since the Hodge structure is given by the class of the symplectic form $[\sigma_I] = [\omega_J] + i[\omega_K]$, the conclusion follows. $\square$

Recall that if $\mathfrak{g}$ is a Lie algebra, the *universal enveloping algebra* of $\mathfrak{g}$ is the smallest associative algebra extending the bracket on $\mathfrak{g}$. It is defined as the quotient of the tensor algebra by the elements of the form:

$$x \otimes y - y \otimes x - [x, y] \quad x, y \in \mathfrak{g}.$$

In particular, if $\mathfrak{g}$ is abelian, then $U\mathfrak{g} = \text{Sym}^*\mathfrak{g}$.

Corollary 3.9. *There is a natural decomposition*

$$U\mathfrak{g}_{\text{tot}}(X) = U\mathfrak{g}_{\text{tot}}(X)_2 \cdot U\mathfrak{g}_{\text{tot}}(X)_0 \cdot U\mathfrak{g}_{\text{tot}}(X)_{-2}.$$

4. Primitive decomposition

In this section, we study the relationship between the actions of $\mathfrak{g}_{\text{tot}}(X)$ and $\mathfrak{g}_{\text{tot}}(X)_0$ on $H^*(X, \mathbb{Q})$, where X is a compact hyperkähler manifold of dimension $\dim(X) = 2n$. The main reference is [**?**], see also [**?**, Theorem 4.4].

Definition 4.1. Let V be a $\mathfrak{g}_{\text{tot}}(X)$-representation. We define the primitive subspace as:

$$\text{Prim}(V) = \{x \in V \mid (\mathfrak{g}_{\text{tot}}(X)_{-2}).x = 0\}.$$

If $V = H^*(X, \mathbb{Q})$ is the standard representation we denote the primitive subspace as $\text{Prim}(X)$.

Remark 4.2. The primitive subspace $\text{Prim}(V)$ is a $\mathfrak{g}_{\text{tot},0}(X)$-subrepresentation. This follows from the fact that $[\mathfrak{g}_{\text{tot}}(X)_0, \mathfrak{g}_{\text{tot}}(X)_{-2}] \subset \mathfrak{g}_{\text{tot}}(X)_{-2}$.

Definition 4.3. The *Verbitsky component* $SH^2(X, \mathbb{Q}) \subseteq H^*(X, \mathbb{Q})$ is the graded subalgebra generated by $H^2(X, \mathbb{Q})$.

Proposition 4.4 ([?, Corollary 1.13 and Corollary 2.3])**.** *The cohomology $H^*(X,\mathbb{Q})$ is generated by* $\mathrm{Prim}(X)$ *as a $SH^2(X,\mathbb{Q})$-module. Moreover, if $W \subset \mathrm{Prim}(X)$ is a $\mathfrak{g}_{\mathrm{tot}}(X)_0$ irreducible subrepresentation, then $SH^2(X,\mathbb{Q}).W \subset H^*(X,\mathbb{Q})$ is an irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$-module.*

Proof. Since $\mathfrak{g}_{\mathrm{tot}}(X)$ is semisimple, we can decompose the cohomology in irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$-representations:

$$H^*(X,\mathbb{Q}) = V_1 \oplus \cdots \oplus V_k.$$

The primitive part is compatible with this decomposition, so we get the decomposition

$$\mathrm{Prim}(X) = \mathrm{Prim}(V_1) \oplus \cdots \oplus \mathrm{Prim}(V_k),$$

of $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representations.

We first want to show that $SH^2(X,\mathbb{Q}).\mathrm{Prim}(V_i) = V_i$. We have

$$SH^2(X,\mathbb{Q}).\mathrm{Prim}(V_i) = U\mathfrak{g}_{\mathrm{tot}}(X)_2.\mathrm{Prim}(V_i) = U\mathfrak{g}_{\mathrm{tot}}(X).\mathrm{Prim}(V_i) \subset V_i, \tag{4.1}$$

where the first equality follows from the fact that $\mathfrak{g}_{\mathrm{tot}}(X)_2$ is abelian, and the second from Corollary 3.9. Thus $SH^2(X,\mathbb{Q}).\mathrm{Prim}(V_i)$ is a $\mathfrak{g}_{\mathrm{tot}}(X)$ subrepresentation of V_i, but V_i is irreducible, so the equality holds. This proves the first part of the proposition.

To prove the second part it is enough to show that each $\mathrm{Prim}(V_i)$ is irreducible as a $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representation. Assume it is not and write $\mathrm{Prim}(V_i) = W_1 \oplus W_2$. The identities (4.1) show that acting with $SH^2(X,\mathbb{Q})$ gives a decomposition $V_i = SH^2(X,\mathbb{Q}).W_1 \oplus SH^2(X,\mathbb{Q}).W_2$. Again, this contradicts the fact that V_i is an irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$-representation. □

Corollary 4.5. *The Verbitsky component $SH^2(X,\mathbb{Q}) \subset H^*(X,\mathbb{Q})$ is an irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$ subrepresentation.*

Proof. By definition we have $SH^2(X,\mathbb{Q}) = SH^2(X,\mathbb{Q}).H^0(X,\mathbb{Q})$, and $H^0(X,\mathbb{Q}) \subset \mathrm{Prim}(X)$. So it is enough to observe that $H^0(X,\mathbb{Q})$ is preserved by $\mathfrak{g}_{\mathrm{tot}}(X)_0$, then we conclude by the previous proposition. □

Lemma 4.6. *The restriction map $\mathfrak{g}_{\mathrm{tot}}(X)_{\mathbb{R}} \longrightarrow \mathfrak{gl}(SH^2(X,\mathbb{R}))$ is injective.*

Proof. Let $K \subset \mathfrak{g}_{\mathrm{tot}}(X)_{\mathbb{R}}$ be the kernel. It is immediate to see that $K \subset \mathfrak{g}_{\mathrm{tot}}(X)'_0$. The action of K is 0 on $H^2(X,\mathbb{R})$, so by (3.1) we get $[K, \mathfrak{g}_{\mathrm{tot}}(X)_{\mathbb{R},2}] = 0$. Taking the Lie group of K and the corresponding adjoint action, we see that $[K, f_a] = 0$ for every $a \in H^2(X,\mathbb{R})$ for which f_a is defined. So K has bracket 0 with $\mathfrak{g}_{\mathrm{tot}}(X)_{\mathbb{R},2}$ and $\mathfrak{g}_{\mathrm{tot}}(X)_{\mathbb{R},-2}$, thus also with $\mathfrak{g}_{\mathrm{tot}}(X)_{\mathbb{R},0}$. Since $\mathfrak{g}_{\mathrm{tot}}(X)$ is semisimple this implies $K{=}0$. □

5. Verbitsky's Theorem

In this section we give a proof of a result by Verbitsky on the structure of the irreducible component $SH^2(X)$. The argument presented was given by Bogomolov in [?].

Theorem 5.1. *There is a natural isomorphism of algebras and $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representations:*

$$SH^2(X,\mathbb{C}) \simeq \mathrm{Sym}^*(H^2(X,\mathbb{C}))/\langle \alpha^{n+1} \mid q(\alpha) = 0\rangle.$$

The key technical fact is the following lemma from representation theory, of which we omit the proof.

Lemma 5.2. *Denote by A the graded $\mathbb{C}$-algebra $\mathrm{Sym}^*(H^2(X,\mathbb{C}))/\langle \alpha^{n+1} \mid q(\alpha) = 0\rangle$. Then we have:*

(1) $A_{2n} \simeq \mathbb{C}$.
(2) *The multiplication map $A_k \times A_{2n-k} \longrightarrow A_{2n}$ induces a perfect pairing.*

Proof of the theorem. From the Local Torelli Theorem we have that $\alpha^{n+1} = 0$ for an open subset of the quadric $\{\alpha \in H^2(X,\mathbb{C}) \mid q(\alpha) = 0\}$. Since the condition $\alpha^{n+1} = 0$ is Zariski closed, we get that it holds for the entire quadric. Consider the multiplication map

$$\mathrm{Sym}^*(H^2(X,\mathbb{C})) \longrightarrow SH^2(X,\mathbb{C}).$$

The kernel contains $\{\alpha^{n+1} \mid q(\alpha) = 0\}$, hence it factors via the ring A. It is an algebra homomorphism by construction, and a map of $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representations because $\mathfrak{g}_{\mathrm{tot}}(X)_0'$ acts via derivations.

The induced map $A \longrightarrow SH^2(X,\mathbb{C})$ is surjective by construction. If it were not injective, by the above lemma, the kernel would contain A_{2n}. But this is impossible, because in top degree the map $A_{2n} \longrightarrow H^{4n}(X,\mathbb{C})$ is non-zero. Indeed if σ is a holomorphic symplectic form, the form $(\sigma + \overline{\sigma})^{2n}$ is non-zero. ☐

Corollary 5.3. *There are natural isomorphisms defined over $\mathbb{Q}$*

$$SH^2(X,\mathbb{Q})_{2k} \simeq \begin{cases} \mathrm{Sym}^k H^2(X,\mathbb{Q}) \text{ if } k \leqslant n, \\ \mathrm{Sym}^{2n-k} H^2(X,\mathbb{Q}) \text{ if } n < k \leqslant 2n. \end{cases}$$

Proof. From Theorem 5.1 it follows that the properties (1) and (2) in Lemma 5.2 hold for $SH^2(X,\mathbb{C})$. Up to changing the isomorphism $SH^2(X,\mathbb{C})_{2n} \simeq \mathbb{C}$, they also hold for $SH^2(X,\mathbb{Q})$. The multiplication map $\mathrm{Sym}^k H^2(X,\mathbb{Q}) \longrightarrow SH^2(X,\mathbb{Q})_{2k}$ is an isomorphism if $k \leqslant n$, because it is so over $\mathbb{C}$. If $k > n$ we have

$$SH^2(X,\mathbb{Q})_{2k} \simeq SH^2(X,\mathbb{Q})^*_{4n-2k} \simeq \mathrm{Sym}^{2n-k} H^2(X,\mathbb{Q})^* \simeq \mathrm{Sym}^{2n-k} H^2(X,\mathbb{Q}),$$

where the last equality is due to the Beauville–Bogomolov–Fujiki form. ☐

Example 5.4. If X is of K3$^{[2]}$-type, for dimensional reasons, the Verbitsky component $SH(X)$ is the only irreducible component in the cohomology. For higher values of n the decomposition of $H^*(X,\mathbb{Q})$ in irreducible components is described in [**?**], for more details on this see [**?**].

6. Spin action

In this section we study how the action of $\mathfrak{so}(H^2(X,\mathbb{Q}),q)$ integrates to an action of the simply connected algebraic group $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$. Recall that there is an exact sequence of algebraic groups

$$1 \longrightarrow \pm 1 \longrightarrow \underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q) \longrightarrow \underline{\mathrm{SO}}(H^2(X,\mathbb{Q}),q) \longrightarrow 1.$$

For more information see [?] and [?].

Proposition 6.1 ([?, Theorem 4.4],[?])**.** *The action of $\mathfrak{so}(H^2(X,\mathbb{Q}),q)$ on $H^*(X,\mathbb{Q})$ integrates to an action of the algebraic group $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ via ring isomorphisms. On the even cohomology it induces an action of $\underline{\mathrm{SO}}(H^2(X,\mathbb{Q}),q)$.*

Proof. The first part of the statement is clear: we can always lift the action because the algebraic group $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ is simply connected. The group $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ acts via ring isomorphisms because the Lie algebra acts via derivations.

To show the second part of the statement we proceed as follows. Fix a hyperkähler metric g and a compatible complex structure I. The Weil operator with respect to I is contained in $(\mathfrak{g}_g)'_0 \simeq \mathfrak{so}(H^2(X,\mathbb{Q}))$. The exponential $\exp(\pi I) \in \underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ acts on the (p,q) part of $H^k(X,\mathbb{C})$ as multiplication by $e^{i(p-q)\pi}$, which is just multiplication by $(-1)^k$. In particular, on $H^2(X,\mathbb{Q})$ it acts as the identity, so $\exp(\pi I) = -1 \in \underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$. We have also shown that $-1 \in \underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ acts on $H^k(X,\mathbb{Q})$ as $(-1)^k$, which means that the action on even cohomology factors through $\underline{\mathrm{SO}}(H^2(X,\mathbb{Q}),q)$. ∎

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ON THE HODGE AND BETTI NUMBERS OF HYPER-KÄHLER MANIFOLDS

PIETRO BERI AND OLIVIER DEBARRE

Abstract. Let X be a compact Kähler manifold of dimension m. One consequence of the Hirzebruch–Riemann–Roch theorem is that the coefficients of the χ_y-genus polynomial

$$p_X(y) := \sum_{p,q=0}^{m} (-1)^q h^{p,q}(X) y^p \in \mathbb{Z}[y]$$

are (explicit) universal polynomials in the Chern numbers of X. In 1990, Libgober–Wood determined the first three terms of the Taylor expansion of this polynomial about $y = -1$ and deduced that the Chern number $\int_X c_1(X)c_{m-1}(X)$ can be expressed in terms of the coefficients of the polynomial $p_X(y)$ (Proposition 2.1).

When X is a hyper-Kähler manifold of dimension $m = 2n$, this Chern number vanishes. The Hodge diamond of X also has extra symmetries which allowed Salamon to translate the resulting identity into a linear relation between the Betti numbers of X (Corollary 2.5).

When X has dimension 4, Salamon's identity gives a relation between $b_2(X)$, $b_3(X)$, and $b_4(X)$. Using a result of Verbitsky's on the injectivity of the cup-product map that produces an inequality between $b_2(X)$ and $b_4(X)$, it is easy to conclude $b_2(X) \leqslant 23$. Guan established in 2001 more restrictions on the Betti numbers (Theorem 3.6).

Contents

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1. Symmetries of the Hodge diamond of a hyper-Kähler manifold

Let X be a compact hyper-Kähler manifold of dimension $2n$ and let σ be a holomorphic symplectic form on X. Apart from the usual symmetries

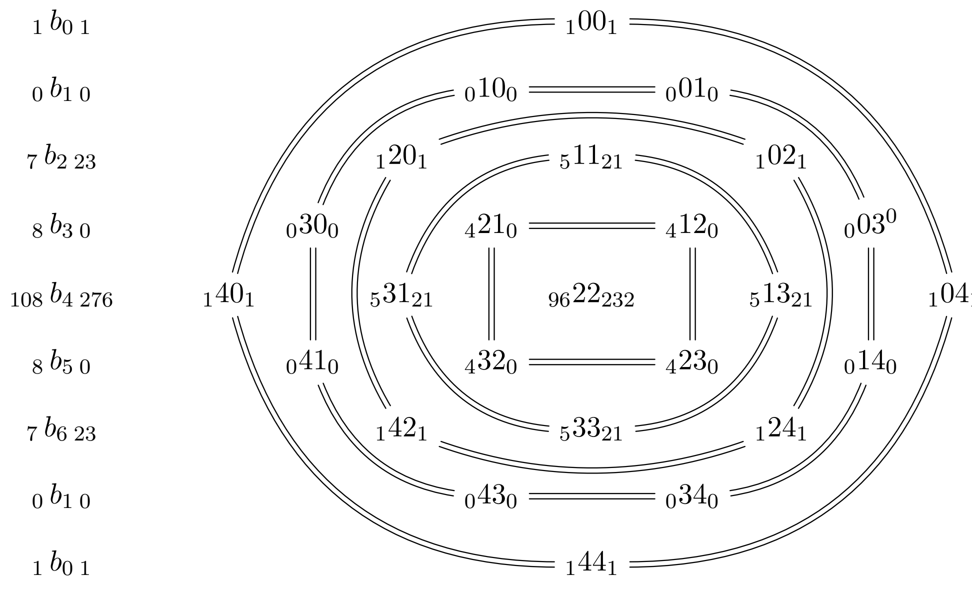
$$h^{p,q}(X) = h^{q,p}(X) = h^{2n-p,2n-q}(X)$$

coming from Kähler theory and Serre duality, there is another symmetry

$$h^{p,q}(X) = h^{2n-p,q}(X) \tag{1.1}$$

coming from the fact that the wedge product $\wedge\,\sigma^{\wedge(n-p)}$ is an isomorphism $\Omega_X^p \xrightarrow{\sim} \Omega_X^{2n-p}$. So the Hodge diamond of X has a D_8-symmetry.

Example 1.1 ($n = 2$)**.** We represent the various symmetries of the Hodge diamond for an irreducible hyper-Kähler fourfold (note that the extra "mirror" symmetry (1.1) is only visible here on the outer edges of the diamond). In the following diagram, the Hodge numbers $h^{p,q}$ of hyper-Kähler fourfolds of Kum_2-type appear as left indices of the pq label and those for the $\mathrm{K3}^{[2]}$-type as right indices.



A priori, there are only three undetermined Hodge numbers: h^{11}, h^{21}, and h^{22}. We will see in Example 2.7 that there is a relation between them.

Example 1.2 ($n = 3$)**.** We represent some of the symmetries of the Hodge diamond of an irreducible hyper-Kähler sixfold. In the following diagram, the Hodge numbers $h^{p,q}$ of hyper-Kähler sixfolds of Kum_3-type appear as left indices of the pq label, those for the $\mathrm{K3}^{[3]}$-type as right indices, and those for the OG6-type as right exponents.

$$
\begin{array}{c|ccccccccccccc}
{}_1\, b_0\, {}^{1}_{1} & & & & & & & {}_1 00^1_1 & & & & & & \\
{}_0\, b_1\, {}^{0}_{0} & & & & & & {}_0 10^0_0 & & {}_0 01^0_0 & & & & & \\
{}_7\, b_2\, {}^{8}_{23} & & & & & {}_1 20^1_1 & & {}_5 11^6_{21} & & {}_1 02^1_1 & & & & \\
{}_8\, b_3\, {}^{0}_{0} & & & & {}_0 30^0_0 & & {}_4 21^0_0 & & {}_4 12^0_0 & & {}_0 03^0_0 & & & \\
{}_{51}\, b_4\, {}^{199}_{299} & & & {}_1 40^1_1 & & {}_6 31^{12}_{22} & & {}_{37} 22^{173}_{253} & & {}_6 13^{12}_{22} & & {}_1 04^1_1 & & \\
{}_{56}\, b_5\, {}^{0}_{0} & & {}_0 50^0_0 & & {}_4 41^0_0 & & {}_{24} 32^0_0 & & {}_{24} 23^0_0 & & {}_4 14^0_0 & & {}_0 05^0_0 & \\
{}_{458}\, b_6\, {}^{1504}_{2554} & {}_1 60^1_1 & & {}_5 51^6_{21} & & {}_{37} 42^{173}_{253} & & {}_{372} 33^{1144}_{2004} & & {}_{37} 24^{173}_{253} & & {}_5 15^6_{21} & & {}_1 06^1_1 \\
{}_{56}\, b_7\, {}^{0}_{0} & & {}_0 61^0_0 & & {}_4 52^0_0 & & {}_{24} 43^0_0 & & {}_{24} 34^0_0 & & {}_4 25^0_0 & & {}_0 16^0_0 &
\end{array}
$$

A priori, there are only six undetermined Hodge numbers: h^{11}, h^{21}, h^{31}, h^{22}, h^{32}, and h^{33}. We will see in Example 2.8 that there is a relation between them.

2. Salamon's results on Betti numbers

2.1. **Hirzebruch–Riemann–Roch.** Let X be a compact Kähler manifold of dimension m. Following [H1], we set

$$\chi^p(X) := \sum_{q=0}^{m} (-1)^q h^{p,q}(X) = \chi(X, \Omega_X^p).$$

By Serre duality, these numbers satisfy

$$\chi^p(X) = (-1)^m \chi^{m-p}(X) \tag{2.1}$$

and we define the χ_y-genus by the formula

$$p_X(y) := \sum_{p=0}^{m} \chi^p(X) y^p = \sum_{p,q=0}^{m} (-1)^q h^{p,q}(X) y^p \in \mathbb{Z}[y]. \tag{2.2}$$

For instance,

- $p_X(0) = \chi^0(X) = \chi(X, \mathscr{O}_X)$,
- $p_X(-1) = \chi_{\mathrm{top}}(X) = e(X)$,
- $p_X(1)$ is the signature of the intersection form on $H^m(X, \mathbb{R})$ (which vanishes when m is odd).

Serre duality translates into the reciprocity property $(-y)^m p_X\big(\frac{1}{y}\big) = p_X(y)$.

One consequence of the Hirzebruch–Riemann–Roch theorem is that $\chi^p(X)$ can be expressed as a universal polynomial $T_{m,p}(c_1,\dots,c_m)$ in the Chern classes of X evaluated on X ([H1, Section IV.21.3, (10)]), that is,

$$p_X(y) = \sum_{p=0}^{m} y^p \int_X T_{m,p}(c_1(X),\dots,c_m(X)) = \int_X T_m(y)(c_1(X),\dots,c_m(X)), \tag{2.3}$$

where $T_m(y) := \sum_{p=0}^{m} T_{m,p}y^p$, a polynomial with coefficients in $\mathbb{Q}[c_1,\dots,c_m]$. One has

- $T_{m,p} = (-1)^m T_{m,m-p}$ and $(-y)^m T_m\big(\frac{1}{y}\big) = T_m(y)$;
- $T_{m,0} = \mathrm{td}_m(c_1,\dots,c_m)$.

Libgober–Wood found in [LW, Lemma 2.2] the first three terms of the Taylor expansion of the polynomial $T_m(y)$ about -1:

$$T_m(y-1) = c_m - \tfrac{1}{2}mc_m y + \tfrac{1}{12}\big(\tfrac{1}{2}m(3m-5)c_m + c_1c_{m-1}\big)y^2 + \cdots \tag{2.4}$$

The following is [LW, Proposition 2.3] (reproved later in [S, Theorem 4.1]).

Proposition 2.1 (Libgober–Wood)**.** *If X is a compact Kähler manifold of dimension m, one has the relation*

$$\int_X c_1(X)c_{m-1}(X) = \sum_{p=0}^{m} (-1)^p\big(6p^2 - \tfrac{1}{2}m(3m+1)\big)\chi^p(X). \tag{2.5}$$

Proof. The Taylor expansion of the polynomial p_X about the point -1 is

$$\begin{aligned} p_X(y-1) &= \sum_{p=0}^{m} \chi^p(X)(y-1)^p \\ &= \sum_{p=0}^{m} (-1)^p\chi^p(X) + y\sum_{p=0}^{m}(-1)^{p-1}\binom{p}{1}\chi^p(X) + y^2\sum_{p=0}^{m}(-1)^p\binom{p}{2}\chi^p(X) + \cdots \end{aligned}$$

Using the Hirzebruch–Riemann–Roch theorem (2.3) and comparing with (2.4), we get, by identifying the coefficients, the relations[1]

$$\begin{aligned} p_X(-1) &= \int_X c_m(X) = \sum_{p=0}^{m}(-1)^p\chi^p(X), \\ p'_X(-1) &= -\tfrac{1}{2}m\int_X c_m(X) = \sum_{p=0}^{m}(-1)^{p-1}p\chi^p(X), \qquad (2.6) \\ p''_X(-1) &= \tfrac{1}{6}\int_X \big(\tfrac{1}{2}m(3m-5)c_m(X) + c_1(X)c_{m-1}(X)\big) = 2\sum_{p=0}^{m}(-1)^p\binom{p}{2}\chi^p(X), \end{aligned}$$

[1] The first two relations are in fact formally equivalent upon using the symmetries (2.1), which give

$$p'_X(-1) = \sum_{p=0}^{m}(-1)^{p-1}(m-p)\chi^p(X) = -mp_X(-1) - p'_X(-1)$$

(see Remark 2.3).

from which it is not difficult to get (2.5). $\square$

The following consequence of Proposition 2.1 was obtained in [G, (1.14) and Proposition 2.4] using modular forms (see also [H2]).[2]

Corollary 2.2 (Gritsenko)**.** *If X is a compact Kähler manifold of dimension m that satisfies $c_1(X)_{\mathbb{R}} = 0$, one has*

$$\tag{2.7} \tfrac{1}{12}\, me(X) = \sum_{p=0}^{m} (-1)^p \big(\tfrac{1}{2}m - p\big)^2 \chi^p(X) = 2 \sum_{0 \leqslant p < m/2} (-1)^p \big(\tfrac{1}{2}m - p\big)^2 \chi^p(X).$$

In particular, when m is even,[3] *$me(X)$ is divisible by* 24.

Proof. The first equality in (2.7) is easily obtained from the relations (2.6), and the second equality from the symmetries (2.1). $\square$

Remark 2.3. Salamon gives in [S, p. 145] the next two terms of the expansion (2.4) (see also [L, Proposition 3.1(4)]):

$$\begin{aligned} T_m(y-1) = c_m - \tfrac{1}{2}mc_m y &+ \tfrac{1}{12}\big(\tfrac{1}{2}m(3m-5)c_m + c_1 c_{m-1}\big)y^2 \\ &- \tfrac{1}{24}(m-2)\big(\tfrac{1}{2}m(m-3)c_m + c_1 c_{m-1}\big)y^3 \\ &+ \tfrac{1}{5760}\big(m(15m^3 - 150m^2 + 485m - 502)c_m + 4(15n^2 - 85n + 108)c_1 c_{m-1} \\ &\qquad + 8(c_1^2 + 3c_2)c_{m-2} - 8(c_1^3 - 3c_1c_2 + 3c_3)c_{m-3}\big)y^4 + \cdots \end{aligned}$$

The y^3-term does not bring any new information since it is in fact a formal consequence of the reciprocity property $(-y)^m T_m\big(\frac{1}{y}\big) = T_m(y)$.

Using this expansion, J. Schmitt was able to find the following analogue of the Libgober–Wood formula (2.5) for a compact Kähler manifold X of dimension m:

$$\tag{2.8} \begin{aligned} &\int_X \big((\tfrac{1}{3}c_1^2(X) + c_2(X))c_{m-2}(X) - (\tfrac{1}{3}c_1^3(X) - c_1(X)c_2(X) + c_3(X))c_{m-3}(X)\big) \\ &\qquad = \sum_{p=0}^{m} (-1)^p \big(10p^4 + (2 - 5m - 15m^2)p^2 + \frac{1}{24}m(5m+1)(15m^2 + 3m - 2)\big)\chi^p(X). \end{aligned}$$

On a hyper-Kähler manifold, where all the odd Chern classes vanish, the left side reduces to $\int_X c_2(X)c_{m-2}(X)$.

[2] Gritsenko also gives in [G, (1.13)] relations between the $\chi^p(X)$ when $m \in \{4, 6, 8, 10\}$, but they are all rewritings of (2.7).

[3] This assumption is missing from [G], but it is necessary: when m is odd and we write $m = 2n+1$, we have

$$\tfrac{m-3}{12} e(X) = 2 \sum_{0 \leqslant p \leqslant n} (-1)^p \big(\big(\tfrac{1}{2}m - p\big)^2 - \tfrac{1}{4}\big)\chi^p(X) = 2 \sum_{0 \leqslant p \leqslant n} (-1)^p \big(n(n+1) - p(2n+1) + p^2\big)\chi^p(X),$$

which is divisible by 4. So what we get is that $\frac{m-3}{2} e(X)$ is divisible by 24.

Remark 2.4. The polynomials T_m can be computed. Setting for simplicity $c_1 = 0$ (the case of interest for us), we have, for even dimensions $m \in \{2, 4, 6\}$ (see [LW] or [D, Section 9]),

$$\begin{aligned}
T_2(y-1) &= c_2 - c_2 y + \tfrac{1}{12} c_2 y^2, \\
T_4(y-1) &= c_4 - 2c_4 y + \tfrac{7}{6} c_4 y^2 - \tfrac{1}{6} c_4 y^3 + \tfrac{1}{720}(3c_2^2 - c_4) y^4, \\
T_6(y-1) &= c_6 - 3c_6 y + \tfrac{13}{4} c_6 y^2 - \tfrac{3}{2} c_6 y^3 + \tfrac{1}{240}(-c_3^2 + c_2 c_4 + 62 c_6) y^4 \\
&\quad + \tfrac{1}{720}(3c_3^2 - 3c_2 c_4 - 6c_6) y^5 + \tfrac{1}{60480}(10 c_2^3 - c_3^2 - 9 c_2 c_4 + 2c_6) y^6.
\end{aligned}$$

Setting $\chi := \mathrm{td}_m$ (this is the constant term and leading coefficient of T_m), we get

$$\begin{aligned}
T_2(y) &= \chi + (2\chi - c_2) y + \chi y^2, \\
T_4(y) &= \chi + (4\chi - \tfrac{1}{6} c_4) y + (6\chi + \tfrac{2}{3} c_4) y^2 + (4\chi - \tfrac{1}{6} c_4) y^3 + \chi y^4.
\end{aligned} \tag{2.9}$$

2.2. **Application to hyper-Kähler manifolds.** Assume now that m is even and that we have the extra "mirror" symmetry $h^{p,q}(X) = h^{m-p,q}(X)$ like we do when X is a hyper-Kähler manifold. We define polynomials

$$h_X(s,t) := \sum_{p,q=0}^{m} h^{p,q}(X) s^p t^q \in \mathbb{Z}[s,t],$$

$$b_X(t) := \sum_{j=0}^{2m} b_j(X) t^j = h_X(t,t).$$

The polynomial h_X is symmetric and $p_X(y) = h_X(-1, y)$. Now we use the evenness of m and the extra symmetry to get

$$\begin{aligned}
\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) &= \sum_{p,q=0}^{m} pq(-1)^{p+q} h^{p,q}(X) \\
&= \sum_{p,q=0}^{m} (m-p)q(-1)^{m-p+q} h^{p,q}(X) \\
&= -\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) + m \sum_{p,q=0}^{m} q(-1)^{p+q} h^{p,q}(X) \\
&= -\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) - m\, \frac{\partial h_X}{\partial t}(-1,-1),
\end{aligned}$$

so that

$$2\, \frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) = -m\, \frac{\partial h_X}{\partial t}(-1,-1) = -m p_X'(-1). \tag{2.10}$$

In terms of the polynomial b_X, we have, by symmetry of h_X,

$$b_X'(t) = 2\, \frac{\partial h_X}{\partial t}(t,t),$$

$$b''_X(t) = 2\,\frac{\partial^2 h_X}{\partial s \partial t}(t,t) + 2\,\frac{\partial^2 h_X}{\partial t^2}(t,t),$$

so that we get, using (2.10),

$$b'_X(-1) = 2p'_X(-1) \qquad , \qquad b''_X(-1) = -mp'_X(-1) + 2p''_X(-1). \tag{2.11}$$

Proceeding as in the proof of Proposition 2.1, we write the Taylor expansion of the polynomial b_X about the point -1:

$$\begin{aligned} b_X(t-1) &= \sum_{j=0}^{2m} b_j(X)(t-1)^j \\ &= \sum_{j=0}^{2m} b_j(X)(-1)^j + t\sum_{j=0}^{2m} b_j(-1)^{j-1}\binom{j}{1} + t^2\sum_{j=0}^{2m} b_j(X)(-1)^j\binom{j}{2} + \cdots \end{aligned}$$

Using (2.11) and (2.6), we get

$$\begin{aligned} \sum_{j=0}^{2m} b_j(-1)^j j &= -b'_X(-1) = -2p'_X(-1) = m\int_X c_m(X), \\ \sum_{j=0}^{2m} b_j(X)(-1)^j\binom{j}{2} &= \tfrac{1}{2}b''_X(-1) = -\tfrac{1}{2}mp'_X(-1) + p''_X(-1) \\ &= \tfrac{1}{4}m^2\int_X c_m(X) + \tfrac{1}{6}\int_X \bigl(\tfrac{1}{2}m(3m-5)c_m(X) + c_1(X)c_{m-1}(X)\bigr). \end{aligned}$$

Putting everything together, we obtain the analogue of (2.5) ([S, Theorem 4.1]):

$$2\int_X c_1(X)c_{m-1}(X) = \sum_{j=0}^{2m}(-1)^j(6j^2 - m(6m+1))b_j(X).$$

Corollary 2.5 (Salamon). *If X is a compact hyper-Kähler manifold of dimension $2n$, one has*[4]

$$\sum_{j=0}^{4n}(-1)^j(3j^2 - n(12n+1))b_j(X) = 0.$$

Using the symmetry $b_j = b_{4n-j}$, one checks that one gets the equivalent relations (in the spirit of (2.7))

$$ne(X) = 6\sum_{j=1}^{2n}(-1)^j j^2 b_{2n-j}(X) \quad , \quad nb_{2n}(X) = 2\sum_{j=1}^{2n}(-1)^j(3j^2-n)b_{2n-j}(X).$$

Example 2.6 ($n=1$)**.** We obtain $b_2(X) = 22$ and $e(X) = 24$.

[4]There is a misprint in [Hu, 24.4.2].

Example 2.7 ($n=2$)**.** Salamon's relation reads

$$b_4(X) = 46 + 10b_2(X) - b_3(X).$$

On an irreducible hyper-Kähler fourfold, because of the symmetries, there are only 3 unkown Hodge numbers: $h^{11}(X)$, $h^{21}(X)$, and $h^{22}(X)$. One has

$$b_2(X) = 2 + h^{11}(X) \quad , \quad b_3(X) = 2h^{21}(X) \quad , \quad b_4(X) = 2 + 2h^{11}(X) + h^{22}(X).$$

Salamon's relation translates into

$$h^{22}(X) = 64 + 8h^{11}(X) - 2h^{21}(X).$$

There are two Chern numbers, $c_4 := \int_X c_4(X) = e(X)$ and $c_2^2 := \int_X c_2(X)^2$. They satisfy

$$3 = \chi(X, \mathscr{O}_X) = T_4(0) = \int_X \mathrm{td}_4(X) = \tfrac{1}{720}(3c_2^2 - c_4). \tag{2.12}$$

But we also have, using (2.9),

$$\chi^1(X) = 12 - \tfrac{1}{6}c_4 \quad , \quad \chi^2(X) = 18 + \tfrac{2}{3}c_4. \tag{2.13}$$

A priori though, the value of c_4 is not enough to determine all the Hodge numbers but, once we know c_4, one Hodge number determines all the others.

The Chern numbers for the two known deformation types of irreducible hyper-Kähler fourfolds are in the following table.

	$\chi_{\mathrm{top}} = e = c_4$	c_2^2
Kum_2	108	756
$\mathrm{K3}^{[2]}$	324	828

Example 2.8 ($n=3$)**.** Salamon's relation reads

$$b_6(X) = 70 + 30b_2(X) - 16b_3(X) + 6b_4(X).$$

Because of the symmetries, there are only 6 undetermined Hodge numbers: $h^{11}(X)$, $h^{21}(X)$, $h^{31}(X)$, $h^{22}(X)$, $h^{32}(X)$, and $h^{33}(X)$. One has

$$\begin{aligned}
b_2(X) &= 2 + h^{11}(X),\\
b_3(X) &= 2h^{21}(X),\\
b_4(X) &= 2 + 2h^{31}(X) + h^{22}(X),\\
b_5(X) &= 2h^{41}(X) + 2h^{32}(X),\\
b_6(X) &= 2 + 2h^{11}(X) + 2h^{22}(X) + h^{33}(X).
\end{aligned}$$

Salamon's relation translates into

$$h^{33}(X) = 140 + 28h^{11}(X) - 32h^{21}(X) + 12h^{31}(X) + 4h^{22}(X).$$

There are three Chern numbers, $c_6 := \int_X c_6(X) = e(X)$, $c_2c_4 := \int_X c_2(X)c_4(X)$, and $c_2^3 := \int_X c_2(X)^3$. They satisfy

$$4 = \chi(X, \mathscr{O}_X) = T_6(0) = \mathrm{td}_6(X) = \tfrac{1}{60480}(10c_2^3 - 9c_2c_4 + 2c_6).$$

The three known examples in dimension 6 are in the following table taken from [N2, Remark 4.13] (see also [N1, Appendix A]) and [MRS, Corollary 6.8].

	$\chi_{\mathrm{top}} = e(X) = c_6$	c_2c_4	c_2^3
Kum_3	448	6784	30208
$\mathrm{K3}^{[3]}$	3200	14720	36800
OG6	1920	7680	30720

3. Guan's bounds for Betti numbers of hyper-Kähler fourfolds

3.1. **Bounds on b_2.** Let X be an irreducible compact hyper-Kähler manifold of complex dimension $m = 2n$. Let σ be a symplectic form on X. One has $b_1(X) = 0$, and $b_2(X) \geqslant 3$ since $H^{2,0}(X) = \mathbb{C}\sigma$, $H^{0,2}(X) = \mathbb{C}\bar{\sigma}$, and $H^{1,1}(X)$ contains the class of any Kähler form.

Our aim is to prove the following upper bound for $b_2(X)$ when $m = 4$ ([Gu, Theorem 1]).

Theorem 3.1 (Guan)**.** *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. Then $3 \leqslant b_2(X) \leqslant 23$. Moreover, if $b_2(X) = 23$, the Hodge numbers of X are the same as the Hodge numbers of the Hilbert square of a K3 surface.*

About the higher Betti numbers, we have the following result ([V, Theorem 1.5], [B, Theorem 1.5]).

Theorem 3.2 (Verbitsky)**.** *Let X be an irreducible compact hyper-Kähler manifold of dimension $2n$. For all $k \leqslant n$, the canonical map $\mathrm{Sym}^k H^2(X,\mathbb{R}) \longrightarrow H^{2k}(X,\mathbb{R})$ given by cup-product is injective. In particular, $b_{2k}(X) \geqslant \binom{b_2(X)+k-1}{k}$.*

We denote by $SH^{2k}(X) \subset H^{2k}(X,\mathbb{R})$ the image of the map above.

Proof of Theorem 3.1. Write b_j for $b_j(X)$. We have $b_3 + b_4 = 46 + 10b_2$ (Example 2.7) and $b_4 \geqslant \frac{b_2(b_2+1)}{2}$ (Theorem 3.2), hence

$$\frac{b_2(b_2+1)}{2} \leqslant b_3 + \frac{b_2(b_2+1)}{2} \leqslant b_3 + b_4 = 46 + 10b_2 \tag{3.1}$$

which can be rewritten as

$$(b_2+4)(b_2-23) \leqslant 0,$$

so $b_2 \leqslant 23$. Assume now $b_2 = 23$. Substituting in the inequality above, we get $b_3 + 276 \leqslant 46 + 230 = 276$, so $b_3 = 0$. This implies $b_4 = 46 + 10b_2 = 276$. So the Betti numbers of X are the same as those of the Hilbert square of a K3 surface. As noted in Example 2.7, this implies that the Hodge numbers are also the same. □

3.2. **Generalized Chern numbers.** For an irreducible compact hyper-Kähler manifold X of dimension $2n$, we have the Beauville–Bogomolov–Fujiki quadratic form q_X on $H^2(X,\mathbb{Q})$ ([F] or [Hu, Section 23]). There exists a positive rational constant c_X such that

$$\forall \beta \in H^2(X,\mathbb{Q}) \qquad \int_X \beta^{2n} = c_X q_X(\beta)^n. \tag{3.2}$$

More generally, let $\alpha \in H^{4j}(X,\mathbb{R})$ be a class that is of type $(2j,2j)$ on all small deformations of X (this is the case for example for the Chern class $c_j(X)$). There is a constant $c_\alpha \in \mathbb{R}$ such that ([Hu, Corollary 23.17])

$$\forall \beta \in H^2(X,\mathbb{R}) \qquad \int_X \alpha\beta^{2(n-j)} = c_\alpha q_X(\beta)^{n-j}. \tag{3.3}$$

For $\alpha = 1$ and $j = 0$, we recover (3.2).

We can now define the generalized Chern numbers.

Definition 3.3. Let $C \in H^{4j}(X,\mathbb{C})$ be a polynomial in the Chern classes. The number

$$N(C) := \frac{\int_X Cu^{2(n-j)}}{\left(\int_X u^{2n}\right)^{\frac{n-j}{n}}}$$

is independent of the choice of $u \in H^2(X,\mathbb{C})$ with $\int_X u^{2n} \neq 0$. We call it a *generalized Chern number* of X.

To see that $N(C)$ does not depend on the choice of u, note that $\int_X Cu^{2(n-j)} = a_C q_X(u)^{n-j}$, where a_C is the sum of the c_α as in (3.3) for all monomials α in C. Moreover, $\int_X u^{2n} = c_X q_X(u)^n$, so $N(C) = a_C c_X^{-\frac{n-j}{n}}$; it is a real number since we can always choose u in $H^{4j}(X,\mathbb{R})$.

In our case, $n = 2$, we are interested in the generalized Chern number $N(c_2(X))$. Guan rewrote [HS, (1)] as follows ([Gu, Lemma 2]).

Lemma 3.4. *Let X be an irreducible compact hyper-Kähler manifold of dimension $2n$. Then*[5]

$$\frac{((2n)!)^{n-1}N(c_2(X))^n}{(24n(2n-2)!)^n} = \int_X \mathrm{td}^{\frac{1}{2}}(X). \tag{3.4}$$

Moreover $N(c_2(X)) > 0$.

[5]Hitchin and Sawon, and then Guan, use the $\hat{A}^{\frac{1}{2}}$-genus instead of $\mathrm{td}^{\frac{1}{2}}$. In general, one has $\hat{A} = e^{c_1/2}\,\mathrm{td}$, so they coincide in our case since $c_1 = 0$.

Proof. For any hyper-Kähler manifold X, one has $\int_X (\sigma+\bar\sigma)^{2n} = c_X q_X(\sigma+\bar\sigma)^n > 0$. Hence we can write

$$N(c_2(X)) = \frac{\int_X c_2(X)(\sigma+\bar\sigma)^{2n-2}}{\left(\int_X (\sigma+\bar\sigma)^{2n}\right)^{\frac{n-1}{n}}}.$$

The lemma therefore follows from the equality

$$\frac{\|R\|^{2n}}{(192\pi^2 n)^n \operatorname{vol}(X)^{n-1}} = \int_X \operatorname{td}^{\frac{1}{2}}(X) \tag{3.5}$$

from [HS, (1)],[6] where

- $\operatorname{vol}(X) = \frac{1}{2^{2n}(2n)!}\int_X (\sigma+\bar\sigma)^{2n}$ is the volume form on X,
- $\|R\|$ is the L^2-norm of the Riemann curvature tensor, given by

$$\|R\|^2 = \frac{8\pi^2}{2^{2n-2}(2n-2)!}\int_X c_2(X)(\sigma+\bar\sigma)^{2n-2}.$$

Note that $\int_X c_2(X)(\sigma+\bar\sigma)^{2n-2}$ is nonnegative, since it is a positive multiple of $\|R\|^2$. If it vanishes, X is flat, hence a torus by the Bieberbach theorem, which is absurd. □

The following proposition is [Gu, Lemma 3].

Proposition 3.5 (Guan)**.** *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. Then*

$$3b_2(X)N(c_2(X))^2 \leqslant (b_2(X)+2)\int_X c_2(X)^2. \tag{3.6}$$

Equality holds if and only if $c_2(X) \in SH^4(X)$.

Proof. The orthogonal complement $SH^4(X)^\perp$ of $SH^4(X)$ in $H^4(X,\mathbb{R})$ with respect to the intersection form consists of primitive classes. Therefore, by the second Hodge–Riemann bilinear relations, the intersection form is positive definite on $SH^4(X)^\perp$ and one has $H^4(X,\mathbb{R}) = SH^4(X) \oplus SH^4(X)^\perp$.

Let us write $c_2(X) = p + r$ with $p \in SH^4(X)$ and $r \in SH^4(X)^\perp$. As noted above, one has $\int_X r^2 \geqslant 0$, with equality if and only if $r = 0$.

For every $\beta \in H^2(X,\mathbb{R})$, one has, using (3.3),

$$\int_X p\beta^2 = \int_X c_2(X)\beta^2 = cq_X(\beta), \tag{3.7}$$

[6] The authors of [HS] and [Gu] use a different convention for exterior products of differential forms. The latter can be seen either as elements of the abstract exterior algebra of the space of 1-forms or as alternating multilinear forms: depending on the point of view, the two definitions of product between differential forms differ by a binomial coefficient. So, if we follow Hitchin and Sawon and we write $\operatorname{vol}(X) = \frac{1}{2^{2n}((n)!)^2}\int_X \sigma^n\bar\sigma^n$ and $\|R\|^2 = \frac{8\pi^2}{2^{2n-2}((n-1)!)^2}\int_X c_2(X)\sigma^{n-1}\bar\sigma^{n-1}$, then (3.4) becomes $\frac{((2n)!)^{n-1}N(c_2(X))^n}{(24n(2n-2)!)^n}\cdot\frac{\binom{2(n-1)}{n-1}^n}{\binom{2n}{n}^{n-1}} = \int_X \operatorname{td}^{\frac{1}{2}}(X)$.

where $c := c_{c_2(X)}$. Write b for $b_2(X)$. Let $(e_1, \dots, e_b)$ be a basis of $H^2(X, \mathbb{C})$ which is orthonormal with respect to q_X. For all $t_1, t_2, t_3, t_4 \in \mathbb{R}$ and pairwise distinct i, j, k, l, we have

$$\int_X (t_1 e_i + t_2 e_j + t_3 e_k + t_4 e_l)^4 = c_X q_X (t_1 e_i + t_2 e_j + t_3 e_k + t_4 e_l)^2 = c_X (t_1^2 + t_2^2 + t_3^2 + t_4^2)^2,$$

which implies

$$\int_X e_i^4 = c_X \quad , \quad \int_X e_i^2 e_j^2 = \frac{1}{3} c_X \quad , \quad \int_X e_i^2 e_j e_k = \int_X e_i e_j e_k e_l = 0. \tag{3.8}$$

Write $p = \sum_{1 \leqslant i \leqslant j \leqslant b} p_{ij} e_i \cdot e_j$. Using (3.7) and (3.8), we obtain, for $i \neq j$,

$$0 = \int_X p e_i e_j = \frac{1}{3} c_X p_{ij},$$

hence $p_{ij} = 0$. Similarly, for each i, we have

$$c = \int_X p e_i^2 = c_X p_{ii} + \frac{1}{3} c_X \sum_{j \neq i} p_{ii}.$$

Summing over $i \in \{1, \dots, b\}$, we obtain

$$bc = c_X \sum_i p_{ii} + \frac{1}{3} c_X (b-1) \sum_i p_{ii} = \frac{c_X (b+2)}{3} \sum_i p_{ii}.$$

Using these relations, we obtain

$$\int_X p^2 = \sum_i p_{ii} \int_X p e_i^2 = c \sum_i p_{ii} = \frac{3bc^2}{c_X (b+2)}.$$

Finally, Definition 3.3 gives

$$N(c_2(X)) = \frac{\int_X c_2(X) e_1^2}{\left(\int_X e_1^4\right)^{1/2}} = \frac{\int_X p e_1^2}{\left(\int_X e_1^4\right)^{1/2}} = c\, c_X^{-1/2}.$$

Putting everything together, we obtain

$$\int_X c_2(X)^2 = \int_X p^2 + \int_X r^2 \geqslant \int_X p^2 = \frac{3bc^2}{c_X (b+2)} = \frac{3b N(c_2(X))^2}{b+2},$$

which is the desired inequality. Equality holds if and only if $\int_X r^2 = 0$. As we saw earlier, this is equivalent to $r = 0$, that is, $c_2(X) \in SH^4(X)$. □

3.3. **Bounds on b_3.** Let again X be an irreducible compact hyper-Kähler manifold of dimension 4. A formal computation shows

$$\int_X \mathrm{td}^{\frac{1}{2}}(X) = \frac{1}{5760} \int_X (7 c_2(X)^2 - 4 c_4(X)). \tag{3.9}$$

The following result is [Gu, Theorem 2].

Theorem 3.6 (Guan)**.** *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. Then*

$$b_3(X) \leqslant \frac{4(23 - b_2(X))(8 - b_2(X))}{b_2(X) + 1}. \tag{3.10}$$

If $b_2(X) > 7$, then $(b_2(X), b_3(X)) \in \{(8, 0), (23, 0)\}$.

Proof. Write b_j for $b_j(X)$, c_2^2 for $\int_X c_2(X)^2$, and c_4 for $\int_X c_4(X)$. We substitute Lemma 3.4, with $n = 2$, in Proposition 3.5 to obtain

$$3b_2\,\frac{(24 \cdot 4)^2}{4!} \int_X \mathrm{td}^{\frac{1}{2}}(X) \leqslant (b_2 + 2)c_2^2.$$

Substituting in (3.9) the expression for c_4 given in (2.12), we get

$$\int_X \mathrm{td}^{\frac{1}{2}}(X) = \frac{1}{5760}\big(7c_2^2 - 4(3c_2^2 - 720 \cdot 3)\big) = \frac{3}{2} - \frac{c_2^2}{1152}.$$

Hence

$$(b_2 + 2)c_2^2 \geqslant 2 \cdot 24^2\, b_2 \int_X \mathrm{td}^{\frac{1}{2}}(X) = 2 \cdot 24^2\, b_2\Big(\frac{3}{2} - \frac{c_2^2}{1152}\Big) = b_2(3 \cdot 24^2 - c_2^2). \tag{3.11}$$

We have $h^{1,1}(X) - 2h^{2,1}(X) = \chi^1(X) = 12 - \frac{c_4}{6}$ (see (2.13)); using

$$b_2 = 2 + h^{1,1}(X) \quad , \quad b_3 = 2h^{1,2}(X),$$

we obtain $c_4 = 3(16 + 4b_2 - b_3)$. We use this in (2.12) to get $c_2^2 = 736 + 4b_2 - b_3$. Then, (3.11) becomes $(b_2 + 1)b_3 \leqslant 4(23 - b_2)(8 - b_2)$ as in the statement of the theorem.

If $b_2 > 7$, the right side of (3.10) is nonpositive because $b_2 \leqslant 23$, so it has to be zero. □

The following is [Gu, Corollary 1].

Corollary 3.7 (Guan)**.** *Let X be an irreducible compact hyper-Kähler manifold of dimension 4. If $b_2(X) \leqslant 7$, one of the following holds:*

- *$b_2(X) = 3$ and $b_3(X) = 4\ell$ with $\ell \leqslant 17$;*
- *$b_2(X) = 4$ and $b_3(X) = 4\ell$ with $\ell \leqslant 15$;*
- *$b_2(X) = 5$ and $b_3(X) = 4\ell$ with $\ell \leqslant 9$;*
- *$b_2(X) = 6$ and $b_3(X) = 4\ell$ with $\ell \leqslant 4$;*
- *$b_2(X) = 7$ and $b_3(X) = 4\ell$ with $\ell \in \{0, 2\}$.*

Proof. By [F, Lemma 1.2], one has $4 \mid b_k$ for k odd. The bounds are obtained using either (3.1) or (3.10). Guan proved in [Gu] that the case $(b_2(X), b_3(X)) = (7, 4)$ cannot occur. □

Remark 3.8. When $b_2(X) = 7$, either $b_3(X) = 0$ or the Hodge numbers of X are the same as the Hodge numbers of a generalized Kummer fourfold.

Remark 3.9. Given $(b_2(X), b_3(X))$, one can compute $N(c_2(X))$ using Lemma 3.4, since the Chern numbers of X are computed in the proof of Theorem 3.6. Then it is possible to check which pairs give an equality in (3.6). Hence, using Proposition 3.5, one can check that $c_2(X) \in SH^4(X)$ if and only if $(b_2(X), b_3(X)) \in \{(5, 36), (7, 8), (8, 0), (23, 0)\}$.

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HYPER-KÄHLER MANIFOLDS OF GENERALIZED KUMMER TYPE AND THE KUGA–SATAKE CORRESPONDENCE

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Abstract. We first describe the construction of the Kuga–Satake variety associated to a (polarized) weight-two Hodge structure of hyper-Kähler type. We describe the classical cases where the Kuga–Satake correspondence between a hyper-Kähler manifold and its Kuga–Satake variety has been proved to be algebraic. We then turn to recent work of O'Grady and Markman which we combine to prove that the Kuga–Satake correspondence is algebraic for projective hyper-Kähler manifolds of generalized Kummer deformation type.

1. Introduction

The Kuga–Satake construction associates to any K3 surface, and more generally to any weight-two Hodge structure of hyper-Kähler type a complex torus which is an abelian variety when the Hodge structure is polarized. This construction allows to realize the Hodge structure on degree-two cohomology of a projective hyper-Kähler manifold as a direct summand of the H^2 of an abelian variety. Although the construction is formal and not known to be motivic, it has been used by Deligne in [2] to prove deep results of a motivic nature, for example the Weil conjecture for K3 surfaces can be deduced from the Weil conjectures for abelian varieties.

Section 2 of the notes is devoted to the description of the original construction, and to the presentation of a few classical examples where the Kuga–Satake correspondence is known to be algebraic, i.e. realized by a correspondence between the hyper-Kähler manifold and its Kuga–Satake variety. In Section 3, we focus on the case of hyper-Kähler manifolds of a generalized Kummer type, and present a few recent results. If X is a (very general) projective hyper-Kähler manifold of generalized Kummer type, the Kuga–Satake variety $\mathrm{KS}(X)$ built on $H^2(X,\mathbb{Z})_{\mathrm{tr}}$ is a sum of copies of an abelian fourfold $\mathrm{KS}(X)_c$ of Weil type. There is another abelian fourfold associated to X, namely the intermediate Jacobian $J^3(X)$ which is defined as the complex torus

$$J^3(X) = H^{1,2}(X)/H^3(X,\mathbb{Z})$$

where $b_3(X) = 8$. Here we use the fact that $H^{3,0}(X) = 0$ and the projectivity of X guarantees that $J^3(X)$ is an abelian variety. O'Grady [11] proves the following result.

Theorem 1.1. *The two abelian varieties $J^3(X)$ and $\mathrm{KS}(X)_c$ are isogenous.*

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We also prove in Section 3.2 a more general statement concerning hyper-Kähler manifolds with $b_3(X) \neq 0$. Section 3.3 is devoted to the question of the algebraicity of the Kuga–Satake correspondence. Following [16], we prove, using a theorem of Markman and Theorem 1.1 above that the Kuga–Satake correspondence is algebraic for hyper-Kähler manifolds of generalized Kummer type.

Theorem 1.2. *There exists a codimension-2n cycle $\mathcal{Z} \in \mathrm{CH}^{2n}(\mathrm{KS}(X)_c \times X)_{\mathbb{Q}}$ such that*

$$[\mathcal{Z}]_* : H_2(\mathrm{KS}(X)_c, \mathbb{Q}) \longrightarrow H_2(X, \mathbb{Q}) \tag{1.1}$$

is surjective.

2. The Kuga–Satake construction

2.1. **Main Construction.** In this section, we recall the construction and some properties of the Kuga–Satake variety associated to a Hodge structure of *hyper-Kähler type*. This construction is due to Kuga and Satake in [6]. For a complete introduction see [5, Ch. 4] and [3].

Definition 2.1. A pair (V, q) is a Hodge structure of *hyper-Kähler type* if the following conditions hold: V is a rational level-two Hodge structure with $\dim V^{2,0} = 1$, and $q : V \otimes V \longrightarrow \mathbb{Q}(-2)$ is a morphism of Hodge structures whose real extension is negative definite on $(V^{2,0} \oplus V^{0,2}) \cap V_{\mathbb{R}}$.

Remark 2.2. Note that if X is a hyper-Käler manifold and q_X is the Beauville-Bogomolov quadratic form, the pair $(H^2(X, \mathbb{Q}), -q_X)$ is indeed a Hodge structure of *hyper-Kähler type.*

Let (V, q) be a Hodge structure of hyper-Kähler type, and let $T(V)$ be the tensor algebra of the underlying rational vector space V:

$$T(V) := \bigoplus_{i \geqslant 0} V^{\otimes i}$$

where $V^{\otimes 0} := \mathbb{Q}$. Then, the *Clifford algebra* of (V, q) is the quotient algebra

$$\mathrm{Cl}(V) := \mathrm{Cl}(V, q) := T(V)/I(q),$$

where $I(q)$ is the two-sided ideal of $T(V)$ generated by elements of the form $v \otimes v - q(v)$ for $v \in V$.

Since $I(q)$ is generated by elements of even degree, the natural $\mathbb{Z}/2\mathbb{Z}$-grading on $T(V)$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cl}(V)$. Write

$$\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V),$$

where $\mathrm{Cl}^+(V)$ is the even part and $\mathrm{Cl}^-(V)$ is the odd part. Note that $\mathrm{Cl}^+(V)$ is still a $\mathbb{Q}$-algebra, it is called *the even Clifford algebra.*

We now use the assumption that (V, q) is a Hodge structure of hyper-Kähler type to define a complex structure on $\mathrm{Cl}^+(V)_\mathbb{R}$.
Consider the decomposition of the real vector space $V_\mathbb{R} = V_1 \oplus V_2$, with

$$V_1 := V^{1,1} \cap V_\mathbb{R}, \quad V_2 := \{V^{2,0} \oplus V^{0,2}\} \cap V_\mathbb{R}.$$

The $\mathbb{C}$-linear span of V_2 is the two-dimensonal vector space $V^{2,0} \oplus V^{0,2}$. Therefore, q is negative definite on V_2.
Pick a generator $\sigma = e_1 + ie_2$ of $V^{2,0}$ with $e_1, e_2 \in V_2$ and $q(e_1) = -1$. Since $q(\sigma) = 0$, we deduce that $q(e_1, e_2) = 0$ and $q(e_2) = -1$, i.e., e_1, e_2 is an orthonormal basis of V_2. From this, it is straightforward to check that $e_1 \cdot e_2 = -e_2 \cdot e_1$ in $\mathrm{Cl}(V)_\mathbb{R}$. Therefore $J := e_1 \cdot e_2$ satisfies the equation $J^2 = -1$ and left multiplication by J induces a complex structure on the real vector space $\mathrm{Cl}(V)_\mathbb{R}$ which preserves the real subspaces $\mathrm{Cl}^+(V)_\mathbb{R}$ and $\mathrm{Cl}^-(V)_\mathbb{R}$. Since giving a complex structure on a real vector space is equivalent to giving a Hodge structure of level one on the rational vector space, we can make the following definition.

Definition 2.3. The *Kuga–Satake Hodge structure* $H^1_{\mathrm{KS}}(V)$ is the Hodge structure of level one on $\mathrm{Cl}^+(V)$ given by

$$\rho\colon \mathbb{C}^* \longrightarrow \mathrm{GL}(\mathrm{Cl}^+(V)_\mathbb{R}), \quad x + yi \longrightarrow x + yJ,$$

where $x + yJ$ acts on $\mathrm{Cl}^+(V)_\mathbb{R}$ via left multiplication.

Therefore, starting from a rational level-two Hodge structure of hyper-Kähler type (V, q), we constructed a rational Hodge structure of level one on $\mathrm{Cl}^+(V)$. This determines naturally a complex torus up to isogeny: Let $\Gamma \subseteq \mathrm{Cl}^+(V)$ be a lattice in the rational vector space $\mathrm{Cl}^+(V)$, then the *Kuga–Satake variety* associated to (V, q) is the (isogeny class of) the complex torus

$$\mathrm{KS}(X) := \mathrm{Cl}^+(V)_\mathbb{R}/\Gamma,$$

where $\mathrm{Cl}^+(V)_\mathbb{R}$ is endowed with the complex structure induced by left multiplication by J.
Notice that if (V, q) is an integral Hodge structure of hyper-Kähler type, then V can be viewed as a lattice in $\mathrm{Cl}^+(V_\mathbb{Q})$. Thus, the natural choice $\Gamma := V$ determines the complex torus $\mathrm{KS}(V)$, and not just up to isogeny.

By construction, one has the following:

$$H^1_{\mathrm{KS}}(V) := H^1(\mathrm{KS}(V), \mathbb{Q}) \simeq \mathrm{Cl}^+(V)^* \simeq \mathrm{Cl}^+(V),$$

where the isomorphism between $\mathrm{Cl}^+(V)$ and its dual is induced by the nondegenerate form q.

Remark 2.4. Consider the case where V can be written as a direct sum of Hodge structures $V = V_1 \oplus V_2$. Since $\dim V^{2,0} = 1$, either V_1, or V_2 has to be pure of type $(1, 1)$. We may then assume that $V_2^{2,0} = 0$. Then, one can check that the Kuga–Satake Hodge structure $\mathrm{Cl}^+(V)$

is isomorphic to the product of 2^{n_2-1} copies of $\mathrm{Cl}^+(V_1) \oplus \mathrm{Cl}^-(V_1)$, with $n_2 := \dim V_2$. In particular:

$$\mathrm{KS}(V_1 \oplus V_2) \sim \mathrm{KS}(V_1)^{2^{n_2}}.$$

Remark 2.5. For any element $w \in \mathrm{Cl}^+(V)$, the right-multiplication morphism

$$r_w \colon \mathrm{Cl}^+(V) \longrightarrow \mathrm{Cl}^+(V), \quad r_w(x) := x \cdot w$$

is a morphism of Hodge structures. This follows from the fact that the Kuga–Satake Hodge structure on $\mathrm{Cl}^+(V)$ is induced by left multiplication by $J \in \mathrm{Cl}^+(V)$ which commutes with right multiplication by elements of $\mathrm{Cl}^+(V)$. Therefore, there is an embedding

$$\mathrm{Cl}^+(V) \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(\mathrm{Cl}^+(V)) \simeq \mathrm{End}(\mathrm{KS}(V)) \otimes \mathbb{Q}.$$

Since the dimension of $\mathrm{Cl}^+(V)$ is $2^{\dim V-1}$, we deduce that the endomorphism algebra of $\mathrm{KS}(V)$ is in general big. This is related with the fact that the Kuga–Satake variety of a Hodge structure of hyper-Kähler type is in general not simple, but it is isogenous to the power of a smaller-dimensional torus.

Remarkably, the Kuga–Satake construction realizes the starting level-two Hodge structure as a sub-Hodge structure of the tensor product of two Hodge structures of level one:

Theorem 2.6. *Let (V, q) be a Hodge structure of hyper-Kähler type. Then there is an embedding of Hodge structures:*

$$V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V),$$

where $\mathrm{Cl}^+(V)$ is endowed with the level-one Hodge structure of Definition 2.3.

Proof. We recall here just the definition of the desired map, for more details we refer to [5, Prop. 3.2.6].
Fix an element $v_0 \in V$ such that $q(v_0) \neq 0$ and consider the following left multiplication map:

$$\begin{aligned} \varphi \colon V &\longrightarrow \mathrm{End}(\mathrm{Cl}^+(V)) \\ v &\longrightarrow f_v \colon w \longmapsto v \cdot w \cdot v_0. \end{aligned}$$

The injectivity of φ follows from the equality $f_v(v' \cdot v_0) = q(v_0)(v \cdot v')$ for any $v' \in V$. See the reference above for the proof of the fact that φ is a morphism of Hodge structures.
Finally, the desired embedding is given by the composition of ϕ and the isomorphisms

$$\mathrm{End}(\mathrm{Cl}^+(V)) \simeq \mathrm{Cl}^+(V)^* \otimes \mathrm{Cl}^+(V) \simeq \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V),$$

where the isomorphism $\mathrm{Cl}^+(V)^* \simeq \mathrm{Cl}^+(V)$ is induced by q. □

Remark 2.7. Note that the morphism of Theorem 2.6 is not canonical, in the sense that it depends on the choice of $v_0 \in V$. Nevertheless, choosing another $v_0' \in V$ changes the embedding by the automorphism of $\mathrm{Cl}^+(V)$ which sends w to $\frac{w \cdot v_0 \cdot v_0'}{q(v_0)}$.

Theorem 2.6 shows that any Hodge structure of hyper-Kähler type can be realized as a Hodge substructure of $W \otimes W$ for some level-one Hodge structure W. On the other hand, in [2, Sec. 7], Deligne proves that for a very general level-two Hodge structure of the same conclusion does not hold. We recall here a version of this fact as presented in [3, Prop. 4.2].

Theorem 2.8. *Let (V, q) be a polarized level-two Hodge structure whose Mumford–Tate group $\mathrm{MT}(V)$ is maximal, that is, equal to $\mathrm{SO}(q)$. Then, if $\dim V^{2,0} > 1$, V cannot be realized as a Hodge substructure of $W \otimes W$ for any level-one Hodge structure W.*

Remark 2.9. One can show in some cases that the technical condition $\mathrm{MT}(V) = \mathrm{SO}(q)$ of Theorem 2.8 is satisfied for a very general Hodge structure, see [2, Sec. 7] and [15, Cor. 4.12]. The proof goes as follows: Given a smooth projective morphism $\pi\colon \mathcal{X} \longrightarrow B$, one shows that for very general $t \in B$, the Mumford–Tate group $\mathrm{MT}(\mathcal{X}_t)$ contains a finite index subgroup of the monodromy group of the base. Then, already in the case of hypersurfaces in a $2r + 1$ dimensional projective space, this shows that for a very general hypersurface X_s, the Mumford Tate group of $H^{2r}(X, \mathbb{Q})$ is maximal in the above sense. Therefore, applying Theorem 2.8, one sees that, for a very general surface X in $\mathbb{P}^3$ of degree $\geqslant 5$, $H^2(X, \mathbb{Q})$ cannot be realized as a Hodge substructure of $W \otimes W$ for any level-one Hodge structure W.

To conclude this section, we recall the fact that if the Hodge structure of hyper-Kähler type is polarized, the resulting Kuga–Satake Hodge structure on the even Clifford algebra is naturally polarized.

Theorem 2.10. *If (V, q) is a Hodge structure of hyper-Kähler type such that q is a polarization for V, then the Kuga–Satake Hodge structure on $\mathrm{Cl}^+(V)$ has a natural polarization. In particular, the Kuga–Satake torus $\mathrm{KS}(V)$ is an abelian variety.*

2.2. **Some examples.** Let X be a hyper-Kähler variety (resp. a two dimensional complex torus). The pair $(H^2(X, \mathbb{Q}), -q_X)$ where q_X is the Beauville-Bogomolov form (resp. the intersection pairing) is a Hodge structure of hyper-Kähler type. Therefore, we can apply the Kuga–Satake construction to it and we get the *Kuga–Satake variety* of X:

$$\mathrm{KS}(X) := \mathrm{KS}(H^2(X, \mathbb{Q})).$$

Since $-q_X$ is not a polarization on the whole $H^2(X, \mathbb{Q})$, the variety $\mathrm{KS}(X)$ is not necessarily an abelian variety, but it is just a complex torus.
On the other hand, if X is projective and l is an ample class on X, the primitive part

$$H^2(X, \mathbb{Q})_p := l^\perp \subseteq H^2(X, \mathbb{Q})$$

is a sub-Hodge structure which is polarized by the restriction of the form $-q_X$. Therefore, by Theorem 2.10, the Kuga–Satake variety of $H^2(X, \mathbb{Q})_p$ is an abelian variety. Moreover, by Remark 2.4, we have

$$\mathrm{KS}(X) := \mathrm{KS}(H^2(X, \mathbb{Q})) \sim \mathrm{KS}(H^2(X, \mathbb{Q})_p)^2.$$

In particular, in the projective case, $\mathrm{KS}(X)$ is an abelian variety. A similar argument can be applied to $H^2(X,\mathbb{Q})_{\mathrm{tr}} \subseteq H^2(X,\mathbb{Q})$, the transcendental lattice of a projective K3 surface, to deduce that $\mathrm{KS}(X)$ is isogenous to some power of the abelian variety $\mathrm{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})$. On the other hand, if X is not projective, the torus $\mathrm{KS}(X)$ need not be polarized.

Theorem 2.11. [10] *Let A a complex torus of dimension two. Then there exists an isogeny*

$$\mathrm{KS}(A) \sim (A \times \hat{A})^4,$$

where $\hat{A}$ is the dual complex torus.
In particular, if A is an abelian surface

$$\mathrm{KS}(A) \sim A^8 \quad \textit{and} \quad \mathrm{KS}(\mathrm{Kum}(A)) \sim A^{2^{19}},$$

where $\mathrm{Kum}(A)$ is the Kummer surface associated to A.

Definition 2.12. Let A be an abelian variety of dimension $2n$ and let d be a positive real number. Then A is called of $\mathbb{Q}(\sqrt{-d})$-*Weil type* if $\mathbb{Q}(\sqrt{-d}) \subseteq \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and if the action of $\sqrt{-d}$ on the tangent space at the origin of A has eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$ both with multiplicity n.

Given an abelian of $\mathbb{Q}(\sqrt{-d})$-Weil type A, then one can associate naturally an element $\delta \in \mathbb{Q}/\mathrm{N}(\mathbb{Q}(\sqrt{-d}))$, where $\mathrm{N}(\mathbb{Q}(\sqrt{-d}))$ is the set of norms of $\mathbb{Q}(\sqrt{-d})$. The element δ is called the *discriminant* of A.
Abelian varieties of Weil type appear often as simple factors of Kuga–Satake varieties; the next result due to Lombardo [7] gives an example of this fact. We recall here the version presented in [3, Thm. 9.2]. In the following, U denotes the hyperbolic plane.

Theorem 2.13. *Let d be a positive real number and let A be an abelian fourfold of $\mathbb{Q}(\sqrt{-d})$-Weil type of discriminant $\delta = 1$. Then A^4 is the Kuga–Satake variety of a polarized Hodge structure of hyper-Kähler type of dimension six (V, q), such that*

$$V \simeq U^{\oplus 2} \oplus \langle -1 \rangle \oplus \langle -d \rangle$$

as quadratic spaces. Conversely, if (V, q) is a Hodge structure of hyper-Kähler type of dimension six as above, its Kuga–Satake variety is isogenous to A^4 for some abelian fourfold of $\mathbb{Q}(\sqrt{-d})$-Weil type.

2.3. **Kuga–Satake Hodge conjecture.** In this section, we analyze the morphism of Hodge structures

$$V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V)$$

of Theorem 2.6, in the case where $V = H^2(X,\mathbb{Q})_{\mathrm{tr}}$, the transcendental lattice of a projective hyper-Kähler variety X.

Using the isomorphism $\mathrm{Cl}^+(H^2(X,\mathbb{Q})_{\mathrm{tr}}) \simeq H^1_{\mathrm{KS}}(H^2(X,\mathbb{Q})_{\mathrm{tr}})$, we can apply the Künneth decomposition and obtain an embedding

$$H^1_{\mathrm{KS}}(H^2(X,\mathbb{Q})_{\mathrm{tr}}) \otimes H^1_{\mathrm{KS}}(H^2(X,\mathbb{Q})_{\mathrm{tr}}) \hookrightarrow H^2(\mathrm{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})^2,\mathbb{Q}).$$

On the other hand, since we the variety X is projective there is a natural projection map $H^2(X,\mathbb{Q}) \twoheadrightarrow H^2(X,\mathbb{Q})_{\mathrm{tr}}$. Composing these morphisms, we then obtain a morphism of Hodge structures

$$H^2(X,\mathbb{Q}) \longrightarrow H^2(\mathrm{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})^2,\mathbb{Q}),$$

which is called the *Kuga–Satake correspondence*. This morphism corresponds via Poincaré duality to a Hodge class

$$\kappa \in H^{2n,2n}(X \times \mathrm{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}}) \times \mathrm{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})),$$

where $2n = \dim X$. The Hodge conjecture applied to this special case gives us the following:

Conjecture 2.14 (Kuga–Satake Hodge conjecture)**.** *Let X be a projective hyper-Kähler variety or a complex projective surface with $h^{2,0} = 1$. Then the class κ is algebraic.*

Remark 2.15. In the case where X is an abelian surface or a Kummer surface, the Kuga–Satake Hodge conjecture can be deduced from Theorem 2.11, using the fact that the Hodge conjecture is known for self-products of any given abelian surface [9].

The Kuga–Satake Hodge conjecture is not known in most cases, already in the case of K3 surfaces. One of the very few examples for which it has been proved is the family of K3 surfaces studied by Paranjape in [12]: Let $L_1,\ldots,L_6$ be six lines in $\mathbb{P}^2$ no three of which intersect in one point, and let $\pi\colon Y \longrightarrow \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along the six lines. Then, Y is a singular surface with simple nodes in the preimages of the intersection points of the lines L_i. Resolving the singularities of π by blowing up the nodes one obtains a K3 surface X. For a general choice of the six lines, the Picard number of X is equal to 16, where a basis of the Néron–Severi group is given by the 15 exceptional divisors over the singular points of Y, together with the pullback of the ample line of $\mathbb{P}^2$ via the map $X \longrightarrow \mathbb{P}^2$. In particular, the transcendental lattice of X is six-dimensional, and hence satisfies the hypotheses of Theorem 2.13. Its Kuga–Satake variety is therefore isogenous to the fourth power of some abelian fourfold. In [12], the author shows that this abelian fourfold is the Prym variety of some 4 : 1 cover $C \longrightarrow E$ where C is a genus 5 curve and E is an elliptic curve, and finds a cycle in the product of X and the Prym variety which realizes the Kuga–Satake correspondence.

The fact that the Kuga–Satake correspondence is algebraic for the family described above has been used by Schlickewei to prove the Hodge conjecture for the square of those K3 surfaces:

Theorem 2.16. [13, Thm. 2] *Let X be a K3 surface which is the desingularization of a double cover of $\mathbb{P}^2$ branched along six lines no three of which intersect in one point. Then, the Hodge conjecture is true for X^2.*

As a part of its PhD thesis, the first author of these notes proves an extension of the statement of Theorem 2.16 and shows that the Hodge conjecture holds for all powers of such K3 surfaces.

In the next section, we review another type of polarized hyper-Kähler manifolds for which the Kuga–Satake Hodge conjecture can be proved: The family of hyper-Kähler manifolds of generalized Kummer type.

3. The case of hyper-Kähler manifolds of generalized Kummer type

3.1. **Cup-product: generalization of a result of O'Grady.** Let X be a hyper-Kähler manifold of dimension $2n$ with $n \geqslant 2$. The Beauville-Bogomolov quadratic form q_X is a non-degenerate quadratic form on $H^2(X,\mathbb{Q})$, whose inverse gives an element of $\mathrm{Sym}^2 H^2(X,\mathbb{Q})$. By Verbitsky [14], the later space imbeds by cup-product in $H^4(X,\mathbb{Q})$, hence we get a class

$$c_X \in H^4(X,\mathbb{Q}). \tag{3.1}$$

The O'Grady map $\phi\colon \bigwedge^2 H^3(X,\mathbb{Q}) \longrightarrow H^{4n-2}(X,\mathbb{Q})$ is defined by

$$\phi(\alpha\wedge\beta) = c_X^{n-2} \cup \alpha \cup \beta. \tag{3.2}$$

The following result was first proved by O'Grady [11] in the case of a hyper-Kähler manifold of generalized Kummer deformation type.

Theorem 3.1. *([11], [16]) Let X be a hyper-Kähler manifold of dimension $2n$. Assume that $H^3(X,\mathbb{Q}) \neq 0$. Then the O'Grady map map $\phi\colon \bigwedge^2 H^3(X,\mathbb{Q}) \longrightarrow H^{4n-2}(X,\mathbb{Q})$ is surjective.*

Proof. We can choose the complex structure on X to be general, so that $\rho(X) = 0$, and this implies that the Hodge structure on $H^2(X,\mathbb{Q})$ (or equivalently $H^{4n-2}(X,\mathbb{Q})$ as they are isomorphic by combining Poincaré duality and the Beauville-Bogomolov form) is simple. As the morphism ϕ is a morphism of Hodge structures, its image is a Hodge substructure of $H^{4n-2}(X,\mathbb{Q})$, hence either ϕ is surjective, or it is 0. Theorem 3.1 thus follows from the next proposition. □

Proposition 3.2. *The map ϕ is not identically* 0.

Sketch of proof. Let $\omega \in H^2(X,\mathbb{R})$ be a Kähler class. Then we know that the ω-Lefschetz intersection pairing $\langle\, ,\, \rangle_\omega$ on $H^3(X,\mathbb{R})$, defined by

$$\langle \alpha,\beta\rangle_\omega := \int_X \omega^{2n-3} \cup \alpha \cup \beta$$

is nondegenerate. This implies that the map

$$\psi\colon \bigwedge^2 H^3(X,\mathbb{Q}) \longrightarrow H^6(X,\mathbb{Q})$$

has the property that $\operatorname{Im}\psi$ pairs nontrivially with the image of $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})$ in $H^{4n-6}(X,\mathbb{Q})$. Note that the Hodge structure on $H^3(X,\mathbb{Q})$ has Hodge level one, so that the Hodge structure on the image of $\operatorname{Im}\psi$ in $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})^*$ is a Hodge structure of level at most two. One checks by a Mumford–Tate group argument (see [16] for more details) that for a very general complex structure on X, the only level-two Hodge substructure of $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})$ is $c_X^{n-2}H^2(X,\mathbb{Q})$, where we see here c_X as an element of $\operatorname{Sym}^2H^2(X,\mathbb{Q})$. It follows that the image of $\operatorname{Im}\psi$ in $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})^*$ pairs nontrivially with $c_X^{n-2}H^2(X,\mathbb{Q})$, which concludes the proof. □

3.2. **Intermediate Jacobian and the Kuga–Satake variety.**

3.2.1. *Universal property of the Kuga–Satake construction.* The following result is proved in [1]. Using the Mumford–Tate group, this is a statement in representation theory of the orthogonal group.

Theorem 3.3. *Let (H^2,q) be a polarized Hodge structure of hyper-Kähler type. Assume that the Mumford–Tate group of the Hodge structure on H^2 is maximal (that is, equal to the orthogonal group of q). Let H be a simple effective weight-one Hodge structure, such that there exists an injective morphism of Hodge structures of bidegree $(-1,-1)$*

$$H^2 \hookrightarrow \operatorname{End}(H).$$

Then H is a direct summand of the Kuga–Satake Hodge structure $H^1_{\mathrm{KS}}(H^2,q)$.

Charles' theorem is in fact stronger, as it proves a similar statement for all tensor powers $H^{\otimes k}\otimes (H^*)^{\otimes k+2r}$. It also addresses the nonpolarized case that appears when dealing with nonprojective hyper-Kähler manifolds. In [4], another version of the universality property is proved. Namely

Theorem 3.4. *Let (H^2,q) be a polarized Hodge structure of hyper-Kähler type. Assume as above that the Mumford–Tate group of the Hodge structure on H^2 is maximal. Let H be a simple effective weight-one Hodge structure, such that there exists an injective morphism of Hodge structures of bidegree $(-1,-1)$*

$$H^2 \hookrightarrow \operatorname{Hom}(H,A),$$

for some weight-one Hodge structure A. Then H is a direct summand of the Kuga–Satake Hodge structure $H^1_{\mathrm{KS}}(H^2,q)$.

Coming back to Theorem 3.3, under the same assumption on the Mumford–Tate group, one knows that the Kuga–Satake weight-one Hodge structure is a power of a simple weight-one

Hodge structure of dimension $\geqslant 2^{\lfloor \frac{b_2-1}{2} \rfloor}$, where $b_2 = \dim H^2$, hence one gets as a consequence an inequality (see [1] for a more precise estimate)

$$\dim H \geqslant 2^{\lfloor \frac{b_2-1}{2} \rfloor}.$$

Proof of Theorem 1.1. Let X be a very general projective hyper-Kähler manifold of generalized Kummer type of dimension $\geqslant 4$. We apply Theorem 3.3 to the O'Grady map (3.2) that we know to be a surjective morphism of Hodge structures by Theorem 3.1, or rather to its dual. We then conclude that $H^3(X, \mathbb{Q})$ contains a direct summand of $H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})$. As $H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})$ is a power of a simple weight-one Hodge structure $H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})_c$ of dimension 8, and $b_3(X) = 8$, we conclude that $H^3(X, \mathbb{Q}) \simeq H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})_c$ as rational Hodge structures. ☐

3.3. **Algebraicity of the Kuga–Satake correspondence for hyper-Kähler manifolds of generalized Kummer type.**

3.3.1. *Markman's result.* For a projective manifold X with $h^{3,0}(X) \neq 0$, it is expected from the Hodge conjecture that there exists a cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)_{\mathbb{Q}}$ such that $[\mathcal{Z}]_* : H_1(J^3(X), \mathbb{Q}) \longrightarrow H^3(X, \mathbb{Q})$ is the natural isomorphism. Indeed, the map $[\mathcal{Z}]_*$ is an isomorphism of Hodge structures, hence provides a degree-4 Hodge class on $J^3(X) \times X$. Equivalently, after replacing $\mathcal{Z}$ by a multiple that makes it integral, the Abel-Jacobi map

$$\Phi_{\mathcal{Z}} \colon J^3(X) \longrightarrow J^3(X),\ \Phi_{\mathcal{Z}} := \Phi_X \circ \mathcal{Z}_*,$$

is a multiple of the identity and in particular Φ_X is surjective.

Theorem 3.5. *(Markman [8]) Let X be a projective hyper-Kähler manifold of generalized Kummer type. Then there exists a codimension-two cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)_{\mathbb{Q}}$ satisfying the property above.*

The proof of this theorem uses a deformation argument starting from a generalized Kummer manifold, using the fact that $J^3(X)$ can be realized as a moduli space of sheaves on X in that case.

We now use Markman's result to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mathcal{Z}$ be the Markman codimension-two cycle of Theorem 3.5. We choose a cycle $C_X \in \mathrm{CH}^2(X)_{\mathbb{Q}}$ of class $[C_X] = c_X$ (it exists by results of Markman [8]). We now consider the cycle

$$\Gamma = \mathcal{Z}^2 \cdot \mathrm{pr}_X^* C_X^{n-2} \in \mathrm{CH}^{2n}(J^3(X) \times X)\mathbb{Q}.$$

One checks using the Künneth decomposition (see [16] for more details) that

$$[\Gamma]_* : H_2(J^3(X), \mathbb{Q}) \longrightarrow H_2(X, \mathbb{Q})$$

is the O'Grady map. By Theorem 1.1, this is also the surjective morphism of Hodge structures (1.1). ☐

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DERIVED CATEGORIES OF HYPER-KÄHLER MANIFOLDS VIA THE LLV ALGEBRA

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Abstract. We mostly review work of Taelman [Tae81] on derived categories of hyper-Kähler manifolds. We study the LLV algebra using polyvector fields to prove that it is a derived invariant. Applications to the action of derived equivalences on cohomology and to the study of their Hodge structures are given.

1. Introduction

In this note we discuss the (bounded) derived category $\mathrm{D}^b(X) := \mathrm{D}^b(\mathrm{Coh}(X))$ and its group of auto-equivalences $\mathrm{Aut}(\mathrm{D}^b(X))$ for projective hyper-Kähler manifolds X. The situation in dimension two, that is for K3 surfaces, is fairly well understood and we refer to [Huy06, Sec. 10] for an overview. Therefore, we will only concentrate on the higher-dimensional case. More precisely, we mainly present the first part of Taelman's paper [Tae81].

These notes are, for the most part, light on derived categories and focus more on a different perspective of the Looijenga–Lunts–Verbitsky (LLV) Lie algebra $\mathfrak{g}(X)$ [Ver96, LL97] which will allow us to show the following.

Theorem 1.1 (Taelman)**.** *A derived equivalence $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ between projective hyper-Kähler manifolds induces naturally a Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(Y).$$

The induced isomorphism of quadratic spaces

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$.

The theorem will be proven in Section 5.

We start these notes by introducing the main objects of study and a collection of known results prior to [Tae81]. Afterwards, we introduce a new Lie subalgebra of the (ungraded) endomorphism algebra $\mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$ which is better suited for the study of derived categories. In the subsequent section we establish Theorem 1.1 via proving that the newly defined Lie subalgebra coincides with the well-known LLV Lie algebra $\mathfrak{g}(X)$. The next three sections will

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draw consequences from this result for the action of derived equivalences on cohomology and for Hodge structures of derived equivalent hyper-Kähler manifolds.

Notation. We work over the complex numbers. Throughout these notes X and Y will be projective hyper-Kähler manifolds of dimension $2n$. All functors will be implicitely derived.

2. Derived categories

2.1. **General theory.** For a thorough introduction to derived categories we recommend [Huy06]. Let us recall one of the most important results in the study of derived equivalences proved by Orlov [Orl97].

Theorem 2.1. *Let Z and T be smooth projective varieties and $\Phi\colon \mathrm{D}^b(Z) \xrightarrow{\sim} \mathrm{D}^b(T)$ be an exact derived equivalence. Then Φ is isomorphic to a Fourier–Mukai functor, i.e. there exists $\mathcal{E} \in \mathrm{D}^b(Z \times T)$ such that*

$$\Phi \simeq \mathrm{FM}_{\mathcal{E}} := p_{T*} \circ (\mathcal{E} \otimes \_\,) \circ p_Z^*.$$

Moreover, a derived equivalence as in the theorem naturally induces isomorphisms of several invariants associated with the varieties such as (topological) K-theory [Huy06, Sec. 5.2]. For us the most important invariant will be singular cohomology. Namely, every derived equivalence $\mathrm{FM}_{\mathcal{E}}$ induces a cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ given by the correspondence $v(\mathcal{E}) \in \mathrm{H}^*(Z\times T)$ where $v = \mathrm{ch}(\_\,)\sqrt{\mathrm{td}}$ is the Mukai vector. These are compatible via the Mukai vector, i.e. the following diagram commutes

$$\begin{tikzcd} \mathrm{D}^b(Z) \arrow[r, "\mathrm{FM}_{\mathcal{E}}"] \arrow[d, "v"] & \mathrm{D}^b(T) \arrow[d, "v"] \\ \mathrm{H}^*(Z,\mathbb{Q}) \arrow[r, "\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}"] & \mathrm{H}^*(T,\mathbb{Q}). \end{tikzcd} \tag{2.1}$$

Hence, the study of derived categories leads naturally to cycles on hyper-Kähler manifolds.

Remark 2.2. Let us mention properties of the cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$.

- Since $v(\mathcal{E}) \in \oplus_p \mathrm{H}^{p,p}(Z\times T)$ is algebraic, the isomorphism $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects the weight-zero Hodge structure on $\mathrm{H}^*(Z)$ (respectively $\mathrm{H}^*(T)$) given by

$$\mathrm{H}^{-i,i}(Z) = \bigoplus_{q-p=i} \mathrm{H}^{p,q}(Z)$$

  for $i \in \mathbb{Z}$ where the Hodge structure on the right-hand side is the usual one.
- The isomorphism $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects the generalized Mukai pairing, see [Căl03].
- The cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects neither the cup product structure on cohomology nor the cohomological grading.

2.2. **Case of hyper-Kähler manifolds.** We know that if a smooth projective variety Z is derived equivalent to a hyper-Kähler manifold X, then the dimensions of X and Z coincide and the canonical bundle ω_Z is trivial [Huy06, Sec. 4]. Huybrechts and Nieper-Wißkirchen [**?**] have proven that Z must in fact also be an irreducible hyper-Kähler manifold.

3. Recollection of the LLV Lie algebra

We quickly recall the definition of the LLV Lie algebra introduced independently by Looijenga–Lunts [LL97] and [Ver96]. For a more thorough discussion we refer to [**?**].

Let X be a hyper-Kähler manifold and $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ be a cohomology class. We attach to it the operator

$$e_\lambda := \lambda \cup \_ \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$$

given by cup product with the class λ. We say that λ has the Hard Lefschetz property, if for all i the maps

$$e_\lambda^i \colon \mathrm{H}^{2n-i}(X, \mathbb{Q}) \longrightarrow \mathrm{H}^{2n+i}(X, \mathbb{Q})$$

are isomorphisms. The class λ is often called a Hard Lefschetz class. We denote by $h \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ the grading operator acting on $\mathrm{H}^i(X, \mathbb{Q})$ via $(i - 2n)\mathrm{id}$. For a Hard Lefschetz class $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$, the triple

$$(e_\lambda, h, f_\lambda),$$

where f_λ is the dual Lefschetz operator, spans a Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ isomorphic to the Lie algebra $\mathfrak{sl}_2$.

Definition 3.1. The LLV Lie algebra $\mathfrak{g}(X)$ is the Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ generated by all $\mathfrak{sl}_2$-triples $(e_\lambda, h, f_\lambda)$ for $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ Hard Lefschetz.

As said in the beginning, we refer to [**?**] or [LL97] for more details and properties of $\mathfrak{g}(X)$. Our main goal is to relate the Lie algebra $\mathfrak{g}(X)$ with $\mathrm{D}^b(X)$. Note that since a cohomological Fourier–Mukai functor does not respect cup product nor grading, which are the defining properties of the LLV algebra, it is a priori not clear how this can be done. The main ingredient for it is the ring of polyvector fields, to be introduced now.

4. Polyvector fields

Definition 4.1. The ring of polyvector fields $\mathrm{HT}^*(X)$ is the graded $\mathbb{C}$-algebra whose degree k part is

$$\mathrm{HT}^k(X) := \oplus_{p+q=k} \mathrm{H}^q(X, \bigwedge^p \mathcal{T}_X).$$

The ring structure is induced from the exterior algebra.

For X a hyper-Kähler manifold we can choose a symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ which induces isomorphisms

$$\bigwedge^p \mathcal{T}_X \simeq \Omega_X^p$$

which, in turn, induce a graded $\mathbb{C}$-algebra isomorphism

$$\mathrm{HT}^*(X) = \mathrm{H}^*(X, \bigwedge^* \mathcal{T}_X) \simeq \mathrm{H}^*(X, \Omega_X^*) \simeq \mathrm{H}^*(X, \mathbb{C}). \tag{4.1}$$

Thus, as a graded $\mathbb{C}$-algebra, the ring of polyvectors is isomorphic to the de Rham cohomology.

In this note, we are mostly interested in another viewpoint of the polyvector fields. Namely, the ring of polyvectors acts on the de Rham cohomology by contraction. That is, given $v \in \mathrm{H}^q(X, \bigwedge^p \mathcal{T}_X)$ and $x \in \mathrm{H}^{q'}(X, \Omega_X^{p'})$ the action is defined as

$$v \lrcorner x \in \mathrm{H}^{q+q'}(X, \Omega_X^{p'-p}).$$

The following is immediate, see also [Tae81, Lem. 2.4].

Lemma 4.2. *For X a hyper-Kähler manifold the de Rham cohomology is a free module of rank one over the polyvector fields generated by a Calabi–Yau form $\sigma^n \in \mathrm{H}^0(X, \Omega_X^{2n})$.*

The reason why the ring of polyvectors is of interest to us is the following crucial result. It relies on the modified Hochschild–Konstant–Rosenberg isomorphism identifying Hochschild (co)homology with polyvectors and the de Rham cohomology [CRVdB12].

Theorem 4.3. *A derived equivalence $\Phi \colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ induces naturally a $\mathbb{C}$-algebra isomorphism $\Phi^{\mathrm{HT}} \colon \mathrm{HT}^*(X) \xrightarrow{\sim} \mathrm{HT}^*(Y)$ such that the action of the polyvector fields is equivariant for the induced isomorphism $\Phi^{\mathrm{H}} \colon \mathrm{H}^*(X, \mathbb{C}) \xrightarrow{\sim} \mathrm{H}^*(Y, \mathbb{C})$.*

Spelling this out, for $v \in \mathrm{HT}^*(X)$ and $x \in \mathrm{H}^*(X, \mathbb{C})$ we have

$$\Phi^{\mathrm{H}}(v \lrcorner x) = \Phi^{\mathrm{HT}}(v) \lrcorner \Phi^{\mathrm{H}}(x) \in \mathrm{H}^*(Y, \mathbb{C}).$$

5. Reinventing the LLV Lie algebra

We will define a new Lie algebra, which will turn out to be isomorphic to $\mathfrak{g}(X)$ with scalars extended to $\mathbb{C}$. This will prove Theorem 1.1 from the introduction.

We consider the holomorphic grading operator h_p and the antihomolorphic grading operator h_q defined by acting on $\mathrm{H}^{k,l}(X)$ via

$$h_p = (k-n)\mathrm{id}, \quad h_q = (l-n)\mathrm{id}.$$

With these definitions the usual grading operator h for the cohomological grading is just $h = h_p + h_q$. We define the Hodge grading operator $h' := h_q - h_p$.

$$
\begin{array}{cccccccc}
 & & & \xleftrightarrow{h'\text{-grading}} & & & & \\
 & & & \mathrm{H}^{0,0} & & & & \\
 & & \mathrm{H}^{1,0} & & \mathrm{H}^{0,1} & & & \\
 & \mathrm{H}^{2,0} & & \mathrm{H}^{1,1} & & \mathrm{H}^{0,2} & & \\
 & & & \vdots & & & & \\
\mathrm{H}^{2n,0} & \mathrm{H}^{2n-1,1} & \dots & \mathrm{H}^{n,n} & \dots & \mathrm{H}^{1,2n-1} & \mathrm{H}^{0,2n} & \Big\updownarrow h\text{-grading} \\
 & & & \vdots & & & & \\
 & \mathrm{H}^{2n,2n-2} & & \mathrm{H}^{2n-1,2n-1} & & \mathrm{H}^{2n,2n-2} & & \\
 & & \mathrm{H}^{2n,2n-1} & & \mathrm{H}^{2n-1,2n} & & & \\
 & & & \mathrm{H}^{2n,2n} & & & &
\end{array}
$$

With this definition the action of the polyvector fields $\mathrm{HT}^*(X)$ on the de Rham cohomology $\mathrm{H}^*(X, \mathbb{C})$ alluded to in Lemma 4.2 has degree two with respect to the grading h'.

For $\mu \in \mathrm{HT}^2(X)$ we define the operator

$$e_\mu := \mu \lrcorner \_ \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{C})).$$

We say that μ is Hard Lefschetz if the operator e_μ satisfies the Hard Lefschetz isomorphisms with respect to the grading operator h'. The Jacobson–Morozov theorem asserts that this is equivalent to the existence of an operator $f_\mu \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ such that

$$(e_\mu, h', f_\mu)$$

generates a Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ isomorphic to $\mathfrak{sl}_2$.

Definition 5.1. The complex Lie algebra $\mathfrak{g}'(X)$ is defined to be the smallest Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ containing all $\mathfrak{sl}_2$-triples (e_μ, h', f_μ) for all Hard Lefschetz $\mu \in \mathrm{HT}^2(X)$.

Equivalently, one could have defined the Lie algebra $\mathfrak{g}'(X)$ as the Lie subalgebra of $\mathrm{End}(\mathrm{HT}^*(X))$ containing all $\mathfrak{sl}_2$-triples with μ Hard Lefschetz. Through the isomorphism

$$\mathrm{HT}^*(X) \lrcorner \sigma^n \simeq \mathrm{H}^*(X, \mathbb{C})$$

these two definitions are identified.

Recall from (4.1) that the choice of a symplectic form produces an abstract graded $\mathbb{C}$-algebra isomorphism

$$\mathrm{HT}^*(X) \simeq \mathrm{H}^*(X, \Omega_X^*) \simeq \mathrm{H}^*(X, \mathbb{C}).$$

Thus, the choice of a symplectic form leads to the following result.

Lemma 5.2. *There is an isomorphism of complex Lie algebras*

$$\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathfrak{g}'(X).$$

We also deduce the following consequence from Theorem 4.3.

Proposition 5.3. *For a derived equivalence between hyper-Kähler manifolds* $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ *the isomorphism*

$$\Phi^{\mathrm{HT}}\colon \mathrm{HT}^2(X) \xrightarrow{\sim} \mathrm{HT}^2(Y)$$

induces naturally a Lie algebra isomorphism

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}'(X) \xrightarrow{\sim} \mathfrak{g}'(Y)$$

such that the induced isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X, \mathbb{C}) \xrightarrow{\sim} \mathrm{H}^*(Y, \mathbb{C})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$.

Spelling this again out means that for $j \in \mathfrak{g}'(X)$ and $x \in \mathrm{H}^*(X, \mathbb{C})$ we have

$$\Phi^{\mathrm{H}}(j.x) = \Phi^{\mathfrak{g}}(j).\Phi^{\mathrm{H}}(x) \in \mathrm{H}^*(Y, \mathbb{C}).$$

The connection between all that has been said so far and the main tool for all the applications we will present is the following main theorem of [Tae81] which was also implicitely proven (but not stated in the form below) by Verbitsky [Ver99].

Theorem 5.4. *The Lie algebras* $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ *and* $\mathfrak{g}'(X)$ *are equal as Lie subalgebras of the Lie algebra* $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$.

Proof. Verbitsky showed that there is an isomorphism of ungraded vector spaces

$$\eta\colon \mathrm{H}^*(X, \mathbb{C}) \xrightarrow{\sim} \mathrm{H}^*(X, \mathbb{C})$$

which conjugates the two Lie algebras, i.e.

$$\eta\,(\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C})\,\eta^{-1} = \mathfrak{g}'(X).$$

Since the isomorphism η is obtained by integrating the action of the Lie algebra $\mathfrak{g}(X)$, one can conclude the proof.

We will, however, follow Taelman's proof. From Lemma 5.2 we infer that it is enough to show only the inclusion

$$\mathfrak{g}'(X) \subset \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

A straightforward calculation shows that

$$(e_\sigma, h_p, e_{\check{\sigma}})$$

is an $\mathfrak{sl}_2$-triple, where $\check{\sigma} \in \mathrm{H}^0(\bigwedge^2(\mathcal{T}_X))$ is the dual symplectic form (note that the Lefschetz operator e_σ acts via cup product, whereas $e_{\check{\sigma}}$ acts by contraction of polyvector fields).

Analogously or using Hodge symmetry, for the complex conjugate form $\bar{\sigma} \in \mathrm{H}^2(X, \mathcal{O}_X)$ the operator $e_{\bar{\sigma}}$ has the Hard Lefschetz property for the grading operator h_q. The Jacobson–Morozov Theorem grants the existence of an operator $g \in \operatorname{End}(\mathrm{H}^*(X, \mathbb{C}))$ such that

$$(e_{\bar{\sigma}}, h_q, g)$$

forms an $\mathfrak{sl}_2$-triple. An easy check shows that all elements from the $\mathfrak{sl}_2$-triple $(e_\sigma, h_p, e_{\check{\sigma}})$ commute with all elements from the $\mathfrak{sl}_2$-triple $(e_{\bar{\sigma}}, h_q, g)$. Thus we obtain two new $\mathfrak{sl}_2$-triples

$$(e_\sigma + e_{\bar{\sigma}}, h, e_{\check{\sigma}} + g), \quad (e_\sigma - e_{\bar{\sigma}}, h, e_{\check{\sigma}} - g).$$

This gives that $e_{\check{\sigma}} \in \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. Since $[e_\sigma, e_{\check{\sigma}}] = h_p$ and $h_p + h_q = h$, we deduce furthermore that h_p, h_q and therefore $h' = h_q - h_p$ are all contained inside $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.

Since evidently $e_{\bar{\sigma}}$ is also contained in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ (the action via contraction of polyvector fields agrees with the cup product), it is left to show that for almost all $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ the operator e_μ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. This follows from the identity

$$[e_{\check{\sigma}}, e_\eta] = e_\mu$$

for $\eta \in \mathrm{H}^1(X, \Omega_X)$ satisfying

$$\mu = \check{\sigma} \lrcorner \eta \in \mathrm{H}^1(X, \mathcal{T}_X)$$

which follows from a straightforward calculation, see [Tae81, Lem. 2.13]. □

The theorem implies that the isomorphism $\Phi^{\mathfrak{g}}$ from Proposition 5.3 is already defined over $\mathbb{Q}$, since the same holds for the induced isomorphism on singular cohomology. We thus have proved Theorem 1.1 which we state her again for the reader's convenience.

Corollary 5.5. *A derived equivalence* $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ *between hyper-Kähler manifolds induces naturally a Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(Y)$$

such that the induced isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y, \mathbb{Q})$$

is equivariant with respect to $\Phi^{\mathfrak{g}}$.

6. Verbitsky component and extended Mukai lattice

We want to draw consequences of Theorem 5.4 for the study of derived equivalences of hyper-Kähler manifolds and their induced actions on cohomology.

Definition 6.1. The Verbitsky component $\mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{H}^*(X, \mathbb{Q})$ is the subalgebra generated by $\mathrm{H}^2(X, \mathbb{Q})$.

It is easy to see that the Verbitsky component is an irreducible representation of the LLV Lie algebra $\mathfrak{g}(X)$ and it is characterized as such as the irreducible representation whose Hodge strucutre attains the maximal possible width. It is equipped with the Mukai pairing b_{SH} defined via

$$b_{\mathrm{SH}}(\lambda_1 \dots \lambda_m, \mu_1 \dots \mu_{2n-m}) := (-1)^m \int_X \lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_{2n-m}$$

for classes $\lambda_i, \mu_j \in \mathrm{H}^2(X, \mathbb{Q})$ which agrees with the generalized Mukai pairing alluded to in Remark 2.2.

Corollary 6.2. *For a derived equivalence $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ between hyper-Kähler manifolds the induced isomorphism Φ^{H} restricts to a Hodge isometry*

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{SH}(Y, \mathbb{Q}).$$

Proof. Since the Verbitsky component is the unique irreducible representation whose Hodge strucutre attains the maximal possible width and by Theorem 1.1 the isomorphism Φ^{H} respects the LLV algebra, we conclude that Φ^{H} must restrict to an isomorphism of the Verbitsky component. The Mukai pairing on the Verbitsky component agrees with the generalized Mukai pairing, which is a derived invariant. □

We want to study the Verbitsky component and the LLV Lie algebra more closely to further refine the study of $\mathrm{Aut}(\mathrm{D}^b(X))$.

Definition 6.3. The rational quadratic vector space defined by

$$\tilde{\mathrm{H}}(X, \mathbb{Q}) := \mathbb{Q}\alpha \oplus \mathrm{H}^2(X, \mathbb{Q}) \oplus \mathbb{Q}\beta.$$

is called the extended Mukai lattice. Its quadratic form $\tilde{b}$ restricts to the Beauville–Bogomolov–Fujiki form b on $\mathrm{H}^2(X, \mathbb{Q})$ [Huy03, Sec. 23] and the two classes α and β are orthogonal to $\mathrm{H}^2(X, \mathbb{Q})$ and satisfy $\tilde{b}(\alpha, \beta) = -1$ as well as $\tilde{b}(\alpha, \alpha) = \tilde{b}(\beta, \beta) = 0$.

Furthermore, we define on $\tilde{\mathrm{H}}(X, \mathbb{Q})$ a grading by declaring α to be of degree -2, $\mathrm{H}^2(X, \mathbb{Q})$ sits in degree zero and β is of degree two. Finally, the extended Mukai lattice is equipped with a weight-two Hodge structure

$$\begin{aligned}
(\tilde{\mathrm{H}}(X, \mathbb{Q}) \otimes \mathbb{C})^{2,0} &:= \mathrm{H}^{2,0}(X)\\
(\tilde{\mathrm{H}}(X, \mathbb{Q}) \otimes \mathbb{C})^{0,2} &:= \mathrm{H}^{0,2}(X)\\
(\tilde{\mathrm{H}}(X, \mathbb{Q}) \otimes \mathbb{C})^{1,1} &:= \mathrm{H}^{1,1}(X) \oplus \mathbb{C}\alpha \oplus \mathbb{C}\beta.
\end{aligned}$$

There exists a graded morphism $\psi\colon \mathrm{SH}(X, \mathbb{Q})[-2n] \longrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ sitting in the following short exact sequence

$$0 \longrightarrow \mathrm{SH}(X, \mathbb{Q})[-2n] \xrightarrow{\psi} \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})) \xrightarrow{\Delta_n} \mathrm{Sym}^{n-2}(\tilde{\mathrm{H}}(X, \mathbb{Q})) \longrightarrow 0.$$

Here, the map Δ_n is the Laplacian operator defined on pure tensors via

$$v_1 \cdots v_n \longmapsto \sum_{i<j} \tilde{b}(v_i, v_j) v_1 \cdots \hat{v_i} \cdots \hat{v_j} \cdots v_n.$$

The map ψ is uniquely determined (up to scaling) by the condition that it is a morphism of $\mathfrak{g}(X)$-modules. The $\mathfrak{g}(X)$-structure of $\tilde{\mathrm{H}}(X, \mathbb{Q})$ is defined by $e_\omega(\alpha) = \omega$, $e_\omega(\mu) = b(\omega, \mu)\beta$ and $e_\omega(\beta) = 0$ for all classes $\omega, \mu \in \mathrm{H}^2(X, \mathbb{Q})$. The n-th symmetric power $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ then inherits the structure of a $\mathfrak{g}(X)$-module by letting $\mathfrak{g}(X)$ act by derivations. We fix once and for all a choice of ψ by setting $\psi(1) = \alpha^n/n!$.

Taelman [Tae81, Sec. 3] showed that the map ψ is an isometry with respect to the Mukai pairing on $\mathrm{SH}(X, \mathbb{Q})$ and the pairing

$$b_{[n]}(x_1 \cdots x_n, y_1 \cdots y_n) = (-1)^n c_X \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} \tilde{b}(x_i, y_{\sigma(i)})$$

on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q}))$, where c_X is the Fujiki constant characterized by the property

$$\int_X \omega^{2n} = c_X \frac{(2n)!}{2^n n!} b(\omega, \omega)^n$$

for all $\omega \in \mathrm{H}^2(X, \mathbb{Q})$. Note that our definition of $b_{[n]}$ differs from Taelman's definition by the Fujiki constant. Ours has the advantage that ψ is always an isometry.

Summing up, the inclusion ψ respects the

- $\mathfrak{g}(X)$-module structure,
- quadratic forms,
- Hodge structures,
- gradings.

7. Action of derived equivalences on the extended Mukai lattice

Recall that we have deduced the existence of a representation

$$\rho^{\mathrm{SH}} \colon \mathrm{Aut}(\mathrm{D}^b(X)) \longrightarrow \mathrm{O}(\mathrm{SH}(X, \mathbb{Q})) \tag{7.1}$$

and the isometries in the image of this representation normalize the action of the LLV algebra $\mathfrak{g}(X)$, i.e. for these $g \in \mathrm{O}(\mathrm{SH}(X, \mathbb{Q}))$ we have

$$g\mathfrak{g}(X)g^{-1} = \mathfrak{g}(X) \subset \mathrm{End}(\mathrm{SH}(X, \mathbb{Q})).$$

Let us study these automorphisms a bit further.

Definition 7.1. The group $\mathrm{Aut}(\mathrm{SH}(X, \mathbb{Q}), b_{\mathrm{SH}}, \mathfrak{g}(X))$ is the group of all isometries of the Verbitsky component that normalize the action of the LLV algebra.

The main representation-theoretic input for our discussion is the following result [Tae81, Sec. 4].

Proposition 7.2. *If n is odd or the second Betti number is odd, then*

$$\mathrm{Aut}(\mathrm{SH}(X,\mathbb{Q}), b_{\mathrm{SH}}, \mathfrak{g}(X)) \simeq \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})).$$

We make this isomorphism more explicit. Let X and Y be deformation-equivalent hyper-Kähler manifolds together with a derived equivalence $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$. Then there exists a unique Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \xrightarrow{\sim} \tilde{\mathrm{H}}(Y,\mathbb{Q})$$

inducing the following commutative diagram

$$\begin{tikzcd} \mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\epsilon(\Phi^{\tilde{\mathrm{H}}})\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"'] & \mathrm{SH}(Y,\mathbb{Q}) \arrow[d, "\psi"] \\ \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(Y,\mathbb{Q})). \end{tikzcd} \tag{7.2}$$

The scalar $\epsilon(\Phi^{\tilde{\mathrm{H}}}) \in \{\pm 1\}$ depends on defining orientations on the vector spaces $\tilde{\mathrm{H}}(X,\mathbb{Q})$ respectively $\tilde{\mathrm{H}}(Y,\mathbb{Q})$ and for $X = Y$ we simply have $\epsilon(\Phi^{\tilde{\mathrm{H}}}) = \det(\Phi^{\tilde{\mathrm{H}}})^{n+1}$. In particular, in the case $X = Y$, the representation (7.1) factors via the commutative diagram

$$\begin{tikzcd} & \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[d] \\ \mathrm{Aut}(\mathrm{D}^b(X)) \arrow[ur, "\rho^{\tilde{\mathrm{H}}}"] \arrow[r, "\rho^{\mathrm{SH}}"'] & \mathrm{O}(\mathrm{SH}(X,\mathbb{Q})). \end{tikzcd}$$

Remark 7.3. In all known examples, derived equivalent hyper-Kähler manifolds are deformation-equivalent, but this is not known in general. Without this assumption, the above proposition has to weakened as we shall demonstrate.

One can, using similitudes, still formulate a version of Proposition 7.2 in the general case. This will be needed in the next section for the application to Hodge structures.

Theorem 7.4. *Let X and Y be arbitrary hyper-Kähler manifolds and $\Phi\colon \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ be a derived equivalence. Then there exists a Hodge similitude $\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \to \tilde{\mathrm{H}}(Y,\mathbb{Q})$ and a scalar $\lambda \in \mathbb{Q}^*$ such that*

$$\begin{tikzcd} \mathrm{SH}(X,\mathbb{Q}) \arrow[r, "\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"'] & \mathrm{SH}(Y,\mathbb{Q}) \arrow[d, "\psi"] \\ \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \arrow[r, "\lambda\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(Y,\mathbb{Q})) \end{tikzcd} \tag{7.3}$$

commutes.

8. Hodge structures

In this section we want to give one application of the results presented so far regarding derived equivalent hyper-Kähler manifolds and their Hodge structures. We first want to recall a recent result of Soldatenkov [Sol21][1], whose statement and proof are similiar in flavour to what we will discuss afterwards for derived equivalences.

Theorem 8.1. *Let X and Y be arbitrary hyper-Kähler manifolds and $\varphi\colon \mathrm{H}^2(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2(Y,\mathbb{Q})$ be an isomorphism of $\mathbb{Q}$-Hodge structures, which is the restriction of a global algebra automorphism $\phi\colon \mathrm{H}^*(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y,\mathbb{Q})$. Then for all $i \in \mathbb{Z}$ the restrictions*

$$\phi\colon \mathrm{H}^i(X,\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^i(Y,\mathbb{Q})$$

are isomorphisms of $\mathbb{Q}$-Hodge structures.

Proof. We briefly sketch the argument. Since ϕ is a graded algebra automorphism, the adjoint action produces an isomorphism

$$\mathrm{ad}(\phi)\colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{g}(Y).$$

The fact that ϕ is graded implies that $\mathrm{ad}(\phi)(h) = h$. Moreover, the restriction of ϕ to $\mathrm{H}^2(X,\mathbb{Q})$ respects the Hodge structures. This implies that $\mathrm{ad}(\phi)(h') = h'$, where again $h' = h_q - h_p$. Indeed, the adjoint action of ϕ is determined by its restriction to the degree two component [Sol21, Prop. 2.11]. As the morphism ϕ respects the Hodge structure on the second cohomology, the claim follows.

Since $h + h' = 2h_q$ and $h - h' = 2h_p$ we deduce $\mathrm{ad}(\phi)(h_p) = h_p$ and $\mathrm{ad}(\phi)(h_q) = h_q$. This is equivalent to ϕ being a morphism of $\mathbb{Q}$-Hodge structures. □

The assertion that the isomorphism of Hodge structures is the restriction of a global algebra automorphism is frequently met. For example, Hodge isometries with positive determinant can be extended to algebra automorphisms of the even cohomology by integrating the LLV action. For more details and examples we refer to [Sol21].

With this in mind, we can now prove the following result of Taelman [Tae81, Sec. 5]. It also establishes a conjecture of Orlov in the case of hyper-Kähler manifolds [Orl05] stating that derived equivalent varieties have the same Hodge numbers.

Theorem 8.2. *Let X and Y be derived equivalent hyper-Kähler manifolds. Then for all $i \in \mathbb{Z}$ we have an isomorphism*

$$\mathrm{H}^i(X,\mathbb{Q}) \simeq \mathrm{H}^i(Y,\mathbb{Q})$$

of $\mathbb{Q}$-Hodge structures.

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[1] We thank Andrey Soldatenkov for a stimulating conversation about his results.

LAGRANGIAN FIBRATIONS

D. HUYBRECHTS & M. MAURI

Abstract. We review the theory of Lagrangian fibrations of hyperkähler manifolds as initiated by Matsushita [Mat99, Mat01, Mat05]. We also discuss more recent work of Shen–Yin [SY18] and Harder–Li–Shen–Yin [HLSY19]. Occasionally, we give alternative arguments and complement the discussion by additional observations.

Assume $f\colon X \longrightarrow B$ is a Lagrangian fibration of a compact hyperkähler manifold X of complex dimension $2n$, and $\pi\colon \mathcal{X} \longrightarrow \Delta$ is a type III degeneration of compact hyperkähler manifolds of complex dimension $2n$. Then the cohomology algebra of $\mathbb{P}^n$ appears naturally in (at least) four different disguises:

(i) As the cohomology algebra of $(0,p)$ resp. $(p,0)$-forms (both independent of f):

$$H^*(\mathbb{P}^n,\mathbb{C}) \simeq H^*(X,\mathcal{O}_X) \text{ and } H^*(\mathbb{P}^n,\mathbb{C}) \simeq H^0(X,\Omega_X^*).$$

(ii) As the cohomology of the base of the fibration:[1]

$$H^*(\mathbb{P}^n,\mathbb{C}) \simeq H^*(B,\mathbb{C}).$$

(iii) As the image of the restriction to the generic fibre X_t of f:

$$H^*(\mathbb{P}^n,\mathbb{C}) \simeq \operatorname{Im}\left(H^*(X,\mathbb{C}) \longrightarrow H^*(X_t,\mathbb{C})\right).$$

(iv) As the cohomology of the dual complex $\mathcal{D}(\mathcal{X}_0)$ of the central fibre $\mathcal{X}_0$ of π:

$$H^*(\mathbb{P}^n,\mathbb{C}) \simeq H^*(\mathcal{D}(\mathcal{X}_0),\mathbb{C}).$$

In this survey we discuss these four situations and explain how they are related. We start by reviewing basic results on Lagrangian fibrations in Section 1, discuss the topology of the base and the restriction to the fibre in Section 2, and then sketch the proof of P=W in Section 3.

Throughout, X denotes a compact hyperkähler manifold of complex dimension $2n$. A fibration of X is a surjective morphism $f\colon X \twoheadrightarrow B$ with connected fibres onto a normal variety B with $0 < \dim(B) < 2n$. A submanifold $T \subset X$ is Lagrangian if the restriction $\sigma|_T \in H^0(T,\Omega_T^2)$ of the holomorphic two-form $\sigma \in H^0(X,\Omega_X^2)$ is zero.

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[1]Here and in (iii) and (iv), one expects isomorphisms of $\mathbb{Q}$-algebras, but this seems not known.

1. Basics on Lagrangian fibrations

We first discuss Lagrangian submanifolds and in particular Lagrangian tori. Then we study the cohomology and the singularities of the base B. Next we show that the fibres, smooth ones as well as singular ones, of any fibration are Lagrangian and conclude that fibrations of hyperkähler manifolds over a smooth base are flat.

At the end, we mention further results and directions without proof: Matshushita's description of the higher direct image sheaves $R^i f_* \mathcal{O}_X$, Beauville's question whether Lagrangian tori are always Lagrangian fibres, smoothness of the base, etc.

1.1. **Lagrangian tori.**

We start with some general comments on Lagrangian manifolds and more specifically on Lagrangian tori.

Proposition 1.1 (Voisin)**.** *Any Lagrangian submanifold $T \subset X$ of a hyperkähler manifold X is projective. In particular, any Lagrangian torus is an abelian variety.*

Proof. We are following the proof as presented in [Cam06]. Since the restriction of any Kähler class on X to T is non-trivial, the restriction $H^2(X,\mathbb{R}) \twoheadrightarrow H^2(T,\mathbb{R})$ is a non-trivial morphism of Hodge structures. On the other hand, as T is Lagrangian, all classes in $H^{2,0}(X) \oplus H^{0,2}(X)$ have trivial restrictions. Hence, the image of $H^2(X,\mathbb{R}) \twoheadrightarrow H^2(T,\mathbb{R})$ is contained in $H^{1,1}(T,\mathbb{R})$. More precisely, the images of $H^2(X,\mathbb{R}) \twoheadrightarrow H^2(T,\mathbb{R})$ and of $H^{1,1}(X,\mathbb{R}) \twoheadrightarrow H^{1,1}(T,\mathbb{R})$ coincide. Therefore, for any Kähler class $\omega \in H^{1,1}(X,\mathbb{R})$ there exists a rational class $\alpha \in H^2(X,\mathbb{Q})$ such that the $(1,1)$-class $\alpha|_T$ comes arbitrarily close to the Kähler class $\omega|_T$. Thus, $\alpha|_T$ is a rational Kähler class and, hence, T is projective. □

Remark 1.2. The normal bundle of a Lagrangian submanifold $T \subset X$ is isomorphic to the cotangent bundle of T, so $\mathcal{N}_{T/X} \simeq \Omega_T$. Hence, the $(1,1)$-part of the restriction map $H^2(X,\mathbb{C}) \twoheadrightarrow H^2(T,\mathbb{C})$ can be identified with the natural map $H^1(X,\mathcal{T}_X) \twoheadrightarrow H^1(T,\mathcal{N}_{T/X})$ that sends a first order deformation of X to the obstruction to deform T sideways with it, see [Voi92]:

$$\begin{array}{ccc} H^{1,1}(X) & \longrightarrow & H^{1,1}(T) \\ \downarrow \simeq & & \downarrow \simeq \\ H^1(X,\mathcal{T}_X) & \longrightarrow & H^1(T,\mathcal{N}_{T/X}). \end{array}$$

Clearly, as T is Lagrangian, the map $(H^{2,0} \oplus H^{0,2})(X) \twoheadrightarrow H^2(T,\mathbb{C})$ is trivial, cf. the proof above. Since the restriction of a Kähler class is again Kähler, $H^{1,1}(X) \twoheadrightarrow H^{1,1}(T)$ is certainly not trivial. Thus, $T \subset X$ deforms with X along a subset of codimension at least one. For smooth fibres of a Lagrangian fibration, so eventually Section 1.5.2 for all Lagrangian tori, the rank of the restriction map and hence the codimension of the image $\mathrm{Def}(T \subset X) \longrightarrow \mathrm{Def}(X)$ is exactly one.[2]

Proposition 1.3. *Assume $T \subset X$ is a Lagrangian torus. Then the restrictions $\mathrm{c}_i(X)|_T \in H^{2i}(T, \mathbb{R})$ of the Chern classes $\mathrm{c}_i(X) \in H^{2i}(X, \mathbb{R})$ are trivial.*

Proof. The normal bundle sequence allows one to compute the restriction of the total Chern class of X to the ones of T. More precisely, $\mathrm{c}(\mathcal{T}_X)|_T = \mathrm{c}(\mathcal{T}_T) \cdot \mathrm{c}(\mathcal{N}_{T/X})$. To conclude, use $\mathcal{N}_{T/X} \simeq \Omega_T$ and the fact that the tangent bundle of a torus is trivial. $\square$

Remark 1.4. (i) In the case that $T \subset X$ is the fibre of a Lagrangian fibration $f\colon X \longrightarrow B$, as it always is, cf. Section 1.5.2, also the restriction of the Beauville–Bogomolov form, thought of as a class $\tilde{q} \in H^4(X, \mathbb{Q})$, is trivial:

$$\tilde{q}|_T = 0.$$

There does not seem to be a direct proof of this fact. However, using that the rank of the restriction map $H^4(X, \mathbb{Q}) \longrightarrow H^4(T, \mathbb{Q})$ is one, see Theorem 2.1, it can be shown as follows. The classes $\tilde{q}$ and c_2 in $H^4(X, \mathbb{Q})$ both have the distinguished property that the homogenous forms $\int_X \tilde{q} \cdot \beta^{2n-2}$ and $\int \mathrm{c}_2(X) \cdot \beta^{2n-2}$ defined on $H^2(X, \mathbb{Z})$ are non-trivial scalar multiples of $q(\beta)^{n-1}$ and, therefore, of each other.[3] If $[T] \in H^{2n}(X, \mathbb{Z})$ is the class of a fibre $f^{-1}(t)$, then up to scaling $[T] = f^*\alpha^n$ for some $\alpha \in H^2(B, \mathbb{Q})$. Hence, for a Kähler class ω on X we find (up to a non-trivial scalar factor)

$$\int_T \tilde{q}|_T \cdot \omega|_T^{n-2} = \int_X \tilde{q} \cdot f^*\alpha^n \cdot \omega^{n-2} = \int_X \mathrm{c}_2(X) \cdot f^*\alpha^n \cdot \omega^{n-2} = \int_T \mathrm{c}_2(X)|_T \cdot \omega|_T^{n-2} = 0.$$

Since $\omega|_T \neq 0$ and $\mathrm{Im}\,(H^*(X, \mathbb{R}) \longrightarrow H^*(T, \mathbb{R}))$ is generated by $\omega|_T$, this proves the claim.

(ii) For other types of Lagrangian submanifolds, the restrictions of the Chern classes of X are not trivial. For example, for a Lagrangian plane $\mathbb{P}^2 \subset X$ one easily computes $\int_{\mathbb{P}^2} \mathrm{c}_2(X)|_{\mathbb{P}^2} = 15$.

As remarked before, the normal bundle of a Lagrangian torus is trivial. The next observation can be seen as a converse, it applies in particular to the smooth fibres of any fibration $f\colon X \longrightarrow B$.

Lemma 1.5. *Assume $T \subset X$ is Lagrangian submanifold with trivial normal bundle. Then T is a complex torus and, therefore, an abelian variety.*

Proof. Since T is Lagrangian, the tangent bundle $\mathcal{T}_T \simeq \mathcal{N}^*_{T/X}$ is trivial. Using the Albanese morphism, one easily proves that any compact Kähler manifold with trivial tangent bundle is a complex torus. $\square$

[2]Is there an a priori reason why this is the case for Lagrangian tori? It fails for general Lagrangian submanifolds; see §4.

[3]The non-triviality of the scalar for $\mathrm{c}_2(X)$ follows from the fact that $\int_X \mathrm{c}_2(X) \cdot \omega^{2n-2} \neq 0$ for any Kähler class.

1.2. **The base of a fibration.** We pass on to (Lagrangian) fibrations.

Proposition 1.6 (Matsushita)**.** *Assume $f\colon X \twoheadrightarrow B$ is a fibration with B smooth. Then B is a simply connected, smooth projective variety of dimension n satisfying $H^{p,0}(B) = H^{0,p}(B) = 0$ for all $p > 0$ and $H^2(B,\mathbb{Q}) \simeq \mathbb{Q}$. In particular,*

$$\mathrm{Pic}(B) \simeq H^2(B,\mathbb{Z}) \simeq \mathbb{Z}.$$

Proof. The smoothness of B implies that the pull-back $f^*\colon H^*(B,\mathbb{Q}) \longrightarrow H^*(X,\mathbb{Q})$ is injective. Next, as $\alpha^{2n} = 0$ for any class $\alpha \in H^2(B,\mathbb{R})$, we have $(f^*\alpha)^{2n} = 0$ and, therefore, $q(f^*\alpha) = 0$. By [Bog96, Ver96], this implies $(f^*\alpha)^{n+1} = 0$ and hence $\alpha^{n+1} = 0$, which yields $\dim(B) \leqslant n$. On the other hand, again by [Bog96, Ver96], $(f^*\alpha)^n \neq 0$ for every class $0 \neq \alpha \in H^2(B,\mathbb{R})$ from which we deduce $n \leqslant \dim(B)$.

If $\alpha \in H^{p,0}(B)$, then $f^*\alpha$ is a non-trivial multiple of some power of σ. Hence, $\alpha = 0$ if p is odd. If $p = 2$, then $f^*\alpha = \lambda \cdot \sigma$ and, hence, $f^*\alpha^n = \lambda^n \cdot \sigma^n$. Since $\sigma^n \neq 0$ and $H^{2n,0}(B) = 0$, one finds $\lambda = 0$. A similar argument can be made to work for all even p and an alternative argument is provided by Theorem 2.1.

Next we show $H^2(B,\mathbb{Q}) \simeq \mathbb{Q}$. Using [Bog96, Ver96], we have

$$S^n f^* H^2(B,\mathbb{Q}) \subset S^n H^2(X,\mathbb{Q}) \subset H^{2n}(X,\mathbb{Q}).$$

On the other hand, the image of $S^n f^* H^2(B,\mathbb{Q})$ is contained in $f^* H^{2n}(B,\mathbb{Q})$ which is just one-dimensional.[4]

Since X is Kähler, also B is, see [Var84]. Using $H^{2,0}(B) = H^{0,2}(B) = 0$, we can conclude that there exists a rational Kähler class on B. Hence, B is projective. According to [Kol95, Prop. 2.10.2], the natural map $\pi_1(X) \longrightarrow \pi_1(B)$ is surjective and, therefore, B is simply connected, as X is.[5] Then, by the universal coefficient theorem, $H^2(B,\mathbb{Z})$ is torsion free, i.e. $H^2(B,\mathbb{Z}) \simeq \mathbb{Z}$. Since $H^{1,0}(B) = H^{2,0}(B) = 0$, the exponential sequence yields $\mathrm{Pic}(B) \xrightarrow{\sim} H^2(B,\mathbb{Z})$. □

Remark 1.7. In fact, as we shall see, $H^{p,q}(B) = 0$ for all $p \neq q$ and $H^{p,p}(B) \simeq H^{p,p}(\mathbb{P}^n)$, i.e. there is an isomorphism of rational Hodge structures

$$H^*(B,\mathbb{Q}) \simeq H^*(\mathbb{P}^n,\mathbb{Q}).$$

There are two proofs of this fact, both eventually relying on the isomorphism $H^*(X,\mathcal{O}_X) \simeq H^*(\mathbb{P}^n,\mathbb{C})$. It seems that unlike $H^2(B,\mathbb{Q}) \simeq \mathbb{Q}$, which above was proved by exploiting the

[4]The traditional proof goes as follows: First one shows that for any non-trivial class $\alpha \in H^2(B,\mathbb{R}) = H^{1,1}(B,\mathbb{R})$ and any Kähler class ω on X one has $\int_X (f^*\alpha) \wedge \omega^{2n-1} \neq 0$. Indeed, otherwise the Hodge index theorem would imply $q(f^*\alpha) < 0$ and, therefore, $(f^*\alpha)^{n+1} \neq 0$, which contradicts $\dim(B) = n$. As a consequence, observe that for any two non-trivial classes $\alpha_1, \alpha_2 \in H^2(B,\mathbb{R})$ there exists a linear combination $\alpha := \lambda_1\alpha_1 + \lambda_2\alpha_2$ with $\int_X (f^*\alpha) \wedge \omega^{2n-1} = 0$, which then implies $\alpha = 0$, i.e. any two classes $\alpha_1, \alpha_2 \in H^2(B,\mathbb{R})$ are linearly dependent.

[5]By Lemma 1.8 below, B is a Fano manifold, which yields an alternative argument for B simply connected.

structure of the subring of $SH^2(X,\mathbb{Q}) \subset H^*(X,\mathbb{Q})$, the higher cohomology groups of B use deeper information about the hyperkähler structure.

(i) The first proof for B smooth and X projective was given by Matsushita [Mat05], as a consequence of the isomorphisms $R^if_*\mathcal{O}_X \simeq \Omega_B^i$, see Section 1.5.1. Combining this isomorphism with the splitting $Rf_*\mathcal{O}_X \simeq \bigoplus R^if_*\mathcal{O}_X[-i]$, see [Kol86b], one finds

$$H^k(X,\mathcal{O}_X) \simeq H^k(B,Rf_*\mathcal{O}_X) \simeq \bigoplus H^{k-i}(B,R^if_*\mathcal{O}_X) \simeq \bigoplus H^{k-i}(B,\Omega_B^i),$$

which proves the claim.[6]

(ii) Another one, which also works for singular B and non-projective X, was given in [SY18] and, roughly, relies on the fact that $H^*(B,\mathbb{C})$ can be deformed into $H^*(X,\mathcal{O}_X)$, see Section 2.2.

Lemma 1.8 (Markushevich, Matsushita)**.** *Under the above assumptions, B is a Fano manifold, i.e. ω_B^* is ample.*

Proof. Since B is dominated by X, we have $\mathrm{kod}(B) \leqslant 0$ by the known case of the Iitaka conjecture; see [Kaw85, Cor. 1.2]. Hence, $\omega_B \simeq \mathcal{O}_B$ or ω_B^* is ample. However, the first case is excluded by $H^{n,0}(B) = 0$.

In [Huy03, Prop. 24.8] the assertion is deduced from the fact that X admits a Kähler–Einstein metric. The case $\omega_B \simeq \mathcal{O}_B$ is excluded, because it would imply $H^{n,0}(B) \neq 0$, which was excluded above. $\square$

Remark 1.9. It turns out that as soon as the base B is smooth, then $B \simeq \mathbb{P}^n$. This result is due to Hwang [Hwa08] and its proof relies on the theory of minimal rational tangents. The results by Matsushita and more recently by Shen and Yin, see Remark 1.7 and Section 2, can be seen as strong evidence for the result. In dimension two, the result is immediate: Any smooth projective surface B with ω_B^* ample and $H^2(B,\mathbb{Q}) \simeq \mathbb{Q}$ is isomorphic to $\mathbb{P}^2$.

It is tempting to try to find a more direct argument in higher dimension, but all attempts have failed so far. For example, according to Hirzebruch–Kodaira [HK57] it suffices to show that $H^*(B,\mathbb{Z}) \simeq H^*(\mathbb{P}^n,\mathbb{Z})$ such that the line bundle L corresponding to the generator of $H^2(B,\mathbb{Z})$ satisfies $h^0(B,L^k) = h^0(\mathbb{P}^n,\mathcal{O}(k))$, see [Li16] for a survey of further results in this direction.

Alternatively, by Kobayashi–Ochai [KO73], it is enough to show that ω_B is divisible by $n+1$, i.e. the Fano manifold B has index $n+1$. As a first step, one could try to show that $f^*\omega_B$ is divisible by $n+1$.

1.3. **Singularities of the base.** It is generally expected that the base manifold B is smooth, but at the moment this is only known for $n \leqslant 2$, cf. [Ou19, BK18, HX19]. The expectation is corroborated by the following computations of invariants of the singularities of B.

[6]By evoking results due to Saito, it should be possible to avoid the projectivity assumption in [Kol86b].

Denote by $IH^*(B,\mathbb{Q})$ the intersection cohomology of the complex variety B with middle perversity and rational coefficients. It is the hypercohomology of the intersection cohomology complex $\mathcal{IC}_B$, i.e. $IH^*(B,\mathbb{Q}) = \mathbb{H}^*(B,\mathcal{IC}_B)$. In particular, if B is smooth or has quotient singularities, cf. [GS93, Prop. 3], then $IH^*(B,\mathbb{Q}) = H^*(B,\mathbb{Q})$.

Proposition 1.10. *Assume $f\colon X \longrightarrow B$ is a fibration over the complex variety B.*

(i) *B is $\mathbb{Q}$-factorial*[7]*, both in the Zariski and in the analytic topology.*

(ii) *The intersection cohomology complex $\mathcal{IC}_B$ of B is quasi-isomorphic to the constant sheaf $\mathbb{Q}_B$. In particular, $IH^*(B,\mathbb{Q}) = H^*(B,\mathbb{Q})$.*

(iii) (Matsushita) *B has log terminal singularities.*

Proof. For (i) and (ii) one only needs that $f\colon X \longrightarrow B$ is a connected and equidimensional morphism from a smooth variety X, while in the proof of (iii) one also needs ω_X trivial.

For any $t \in B$, choose a chart $\varphi\colon U_x \subset X \longrightarrow \mathbb{C}^{2n}$, centred at x, and the analytic subset $S := \varphi^{-1}(\Lambda)$, where $\Lambda \subseteq \mathbb{C}^{2n}$ is an n-dimensional affine subspace intersecting the fibre $\varphi(f^{-1}(t))$ transversely. Since f is equidimensional, the restriction $f|_S\colon S \longrightarrow B$ is finite over an analytic neighbourhood U of t. Therefore, U is $\mathbb{Q}$-factorial by the elementary [KM98, Lem. 5.16].

Denote $S^\circ := S \cap f^{-1}(U)$. By the decomposition theorem[8], $\mathcal{IC}_U$ is a direct summand of $R(f|_{S^\circ})_*\mathbb{Q}_{S^\circ}$. Taking stalks at t, we have

$$\mathcal{H}^0(\mathcal{IC}_B)_t \simeq \mathbb{Q}_{B,t} \qquad \mathcal{H}^i(\mathcal{IC}_U)_t \subseteq \mathcal{H}^i(R(f|_{S^\circ})_*\mathbb{Q}_{S^\circ})_t = 0,$$

because of the finiteness of $f|_{S^\circ}$. Thus, the natural map $\mathbb{Q}_B \longrightarrow \mathcal{IC}_B$ is a quasi-isomorphism in the constructible derived category $D^b_c(B)$ with rational coefficients.

By the canonical bundle formula, there exists a $\mathbb{Q}$-divisor $\Delta \subset B$ such that the pair (B,Δ) is log terminal; see [Kol07a, Thm. 8.3.7.(4)] and [Nak88, Thm. 2]. By the $\mathbb{Q}$-factoriality, B has log terminal singularities too. □

Remark 1.11 (Quotient singularities)**.** The finiteness of the restriction $f|_S\colon S \longrightarrow B$ over b suggests that B should have at worst quotient singularities. This would follow from the following conjecture[9].

Conjecture 1.12. [Kol07b, §2.24] *Let $f\colon X \longrightarrow Y$ be a finite and dominant morphism from a smooth variety X onto a normal variety Y. Then Y has quotient singularities.*

This is known for $n = 2$ by [Bri68, Lem. 2.6], but it is open in higher dimension. One of the main issue is that f itself need not be a quotient map, not even locally.

Corollary 1.13. *The pullback $f^*\colon H^*(B,\mathbb{Q}) \longrightarrow H^*(X,\mathbb{Q})$ is injective.*

[7] Are the singularities of B actually factorial?

[8] Alternatively, note that the trace map $R(f|_{S^\circ})_*\mathbb{Q}_{S^\circ} \longrightarrow \mathcal{IC}_U$ splits the natural morphism $\mathcal{IC}_U \longrightarrow R(f|_{S^\circ})_*\mathbb{Q}_{S^\circ}$.

[9] Thanks to Paolo Cascini to bring this conjecture to our attention.

Proof. By Proposition 1.10 this follows from the inclusion $IH^*(B,\mathbb{Q}) \hookrightarrow H^*(X,\mathbb{Q})$ coming from the decomposition theorem. $\square$

Remark 1.14. Let $f\colon M \longrightarrow N$ be a surjective holomorphic map between compact complex manifold, with M Kähler. By [Voi07, Lem. 7.28], the pullback $f^*\colon H^*(N,\mathbb{Q}) \longrightarrow H^*(M,\mathbb{Q})$ is injective. However, this may fail if N is singular, e.g. if f is a normalization of a nodal cubic, even if N has $\mathbb{Q}$-factorial log terminal singularities, e.g. [Mau21, Thm. 5.11].

Remark 1.15. Assume that B is projective. By Corollary 1.13, the smoothness of B can be dropped from the assumptions of Proposition 1.6 and Lemma 1.8.

1.4. **The fibres of a fibration.** Next we present Matsushita's result that any fibration of a compact hyperkähler manifold is a Lagrangian fibration.

Lemma 1.16 (Matsushita)**.** *Assume $f\colon X \longrightarrow B$ is a fibration. Then every smooth fibre $T := X_t \subset X$ is a Lagrangian torus and in fact an abelian variety.*

Proof. Comparing the coefficients of $x^{n-2}y^n$ in the polynomial (in x and y) the equation

$$q(\sigma + \bar{\sigma} + x \cdot \omega + y \cdot f^*\alpha)^n = c_X \cdot \int_X (\sigma + \bar{\sigma} + x \cdot \omega + y \cdot f^*\alpha)^{2n}$$

shows $\int_X (\sigma\bar{\sigma}) \wedge \omega^{n-2} \wedge f^*(\alpha^n) = 0$ for all $\omega \in H^2(X,\mathbb{R})$ and all $\alpha \in H^2(B,\mathbb{R})$. Since $[T] = f^*(\alpha^n)$ for some class α, this yields $\int_F (\sigma\bar{\sigma})|_T \wedge \omega^{n-2}|_T = 0$, which for a Kähler class ω and using that $\sigma \wedge \bar{\sigma}$ is semi-positive implies $\sigma|_T = 0$. Then conclude by Lemma 1.5. $\square$

Lemma 1.17 (Matsushita)**.** *The symplectic form $\sigma \in H^{2,0}(X)$ is trivial when restricted to any subvariety $T \subset X$ contracted to a point t under f. In particular, all fibres of f are of dimension n, i.e. f is equidimensional, and if B is smooth, f is flat.*

Proof. A theorem due to Kollár [Kol86a, Thm. 2.1] and Saito [Sai90, Thm. 2.3, Rem. 2.9.] says that $R^2f_*\omega_X$ is torsion free. Since in our case $\omega_X \simeq \mathcal{O}_X$, this shows that $R^2f_*\mathcal{O}_X$ is torsion free. Let $\bar{\sigma} \in H^2(X,\mathcal{O}_X)$ be the conjugate of the symplectic form, and ρ be its image in $H^0(B,R^2f_*\mathcal{O}_X)$. Since the general fibre is Lagrangian, ρ must be torsion and hence zero. If $\widetilde{T} \longrightarrow T$ is a resolution of T, then the image of $\bar{\sigma}$ in $H^2(\widetilde{T},\mathcal{O}_{\widetilde{T}})$ is contained in the image of

$$R^2f_*\mathcal{O}_X \otimes k(t) \longrightarrow H^2(T,\mathcal{O}_T) \longrightarrow H^2(\widetilde{T},\mathcal{O}_{\widetilde{T}})$$

and hence trivial. This implies that the image of σ in $H^0(\widetilde{T},\Omega^2_{\widetilde{T}})$ is trivial, i.e. $\sigma|_T = 0$. By semi-continuity of the dimension of the fibres, $\dim T \geqslant n$, and so T is Lagrangian.

The flatness follows from the smoothness of X and B, see [Har77, Exer. III.10.9]. $\square$

Remark 1.18. Note that the conclusion that f is flat really needs the base to be smooth. In fact, by miracle flatness, f is flat if and only if B is smooth.

1.5. **Further results.** We summarize a few further results without proof.

1.5.1. *Higher direct images.* The first one is the main result of [Mat05].

Theorem 1.19 (Matsushita)**.** *Assume $f\colon X \longrightarrow B$ is a fibration of a projective*[10] *hyperkähler manifold over a smooth base. Then*

$$R^i f_* \mathcal{O}_X \simeq \Omega_B^i.$$

On the open subset $B^\circ \subset B$ over which $f^\circ := f|_{f^{-1}(B^\circ)}\colon X^\circ \longrightarrow B^\circ$ is smooth, the result can be obtained by dualising the isomorphism

$$f_*^\circ \Omega^1_{X^\circ/B^\circ} \simeq T_{B^\circ},$$

which holds because the smooth fibres of f are Lagrangian. A relative polarization is used to show that $R^1 f_*^\circ \mathcal{O}_{X^\circ}$ and $f_*^\circ \Omega^1_{X^\circ/B^\circ}$ are dual to each other. To extend the result from B° to the whole B, Theorem 1.19 uses a result of Kollár [Kol86a, Thm. 2.1] saying that $R^i f_* \omega_X$ are torsion free, which for X hyperkähler translates into $R^i f_* \mathcal{O}_X$ being torsion free.

As mention in Remark 1.7, the theorem implies $H^*(B,\mathbb{Q}) \simeq H^*(\mathbb{P}^n,\mathbb{Q})$.

1.5.2. *Lagrangian tori are Lagrangian fibres.* In [Bea11] Beauville asked whether every Lagrangian torus $T \subset X$ is the fibre of a Lagrangian fibration $X \longrightarrow B$. The question has been answered affirmatively:

(i) Greb–Lehn–Rollenske in [GLR13] first dealt with the case of non-projective X and later showed in [GLR14] the existence of an almost holomorphic Lagrangian fibration in dimension four.

(ii) A different approach to the existence of an almost holomorphic Lagrangian fibration with T as a fibre was provided by Amerik–Campana [AC13]. The four-dimensional case had been discussed before by Amerik [Ame12].

(iii) Hwang–Weiss [HW13] deal with the projective case and proved the existence of an almost Lagrangian fibration with fibre T. Combined with techniques of [GLR13] this resulted in a complete answer.

2. Cohomology of the base and cohomology of the fibre

The aim of this section is to prove the following result.

Theorem 2.1. *Assume $X \longrightarrow B$ is a fibration and let X_t be a smooth fibre. Then*

$$H^*(\mathbb{P}^n,\mathbb{Q}) \simeq H^*(B,\mathbb{Q}) \text{ and } H^*(\mathbb{P}^n,\mathbb{Q}) \simeq \mathrm{Im}\left(H^*(X,\mathbb{Q}) \longrightarrow H^*(X_t,\mathbb{Q})\right).$$

The first isomorphism for X projective and B smooth is originally due to Matsushita [Mat05], see Remark 1.7. The proof we give here is a version of the one by Shen and Yin [SY18] that works without assuming X projective. Note also that we do not assume that the base B is smooth.

[10] Again, the projectivity assumption can presumably be dropped by applying results of Saito.

The second isomorphism in degree two is essentially due to Oguiso [Ogu09], relying on results of Voisin [Voi92]. The paper by Shen and Yin [SY18] contains two proofs of the general result, one using the $\mathfrak{sl}_2$-representation theory of the perverse filtration and another one, due to Voisin, relying on classical Hodge theory.

The proof we shall give avoids the perverse filtration as well as the various $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-actions central for the arguments in [SY18]. The discussion below also proves the second result in [SY18, Thm. 0.2], namely the equality

$$^{\mathfrak{p}}h^{i,j}(X) = h^{i,j}(X)$$

between the classical and perverse Hodge numbers, see Section 2.3. How it fits into the setting of P=W is explained in Section 3.

2.1. **Algebraic preparations.** To stress the purely algebraic nature of what follows we shall use the shorthand $H^* := H^*(X, \mathbb{C})$ and consider it as a graded $\mathbb{C}$-algebra.

Consider a non-trivial, isotropic element β of degree two, i.e. $0 \neq \beta \in H^2$ with $q(\beta) = 0$. Then, according to Verbitsky and Bogomolov [Bog96, Ver96], one has

$$\beta^n \neq 0 \text{ and } \beta^{n+1} = 0.$$

In particular, multiplication with β defines on H^* the structure of a graded $\mathbb{C}[x]/(x^{n+1})$-algebra with x of degree two.

All that is needed in the geometric applications is then put into the following statement.

Proposition 2.2. *For every two non-zero, isotropic elements $\beta, \beta' \in H^2$, the induced graded $\mathbb{C}[x]/(x^{n+1})$-algebra structures on H^* are isomorphic.*

Proof. Note that due to the existence of the isotropic planes, we know that the Beauville–Bogomolov form q is indefinite.

Consider the complex algebraic group of automorphisms $\mathrm{Aut}(H^*)$ of the graded $\mathbb{C}$-algebra H^* and its image G under $\mathrm{Aut}(H^*) \longrightarrow \mathrm{Gl}(H^2)$. Clearly, the assertion holds if $\beta, \beta' \in H^2$ are contained in the same G-orbit. As any two non-zero isotropic classes β, β' are contained in the same orbit of the complex orthogonal group $\mathrm{O}(H^2, q)$, it suffices to show that $\mathrm{O}(H^2, q) \subset G$.

Now, monodromy defines a discrete subgroup in $\mathrm{Aut}(H^*)$ and its image in $\mathrm{Gl}(H^2)$ contains a finite index subgroup of the integral orthogonal group $\mathrm{O}(H^2(X, \mathbb{Z}))$. Since by [Bor66] the latter is Zariski dense in $\mathrm{O}(H^2, q)$ when q is indefinite, we indeed have $\mathrm{O}(H^2, q) \subset G$. □

Remark 2.3. The arguments can be adapted to prove the following statement: Assume $\beta, \beta' \in H^2$ satisfy $q(\beta) = q(\beta') \neq 0$. Then the induced graded $\mathbb{C}[x]/(x^{2n+1})$-algebra structures on H^*, given by letting x act by multiplication with β resp. β', are isomorphic.

For $0 \neq \beta \in H^2$ with $q(\beta) = 0$ we let

$$H^d_{\beta\text{-pr}} := \mathrm{Ker}\left(\beta^{n-d+1} \colon H^d \longrightarrow H^{2n-d+2} \right),$$

which is called the space of β-primitive forms. Note, however, that β does not satisfy the Hard Lefschetz theorem for which we would need to define primitive classes as elements in the kernel of β^{2n-d+1}.

We will also need the two spaces

$$(2.1)\quad P_0H^d := \mathrm{Im}\left(\bigoplus_i \beta^i \cdot H^{d-2i}_{\beta\text{-pr}} \longrightarrow H^d\right) \text{ and } \bar{P}_0H^d := \mathrm{Coim}\left(\beta^n\colon H^d \longrightarrow H^{d+2n}\right).$$

It turns out that the map in the definition of P_0 is injective, but this is not needed for the argument. Note that $P_0H^d \subset \mathrm{Ker}(\beta^n) \subset H^d$ for all $d > 0$.

Corollary 2.4. *The dimensions of the spaces P_0H^d and $\bar{P}_0H^d$ are independent of the choice of the non-trivial, isotropic class $\beta \in H^2$.* □

2.2. **Geometric realizations.** Let us begin by looking at the obvious choice for β provided by the symplectic form $\sigma \in H^0(X, \Omega_X^2) \subset H^2(X, \mathbb{C})$.

Lemma 2.5. *For $\beta = \sigma$ one has*

$$P_0H^d = H^0(X, \Omega_X^d) \subset H^d(X, \mathbb{C}) \text{ and } P_0H^* \simeq H^*(\mathbb{P}^n, \mathbb{C})$$

and

$$\bar{P}_0H^d \simeq H^d(X, \mathcal{O}_X) \text{ and } \bar{P}_0H^* \simeq H^*(\mathbb{P}^n, \mathbb{C}).$$

Proof. Concerning the first equality, one inclusion is obvious: Since $H^0(X, \mathcal{O}_X) = H^0(X, \mathbb{C}) = H^0_{\sigma\text{-pr}}$, we have $H^0(X, \Omega_X^d) = \mathbb{C} \cdot \sigma^{d/2} \subset P_0H^d$ for d even and $H^0(X, \Omega_X^d) = 0$ for d odd. For the other direction, use that $\sigma^{n-p}\colon \Omega_X^p \xrightarrow{\sim} \Omega_X^{2n-p}$ is an isomorphism and that, therefore, for $q > 0$ the composition

$$(2.2)\qquad H^{p,q}(X) \xrightarrow{\sigma^{n-d+1}} H^{2n-p-2q+2,q}(X) \xrightarrow{\sigma^{q-1}} H^{2n-p,q}(X)$$

is injective. Hence, σ^{n-d+1} is injective, i.e. $H^{p,q}(X) \cap H^d_{\sigma\text{-pr}} = 0$ for $q > 0$, which is enough to conclude.

For the second part observe that $\mathrm{Ker}(\sigma^n) \cap \bigoplus H^{p,q}(X) = \bigoplus_{p>0} H^{p,q}(X)$. □

As an immediate consequence of Corollary 2.4 one then finds.

Corollary 2.6. *For any non-trivial, isotropic class $\beta \in H^2$ there exist isomorphisms*

$$P_0H^* \simeq H^*(\mathbb{P}^n, \mathbb{C}) \text{ and } \bar{P}_0H^* \simeq H^*(\mathbb{P}^n, \mathbb{C})$$

of graded vector spaces. □

Next let us consider a Lagrangian fibration $f\colon X \longrightarrow B$. We consider the class $\beta := f^*\alpha$, which is isotropic since $\alpha^{n+1} = 0$ for dimension reasons.

Lemma 2.7. *For $\beta = f^*\alpha$ there exists an inclusion*

$$f^*H^*(B, \mathbb{C}) \subset P_0H^*(X, \mathbb{C}).$$

Proof. The assertion follows from the Lefschetz decomposition

$$H^d(B,\mathbb{C}) = IH^d(B,\mathbb{C}) = \bigoplus_i \alpha^i \cdot IH^{d-2i}(B,\mathbb{C})_{\mathrm{pr}}$$

on B, with respect to the unique ample class $\alpha \in H^2(B,\mathbb{Q})$, cf. [dCM05, Thm. 2.2.3.(c)], and the observation that pull-back via f maps $IH^{d-2i}(B,\mathbb{C})_{\mathrm{pr}}$ into $H^{d-2i}_{\beta\text{-pr}}$. $\square$

Corollary 2.4 then immediately yields

$$H^*(B,\mathbb{C}) \simeq P_0H^* \simeq H^*(\mathbb{P}^n,\mathbb{C}).$$

cf. Remark 1.7, which proves the first part of Theorem 2.1.

We keep the isotropic class $\beta = f^*\alpha$ and observe that the natural inclusion

$$\text{(2.3)} \qquad \mathrm{Ker}\left(H^d(X,\mathbb{Q}) \longrightarrow H^d(X_t,\mathbb{Q})\right) \subset \mathrm{Ker}\left([X_t] \colon H^d(X,\mathbb{Q}) \longrightarrow H^{d+2n}(X,\mathbb{Q})\right).$$

is actually an isomorphism.

Lemma 2.8 (Voisin)**.** *Let $\beta = f^*\alpha$ be as before and $X_t \subset X$ a smooth fibre of f. Then*

$$\mathrm{Ker}(\beta^n) \subset \mathrm{Ker}\left(H^d(X,\mathbb{Q}) \longrightarrow H^d(X_t,\mathbb{Q})\right).$$

Proof. The result is proved in [SY18, App. B]. The assertion is shown to be equivalent to the statement that the intersection pairing on the fibre is non-degenerate on the image of the restriction map, which in turn is deduced from Deligne's global invariant cycle theorem. $\square$

The result yields a surjection

$$\pi \colon \bar{P}_0H^* \twoheadrightarrow \mathrm{Im}\left(H^*(X,\mathbb{C}) \longrightarrow H^*(X_t,\mathbb{C})\right).$$

Since $\bar{P}_0H^* \simeq H^*(\mathbb{P}^n,\mathbb{C})$ by Corollary 2.6, its image in $H^*(X_t,\mathbb{C})$ is the subring generated by the restriction of a Kähler class. Hence, π is an isomorphism, which proves the second isomorphism in Theorem 2.1. However, it is easier to argue directly, as the equality holds in Lemma 2.8 by (2.3).

2.3. As in Section 2.1, we consider the abstract algebraic situation provided by $H^* := H^*(X,\mathbb{C})$ and the additional structure induced by the choice of a non-zero isotropic class $\beta \in H^2$. The two spaces P_0H^d and $\bar{P}_0H^d$ defined there, both depending on β, are part of a filtration

$$P_0H^* \subset P_1H^* \subset \cdots \subset P_{2n-1}H^* \subset P_{2n}H^* = H^*,$$

where P_0H^d is as defined before and $\bar{P}_0H^d = H^d/P_{d-1}H^d$.

In general, one defines

$$\text{(2.4)} \qquad P_kH^d := \sum_{i \geqslant 0} \beta^i \cdot \mathrm{Ker}\left(\beta^{n-(d-2i)+k+1} \colon H^{d-2i} \longrightarrow H^{2n-d+2i+2k+2}\right).$$

If we want to stress the dependence of β, we write $P^\beta_kH^d$. The graded objects of this filtration

$$\mathrm{Gr}^P_i H^* := P_iH^*/P_{i-1}H^*,$$

in particular $\mathrm{Gr}_d H^d = \bar{P}_0 H^d$, are used to define the *Hodge numbers* of the filtration as

$$^{P}h^{i,j} := \dim \mathrm{Gr}_i^P H^{i+j}.$$

As a further consequence of Proposition 2.2, one has

Corollary 2.9. *The Hodge numbers* $^{P}h^{i,j}$ *of the filtration* $P_i H^*$ *are independent of the choice of the isotropic class* $\beta \in H^2$. □

Let us quickly apply this to two geometric examples.

(i) First, consider $\beta = \bar{\sigma} \in H^2(X, \mathcal{O}_X) \simeq H^{0,2}(X) \subset H^2(X, \mathbb{C})$, the anti-holomorphic symplectic form. Then the filtration gives back the Hodge filtration, i.e.

$$P_k^{\bar{\sigma}} H^d = \bigoplus_{p \leqslant k} H^{p,d-p}(X).$$

To see this, one needs to use the Lefschetz decomposition with respect to $\bar{\sigma}$:

$$H^q(X, \Omega_X^p) = \bigoplus_{q-\ell \geqslant (q-n)^+} \bar{\sigma}^{q-\ell} \cdot H^{2\ell-q}(X, \Omega_X^p)_{\bar{\sigma}\text{-pr}}.$$

Note that from this example one can deduce that indeed for any choice of β one has $P_k^{\beta} H^d = 0$ for $k < 0$ and $P_k^{\beta} H^d = H^d$ for $k \geqslant d$.

(ii) For the second example consider a Lagrangian fibration $f\colon X \longrightarrow B$ and let β be the pull–back of an ample class $\alpha \in H^2(B, \mathbb{Q})$. The induced filtration is called the *perverse filtration*[11] and the Hodge numbers are denoted $^{\mathfrak{p}}h^{i,j}(X)$.

Then [SY18, Thm. 0.2] becomes the following immediate consequence of Proposition 2.2 or Corollary 2.9.

Corollary 2.10 (Shen–Yin)**.** *For any Lagrangian fibration* $f\colon X \longrightarrow B$ *the Hodge numbers of the perverse filtration equal the classical Hodge numbers:*

$$^{\mathfrak{p}}h^{i,j}(X) = h^{i,j}(X).$$

3. P=W

P=W for compact hyperkähler manifolds asserts that the perverse filtration associated with a Lagrangian fibration can be realised as the weight filtration of a limit mixed Hodge structure of a degeneration of compact hyperkäher manifolds. It boils down to the observation that the cup product by a semiample not big class and a logarithmic monodromy operator define nilpotent endomorphisms in cohomology which are not equal, but up to renumbering induce the same filtration. Inspired by P=W, we provide some geometric explanation or conjecture concerning the appearance of the cohomology of $\mathbb{P}^n$ in the introduction and in Theorem 2.1.

[11] The classical definition of the perverse filtration for the constructible complex $Rf_*\mathbb{Q}_X$ due to [BBDG18] or [dCM05, Def. 4.2.1] coincides with the present one; see [dCM05, Prop. 5.2.4.(39)].

3.1. **The weight filtration of a nilpotent operator.**

Definition 3.1. Given a nilpotent endomorphism N of a finite dimensional vector space H^* of index l, i.e. $N^l \neq 0$ and $N^{l+1} = 0$, the *weight filtration of N centred at l* is the unique increasing filtration

$$W_0H^* \subset W_1H^* \subset \cdots \subset W_{2l-1}H^* \subset W_{2l}H^* = H^*,$$

with the property that (1) $NW_k \subseteq W_{k-2}$, and denoting again N the induced endomorphism on graded pieces, (2) $N^k \colon \mathrm{Gr}^W_{l+k}H^* \simeq \mathrm{Gr}^W_{l-k}H^*$ for every $k \geqslant 0$, see [Del80, §1.6].

The weight filtration of N on H^* can be constructed inductively as follows: first let $W_0 := \mathrm{Im}N^l$, and $W_{2l-1} := \ker N^l$. We can replace H^* with W_{2l-1}/W_0, on which N is still well-defined and $N^l = 0$. Then define

$$W_1 := \text{inverse image in } W_{2l-1} \text{ of } \mathrm{Im}N^{l-1} \text{ in } W_{2l-1}/W_0,$$
$$W_{2l-2} := \text{inverse image in } W_{2l-1} \text{ of } \ker N^{l-1} \text{ in } W_{2l-1}/W_0.$$

Continuing inductively, we obtain the unique (!) filtration on H^* satisfying (1) and (2).

By the Jacobson–Morozov theorem, the nilpotent operator N can be extended to an $\mathfrak{sl}_2$-triple with Cartan subalgebra generated by an element H^N which is unique up to scaling. By the representation theory of $\mathfrak{sl}_2$-triple, there exists a decomposition of H^*

$$H^* = \bigoplus_{\lambda=-l}^{l} H^*_\lambda,$$

called *weight decomposition*, with the property that $H^N(v) = \lambda v$ for all $v \in H^*_\lambda$. In particular, the decomposition splits the weight filtration of N

$$W_kH^* = \bigoplus_{\lambda=-l}^{-l+k} H^*_\lambda.$$

Let us apply this to some geometric examples.

(i) Any cohomology class $\omega \in H^2(X,\mathbb{C})$ define a nilpotent operator L_ω on $H^* := H^*(X,\mathbb{C})$ by cup product. If ω is Kähler, then the Hard Lefschetz theorem implies that the weight filtration on H^* centred at $2n$ is

$$W^\omega_k H^* = \bigoplus_{i \geqslant 4n-k} H^i(X,\mathbb{C}).$$

(ii) Consider a Lagrangian fibration $f\colon X \longrightarrow B$ and let β be the pull–back of an ample class $\alpha \in H^2(B,\mathbb{Q})$. Up to renumbering, the weight filtration associated with the class β on H^* centred at n coincides with the perverse filtration, see Section 2.3

$$W^\beta_k H^d(X,\mathbb{Q}) = P_{d+k-2n}H^d(X,\mathbb{Q}).$$

Indeed, the action of β gives the morphisms

$$\beta\colon P_kH^d(X,\mathbb{Q}) \longrightarrow P_kH^{d+2}(X,\mathbb{Q}) \qquad \beta^j\colon \mathrm{Gr}^P_i H^{n+i-j} \simeq \mathrm{Gr}^P_i H^{n+i+j}.$$

The isomorphism is called *perverse Hard Lefschetz theorem* [dCM05, Prop. 5.2.3]. By Proposition 2.2, this corresponds to the isomorphism $\bar{\sigma}^j \colon H^{n-j}(X, \Omega_X^i) \simeq H^{n+j}(X, \Omega_X^i)$.

(iii) Let $\pi \colon \mathcal{X} \longrightarrow \Delta$ be a projective degeneration of hyperkähler manifolds over the unit disk which we assume to be semistable, i.e. the central fibre $\mathcal{X}_0$ is reduced and snc. For $t \in \Delta^*$, let N denote the logarithmic monodromy operator on $H^*(\mathcal{X}_t, \mathbb{Q})$. The weight filtration of N centred at d on $H^d(\mathcal{X}_t, \mathbb{Q})$, denoted $W_k H^d(\mathcal{X}_t, \mathbb{Q})$, is the weight filtration of the limit mixed Hodge structure associated to π, see [PS08, Thm. 11.40].

The degeneration $\pi \colon \mathcal{X} \longrightarrow \Delta$ is called of type III if $N^2 \neq 0$ and $N^3 = 0$ on $H^2(\mathcal{X}_t, \mathbb{Q})$. In this case, the limit mixed Hodge structure is of Hodge–Tate type by [Sol20, Thm. 3.8], and in particular $\mathrm{Gr}^W_{2i+1} H^*(\mathcal{X}_t, \mathbb{Q}) = 0$. Then the even graded pieces of the weight filtration are used to define the *Hodge numbers*

$$^{\mathfrak{w}}h^{i,j}(\mathcal{X}) \coloneqq \dim \mathrm{Gr}^W_{2i} H^{i+j}(\mathcal{X}_t, \mathbb{Q}).$$

The Hodge numbers $^{\mathfrak{w}}h^{0,j}(\mathcal{X})$ have a clear geometric description. The dual complex of $\mathcal{X}_0 = \sum \Delta_i$, denoted $D(\mathcal{X}_0)$, is the CW complex whose k-cells are in correspondence with the irreducible components of the intersection of $(k+1)$ divisors Δ_i. The Clemens–Schmid exact sequence then gives

$$^{\mathfrak{w}}h^{0,j}(\mathcal{X}) = \dim H^j(D(\mathcal{X}_0), \mathbb{Q}), \tag{3.1}$$

see for instance [Mor84, §3, Cor. 1 & 2].

In order to show P=W, namely that the filtrations (ii) and (iii) can be identified, we need the notion of hyperkähler triples with their associated $\mathfrak{so}(5, \mathbb{C})$-action.

3.2. **Hyperkähler triples.** A hyperkähler manifold is a Riemannian manifold (X, g) which is Kähler with respect to three complex structures I, J, and K, satisfying the standard quaternion relations $I^2 = J^2 = K^2 = IJK = -\mathrm{Id}$. The corresponding hyperkähler triple is the triple of Kähler classes in $H^2(X, \mathbb{C}) \times H^2(X, \mathbb{C}) \times H^2(X, \mathbb{C})$ given by

$$(\omega_I, \omega_J, \omega_K) \coloneqq (g(I\cdot, \cdot), g(J\cdot, \cdot), g(K\cdot, \cdot)).$$

The set of all hyperkähler triples on X form a Zariski-dense subset in

$$D^\circ = \{(x, y, z) \colon\ q(x) = q(y) = q(z) \neq 0, q(x, y) = q(y, z) = q(z, x) = 0\}.$$

In particular, all algebraic relations that can be formulated for triples in D° and which hold for triples of the form $(\omega_I, \omega_J, \omega_K)$ hold in fact for all $(x, y, z) \in D^\circ$, see [SY18, Prop. 2.3].

3.3. **The $\mathfrak{so}(5, \mathbb{C})$-action.** Recall the scaling operator

$$H \colon H^i(X, \mathbb{C}) \longrightarrow H^i(X, \mathbb{C}) \qquad H(v) = (i - 2n)v.$$

By the Jacobson–Morozov theorem, to any $\omega \in H^2(X,\mathbb{C})$ of Lefschetz type we can associate a $\mathfrak{sl}_2$-triple $(L_\omega, H, \Lambda_\omega)$. Let $p = (x,y,z) \in D^\circ$. The $\mathfrak{sl}_2$-triples associated to x, y and z generate the Lie subalgebra $\mathfrak{g}_p \subset \mathrm{End}(H^*(X,\mathbb{C}))$, isomorphic to $\mathfrak{so}(5,\mathbb{C})$, with Cartan subalgebra

$$\mathfrak{h} = \langle H, H'_p := \sqrt{-1}[L_y, \Lambda_z]\rangle. \tag{3.2}$$

There is an associated weight decomposition

$$H^*(X,\mathbb{C}) = \bigoplus_{i,j} H^{i,j}(p) \tag{3.3}$$

such that for all $v \in H^{i,j}(p)$ we have

$$H(v) = (i+j-2n)v \qquad H'_p(v) = (j-i)v.$$

The following $\mathfrak{sl}_2$-triples in $\mathfrak{g}_p$

$$E_p := \frac{1}{2}(L_y - \sqrt{-1}L_z) \qquad F_p := \frac{1}{2}(\Lambda_y + \sqrt{-1}\Lambda_z) \qquad H_p := \frac{1}{2}(H + H'_p), \tag{3.4}$$

$$E'_p := [E_p, \Lambda_x] \qquad F'_p := [L_x, F_p] \qquad H'_p \tag{3.5}$$

induce the same weight decomposition, since for any $v \in H^{i,j}(p)$ we have

$$H_p(v) = (j-n)v \qquad H'_p(v) = (j-i)v.$$

Remark 3.2. The previous identities for hyperkähler triples are due to Verbitsky. The result for a general pair $p = (x,y,z) \in D^\circ$ follows from the density of hyperkähler triples in D°, and the fact that the $\mathfrak{sl}_2$-representation $H^*(X,\mathbb{C})$ associated to x, y and z have the same weights, since x, y, and z are all of Lefschetz type, see [SY18, §2.4].

3.4. **P=W.** The main result of [HLSY19] is the following

Theorem 3.3 (P=W)**.** *For any Lagrangian fibration $f\colon X \longrightarrow B$, there exists a type III projective degeneration of hyperkähler manifolds $\pi\colon \mathcal{X} \longrightarrow \Delta$ with $\mathcal{X}_t$ deformation equivalent to X for all $t \in \Delta^*$, together with a multiplicative isomorphism $H^*(X,\mathbb{Q}) \simeq H^*(\mathcal{X}_t,\mathbb{Q})$, such that*

$$P_k H^*(X,\mathbb{Q}) = W_{2k} H^*(\mathcal{X}_t,\mathbb{Q}) = W_{2k+1} H^*(\mathcal{X}_t,\mathbb{Q}).$$

Proof. Let $\beta = f^*\alpha$ be the pullback of an ample class $\alpha \in H^2(B,\mathbb{Q})$, and $\eta \in H^2(X,\mathbb{Q})$ with $q(\eta) > 0$. Since $\beta^{n+1} = 0$, we have $q(\beta) = 0$. Up to replacing η with $\eta + \lambda\beta$ for some $\lambda \in \mathbb{Q}$, we can suppose that $q(\eta) = 0$. Set

$$y = \beta + \eta \qquad z = -\sqrt{-1}(\eta - \beta).$$

By scaling a nonzero vector $x \in H^2(X,\mathbb{C})$ perpendicular to y and z with respect to q, we obtain $p(f) = (x,y,z) \in D^\circ$ with

$$\beta = \frac{1}{2}(y - \sqrt{-1}z).$$

Soldatenkov showed that the nilpotent operator $E'_{p(f)}$ is the logarithmic monodromy N of a projective type III degeneration $\pi\colon \mathcal{X} \longrightarrow \Delta$ of compact hyperkähler manifolds deformation equivalent to X, see [Sol20, Lem. 4.1, Thm. 4.6][12].

The weight decomposition for the $\mathfrak{sl}_2$-triple (3.4) splits the perverse filtration associated to f, since $E_{p(f)}$ acts in cohomology via the cup product by β. The weight decomposition for the $\mathfrak{sl}_2$-triple (3.5) splits the weight filtration of the limit mixed Hodge structure associated to π, because $E'_{p(f)} = N$. Hence, by Section 3.3, this yields P=W. $\square$

P=W also yields alternative proofs of Corollary 2.10 and Theorem 2.1.

Corollary 3.4 (Numerical P=W)**.** ${}^{\mathfrak{p}}h^{i,j}(X) = {}^{\mathfrak{w}}h^{i,j}(\mathcal{X}) = h^{i,j}(X)$.

Proof. By Theorem 3.3 we obtain ${}^{\mathfrak{p}}h^{i,j}(X) = {}^{\mathfrak{w}}h^{i,j}(\mathcal{X})$. The equality ${}^{\mathfrak{p}}h^{i,j}(X) = h^{i,j}(X)$ is Corollary 2.10.

Alternatively, one can argue as follows. By [Sol20, Thm. 3.8], the limit mixed Hodge structure $(H^*_{\lim}(\mathcal{X}_t, \mathbb{Q}) \simeq H^*(\mathcal{X}_t, \mathbb{C}), W_*, F_*)$ associated to π is of Hodge–Tate type, and so ${}^{\mathfrak{w}}h^{i,j}(\mathcal{X}) = \dim_{\mathbb{C}} \mathrm{Gr}^F_i H^{i+j}_{\lim}(\mathcal{X}_t, \mathbb{C})$. By the classical result [PS08, Cor. 11.25], we have $\dim_{\mathbb{C}} \mathrm{Gr}^F_i H^{i+j}_{\lim}(\mathcal{X}_t, \mathbb{C}) = h^{i,j}(\mathcal{X}_t)$. We conclude that ${}^{\mathfrak{p}}h^{i,j}(X) = h^{i,j}(\mathcal{X}_t) = h^{i,j}(X)$. $\square$

Corollary 3.5. *At the boundary of the Hodge diamond of X, P=W gives*[13]

$$\dim H^j(B, \mathbb{Q}) = {}^{\mathfrak{p}}h^{0,j}(X) = h^{0,j}(X) = \dim H^j(\mathbb{P}^n, \mathbb{Q}),$$
$$\dim H^j(D(\mathcal{X}_0), \mathbb{Q}) = {}^{\mathfrak{w}}h^{0,j}(\mathcal{X}) = h^{0,j}(X) = \dim H^j(\mathbb{P}^n, \mathbb{Q}),$$
$$\dim \mathrm{Im}(H^i(X, \mathbb{Q}) \longrightarrow H^i(X_t, \mathbb{Q})) = {}^{\mathfrak{p}}h^{i,0}(X) = h^{i,0}(X) = \dim H^i(\mathbb{P}^n, \mathbb{Q}).$$

In the following, we provide conjectural conceptual explanations for these identities.

3.5. **A conjectural explanation I.** Assume that $\mathcal{X}$ is Calabi–Yau. This can be always achieved via a MMP, at the cost of making $\mathcal{X}_0$ mildly singular (precisely divisorial log terminal), see [Fuj11]. Under this assumption the homeomorphism class of $D(\mathcal{X}_0)$ is well-defined.

Then the SYZ conjecture predicts that $\mathcal{X}_t$ carries a special Lagrangian fibration $f\colon \mathcal{X}_t \longrightarrow D(\mathcal{X}_0)$ with respect to a hyperkähler metric. By hyperkähler rotation [Hit00, §3], f should become a holomorphic Lagrangian fibration $f\colon X \longrightarrow B$ on a hyperkähler manifold X deformation equivalent to $\mathcal{X}_t$. It is conjectured that the base of a Lagrangian fibration on X is a projective space. So in brief, we should have the homeomorphisms

$$D(\mathcal{X}_0) \simeq \mathbb{P}^n \simeq B. \tag{3.6}$$

[12]One can use the Lie algebra structure of the LLV algebra to compare the present description of $E'_{p(f)}$ with that of [Sol20, Lem. 4.1], cf. [KSV19, Lem. 3.9]. Mind that Soldatenkov's existence result is not constructive: it relies on lattice theory and the geometry of the period domain, and does not produce an explicit type III degeneration.

[13]The identity $\dim H^j(D(\mathcal{X}_0), \mathbb{Q}) = \dim H^j(\mathbb{P}^n)$ has been first proved in [KLSV18, Thm. 7.13].

The latter equality is known to hold if $n \leqslant 2$, see §1.3, or conditional to the smoothness of the base [Hwa08]. The former equality is known for degenerations of Hilbert schemes or generalised Kummer varieties [BM19]. In both case, the most delicate problem is to assess the smoothness of $D(\mathcal{X}_0)$ or B. From this viewpoint, the identity

$$\dim H^j(D(\mathcal{X}_0), \mathbb{Q}) = \dim H^j(\mathbb{P}^n, \mathbb{Q}) = \dim IH^j(B, \mathbb{Q}) = \dim H^j(B, \mathbb{Q}).$$

is a weak cohomological evidence for the conjecture (3.6).

3.6. **A conjectural explanation II.** We conjecture that the equality ${}^{\mathfrak{p}}h^{i,0}(\mathcal{X}) = {}^{\mathfrak{w}}h^{i,0}(\mathcal{X})$ is the result of the identification of two Lagrangian tori up to isotopy.

Definition 3.6. Let x be a zero-dimensional stratum of $\mathcal{X}_0$. Choose local coordinates $z_0, \ldots, z_{2n}$ centered at x with $\pi(z) = z_0 \cdot \ldots \cdot z_{2n}$. For fixed radius $0 < r_i \ll 1$ and $t = \prod_{i=0}^{2n} r_i$, a *profound torus* $\mathbb{T} \subset \mathcal{X}_t$ is

$$\mathbb{T} = \{(r_0 e^{i\theta_0}, \ldots, r_{2n} e^{i\theta_{2n}}) \colon \theta_0, \ldots, \theta_{2n} \in [0, 2\pi),\ \theta_0 + \cdots + \theta_{2n} - \arg(t) \in \mathbb{Z}\}.$$

Remark 3.7. The ambient-isotopy type of $\mathbb{T} \subset \mathcal{X}_t$ does not depend on the choice of the coordinates: $\mathbb{T}$ is homotopic to $U_x \cap \mathcal{X}_t$, where U_x is a neighbouhood of x in $\mathcal{X}$. More remarkably, if $\mathcal{X}$ is Calabi–Yau, then the isotopy class of $\mathbb{T}$ in $\mathcal{X}_t$ is independent of x. This follows at once from Kollár's notion of $\mathbb{P}^1$-link (see [Kol13, Prop. 4.37]or [Har19, Lem. 3.10]), or equivalently because profound tori are fibre of the same smooth fibration, by adapting [EM21, Prop. 6.12.]

Conjecture 3.8 (Geometric P=W)**.** *For any Lagrangian fibration $f \colon X \longrightarrow B$ with general fibre T, there exists a projective minimal dlt type III degeneration of hyperkähler manifolds $\pi \colon \mathcal{X} \longrightarrow \Delta$ with $\mathcal{X}_t$ deformation equivalent to X for all $t \in \Delta^*$, such that T is isotopic to a profound torus $\mathbb{T}$.*

The conjecture is inspired by the geometric P=W conjecture for character varieties, see the new version of [MMS18] (to appear soon). Lemma 2.8 and (2.1) give

$$P_{d-1}H^d(X, \mathbb{Q}) = \operatorname{Ker}\left(H^d(X, \mathbb{Q}) \longrightarrow H^d(T, \mathbb{Q})\right).$$

If $\mathcal{X}_0$ is snc (or ideally adapting [Har19, Thm. 3.12] to the dlt setting), one obtain that

$$W_{2d-1}H^d(\mathcal{X}_t, \mathbb{Q}) = \operatorname{Ker}\left(H^d(\mathcal{X}_t, \mathbb{Q}) \longrightarrow H^d(\mathbb{T}, \mathbb{Q})\right).$$

Therefore, Conjecture 3.8 would give a geometric explanation of P=W at the highest weight

$$P_{d-1}H^d(X, \mathbb{Q}) = W_{2d-1}H^d(\mathcal{X}_t, \mathbb{Q}).$$

It is not clear what should be a geometric formulation of P=W which can explain the cohomological statement in all weights.

Recent advance in the SYZ conjecture due to Yang Li [Li20] suggests that profound tori can be made special Lagrangian, up to a conjecture in non-archimedean geometry. Since few months ago, the existence of a single special Lagrangian torus on $\mathcal{X}_t$ was a complete mystery, see

[Gro13, §5, p.152]. Note also that Li's result is compatible with the expectation in symplectic geometry [Aur07, Conj. 7.3]. Profound tori appear as general fibres of the SYZ fibration that Li constructed onto an open set which contains an arbitrary large portion of the mass of $\mathcal{X}_t$ with respect to a Calabi–Yau metric, still modulo the non-archimedean conjecture. It is curious (but maybe not surprising) that also the previously quoted results [HX19] and [BM19] highly rely on non-archimedean techniques.

3.7. **Multiplicativity of the perverse filtration.** P=W implies that the perverse filtration on $H^*(X, \mathbb{Q})$ is compatible with cup product.

Corollary 3.9 (Multiplicativity of the perverse filtration)**.** *Assume $f\colon X \longrightarrow B$ is a fibration. Then the perverse filtration on $H^*(X, \mathbb{Q})$ is multiplicative under cup product, i.e.*

$$\cup\colon P_k H^d(X, \mathbb{Q}) \times P_{k'} H^{d'}(X, \mathbb{Q}) \longrightarrow P_{k+k'} H^{d+d'}(X, \mathbb{Q}).$$

Proof. By P=W, it is sufficient to show that the weight filtration is multiplicative. To this end, endow the tensor product $H^*(\mathcal{X}_t, \mathbb{Q}) \otimes H^*(\mathcal{X}_t, \mathbb{Q})$ with the nilpotent endomorphism $N^{\otimes} := N \otimes 1 + 1 \otimes N$, and call $W^{\otimes}$ the weight filtration of $N^{\otimes}$. Since the monodromy operator e^N is an algebra homomorphism of $H^*(\mathcal{X}_t, \mathbb{Q})$, N is a derivation, i.e.

$$N(x \cup y) = Nx \cup y + x \cup Ny = \cup(N^{\otimes}(x \otimes y)).$$

As a consequence, the construction of the weight filtration (cf. Section 3.1) yields

$$\cup(W_k^{\otimes}(H^i(\mathcal{X}_t, \mathbb{Q}) \otimes H^j(\mathcal{X}_t, \mathbb{Q}))) \subseteq W_k H^{i+j}(\mathcal{X}_t, \mathbb{Q}).$$

Together with [Del80, 1.6.9.(i)] which says that

$$W_k^{\otimes}(H^i(\mathcal{X}_t, \mathbb{Q}) \otimes H^j(\mathcal{X}_t, \mathbb{Q})) = \bigoplus_{a+b=k} W_a H^i(\mathcal{X}_t, \mathbb{Q}) \otimes W_b H^j(\mathcal{X}_t, \mathbb{Q}),$$

we conclude that the weight filtration is multiplicative. Alternatively see [HLSY19, §5]. □

Remark 3.10. For an arbitrary morphism of projective varieties or Kähler manifolds, the perverse filtration is not always multiplicative [Zha17, Exa. 1.5], but it is so for instance if it coincides with the Leray filtration, or if P=W holds. Indeed, the Leray filtration and the weight filtration of the limit mixed Hodge structure are multiplicative.

It is natural to ask whether the multiplicativity holds at a sheaf theoretic level, for $Rf_*\mathbb{Q}_X$, or over an affine base. The motivation for this comes from the celebrated P=W conjecture for twisted character varieties [dCHM12], which has been proved to be equivalent to the conjectural multiplicativity of the perverse filtration of the Hitchin map, namely a proper holomorphic Lagrangian fibration over an affine base, see [dCMS19, Thm. 0.6]. From this viewpoint, it is remarkable that Shen and Yin give a proof of the multiplicativity in the compact case [SY18, Thm. A.1] which uses only the representation theory of $\mathfrak{sl}(2)$-triples, with no reference to the weight filtration.

3.8. **Nagai's conjecture for type III degenerations.** Let $\pi\colon \mathcal{X} \longrightarrow \Delta$ be a projective degeneration of hyperkähler manifolds with unipotent monodromy T_d on $H^d(\mathcal{X}_t, \mathbb{Q})$. The *index of nilpotence* of $N_d := \log T_d$ is

$$\mathrm{nilp}(N_d) = \max\{i\colon N_d^i \neq 0\},$$

and $\mathrm{nilp}(N_d) \leqslant d$ by [Gri15, Ch. IV]. It is known that $H^2(\mathcal{X}_t, \mathbb{Q})$ determines the Hodge structure of $H^d(\mathcal{X}_t, \mathbb{Q})$ by means of the LLV representation, see [Sol19]. The Nagai's conjecture investigates to what extent $\mathrm{nilp}(N_2)$ determines $\mathrm{nilp}(N_d)$. The ring structure of the subalgebra generated by H^2 implies the inequality $\mathrm{nilp}(N_{2k}) \geqslant k \cdot \mathrm{nilp}(N_2)$, see [**?**, Lem. 2.4], but equality is expected.

Conjecture 3.11 (Nagai)**.** $\mathrm{nilp}(N_{2k}) = k \cdot \mathrm{nilp}(N_2)$ *for* $k \leqslant 2n$*.*

The previous inequalities imply Nagai's conjecture for type III degenerations, i.e. $\mathrm{nilp}(N_2) = 2$. Remarkably, P=W explains Nagai's conjecture in term of the level of the Hodge structure $H^d(\mathcal{X}_t, \mathbb{Q})$, and determines $\mathrm{nilp}(N_d)$ even for d odd. Recall that the *level* of a Hodge structure $H = \oplus H^{p,q}$, denoted $\mathrm{level}(H)$, is the largest difference $|p-q|$ for which $H^{p,q} \neq 0$, or equivalently the length of the Hodge filtration on H.

Proposition 3.12. *Let* $\pi\colon \mathcal{X} \longrightarrow \Delta$ *be a type III projective degeneration of hyperkähler manifolds with unipotent monodromy. Then*

$$\mathrm{nilp}(N_d) = \mathrm{level}(H^d(\mathcal{X}_t, \mathbb{C})).$$

For $k \leqslant 2n$*, the following identities hold:*

(i) $\mathrm{nilp}(N_{2k}) = 2k = k \cdot \mathrm{nilp}(N_2)$*,*
(ii) $\mathrm{nilp}(N_{2k+1}) = 2k - 1$*, if* $H^3(\mathcal{X}_t, \mathbb{C}) \neq 0$*.*

Remark 3.13. The statement (ii) is proved in [Sol20, Prop. 3.15]. Here we present an alternative simple proof of (ii) which avoids the LLV representation.

Nagai's conjecture is known to hold for degenerations of type I and III, i.e. $\mathrm{nilp}(N_2) = 0$ and 2, see [KLSV18, Thm. 6.5]. In order to establish Nagai's conjecture in full, only the case of type II degenerations remains open, i.e. $\mathrm{nilp}(N_2) = 1$. For type II there are partial results: $k \leqslant \mathrm{nilp}(N_{2k}) \leqslant 2k - 2$ for $2 \leqslant k \leqslant n - 1$, see [KLSV18, Thm. 6.5], and $\mathrm{nilp}(N_{2n}) = n$, see [**?**, Thm. 1.2]. The full conjecture holds for all the known deformation families of hyperkähler manifolds by [GKLR19, Thm. 1.13]. Further comments on Nagai's conjecture for type II can be found in [GKLR19, Har67, **?**].

Proof. Let l_d be half of the length of the weight filtration of N_d, i.e. $l_d := \min\{i\colon W_{2i}H^d(\mathcal{X}_t, \mathbb{Q}) = H^d(\mathcal{X}_t, \mathbb{Q})\}$. By Definition 3.1, we have $\mathrm{nilp}(N_d) = l_d$.

For any type III degeneration of Hodge structure of hyperkähler type with unipotent monodromy, we know by the proof of Theorem 3.3, that the logarithmic monodromy N_* is of the form $E'_p = [\beta, \Lambda_x]$ for some β and x in $H^2(X, \mathbb{Q})$ with $q(\beta) = 0$ (here we use the assumption $b_2(\mathcal{X}_t) \geqslant 5$, see [Sol20, §4.1]). Then, by Corollaries 2.9 and 3.4, we have $l_d = \mathrm{level}(H^d(\mathcal{X}_t, \mathbb{C}))$. Hence, $\mathrm{nilp}(N_d) = \mathrm{level}(H^d(\mathcal{X}_t, \mathbb{C}))$.

Finally, statements (i) and (ii) are equivalent to (i) $H^{2k,0}(\mathcal{X}_t) = \mathbb{C}\sigma \neq 0$, and (ii) $H^{2k,1}(\mathcal{X}_t) \neq 0$ if $H^{2,1}(\mathcal{X}_t) \neq 0$, which follows from (2.2). $\square$

4. Examples and counterexamples

Example 4.1. [14] In [Nam01, Ex. 1.7.(iv)] Namikawa exhibits an example of a submanifold T of a hyperkähler manifold X which is isomorphic to a complex torus, but is not Lagrangian (actually it is symplectic).

Let E, F be elliptic curve defined by the cubic equation f and g respectively, and let $Y \subseteq \mathbb{P}^5$ be the cubic fourfold given by the equation $h := f(x_0, x_1, x_2) + g(y_0, y_1, y_2) = 0$. The cyclic group $G := \mathbb{Z}/3\mathbb{Z}$ acts on Y by

$$\phi_\zeta \colon [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \longmapsto [x_0 : x_1 : x_2 : \zeta y_0 : \zeta y_1 : \zeta y_2],$$

where ζ is a third root of unity. The induced action on the Fano variety of lines X is symplectic, i.e. $\phi_\zeta^* \sigma = \sigma$ for $\sigma \in H^0(X, \Omega_X^2)$. Indeed, by [BD85] there is a G-equivariant isomorphism $H^0(X, \Omega_X^2) \simeq H^1(Y, \Omega_Y^3)$. Denoting Ω the canonical section of $H^0(\mathbb{P}^5, K_{\mathbb{P}^5}(6))$, $H^1(Y, \Omega_Y^3)$ is generated by the G-invariant residue $\mathrm{Res}_Y(\Omega/h^2)$, and so the action is symplectic. In particular, the fixed locus T of the G action on X is a symplectic submanifold. T is given by the set of lines which join two points on $Y \cap \{y_0 = y_1 = y_2 = 0\} \simeq E$ and $Y \cap \{x_0 = x_1 = x_2 = 0\} \simeq F$ respectively. Hence, $T \simeq E \times F$. We conclude that T is a symplectic torus embedded in the hyperkähler manifold X.

Example 4.2. There exists a Lagrangian submanifold L of a hyperkähler manifold X with

$$\mathrm{Im}(H^2(X, \mathbb{Q}) \longrightarrow H^2(L, \mathbb{Q})) \not\simeq \mathbb{Q}.$$

Proof. Let $f \colon S \longrightarrow \mathbb{P}^1$ be an elliptic K3 surface with smooth fibre E. Define $L \subseteq X := S^{[2]}$ to be the locus of non-reduced length-two subschemes of S supported on E, which is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\Omega_S^1|_E)$ over E. Then, L is an irreducible component of the fibre of the Lagrangian fibration $f^{[2]} \colon S^{[2]} \longrightarrow S^{(2)} \longrightarrow \mathbb{P}^2$, thus L is Lagrangian. The exceptional divisor Exc of the Hilbert–Chow morphism $S^{[2]} \longrightarrow S^{(2)}$ restricts to a multiple of the tautological line bundle $\mathcal{O}_{\mathbb{P}(\Omega_S^1|_E)}(-1)$ on L. Therefore, the second cohomology group $H^2(L)$ is generated by the restriction of Exc and the pullback of an ample line bundle of $S^{(2)}$. $\square$

Example 4.3. There exists a Lagrangian submanifold L of a hyperkähler manifold X with

$$\mathrm{Im}(H^2(X, \mathbb{Q}) \longrightarrow H^2(L, \mathbb{Q})) \simeq \mathbb{Q} \quad \text{and} \quad \mathrm{Im}(H^*(X, \mathbb{Q}) \longrightarrow H^*(L, \mathbb{Q})) \not\simeq H^*(\mathbb{P}^n, \mathbb{Q}).$$

[14] Thanks to Thorsten Beckmann for pointing out this example.

Proof. Let C be a smooth curve of genus two in an abelian surface A. Consider $M_{\text{odd}}(A)$ a moduli space of stable 1-dimensional sheaves on A supported on the curve class

$$2[C] \in H_2(A, \mathbb{Z})$$

and Euler characteristic -1. The fibre of the albanese morphism $M_{\text{odd}}(A) \twoheadrightarrow A \times \widehat{A}$ is a compact hyperkähler manifold X deformation equivalent to a generalised Kummer variety of dimension six. Taking Fitting supports defines a Lagrangian fibration

$$X \twoheadrightarrow \mathbb{P}^3 = |2C|.$$

The fibre over the curve $2C$ contains as irreducible component the moduli space L of rank-two vector bundles on C of degree one, which is isomorphic to the intersection of two quadrics in $\mathbb{P}^5$. The cohomology $H^*(X)$ is generated by so-called tautological classes, and $H^*(L)$ is generated by their restrictions, see [Mar02] and [New72, Thm. 1]. Therefore, we have

$$H^*(X, \mathbb{Q}) \twoheadrightarrow H^*(L, \mathbb{Q}) \simeq H^*(\mathbb{P}^3, \mathbb{Q}) \oplus \mathbb{Q}^4[-3] \not\simeq H^*(\mathbb{P}^3, \mathbb{Q}).$$

□

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THE LLV DECOMPOSITION OF HYPERKÄHLER COHOMOLOGY AND APPLICATIONS TO THE NAGAI CONJECTURE (AFTER GREEN–KIM–LAZA–ROBLES)

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Abstract. Following work of Green, Kim, Laza, and Robles, we discuss the structure and known cases of the decomposition of the cohomology of hyperkähler varieties under the Looijenga–Lunts–Verbitsky algebra. This has applications to the Nagai conjecture concerning degenerations of hyperkähler varieties.

1. Introduction

Given a compact hyperkähler manifold X, the rational second cohomology group $H^2(X,\mathbb{Q})$ is equipped with the Beauville–Bogomolov–Fujiki form q_X. Following [GKLR], we let

$$(V,q) \coloneqq \left(H^2(X,\mathbb{Q}) \oplus \mathbb{Q}^2, q_X \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$$

denote its *Mukai completion*. Let also $h \in \operatorname{End} H^*(X,\mathbb{Q})$ be the degree operator defined by

$$h|_{H^k(X,\mathbb{Q})} = (k - \dim X)\operatorname{Id}$$

such that the degrees are centered at the middle cohomology.

The *Looijenga–Lunts–Verbitsky* (LLV) *algebra* $\mathfrak{g}$ is the subalgebra of $\operatorname{End} H^*(X,\mathbb{Q})$ generated by all $\mathfrak{sl}_2$*-triples* (L_a, h, Λ_a), where L_a is the operator of cup product with a class $a \in H^2(X,\mathbb{Q})$. We refer to [A] for an introduction. In particular, the fundamental theorem about this algebra is the following:

Theorem 1.1 (Looijenga–Lunts [LL], Verbitsky [V])**.**
(1) $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(V,q)$;
(2) $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_k$ acts by degree k;
(3) $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus \mathbb{Q}h$, and the reduced part $\mathfrak{g}_0' \coloneqq [\mathfrak{g}_0, \mathfrak{g}_0]$ is isomorphic to $\mathfrak{so}(H^2(X,\mathbb{Q}), q_X)$.

The cohomology $H^*(X,\mathbb{Q})$ is a $\mathfrak{g}$-module by construction. By semisimplicity, the cohomology hence splits into a direct sum of irreducible $\mathfrak{g}$-submodules V_λ,

$$H^*(X,\mathbb{Q}) \cong \bigoplus_\lambda V_\lambda^{m_\lambda},$$

called the LLV decomposition; here $m_\lambda \in \mathbb{N}$ are the multiplicities of the components. The main goal of this note is to discuss the general structure of this decomposition. One has the following basic results (the first holds since $\mathfrak{g}$ is of even degree).

Proposition 1.2. *$H^*(X,\mathbb{Q})$ decomposes into $H^*_{\mathrm{even}}(X,\mathbb{Q}) \oplus H^*_{\mathrm{odd}}(X,\mathbb{Q})$ as $\mathfrak{g}$-modules.*

Theorem 1.3 ([V])**.** *The subalgebra $SH^2(X,\mathbb{Q}) \subset H^*(X,\mathbb{Q})$ generated by $H^2(X,\mathbb{Q})$ is an irreducible $\mathfrak{g}$-submodule. It is isomorphic to $\operatorname{Sym}^*(H^2(X,\mathbb{Q}))/\langle a^{n+1} \mid q_X(a) = 0\rangle$ as algebra and $\mathfrak{g}_0'$-module.*

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The branching rules for $\mathfrak{g}_0' \subset \mathfrak{g}$ show that $SH^2(X,\mathbb{Q})$ is isomorphic to $V_{(n)}$ as a $\mathfrak{g}$-module (see below for notation). Hence there is always an irreducible submodule in $H^*(X,\mathbb{Q})$ that is known (and also quite big), and referred to as the *Verbitsky component*. The structure of the remaining components is however still mysterious. The so far strongest conjectural bound on their weights will be given in Remark 5.8.

The plan of this note is as follows: In Section 2 we recall useful facts about the representation theory of Lie algebras of type B and D. In Section 3 we discuss the connection of the LLV algebra to the Hodge structure which will be sufficient to determine the LLV decomposition for the OG10 class. In Section 4 we introduce the Mumford–Tate algebra, and in Section 5 we almost give full details in the computation of the LLV decomposition in K3$^{[n]}$-type. The remaining cases of generalized Kummer varieties and OG6 are sketched in Section 6. Starting with Section 7 the last two sections will consider applications of the LLV decomposition to the Nagai conjecture which concerns the question, how the nilpotency indices of degenerations of hyperkähler varieties are related in different degrees.

Acknowledgements: This note originated from a joint talk at the University Bonn/Paris reading seminar on hyperkähler varieties in the Spring of 2021. The main source is the beautiful paper [GKLR] by Green, Kim, Laza, and Robles, and aside from streamlining a few arguments we do not claim any originality. We thank Daniel Huybrechts for organizing the seminar and inviting us to contribute this note.

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2. Representation theory

We introduce the necessary notions for the representation theory of $\mathfrak{g}$. For this section, we let $\mathfrak{g} \coloneqq \mathfrak{so}(V,q)$ denote a Lie algebra of type B_r or D_r defined over $\mathbb{Q}$, where $\dim V = 2r+1$ or $\dim V = 2r$. For references, see the Appendices of [GKLR] and the book [FH].

Type B. Let $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ be a Cartan subalgebra. The standard representation V decomposes as

$$V = V(0) \oplus V(\varepsilon_1) \oplus V(-\varepsilon_1) \oplus \cdots \oplus V(\varepsilon_r) \oplus V(-\varepsilon_r),$$

for some $0, \pm\varepsilon_1, \ldots, \pm\varepsilon_r \in \mathfrak{h}^\vee$ which are called the weights of V. An element $h \in \mathfrak{h}$ acts as the scalar $\varepsilon(h)$ on $V(\varepsilon)$. We choose a positive Weyl chamber generated by the fundamental weights

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \le i \le r-1, \quad \varpi_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r).$$

They correspond to the highest weight of $\bigwedge^i V$ for $1 \le i \le r-1$ and the spin module respectively. The set of dominant weights is the following

$$\Lambda^+ = \left\{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_1 \ge \cdots \ge \lambda_r \ge 0 \\ \lambda_i \in \frac{1}{2}\mathbb{Z}, \lambda_i - \lambda_j \in \mathbb{Z} \text{ for all } i,j \end{array} \right\}.$$

Over $\mathbb{C}$, irreducible $\mathfrak{g}_{\mathbb{C}}$-modules are classified by their highest weight.

Over $\mathbb{Q}$, the Schur–Weyl construction for a $\mathfrak{g}$-module with integral highest weight is still available: let λ be a dominant weight with $\sum \lambda_i = d$. Then one defines

$$V_\lambda \coloneqq \mathbf{S}_\lambda V \cap V^{[d]},$$

where $\mathbf{S}_\lambda$ is the Schur functor, and $V^{[d]}$ is the intersection of all the kernels $\ker(V^d \xrightarrow{q} V^{d-2})$ given by contracting any two components with q. On the other hand, modules with half-integer highest weight are not necessarily defined over $\mathbb{Q}$.

Example. We have $V_{(1,\ldots,1)} = \bigwedge^k V$, and $V_{(k)} = \ker(\operatorname{Sym}^k V \xrightarrow{q} \operatorname{Sym}^{k-2} V)$.

Type D. The standard representation V has weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_r \in \mathfrak{h}^\vee$. The fundamental weights are given by

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \text{ for } 1 \le i \le r-2, \quad \varpi_{r-1} = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r), \quad \varpi_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r),$$

corresponding to the highest weight of $\bigwedge^i V$ for $1 \le i \le r-2$ and the two half-spin modules respectively. The set of dominant weights is the following

$$\Lambda^+ = \left\{ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r \;\middle|\; \begin{array}{c} \lambda_1 \ge \cdots \ge \lambda_{r-1} \ge |\lambda_r| \ge 0 \\ \lambda_i \in \frac{1}{2}\mathbb{Z}, \lambda_i - \lambda_j \in \mathbb{Z} \text{ for all } i, j \end{array} \right\}.$$

Again, all the representations with integral highest weight are defined over $\mathbb{Q}$ via the Schur–Weyl construction, which is not necessarily the case for those with half-integer highest weight.

For both type B and type D, the dimension of each V_λ can be obtained using Weyl dimension formula, which we won't state here. We will however need the following corollary of the dimension formula.

Lemma 2.1. *Let λ and $\mu \neq 0$ be dominant integral weights of $\mathfrak{g}$, then* $\dim V_{\lambda+\mu} > \dim V_\lambda$.

Weyl character. We review results on the Weyl character ring for any reductive rational Lie algebra $\mathfrak{g}$, although our main interest remains in type B and D. Let $\mathrm{Rep}(\mathfrak{g})$ be the category of finite dimensional rational $\mathfrak{g}$-modules. Complexification gives a functor

$$\mathrm{Rep}(\mathfrak{g}) \longrightarrow \mathrm{Rep}(\mathfrak{g}_\mathbb{C})$$

to the category of $\mathfrak{g}_\mathbb{C}$-modules, which induces an *injective* morphism

$$K(\mathfrak{g}) \hookrightarrow K(\mathfrak{g}_\mathbb{C})$$

at the level of *representation rings*, that is, the Grothendieck rings of the corresponding categories.

The Weyl character of a $\mathfrak{g}_\mathbb{C}$-module $V = \bigoplus_\mu V(\mu)$ is given by $\mathrm{ch}\, V \coloneqq \sum \dim V(\mu) e^\mu$ with value in the group ring $\mathbb{Z}[\Lambda]$, where e^μ is the element corresponding to the weight μ. The character map factors through the representation ring $K(\mathfrak{g}_\mathbb{C})$ and has image in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, the $\mathfrak{W}$-invariant subring.

Theorem 2.2. *The character map* $\mathrm{ch}\colon K(\mathfrak{g}_\mathbb{C}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}$ *is a ring isomorphism.*

We describe the Weyl character ring $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ for $\mathfrak{g}$ of type B_r and D_r.

Proposition 2.3.

(1) *When $\mathfrak{g}$ is of type* B_r, *write $x_i \coloneqq e^{\varepsilon_i}$. Then*

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\pm\frac{1}{2}}].$$

The Weyl group $\mathfrak{W}_{2r+1}$ is isomorphic to $\mathfrak{S}_r \ltimes (\mathbb{Z}/2)^r$, where $\mathfrak{S}_r$ acts as permutations on $x_1, \ldots, x_r$ and the i-th $\mathbb{Z}/2$ acts as $x_i \mapsto x_i^{-1}$.

(2) *When $\mathfrak{g}$ is of type* D_r, *the group ring $\mathbb{Z}[\Lambda]$ is the same as above for* B_r, *while the Weyl group $\mathfrak{W}_{2r}$ is the index-2 subgroup of $\mathfrak{W}_{2r+1}$ consisting of elements with an even number of non-trivial components in $(\mathbb{Z}/2)^r$.*

We have the following result that relates the two.

Proposition 2.4. *Let (V, q) be a rational quadratic space of dimension* $\dim V = 2r+1$, *and $W \subset V$ a non-degenerate subspace of dimension* $\dim W = 2r$. *Let $\mathfrak{g} = \mathfrak{so}(V, q)$ and $\mathfrak{m} = \mathfrak{so}(W, q|_W)$. Then the restriction functor* $\mathrm{Res}\colon \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(\mathfrak{m})$ *induces an injective morphism for the character rings, and consequently, the (rational) representation rings. We have the following diagram*

$$\begin{tikzcd}
K(\mathfrak{g}) \arrow[r, hook, "\mathrm{Res}"] \arrow[d, hook] & K(\mathfrak{m}) \arrow[d, hook] \\
K(\mathfrak{g}_{\mathbb{C}}) \arrow[r, hook, "\mathrm{Res}"] \arrow[d, "\mathrm{ch}"', "\simeq"] & K(\mathfrak{m}_{\mathbb{C}}) \arrow[d, "\mathrm{ch}"', "\simeq"] \\
\mathbb{Z}[\Lambda]^{\mathfrak{W}_{2r+1}} \arrow[r, hook] & \mathbb{Z}[\Lambda]^{\mathfrak{W}_{2r}}.
\end{tikzcd}$$

In particular, for an arbitrary $\mathfrak{g}$-module, if one can obtain its decomposition as an $\mathfrak{m}$-module via restriction, then its Weyl character is uniquely determined and hence so is its $\mathfrak{g}$-module structure.

Remark 2.5. In the hyperkähler setting, the LLV algebra $\mathfrak{g}$ is of type B_{r+1} or D_{r+1}, and its reduced part $\mathfrak{g}'_0$ is of type B_r or D_r, so the proposition does not apply directly for $\mathfrak{g}'_0 \subset \mathfrak{g}$. Instead, in the $\mathrm{K3}^{[n]}$-case, we will take the subalgebra $\mathfrak{m}$ to be $\mathfrak{g}(S)$, the LLV algebra of a K3 surface S.

3. Hodge structures

From this section on, we let $r := \lfloor b_2(X)/2 \rfloor$, so that $\mathfrak{g}$ is of type B_{r+1} or D_{r+1}, and $\mathfrak{g}'_0$ is of type B_r or D_r. The weights of $\mathfrak{g}$ will be denoted as $\lambda = \lambda_0 \varepsilon_0 + \cdots + \lambda_r \varepsilon_r$.

The LLV decomposition is a diffeomorphism invariant, but we can obtain more information using a complex structure. Let $f \in \operatorname{End} H^*(X, \mathbb{R})$ be the Weil operator

$$f|_{H^{p,q}(X)} = i(q-p)\,\mathrm{Id}\,.$$

We will use this operator to define Hodge structures on each irreducible component V_λ, and obtain some conditions on the dominant weight λ that can appear.

Proposition 3.1. *We have $f \in (\mathfrak{g}'_0)_{\mathbb{R}}$.*

Proof. Denote by I, J, K three complex structures coming from a hyperkähler metric g where I is the complex structure that we are using. We have three Kähler classes $\omega_I = g(I-,-)$, $\omega_J = g(J-,-)$, and $\omega_K = g(K-,-)$, hence three $\mathfrak{sl}_2$-triples

$$(L_I, h, \Lambda_I),\ (L_J, h, \Lambda_J),\ (L_K, h, \Lambda_K). \tag{1}$$

These are all operators on $H^*(X, \mathbb{R})$ and lie in $\mathfrak{g}_{\mathbb{R}}$ by construction.

By working pointwise on tangent spaces and then using harmonic forms Verbitsky showed that the Weil operator $f = f_I$ for the complex structure I satisfies

$$f_I = -[L_J, \Lambda_K] = -[L_K, \Lambda_J],$$

so $f_I \in (\mathfrak{g}_0)_{\mathbb{R}}$. One may consider Weil operators f_J and f_K for the other two complex structures, and verify that

$$[f_J, f_K] = -2f_I.$$

So f_I indeed lies in $[(\mathfrak{g}_0)_{\mathbb{R}}, (\mathfrak{g}_0)_{\mathbb{R}}] = (\mathfrak{g}'_0)_{\mathbb{R}}$. □

Remark 3.2. Recall that the real subalgebra $\mathfrak{g}_g$ generated by the three $\mathfrak{sl}_2$-triples (1) is isomorphic to $\mathfrak{so}(4,1)$: an explicit basis over $\mathbb{R}$ is given by

$$\Lambda_I, \Lambda_J, \Lambda_K, \quad f_I, f_J, f_K, h, \quad L_I, L_J, L_K.$$

In particular, the degree-0 part is generated by h and the three Weil operators.

Under the action of f, the standard representation V decomposes as

$$V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2},$$

where f acts as $-2i$, 0, and $2i$ respectively. Similarly, we have another decomposition under the action of h

$$V = V_{-2} \oplus V_0 \oplus V_2,$$

where h acts as -2, 0, and 2 respectively. Hence if we take $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ a Cartan subalgebra that contains both h and f, then h and if are among the $\pm\varepsilon_i^\vee$. Up to the choice of a Weyl chamber, we may hence suppose that $h = \varepsilon_0^\vee$ and $if = \varepsilon_1^\vee$. Under this choice, we can also identify $\varepsilon_1, \dots, \varepsilon_r$ as the weights of $\mathfrak{g}_0'$.

For a $\mathfrak{g}$-module V_λ that appears in $H^*(X, \mathbb{Q})$, we take its weight decomposition with respect to the chosen Cartan subalgebra $\mathfrak{h}$: $(V_\lambda)_{\mathbb{C}} = \bigoplus_\mu V_\lambda(\mu)$, where $V_\lambda(\mu)$ is the component of weight $\mu = \mu_0\varepsilon_0 + \cdots + \mu_r\varepsilon_r$. Then h acts as $2\mu_0$ and if acts as $2\mu_1$ on $V_\lambda(\mu)$. We find

$$V_\lambda(\mu) \subset H^{p,q}(X)$$

where

$$\begin{cases} 2\mu_0 = p + q - 2n \\ 2\mu_1 = i \cdot i(q-p) = p - q \end{cases} \Rightarrow \quad \begin{cases} p = \mu_0 + \mu_1 + n \\ q = \mu_0 - \mu_1 + n \end{cases} \tag{2}$$

In other words, $V_\lambda \subset H^*(X, \mathbb{Q})$ is a sub-Hodge structure.

More generally, there is a naturally defined Hodge structure on any $\mathfrak{g}$-module V_λ determined by the actions of h and f. We simply set

$$(V_\lambda)_{\mathbb{C}}^{p,q} := \bigoplus_{\mu \text{ satisfying } (2)} V_\lambda(\mu).$$

The Hodge numbers $h^{p,q}$ count the multiplicities of suitable weights.

Remark 3.3. The Hodge numbers only give information about λ_0, λ_1 in the representation V_λ and do not necessarily determine the $\mathfrak{g}$-module structure (such an example will show up in the case of OG6).

Example. The Verbitsky component $SH^2(X, \mathbb{Q})$ contains a non-trivial $H^{2n,2n}$-part. We have $p = 2n, q = 2n$ so $\mu_0 = n, \mu_1 = 0$ which must be the highest weight (n). (In fact this can be used to prove that $SH^2(X, \mathbb{Q}) \simeq V_{(n)}$: we just saw that the highest weight of $SH^2(X, \mathbb{Q})$ dominates (n); on the other hand, we have $\dim SH^2(X, \mathbb{Q}) = \dim V_{(n)}$ due to the description of Verbitsky, so by Lemma 2.1, the highest weight must be exactly (n).) In particular, since $H^{2n,2n}(X)$ is one-dimensional, the component $V_{(n)}$ appears with multiplicity 1 in $H^*(X, \mathbb{Q})$. By using the explicit description of $V_{(n)}$ one also sees that it exhausts all the outermost Hodge numbers $h^{2k,0} = 1$.

Corollary 3.4.

(1) Each component V_λ of $H^*_{\text{even}}(X, \mathbb{Q})$ has integral highest weight λ;
(2) Each component V_λ of $H^*_{\text{odd}}(X, \mathbb{Q})$ has half-integer highest weight λ;
(3) Each component V_λ other than the Verbitsky component satisfies $\lambda_0 + \lambda_1 \leq n - 1$ and $\lambda_0 \leq n - \frac{3}{2}$.

Proof. For statements (1) and (2), we look at the component $V_\lambda(\lambda)$ and get

$$p + q = 2\lambda_0 + 2n,$$

which allows us to conclude that λ_0 is an integer or a half-integer in the two cases.

For statement (3), since V_λ is not the Verbitsky component, it cannot have a $H^{2n,0}$-part, so by looking at the component $V_\lambda(\lambda)$ we get

$$\lambda_0 + \lambda_1 + n = p \leq 2n - 1,$$

which gives the first inequality. By definition, the Verbitsky component exhausts the second cohomology $H^2(X, \mathbb{Q})$ and hence $H^{4n-2}(X, \mathbb{Q})$ by Hodge symmetry, so we also have

$$3 \leq p + q = 2\lambda_0 + 2n \leq 4n - 3,$$

which gives the second inequality. □

Remark 3.5.

(1) The two inequalities in (3) are tight: for generalized Kummer varieties Kum_n with $n \geq 2$ (whose LLV algebra is of type B_4), we can have the component $V_{(n-\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}$. In fact, the existence of this component is equivalent to the non-vanishing of $b_3(X)$.

(2) When $n = 2$, the statement (3) shows that $\lambda = (2)$ or $\lambda_0 \leq \frac{1}{2}$, so all the possible weights are $(2), (\frac{1}{2}, \cdots, \frac{1}{2}), (0)$, plus $(\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ in the type D case. The half-integer weight component generates $H^*_{\mathrm{odd}}(X, \mathbb{Q})$.

   If we assume the conjecture of [GKLR] (see Remark 5.8 below), the sum of each weight would then be bounded by $n = 2$, so the odd cohomologies must vanish entirely when $b_2(X) \geq 8$. This is indeed the case, by a theorem of Guan (see [BD, Theorem 3.6] of this volume).

(3) When $n = 3$ and V_λ is a component of $H^*_{\mathrm{even}}(X, \mathbb{Q})$ other than $V_{(3)}$, we get $\lambda_0 \leq 1$ so λ is a sequence of ones, and V_λ must be a wedge product $\bigwedge^k V$.[1]

The corollary gives some contraints on the irreducible components that can appear. For O'Grady's 10-dimensional example, this is already enough to determine the full decomposition.

Proposition 3.6. *Let X be a hyperkähler manifold of dimension* 10 *such that $b_2(X) = 24$, $e(X) = 176904$, and $H^*_{\mathrm{odd}}(X) = 0$. Then we have the following decomposition of $\mathfrak{g}$-modules*

$$H^*(X, \mathbb{Q}) = V_{(5)} \oplus V_{(2,2)}.$$

In particular, O'Grady's example OG10 satisfies these numerical conditions, so we have obtained its LLV decomposition.

Proof. The LLV algebra $\mathfrak{g}$ is of type D_{13}. Write $H^*(X, \mathbb{Q}) = H^*_{\mathrm{even}}(X, \mathbb{Q}) = V_{(5)} \oplus V'$. We have $\dim V' = e(X) - \dim V_{(5)} = 37674$. By using the inequalities in Corollary 3.4 and by considering the dimension bound and Lemma 2.1, the only possible dominant weights that can appear are

$$\{(3), (2, 2), (2, 1), (2), (1, 1, 1, 1), (1, 1, 1), (1, 1), (1), (0)\}.$$

Each V_λ carries a Hodge structure and therefore has its own Betti numbers. Using Salamon's result on the Betti numbers of a hyperkähler manifold (see [BD, Section 2] of this volume), one can verify that the only possible solution is one copy of $V_{(2,2)}$. □

[1] In the article of Sawon on the bound of $b_2(X)$, he wrongly assumed that only $\bigwedge^2 V$ can appear. Even if the general conjecture holds, that is, the sum of λ is bounded by 3, we can still have $\bigwedge^3 V$.

4. Mumford–Tate algebra

Definition 4.1. Let W be a rational Hodge structure. Let f be the Weil operator

$$f|_{W^{p,q}} = i(q-p)\,\mathrm{Id}\,.$$

The special Mumford–Tate algebra $\mathfrak{m} = \mathfrak{m}(W)$ is the smallest rational subalgebra of $\mathrm{End}(W)$ such that $f \in \mathfrak{m}_{\mathbb{R}}$. The (full) Mumford–Tate algebra is $\mathfrak{m} \oplus \mathbb{Q}h$ where h is the degree operator $h|_{W^{p,q}} = (p+q)\,\mathrm{Id}$. It coincides with the associated Lie algebra of the Mumford–Tate group of W. (This degree operator differs from the one that we defined earlier, so we need to take a Tate twist $H^*(X,\mathbb{Q})(\dim X)$.)

When W is the cohomology $H^*(X,\mathbb{Q})$ of a hyperkähler manifold X, by Proposition 3.1 we see that $\mathfrak{m}$ is a subalgebra of $\mathfrak{g}_0'$. Conversely, we have the following result.

Proposition 4.2. *For X a very general hyperkähler manifold, the Mumford–Tate algebra $\mathfrak{m}$ is equal to $\mathfrak{g}_0'$.*

Proof. Consider the restriction map

$$\rho\colon\ \mathrm{End}\,H^*(X,\mathbb{Q}) \longrightarrow \mathrm{End}\,H^2(X,\mathbb{Q}).$$

The Weil operator f_2 on $H^2(X,\mathbb{Q})$ is the restriction of f. Since $\mathfrak{m}$ satisfies $f \in \mathfrak{m}_{\mathbb{R}}$, its restriction $\rho(\mathfrak{m})$ will satisfy $f_2 = \rho(f) \in \rho(\mathfrak{m})_{\mathbb{R}}$. Thus by definition, $\rho(\mathfrak{m})$ contains the special Mumford–Tate algebra $\mathfrak{m}(H^2(X,\mathbb{Q}))$ for the second cohomology. By the local Torelli theorem, the latter is equal to $\mathfrak{so}(H^2(X,\mathbb{Q}),q_X) \simeq \mathfrak{g}_0'$ for X very general. So $\rho(\mathfrak{m}) \simeq \mathfrak{g}_0'$, which shows that $\mathfrak{m}$ must coincide with $\mathfrak{g}_0'$. ☐

Consequently, for a very general X, the decomposition of $H^*(X,\mathbb{Q})$ into $\mathfrak{g}_0'$-modules is the same as decomposition into sub-Hodge structures.

Example. For X of $\mathrm{K3}^{[2]}$-type, by a dimension count we have $H^*(X,\mathbb{Q}) = V_{(2)}$ as $\mathfrak{g}$-module. Write H for the second cohomology group as a $\mathfrak{g}_0'$-module. Using the description by Verbitsky, we get an isomorphism of $\mathfrak{g}_0'$-modules

$$\begin{array}{rccccccccc} H^*(X,\mathbb{Q}) = & \mathbb{Q} & \oplus & H & \oplus & \mathrm{Sym}^2 H & \oplus & H & \oplus & \mathbb{Q} \\ = & \mathbb{Q} & \oplus & H & \oplus & (H_{(2)} \oplus \mathbb{Q}) & \oplus & H & \oplus & \mathbb{Q} \end{array},$$

where $H_{(2)}$ is an irreducible $\mathfrak{g}_0'$-module obtained as $\ker(\mathrm{Sym}^2 H \xrightarrow{q_X} \mathbb{Q})$. The 1-dimensional component $\mathbb{Q} \subset H^4(X,\mathbb{Q})$ is generated by the dual of q_X, which is also proportional to $c_2(X)$.

For a Hodge special X, the Mumford–Tate algebra $\mathfrak{m}$ becomes smaller, so $H^*(X,\mathbb{Q})$ may decompose further into smaller components. This is the key idea for determining the LLV decomposition for the other three types of hyperkähler manifolds.

5. $\mathrm{K3}^{[n]}$-type

In the $\mathrm{K3}^{[n]}$-type case, there is a natural choice of a Hodge special locus: when $X = S^{[n]}$ is actually the Hilbert scheme of a K3 surface S (not necessarily algebraic). We have a decomposition

$$(H^2(X,\mathbb{Q}),q_X) = (H^2(S,\mathbb{Q}),q_S) \oplus \langle -2(n-1)\rangle.$$

Hence $\mathfrak{g}(S)$ is naturally realized as a subalgebra of $\mathfrak{g} = \mathfrak{g}(X)$, and $\mathfrak{m}(S) = \mathfrak{m}(H^2(S,\mathbb{Q}))$ as a subalgebra of $\mathfrak{m} = \mathfrak{m}(H^2(X,\mathbb{Q}))$. We write $W := H^*(S,\mathbb{Q})$, which coincides with the Mukai completion of $H^2(S,\mathbb{Q})$ and is therefore the standard representation for $\mathfrak{g}(S)$. When S is non-algebraic and very general, $\mathfrak{m}(S)$ coincides with $\mathfrak{g}_0'(S) = \mathfrak{so}(H^2(S,\mathbb{Q}), q_S)$ and is of type D_{11}. We have the diagram

$$\begin{tikzcd} \mathfrak{g} & \mathfrak{g}_0' \arrow[l, hook'] \\ \mathfrak{g}(S) \arrow[u, hook] & \mathfrak{g}_0'(S) = \mathfrak{m}(S) \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

The Hodge structure on $H^*(S^{[2]},\mathbb{Q})$ is described by Göttsche–Soergel [GS] (stated for algebraic ones only; the general case is due to de Cataldo–Migliorini).

Theorem 5.1. *Let S be a K3 surface, not necessarily algebraic. We have an isomorphism of Hodge structures*

$$H^*(S^{[n]},\mathbb{Q})(n) \simeq \bigoplus_{\alpha\vdash n} H^*(S^{(a_1)}\times\cdots\times S^{(a_n)},\mathbb{Q})(a_1+\cdots+a_n).$$

The sum is taken over all partitions α of n, where $\alpha = (a_1,\cdots,a_n)$ satisfies $a_1\cdot 1+\cdots+a_n\cdot n = n$. Here $S^{(a)}$ denotes the a-th symmetric power $S^a/\mathfrak{S}_a$ of S, and we have an isomorphism of Hodge structures

$$H^*(S^{(a)},\mathbb{Q}) \simeq \mathrm{Sym}^a H^*(S,\mathbb{Q}).$$

Remark 5.2. We can omit all the Tate twists by considering the grading h on the cohomologies centered at the middle cohomology.

In other words, we have obtained the decomposition of $H^*(X,\mathbb{Q})$ as an $\mathfrak{m}(S)$-module. To deduce the $\mathfrak{g}$-module structure, we first lift this as a $\mathfrak{g}(S)$-module decomposition, and then apply Proposition 2.4.

Theorem 5.3. *We have an isomorphism*

$$H^*(S^{[n]},\mathbb{Q}) \simeq \bigoplus_{\alpha\vdash n}\bigotimes_{i=1}^{n} \mathrm{Sym}^{a_i} H^*(S,\mathbb{Q}) \tag{3}$$

of $\mathfrak{g}(S)$-modules. Consequently, the Weyl character of $H^*(S^{[n]},\mathbb{Q})$ as a $\mathfrak{g}(S)$-module is equal to

$$\mathrm{ch}\, H^*(S^{[n]},\mathbb{Q}) = \sum_{\alpha\vdash n}\prod_{i=1}^{n} \mathrm{ch}\,\mathrm{Sym}^{a_i} W.$$

In view of Proposition 2.4, this gives the Weyl character of $H^*(X,\mathbb{Q})$ as a $\mathfrak{g}$-module.

Proof. The $\mathfrak{g}(S)$-module structure is a diffeomorphism invariant, so we may assume that S is very general and non-algebraic. Recall that in this case, the special Mumford–Tate algebra $\mathfrak{m}(S)$ coincides with $\mathfrak{g}_0'(S) = \mathfrak{so}(H^2(S,\mathbb{Q}), q_S)$. So the isomorphism of Hodge structures gives an isomorphism of $\mathfrak{g}_0'(S)$-modules.

Since $\mathfrak{g}_0(S) = \mathfrak{g}_0'(S)\oplus\mathbb{Q}h$ and the decomposition respects the grading h, we can lift it to an isomorphism of $\mathfrak{g}_0(S)$-modules. Finally, the weight lattice of $\mathfrak{g}_0(S)$ is the same as that of $\mathfrak{g}(S)$, so this also is an isomorphism of $\mathfrak{g}(S)$-modules. ☐

Remark 5.4. Alternatively, one can prove the $\mathfrak{g}(S)$-equivariance of the isomorphism (3) using Nakajima operators [N] and the explicit description of the LLV action in the Nakajima basis given in [O]. By [dCM] the isomorphism (3) matches the Nakajima description.

Example 5.5. We consider again the $\mathrm{K3}^{[2]}$-type case. The isomorphism is given as

$$H^*(S^{[2]}, \mathbb{Q}) \simeq H^*(S^{(2)} \times S^{(0)}, \mathbb{Q}) \oplus H^*(S^{(0)} \times S^{(1)}, \mathbb{Q}) = \mathrm{Sym}^2 H^*(S, \mathbb{Q}) \oplus H^*(S, \mathbb{Q}).$$

The right hand side decomposes into 3 irreducible $\mathfrak{g}(S)$-modules, and further into 10 irreducible $\mathfrak{g}_0'(S)$-modules.

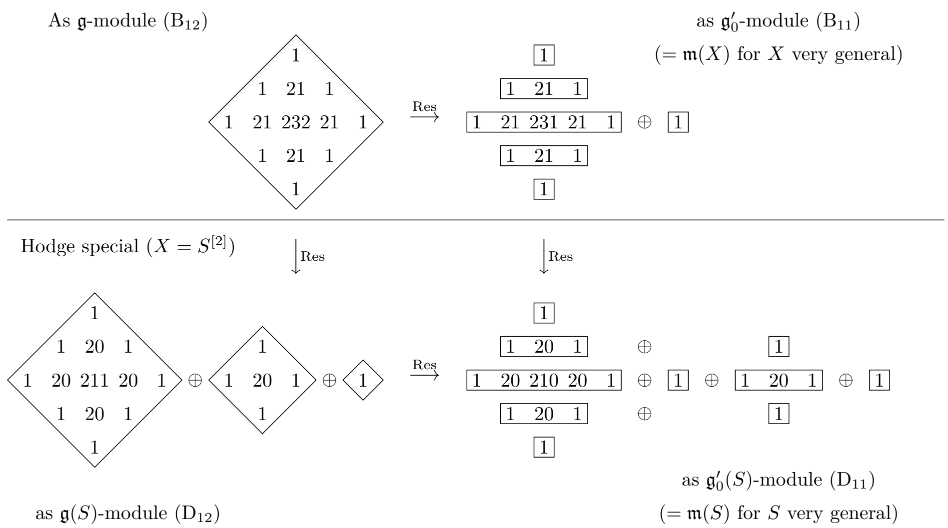

FIGURE 1. Decompositions of the Hodge diamond of $H^*(\mathrm{K3}^{[2]}, \mathbb{Q})$

We may write the formula for the characters of $H^*(\mathrm{K3}^{[n]}, \mathbb{Q})$ in a more succinct fashion by considering all Hilbert powers at the same time. Note that the LLV algebras are a priori not the same in different dimensions. But since we are considering Weyl characters, we only need the complexification $\mathfrak{g}_{\mathbb{C}}$ which is always isomorphic to $\mathfrak{so}(25)$.

Proposition 5.6 ([GKLR])**.** *Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}(25)$. The generating series of the characters of the $\mathfrak{g}$-modules $H^*(\mathrm{K3}^{[n]})$ for $n \geq 2$ is given by*

$$\sum_{n=0}^{\infty} \mathrm{ch}\, H^*(\mathrm{K3}^{[n]}) q^n = \prod_{n=1}^{\infty} \prod_{i=0}^{11} \frac{1}{(1 - x_i q^n)(1 - x_i^{-1} q^n)}. \tag{4}$$

The identity lives inside the formal power series ring $A[[q]]$ where

$$A := \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[x_0^{\pm 1}, \dots, x_{11}^{\pm 1}, (x_0 \cdots x_{11})^{\pm \frac{1}{2}}]^{\mathfrak{W}_{25}}$$

is the Weyl character ring of type B_{12}. Note that when $n = 1$, the cohomology $H^*(\mathrm{K3})$ does not admit a structure of $\mathfrak{g}$-module, so we write formally

$$\mathrm{ch}\, H^*(\mathrm{K3}^{[1]}) := \sum_{i=0}^{11} (x_i + x_i^{-1}).$$

Corollary 5.7. *Let X be a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type. Any irreducible component V_λ of the LLV decomposition of $H^*(X, \mathbb{Q})$ with highest weight $\lambda = \lambda_0 \varepsilon_0 + \cdots + \lambda_{11} \varepsilon_{11}$ satisfies*

$$\lambda_0 + \cdots + \lambda_{11} \leq n.$$

Proof. The weight λ corresponds to the monomial $x_0^{\lambda_0} \cdots x_{11}^{\lambda_{11}}$ in the character ring. When we expand the right hand side of (4) we get

$$\prod_{n=1}^{\infty} \prod_{i=0}^{11} \left(\sum_{j \geq 0} (x_i q^n)^j \right) \left(\sum_{k \geq 0} (x_i^{-1} q^n)^k \right).$$

For each term of this product, its degree in x_i is bounded by its degree in q. So each monomial that appears in the coefficient of q^n has degree $\leq n$, which gives the inequality. $\square$

Remark 5.8. More generally, for any hyperkähler manifold X of dimension $2n$ with $r = \lfloor b_2(X)/2 \rfloor$, Green–Kim–Laza–Robles [GKLR] conjecture the inequality

$$\lambda_0 + \cdots + \lambda_{r-1} + |\lambda_r| \leq n$$

for each irreducible component V_λ of the LLV decomposition of $H^*(X, \mathbb{Q})$. This conjecture holds for all known examples of hyperkähler varieties.

Remark 5.9. Once the character of the $\mathfrak{g}$-module structure is known, one can use computer algebra to recover the actual decomposition. One implementation in `Sage` can be found on the second author's webpage.

6. Generalized Kummer varieties and OG6

After having treated the $\mathrm{K3}^{[n]}$ case in the previous section and OG10 in Proposition 3.6, we briefly remark on the remaining two cases. See [GKLR] for details and references.

Generalized Kummer varieties. The LLV algebra $\mathfrak{g}$ is of type B_4.

Similar to the case of $\mathrm{K3}^{[n]}$-type, we consider Hodge special members of the family: we specialize X to an actual generalized Kummer variety associated to a very general complex torus A of dimension 2. The results of Göttsche–Soergel give a complete description of the Hodge structure of $H^*(X)$ in terms of the Hodge structures on $H^*(A)$, which can be seen as a decomposition of $\mathfrak{m}(A)$-modules (of type D_3). We can similarly lift it to a $\mathfrak{g}(A)$-module decomposition (of type D_4) and apply Proposition 2.4 to obtain the character of $H^*(X)$ as a $\mathfrak{g}$-module.

OG6. This last case is more complicated. The LLV algebra $\mathfrak{g}$ is of type D_5.

Using the Hodge numbers of OG6 and the Hodge numbers of the $\mathfrak{g}$-modules, we may obtain two possible decompositions for $H^*(X, \mathbb{Q})$. To determine which case we are in, we specialize X to a Hodge special member with an explicit geometric construction given by Rapagnetta. In this situation, the Mumford–Tate algebra $\mathfrak{m}$ is of type B_2 (that of a very general abelian surface A), and the geometric construction gives a description of the Hodge structure of $H^*(X)$ in terms of $\mathfrak{m} = \mathfrak{m}(A)$-modules (Mongardi–Rapagnetta–Saccà). Then by comparing the restrictions to $\mathfrak{m}$ of the two possible $\mathfrak{g}$-module decompositions, only one agrees with the $\mathfrak{m}$-module decomposition obtained from geometry, so we may conclude.

7. Application: The Nagai conjecture

Let $\pi : \mathcal{X} \to \Delta$ be a 1-paramer projective degeneration of hyperkähler manifolds over the disc $\Delta = \{t \in \mathbb{C} | \, |t| \leq 1\}$. We assume that the fibers $X_t = \pi^{-1}(t)$ are smooth for $t \neq 0$. Let

$$T : H^*(X_{t_0}, \mathbb{Z}) \longrightarrow H^*(X_{t_0}, \mathbb{Z})$$

be the monodromy operator for a fixed basepoint $t_0 \neq 0$. It is well-known (see e.g. [KK]) that T is quasi-unipotent, i.e. there exists $m, n > 0$ such that $(T^m - 1)^n = 0$. For any such m, we hence can define the nilpotent operator

$$N = \frac{1}{m}\log(T^m).$$

Let also $N_k = N|_{H^k(X_0,\mathbb{Z})}$ be the restriction to degree k and define its nilpotency:

$$\nu_k := \min\left\{r \in \mathbb{Z}_{\geq 1} | N_k^r \neq 0, N_k^{r+1} = 0\right\}.$$

By a classical result of Schmid we always have $\nu_k \leq k$, and the precise value of ν_k should be viewed as measuring the change of topology under the degeneration.

For hyperkähler varieties the *type* of the degeneration is determined by ν_2 according to the following table:

Type	ν_2
I	0
II	1
III	2

For example, a quartic K3 surface S degenerating to a nodal K3 is of type I, the degeneration of S to the union of two quadrics $Q_1 \cup_E Q_2$ is type II, and breaking S into the union of 4 hyperplanes is type III. More generally, Kulikov classifies all the limits of semistable degenerations of K3 surfaces according to type.

In higher dimension a priori we need to consider all the nilpotencies, the even $\nu_2, \nu_4, \ldots, \nu_{4n-2}$ and the odd ones $\nu_3, \nu_5, \ldots, \nu_{4n-3}$. However, Nagai made the following prediction for the even degeneracies (the precise behaviour of the odd remains an open question).

Conjecture 7.1 (Nagai)**.** *$\nu_{2k} = k\nu_2$.*

Using the LLV decomposition one finds the following results:

Theorem 7.2 ([KLSV, GKLR])**.** *Nagai's conjecture holds if $\nu_2 = 0$ or $\nu_2 = 2$.*

Theorem 7.3 ([GKLR])**.** *Let $H^*(X,\mathbb{Q}) = \bigoplus_\lambda V_\lambda^{m_\lambda}$ be the LLV decomposition. If*

$$\lambda_0 + \lambda_1 + |\lambda_2| \leq n$$

for all λ, then Nagai's conjecture holds.

Corollary 7.4. *Nagai's conjecture holds if $\dim X \leq 8$. It holds for all known examples of hyperkähler manifolds. It holds if the conjectural description of the LLV decomposition in Remark 5.8 is satisfied.*

We give a sketch of the proof below. The main geometric step is to relate the nilpotent matrices N_2 and N_k as follows. Let $\rho_k : \mathfrak{g}_0' \to \operatorname{End} H^k(X,\mathbb{Q})$ be the restriction of the LLV action to degree k. Moreover, since $T|_{H^2}$ preserves the Beauville–Bogomolov form, the nilpotent matrix $N_2 \in \operatorname{End} H^2(X,\mathbb{Q})$ lies in $\mathfrak{g}_0'$. Then one has:

Theorem 7.5 (Soldatenkov)**.** *$N_k = \rho_k(N_2)$*

From this, the cases $\nu_2 \in \{0, 2\}$ are fairly straightforward and follow essentially from the description of the Verbitsky component. The critical range of the Nagai conjecture is $\nu_2 = 1$.

8. Sketch of proof for Theorems 7.3 and 7.5

We first give a sketch of the result of Soldatenkov following [GKLR].

Sketch of proof for Theorem 7.5. We divide the proof into two steps.

Step 1. Let $\mathcal{X}/S$ be a fixed degeneration of hyperkähler manifolds over a smooth base S, let $t_0 \in S$ be a base point, and let $\widetilde{S} \to S$ be the universal cover. Define the (extended) period domains parametrizing Hodge structures in degree $k > 2$ and 2 respectively:

$$\widehat{D}_k = \mathrm{Flag}(H^k(X_{t_0}, \mathbb{C}), f^\bullet),$$
$$\widehat{D}_2 = \mathbb{P}(H^2(X_{t_0}, \mathbb{C})).$$

Here $f^\bullet$ is the dimension vector of the Hodge filtration.

Proposition 8.1. *There exists a canonical morphism $\psi_k : \widehat{D}_2 \to \widehat{D}_k$ such that the diagram*

$$\begin{tikzcd} \widetilde{S} \arrow[rr, "\widetilde{\Phi}_2"] \arrow[dr, "\widetilde{\Phi}_k"'] & & \widehat{D}_2 \arrow[dl, "\psi_k"] \\ & \widehat{D}_k & \end{tikzcd} \tag{5}$$

commutes, where $\widetilde{\Phi}_k$ is the period mapping $t \mapsto \mathsf{pt}_{t_0,t}(F^\bullet H^k(X_t))$.

Here $\mathsf{pt}_{t_0,t}$ is obtained from the parallel transport map $H^*(X_t, \mathbb{Z}) \to H^*(X_{t_0}, \mathbb{Z})$ along any path from t to t_0 by tensoring with $\mathbb{C}$, or equivalently it is the parallel transport with respect to the Gauss–Manin connection. It is well-defined since $\widetilde{S}$ is simply connected.

Proof (Sketch). Recall from Proposition 3.1 that the Weil operator $f \in \mathrm{End}(H^*(X, \mathbb{R}))$ defined by $f|_{H^{p,q}(X)} = i(q-p)\,\mathrm{Id}$ lies in $(\mathfrak{g}_0')_\mathbb{R}$ and hence satisfies

$$f|_{H^k} = \rho_k(f|_{H^2}). \tag{6}$$

This motivates the following:

Construction of ψ_k: Given $\mathfrak{o}_2 \in \widehat{D}_2$, there exists a unique semisimple $f_{\mathfrak{o}_2} \in (\mathfrak{g}_0')_\mathbb{R}$ such that $f_{\mathfrak{o}_2}$ induces $\mathfrak{o}_2$, i.e.

$$f_{\mathfrak{o}_2}(x) = (q-p)ix \text{ for all } x \in H^{p,q}_{\mathfrak{o}_2}(X).$$

Define $\psi_k(\mathfrak{o}_2)$ to be the filtration induced by $\rho_k(\mathfrak{o}_2)$.

Proof of Commutativity of (5)**:** Let $\psi_k^t : \mathbb{P}(H^2(X_t, \mathbb{C})) \to \widehat{D}_k^t = \mathrm{Flag}(H^k(X_{t_0}, \mathbb{C}), f^\bullet)$ be the map ψ_k above with respect to the base point t. If $\mathfrak{o}_2(t), \mathfrak{o}_k(t)$ are the elements determined by the Hodge structure of $H^*(X_t)$ then by construction of ψ_k^t and (6) we have that

$$\psi_k^t(\mathfrak{o}_2(t)) = \mathfrak{o}_k(t).$$

Parallel transport naturally intertwines the LLV algebra, that is

$$\mathsf{pt}_{t_0,t} \circ \rho_k(\alpha) = \rho_k(\mathsf{pt}_{t_0,t}(\alpha)) \circ \mathsf{pt}_{t_0,t}, \quad \text{hence} \quad \psi_k^{t_0} \circ \mathsf{pt}_{t_0,t} = \mathsf{pt}_{t_0,t} \circ \psi_k^t$$

We conclude:

$$\begin{aligned} \psi_k\left(\widetilde{\Phi}_2(t)\right) &= \psi_k^{t_0}(\mathsf{pt}_{t_0,t}(\mathfrak{o}_2(t))) \\ &= \mathsf{pt}_{t_0,t}(\psi_k^t \mathfrak{o}_2(t)) \\ &= \mathsf{pt}_{t_0,t}(\mathfrak{o}_k(t)) \\ &= \widetilde{\Phi}_k(t). \end{aligned}$$

□

Step 2. We prove Theorem 7.5. For that we require the degeneration $\mathcal{X}/S$ to be projective. We let $D_2 \subset \widehat{D}_2$ and $D_k \subset \widehat{D}_k$ be the period domains with respect to the choosen polarization, that is we define

$$D_2 = \{x \in \mathbb{P}(H^2_{\mathrm{prim}}(X_{t_0}, \mathbb{C}))|x \cdot x = 0, x \cdot \overline{x}\}$$

and we let

$$D_k \subset \mathrm{Flag}(H^k_{\mathrm{prim}}(X_{t_0}, \mathbb{C}))$$

be the orbit of $\mathfrak{o}_k(t_0)$ under $\mathfrak{mt}_{k,\mathbb{R}}$, where $\mathfrak{mt}_k$ is the generic special Mumford–Tate algebra of $H^k_{\mathrm{prim}}(X_{t_0}, \mathbb{C})$ (i.e. the special Mumford–Tate algebra $\mathfrak{m}(H^k_{\mathrm{prim}}(X', \mathbb{C}))_{\mathbb{R}}$ of a generic projective deformation X' of X_{t_0}, see [GKLR]). For the projective degeneration, the period mappings $\widetilde{\Phi}_k$ take values in D_k; we write Φ_k in this case. The key point is now:

- $\mathfrak{mt}_k = \rho_k(\mathfrak{mt}_2) = \rho_k(\mathfrak{g}_0')$ and hence ψ_k defines a morphism $D_2 \to D_k$;
- $D_k \cong (\mathfrak{mt}_{k,\mathbb{R}} \cdot \mathfrak{o}_k(t_0))/K$ where K is a compact subgroup (see e.g. [KK]).

It follows that if $T_k = T|_{H^k(X_{t_0},\mathbb{Z})}$ is the monodromy along the loop γ, then in D_k we have

$$\begin{aligned} T_k^m \Phi_k(t_0) &= \Phi_k(\gamma^m \cdot t_0) \\ &= \psi_k(\Phi_2(\gamma^m \cdot t_0) \\ &= \psi_k(T_2^m \Phi_2(t_0)) \\ &= \rho_k(T_2)^m \psi_k(\Phi_2(t_0)) \\ &= \rho_k(T_2)^m \Phi_k(t_0) \end{aligned}$$

for all $m \geq 1$ hence

$$T_k^{-m} \rho_k(T_2)^m \in K \cap \mathrm{GL}(H^k_{\mathrm{prim}}(X, \mathbb{Z}))$$

Since the intersection on the right is a finite subgroup, there exists m_1, m_2 such that $T_k^{-m_1} \rho_k(T_2)^{m_1} = T_k^{-m_2} \rho_k(T_2)^{m_2}$, and hence $T_k^{m_1 - m_2} = \rho_k(T_2)^{m_1 - m_2}$. Taking log on both sides, we get the desired equality $N_k = \rho_k(N_2)$.

We are ready to give an idea of the proofs of Theorem 7.2 and 7.3.

Sketch of proof for Theorems 7.2 and 7.3. We first state the following lemma.

Lemma 8.2. *If a nilpotent operator $N : V \to V$ has nilpotency ν, then $\mathrm{Sym}^k(N) : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(V)$ has nilpotency $k \cdot \nu$.*

Proof. Since $N^{\nu+1}$ is zero, for all elements $x_1, \ldots, x_k \in V$ we have $\mathrm{Sym}^k(N)^{k\nu}(x_1 \cdots x_k) = (N^\nu x_1) \cdots (N^\nu x_k)$ and $\mathrm{Sym}^k(N)^{k\nu+1}(x_1 \cdots x_k) = 0$. On the other hand, since N^ν is not zero, we may pick $x \in V$ such that $N^\nu x \neq 0$, and therefore $\mathrm{Sym}^k(N)^{k\nu}(x^k) = (N^\nu x)^k \neq 0$. $\square$

For each $k \geq 1$, $H^{2k}(X, \mathbb{Q})$ contains the Verbitsky component $\mathrm{Sym}^k H^2(X, \mathbb{Q})$ as a sub $\mathfrak{g}_0'$-module. Hence the nilpotency ν_{2k} of $N_k = \rho_k(N_2)$ is bounded below by the nilpotency of $\mathrm{Sym}^k N_2$, which is equal to $k \cdot \nu_2$ by the lemma. Therefore we obtain $\nu_{2k} \geq k \cdot \nu_2$.

- If $\nu_2 = 0$, then $N_2 = 0$, so $N_k = \rho_k(N_2) = 0$ is identically zero, and $\nu_{2k} = 0$.
- If $\nu_2 = 2$, then we already have the classical bound $\nu_{2k} \leq 2k$ so we may conclude that $\nu_{2k} = 2k$.
- If $\nu_2 = 1$, then we have $\nu_{2k} \geq k$. On the other hand, the nilpotency of the action of N_2 on V_λ can be bounded above by the weights, see [GKLR]. In particular, the nilpotency of $N_k = \rho_k(N_2)$ is bounded by all the weights λ that appear in the $\mathfrak{g}_0'$-decomposition of $H^k(X_{t_0}, \mathbb{Q})$. Using the branching rules this can be expressed by the weights of the LLV decomposition of $H^*(X_{t_0}, \mathbb{Z})$. It is fortunate that this yields the rather simple weight bound of the claim.

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