Hodge line bundle

Disclaimer:

- Only work with open moduli spaces M_g, A_g, F_g (not $\overline{M}_g, \overline{A}_g, \overline{F}_g^*$)
- Pretend all moduli are smooth & fine.

Goal:

- Motivation: M_g, A_g
- "Small tautological ring" $\simeq \mathbb{Q}[x]/(x^{18})$ (vd Geer – Katsura)

§1 M_g : $\pi : C \to M_g$, $\dim M_g = 3g-3$, $K := c_1(\omega_\pi)$

$\kappa_i := \pi_*(K^{i+1})$, $\lambda_i := c_i(\mathbb{E} := \pi_* \omega_\pi) \in A^i(M_g)$ ($\mathbb{Q}$-coeff)

E.g. $\kappa_0 = (2g-2)[M_g]$, rk $\mathbb{E} = g$, $\lambda_{i>g} = 0$

Note: - GV-duality: $R\pi_* \omega_\pi[1] = (R\pi_* \mathcal{O})^*$, $R^1\pi_*\omega_\pi = \mathcal{O}$, $\mathbb{E}^\vee = R^1\pi_*\mathcal{O}$

- Harer: $\mathrm{Pic}(M_g)_\mathbb{Q} = \mathbb{Q}\lambda_1$; - Franchetta: $\mathrm{Pic}(C)_\mathbb{Q} = \mathrm{Pic}(M_g)_\mathbb{Q} \oplus \mathbb{Q}\omega_\pi$

3 tricks: (i) GRR ; (ii) $c_{i>n} = 0$, (iii) Hodge theory

(i) $\mathrm{ch}(\pi_*(\ )) = \pi_*(\mathrm{ch}(\ )\,\mathrm{td}(T_\pi))$ $\mathrm{td}\, T_\pi = \dfrac{-K}{1-e^K}$

Apply to ω_π: $\Rightarrow \lambda_i \in \mathbb{Q}[\kappa_0, \kappa_1, \dots] =: R(M_g) \subset A^*(M_g)$

$$\begin{array}{ccc} R(M_g) & \subset & A^*(M_g) \\ \Downarrow \ \simeq ? & & \\ R^{H}(M_g) & \subset & H^*(M_g) \end{array}$$

Eg: $\lambda_1 = \frac{1}{12}\kappa_1$, $\lambda_2 = \frac{1}{288}\kappa_1^2$

Also $\mathrm{ch}_{2i}(\mathbb{E}) = 0$
$\quad\quad {}_{i>0}$

Apply to $\omega_\pi^\times \Rightarrow c_1(\mathbb{J}_{M_g} = R^1\pi_* \omega_\pi^\times) \in R(M_g)$

e.g. $c_1(\omega_{M_g}) = 13\lambda_1$ cycle   Q: M_g gen. type? de $g \geq 24, 22$

(ii) $c_{i>rk} = 0$ Apply to $Q := \ker(\pi^* \mathbb{E} \twoheadrightarrow \omega_\pi)$, $rk\, Q = g-1$

$\Rightarrow c(Q) = (1+K)^{-1} \cdot \pi^* c(\mathbb{E}) = (1 - K + K^2 \pm \dots)\, \pi^*(1 + \lambda_1 + \lambda_2 + \dots)$

$0 = \pm c_{g+2}(Q) = K^{g+2} - K^{g+1}\pi^*\lambda_1 + \dots \quad + (-1)^g \pi^* \lambda_{g+1}$

π_* $0 = \kappa_{g+1} - \kappa_g \lambda_1 \pm \dots \quad + (-1)^{g-1} \lambda_{g+1}$

$\Rightarrow \kappa_{g+1} \in \mathbb{Q}[\kappa_0 \dots \kappa_g, \lambda_1 \dots \underbrace{\lambda_{g+1}}_{=0}]$ $rk\, \mathbb{E} = g$

$R(M_g) = \mathbb{Q}[\kappa_0 \dots \kappa_g]$

$\lambda_g = \lambda_{g-1} = 0 \Rightarrow \quad = \mathbb{Q}[\kappa_0 \dots \kappa_{g-2}]$ (Mumford)

(iii) Hodge theory $\quad 0 \to \underbrace{\pi_* \omega_\pi}_{\cong \mathbb{E}} \to R^1\pi_* \mathbb{C} \otimes \mathcal{O} \to \underbrace{R^1\pi_* \mathcal{O}}_{\cong \mathbb{E}^\vee} \to 0$

$$\Rightarrow \quad c(\mathbb{E}) \cdot c(\mathbb{E}^\vee) = c(R^1\pi_* \mathbb{C} \otimes \mathcal{O}) = 1 \qquad \underline{\text{in } H^*}$$

$$ch(\mathbb{E}) + ch(\mathbb{E}^\vee) = 2g \qquad (\text{also in } A^*)$$

$$\Rightarrow \quad ch_{2i>0}(\mathbb{E}) = 0$$

Further results: • Morita, Ionel: $R(M_g) = \mathbb{Q}[\kappa_0 \ldots \kappa_{\lfloor g/3 \rfloor}]$

• Boldsen: No relations in degree $\leq g/2$

• Mumford Conj: $\lim_{g\to\infty} R^*(M_g) = H^{2*}(M_g)$ ✓

• Faber Conj: $R^k \times R^{g-2-k} \xrightarrow[\text{perfect}]{} R^{g-2} \underset{\checkmark}{\cong} \mathbb{Q} \quad (\lambda_1^{g-2} \neq 0 \neq \kappa_{g-2})$

Looijenga: $R^k = 0 \quad k > g-2$

OK: $g \leq 23$ $\qquad$ Not compatible with Pixton $g \geq 24$

§ A_g: $\quad (Y,\mathcal{L}) \xrightarrow[\pi]{} A_g$, $\quad \dim A_g = \frac{1}{2} g(g+1)$

· $\omega_\pi \in \pi^* \mathrm{Pic}(A_g)\otimes\mathbb{Q} \quad \Rightarrow \quad \pi_*(\kappa^{i+g}) = 0$

· $\Omega_\pi = \pi^*(\mathbb{E} := \pi_*\Omega_\pi)$, $\quad \mathrm{rk}\,\mathbb{E} = g$, $\quad \lambda_i = c_i(\mathbb{E})$

<u>3 tricks</u> (1) GRR

Apply to $\mathcal{O}$, Ω_π: $\quad \mathrm{ch}(R\pi_*\mathcal{O}) = \pi_*(\mathrm{td}\,T_\pi) = \pi_*(\pi^*\mathrm{td}\,\mathbb{E}^\vee)) = 0 \quad$ ⊛

$$\mathrm{ch}(R\pi_*\Omega_\pi) = 0$$

& $R^1\pi_*\mathcal{O} \cong \mathbb{E}^\vee$ (use pol.) & $R^i\pi_*\mathcal{O} \cong \Lambda^i R^1\pi_*\mathcal{O} \cong \Lambda^i\mathbb{E}^\vee$

$$0 = \sum(-1)^i \mathrm{ch}(\Lambda^i\mathbb{E}^\vee) \underset{[BS]}{=} \underbrace{c_g(\mathbb{E}^\vee)}_{\lambda_g}\cdot\mathrm{td}(\mathbb{E}^\vee)^{-1} \Rightarrow \lambda_g = 0$$

Apply to pol. $\mathcal{L}$ $\quad \mathrm{ch}(\pi_*\mathcal{L}^n) = \pi_*(\exp(n\Theta)\mathrm{td}(\mathbb{E}^\vee)) \qquad \Theta = c_1(\mathcal{L})$

$$= \frac{n^g}{g!}\pi_*(\Theta^g)\mathrm{td}(\mathbb{E}^\vee) + n^{g+1}(\underbrace{\qquad\qquad}_{=0})$$

$n=1 \quad \Rightarrow \quad \underbrace{c_1(\pi_*\mathcal{L})}_{\in \mathrm{Pic}} = \mathrm{td}(\mathbb{E}^\vee)_1 = -\lambda_1/2$

$$\exp(\quad) = \overbrace{\pi_*(\Theta^g)}^{=1}\mathrm{td}(\mathbb{E}^\vee) \quad (\Rightarrow \lambda_i \in \mathbb{Q}[\lambda_1])$$

Note: $\left.\begin{array}{l} 0 \to \underset{\cong S^2(\mathbb{E}^\vee)}{T_{A_g}} \to \underset{\cong \mathbb{E}^\vee \otimes \mathbb{E}^\vee}{R^1\pi_* T_\pi} \to \underset{\cong \Lambda^2 \mathbb{E}^\vee}{R^2\pi_* \mathcal{O}} \to 0 \end{array}\right\} c_i(A_g) \in \mathbb{Q}[\lambda_0, \lambda_1 \ldots]$

e.g. $c_1(\omega_{A_g}) = +\lambda_1 \quad \rightsquigarrow \quad \underline{Q}$. A_g of general type? Ok $g \geq 7$
(open $g = 6$)

(ii) $c_{i > n} = 0$??

(iii) Hodge theory: $0 \to \mathbb{E} \to R^1\pi_* \mathbb{C} \otimes \mathcal{O} \to \mathbb{E}^\vee \to 0$

$\Rightarrow c(\mathbb{E}) \cdot c(\mathbb{E}^\vee) = 1$, i.e. $(1 + \lambda_1 + \lambda_2 \ldots) \cdot (1 - \lambda_1 + \lambda_2 \ldots) = 1$

$ch(\mathbb{E}) + ch(\mathbb{E}^\vee) = 2g$, eg. $ch_{2i>0}(\mathbb{E}) = 0$ in H^* and A^*

Special case: $g = 2$, $\lambda_2 = 0$, $\lambda_1^2 = 0$

Tautological ring $R(A_g) := \mathbb{Q}[\lambda_1, \lambda_2 \ldots, \lambda_{g-1}]$

$\uparrow$ "$\simeq$" vd Geer

$\mathbb{Q}[x_1 \ldots x_{g-1}] / (1 + x_1 + \ldots)(1 - x_1 + x_2 \ldots)$

§3 F_g : $\pi : (S, \mathcal{L}) \longrightarrow F_g$, $\dim F_g = 19$, $\omega_\pi \in \pi^* \mathrm{Pic}(F_g)$

$\Rightarrow \quad \pi_*(K^{i+2}) = 0 \qquad K = c_1(\omega_\pi) = \pi^* \lambda$

- $\Omega_\pi \neq \pi^*(\quad) \rightsquigarrow c_1(\Omega_\pi), c_2(\Omega_\pi)$, $c_1(\Omega_\pi) = c_1(\omega_\pi) = \pi^* \lambda$

$\rightsquigarrow \qquad \pi_*(c_1^i \cdot c_2^{j+1}) \in A^{i+2j}(F_g) \qquad \lambda = c_1(\pi_* \omega_\pi)$

$= \lambda^i \underbrace{\pi_*(c_2^{j+1})}_{=: \kappa_j \in A^{2j}(F_g)}$ (Next week: More general $\kappa_{i,j}$)

- $\mathbb{E} := R^1 \pi_* \Omega_\pi$, $rg = 20$, $\mathbb{E} \simeq \mathbb{E}^\vee$, $\lambda_i := c_i(\mathbb{E}) \quad = 0 \; i > 20$ or odd
- $\pi_* \Omega_\pi = 0$, $R^2 \pi_* \Omega_\pi = 0$, $R^1 \pi_* \mathcal{O} = 0$, $R^2 \pi_* \mathcal{O} \simeq (\pi_* \omega_\pi)^\vee$

- $\pi_*(\mathcal{L}^n) = ?$

3 tricks : (i) GRR

Apply to $\mathbb{E}$ $\quad ch(\mathbb{E}) = -ch(R\pi_* \Omega_\pi) = \pi_*(ch(\Omega_\pi)\, td(T_\pi))$

$\Rightarrow \quad \lambda_i \in \mathbb{Q}[\lambda, \kappa_j]$

Apply to G: $1 + \exp(-\lambda) = \mathrm{ch}(R\pi_* G) = \pi_*(\mathrm{td}\, T_\pi)$

$\Rightarrow \quad -\lambda = -\lambda \dfrac{\pi_* c_2}{24} = -\lambda \quad \checkmark$

$$\pi_*(\mathrm{td}\, T_\pi) = 1 + \frac{1}{2}c_1 + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24}$$

(Pappas: Moduli spaces of Enriques is affine)

$$\frac{\lambda^2}{2} = \frac{4}{720}\lambda^2 \pi_* c_2 + \frac{3}{720}\kappa_1 \ \dots$$

$\Rightarrow \quad \kappa_i \in \mathbb{Q}[\lambda]$ = "small tautological ring"

(ii) $c_{i>m} = 0$ Q: Are there interesting bundles of rk < 19 on F_g? (< 21 on $\mathcal{F}$)

(iii) Hodge theory $[R^2\pi_* \mathbb{C} \otimes G] = [R^1\pi_* \Omega_\pi] + [R^2\pi_* \mathcal{O}] + [\pi_* \omega_\pi]$

$R^1\pi_*\Omega_\pi = \mathbb{E}$, $R^2\pi_*\mathcal{O} = (\pi_*\omega_\pi)^\vee$, $\lambda = c_1(\pi_*\omega_\pi)$

$\Rightarrow \quad \mathrm{ch}(\mathbb{E}) = \underbrace{\mathrm{ch}(R^2\pi_* \mathbb{C} \otimes G)}_{} - \exp(-\lambda) - \exp(\lambda)$

in $H^*(F_g)$ $= 22$

$\mathrm{ch}_{2i+1}(\mathbb{E}) = 0$

$\mathrm{ch}_{2i}(\mathbb{E}) = \dfrac{-2\lambda^{2i}}{(2i)!}$

Q: How is A^*?

$$0 \to \mathbb{J}_{\mathcal{F}_g} \to R^1\pi_* \mathbb{J}_\pi \to R^2\pi_* \mathcal{O} \to 0 \qquad \text{ses}$$

$$\cong \mathbb{E} \otimes (\pi_* \omega_\pi)^* \qquad \Lambda^2 (\pi_* \omega_\pi)^*$$

$$\mathrm{ch}(\mathbb{J}_{\mathcal{F}_g}) = \exp(-\lambda)\,(\mathrm{ch}(\mathbb{E}) - 1) = \exp(-\lambda)\left(19 - 2\left(\frac{\lambda^2}{2} + \frac{\lambda^4}{4!} + \dots\right)\right)$$

$$\Rightarrow \; c_i(\mathbb{J}_{\mathcal{F}_g}) \in \mathbb{Q}[\lambda], \quad c_1(\omega_{\mathcal{F}_g}) = 19\lambda \quad \text{ample}$$

O'Grady, v.d. Geer

$\rightsquigarrow$ $\underline{Q}$: General type? — Ok $g \geq 54$

Polarization: $\mathrm{ch}(\pi_* \mathcal{L}^n) = \pi_*(\exp(n\ell) \cdot \mathrm{td}(\mathbb{J}_\pi))$ $\qquad \ell = c_1(\mathcal{L})$

$$= \underbrace{\pi_*(\mathrm{td}\, \mathbb{J}_\pi)}_{1+\exp(-\lambda)} + n\, \pi_*(\ell \cdot \mathrm{td}\, \mathbb{J}_\pi) + \frac{n^2}{2} \pi_*(\ell^2 \,\mathrm{td}(\mathbb{J}_\pi))$$

$$+\; n^3 \big(\underbrace{\qquad\qquad}_{=0} \big)$$

$$\mathrm{ch}(\underbrace{\pi_*\mathcal{L}}_{\text{Verlinde bdl}}) = 1 + \exp(-\lambda) + \pi_*(\underline{\ell\, \mathrm{td}\, \mathbb{J}_\pi}) + \frac{1}{2}\pi_*(\underline{\ell^2\, \mathrm{td}\, \mathbb{J}_\pi})$$

$\rightsquigarrow$ Chow $\varkappa_{ij}$

$\underline{Q}$: $\overset{?}{\Longrightarrow}$ $\pi_*\mathcal{L}$ not projflat

<u>Prop</u> (vd Geer - Katsura)

(i) $\lambda^{17} \neq 0$, (ii) $\lambda^{18} = 0$ $\left(\rightsquigarrow \mathbb{Q}[x]/_{x^{18}} \cong \text{subalg. gen} \subset A(F_g) \right)$

Q: $F_g^\circ \subset F_g$ moduli of pol. $\lambda^{17}_{|F_g^\circ} \neq 0$? <u>Q</u>: $F_g^\circ \subset$ affine ?

q-pol.

<u>Corollary</u>: $Z \subset F_g$ complete $\Rightarrow \dim Z \leq 17$

<u>Proof</u> (i) $F_g \subset F_g^*$ Borel comp. $\dim \partial F_g \leq 1$

proj $\Rightarrow \exists Z \subset F_g$ proj, $\dim Z = 17$

$\Rightarrow \int_Z \lambda^{17} \neq 0 \Rightarrow \lambda^{17} \neq 0$

(ii) General: $\underline{\Lambda}$ lattice, sign $\underline{\Lambda} = (2, r)$, $\Gamma \subset O(\underline{\Lambda})$ finite index

$\rightsquigarrow D \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ period domain , $\dim D = r$

D/Γ q-proj. <u>Claim</u> $\lambda^{r-1} = 0$ (K3: $r = 19$)

Hodge lb $\hat{=} \mathcal{O}(-1)$ on $\mathbb{P}(\Lambda \otimes \mathbb{C})$

Observe: Independent of Γ

and of Λ

$$\begin{tikzcd}[column sep=small] & D/\Gamma_1 \cap \Gamma_2 \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ D/\Gamma_1 & & D/\Gamma_2 \end{tikzcd}$$

Only: $\Delta \otimes \mathbb{Q}$ matters

Induction over r: $\Lambda' \subset \Lambda \rightsquigarrow D'/\Gamma' \xrightarrow{f} D/\Gamma$

$(2, r-1) \quad (2, r)$ image is NL divisor

$$\lambda'^{\,r-2} = 0 \Rightarrow \lambda^{r-2} \cdot [D'/\Gamma'] = 0 \quad \forall \Lambda' < \Lambda$$

$\lambda' = f^* \lambda$ Write $\lambda = \sum u_i [D^i/\Gamma^i]$ (Hodge bundle is lin. comb of NL divisors.)

$$\Rightarrow \lambda^{r-1} = \lambda^{r-2} \left(\sum u_i [\quad] \right) = 0$$

Induction hypothesis: $r = 3$

A_2: $(A_1, \Theta_1) \cong (A_2, \Theta_2) \Leftrightarrow H^2(A_1, \mathbb{Z}) \cong H^2(A_2, \mathbb{Z})$ Hodge isom.

GT $\quad \Theta_1 \longmapsto \Theta_2$

$\Rightarrow \quad A_2 \simeq D/\Gamma \qquad D \subset \mathbb{P}(\Lambda \otimes \mathbb{C}) \qquad \Lambda = H^2(A, \mathbb{Z})_{pr}$

sign $= (2, 3)$

Compare λ on D/Γ to λ_1 on A_2

At $[A] \in A_2$: $\quad \lambda$ has fiber $H^{2,0}(A)$

$\lambda_1 = c_1(\mathbb{E})$, fiber of $\mathbb{E}$ at $[A]$: $H^{1,0}(A)$

$$H^{2,0}(A) \simeq \Lambda^2 H^{1,0}(A)$$

$$\Rightarrow \quad \lambda = \lambda_1$$

A_2:

$$\lambda_1^2 = 0 \quad \Rightarrow \quad \lambda^2 = 0$$

Corollary (Petersen): $\quad NL \subset H^*(F_g)$ only in col.

$= 0$ in degree $k > 34$