A reconstruction theorem for varieties

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1 Introduction

In general, the underlying topological space $|X|$ of a variety carries little information about X. For example, any bijection between irreducible curves that preserves the generic point is a homeomorphism between the underlying Zariski topologies. In higher dimensions, the situation is less dramatic. For instance, observe that one can distinguish the birationally equivalent varieties $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ by looking at intersections of irreducible closed subsets.

It is natural to ask what additional structures suffice to tell apart varieties with homeomorphic underlying topological spaces. Theorems of Torelli type (cf. [BKT10] and [Zil12]) state that, under certain conditions, adding the datum of the Jacobian to the underlying topological space of a curve C determines C up to isomorphism. Based on [KLOS21], the aim of this thesis is to present a proof of the following theorem:

Theorem 1.1 ([KLOS21, Main Theorem]). *Let X be a proper normal integral variety of dimension at least two over an algebraically closed field. Then X is determined as a scheme by the pair*

$$(|X|,\ c\colon X^{(1)} \to \mathrm{Cl}(X)),$$

where $X^{(1)}$ is the set of codimension one points of X and c is the map sending a codimension one point of X to its class in the divisor class group.

A more precise formulation of this theorem is given in Section 3.2.

The key ingredient for proving Theorem 1.1 is a variant of the Fundamental Theorem of Projective Geometry (cf. [Art57, Thm. 2.26]), which roughly states that a bijection between projective spaces that maps lines to lines, lifts to an isomorphism of the underlying vector spaces. This is discussed in Section 2. Section 3 begins with a review of the theory of divisors on normal varieties. In particular, the notion of *divisorial structures* will be introduced, allowing us to give a more precise formulation of Theorem 1.1. Afterwards, we give arguments to reduce the proof to the case of locally-factorial (quasi-)projective varieties. In this situation, we see that linear systems associated with very ample invertible sheaves are projective spaces that satisfy the assumptions of the variant Fundamental Theorem of Projective Geometry. In Section 4, these findings are then combined to finally prove Theorem 1.1. We conclude the section by briefly sketching how to extend the results to varieties over non-algebraically closed fields.

In Section 5, we discuss examples and stronger results related to Theorem 1.1. At first, we consider the case of one-dimensional varieties, i.e., curves. After presenting a counterexample to Theorem 1.1 in the case of curves over non-algebraically closed fields, we apply a theorem of Torelli type [Zil12] to obtain a variant of Theorem 1.1 for curves over algebraically closed fields.

One might raise the question, whether, under the assumptions of Theorem 1.1, the topological space $|X|$ alone suffices to recover the variety X up to isomorphism of schemes. It turns out that the answer to this question depends on the characteristic of the ground field. We present counterexamples in positive characteristic. In particular, we generalize an example given in [KLOS21], which makes use of elliptic curves, to hyperelliptic curves of arbitrary genus. On the other hand, J. Kollár [Kol20] has shown that, in characteristic zero, the answer to the question above is indeed positive. Recall that smooth varieties over $\mathbb{C}$ may be endowed with a so-called *analytic topology*, realizing them as complex manifolds. We conclude this thesis by a brief comparison of the analytic and Zariski topologies.

Deutsche Zusammenfassung

Das Hauptziel dieser Arbeit ist, einen Beweis des folgenden Theorems zu präsentieren:

Theorem ([KLOS21])**.** *Sei X eine eigentliche, normale, integrale Varietät von Dimension größer gleich zwei über einem algebraisch abgeschlossenen Körper. Dann ist X als Schema eindeutig bestimmt durch das Paar*

$$(|X|, c\colon X^{(1)} \to \mathrm{Cl}(X)),$$

wobei $X^{(1)}$ die Menge der Punkte in Kodimension eins und c die Abbildung ist, die die Punkte in Kodimension eins auf die zugehörige Klasse in der Klassengruppe von X schickt.

Eine präzisere Formulierung des Theorems findet sich in Abschnitt 3.2.

Der Hauptbestandteil des Beweises ist eine Variante des Hauptsatzes der projektiven Geometrie [Art57, Thm. 2.26], der in seiner klassischen Form besagt, dass jede Bijektion zwischen projektiven Räumen, die Geraden auf Geraden schickt, auf eine lineare Abbildung zwischen den zugrundeliegenden Vektorräumen zurückzuführen ist. Dies wird in Abschnitt 2 diskutiert. Abschnitt 3 beginnt mit einer Rekapitulation der Theorie von Divisoren auf normalen Varietäten. Es werden insbesondere sogenannte *divisorielle Strukturen* eingeführt, die eine präzisere Formulierung des Haupttheorems ermöglichen. Danach liefern wir Argumente, um den Beweis des Theorems auf den Fall von lokal faktoriellen, (quasi-)projektiven Varietäten zu reduzieren. In dieser Situation zeigen wir nun, dass die linearen Systeme, die mit sehr amplen invertierbaren Garben assoziiert sind, projektive Räume sind, welche die Voraussetzungen der Variante des Hauptsatzes der projektiven Geometrie erfüllen. Diese Beobachtungen werden dann in Abschnitt 4 zu einem Beweis des Haupttheorems zusammengeführt.

In Abschnitt 5 werden Beispiele und Verstärkungen des Theorems erörtert. Nach der Präsentation eines Gegenbeispiels im Fall von Kurven über nicht algebraisch abgeschlossenen Körpern, verwenden wir einen Satz des Torelli-Typs [Zil12], um eine Variante des obigen Theorems für Kurven über algebraisch abgeschlossenen Körper zu erhalten.

Man kann sich die Frage stellen, ob unter den Voraussetzungen des Haupttheorems schon allein der topologische Raum $|X|$ genügt, um die Varietät X bis auf Schema-Isomorphie zu bestimmen. Es stellt sich heraus, dass die Antwort von der Charakteristik des Grundkörpers abhängt. Wir präsentieren mehrere Gegenbeispiele in positiver Charakteristik. Insbesondere verallgemeinern wir ein Beispiel aus [KLOS21], welches elliptische Kurven verwendet, auf hyperelliptische Kurven beliebigen Geschlechts. Andererseits hat J. Kollár in [Kol20] gezeigt, dass die Antwort auf die obige Frage positiv ist, wenn man Körper mit der Charakteristik Null betrachtet. Dieses stärkere Resultat nutzen wir, um Beispiele von glatten projektiven Varietäten über $\mathbb{C}$ anzugeben, die diffeomorph als reelle differenzierbare Mannigfaltigkeiten sind, deren zugrundeliegenden Zariski-Topologien jedoch nicht homöomorph sind.

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2 The Fundamental Theorem of Projective Geometry

Let V and W be vector spaces over a field. In general, a bijection $\varphi\colon \mathbb{P}(V) \to \mathbb{P}(W)$ does not yield an isomorphism $V \cong W$ of the underlying vector spaces. However, the *Fundamental Theorem of Projective Geometry* asserts that if φ preserves lines, one can indeed construct an isomorphism $V \cong W$ compatible with the map φ. In this section, we establish a generic variant of this theorem, which is then used in the reconstruction of varieties as described in Sections 3 and 4. Similar methods are used in [BT08] to reconstruct function fields of surfaces over the algebraic closures of finite fields.

Let us first recall some basic definitions.

Definition 2.1. Let V be a vector space over a field k. The *projective space* associated to V is the quotient

$$\mathbb{P}(V) \coloneqq (V \setminus \{0\})/k^\times.$$

If V is finite-dimensional, then the *dimension* of $\mathbb{P}(V)$ is $\dim_k V - 1$. A projective space of dimension one is called a *line*. If $W \subseteq V$ is a linear subspace, then there is a natural inclusion $\mathbb{P}(W) \subseteq \mathbb{P}(V)$ of projective spaces. The subsets of $\mathbb{P}(V)$ arising in this way are called *(linear) subspaces*.

The following definition generalizes the notion of linear maps between vector spaces.

Definition 2.2. Let $\sigma\colon k_1 \to k_2$ be an isomorphism of fields. Let V_1 (resp. V_2) be a vector space over k_1 (resp. k_2). A map $\gamma\colon V_1 \to V_2$ is called *σ-linear* if γ satisfies

$$\gamma(v + w) = \gamma(v) + \gamma(w) \text{ and } \gamma(a \cdot v) = \sigma(a) \cdot \gamma(v)$$

for all $a \in k_1$ and $v, w \in V_1$.

Note that every injective σ-linear map $\gamma\colon V_1 \to V_2$ induces a map

$$\mathbb{P}(\gamma)\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2).$$

We call this map the *projectivization* of γ. The Fundamental Theorem of Projective Geometry yields a partial converse to this construction.

Theorem 2.3 (Fundamental Theorem of Projective Geometry, [Art57, Thm. 2.26])**.** *Let V_1 (resp. V_2) be a vector space over a field k_1 (resp. k_2) of dimension at least three. If*

$$\varphi\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2)$$

is a bijection that induces a bijection on the sets of lines of $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$, then there is an isomorphism of fields $\sigma\colon k_1 \xrightarrow{\sim} k_2$ and a σ-linear isomorphism $\gamma\colon V_1 \xrightarrow{\sim} V_2$ such that the induced map $\mathbb{P}(\gamma)\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2)$ agrees with φ.

Remark 2.4. Note that for isomorphisms of fields $\sigma, \sigma'\colon k_1 \xrightarrow{\sim} k_2$ and σ-(resp. σ'-)linear injective maps $\gamma, \gamma'\colon V_1 \to V_2$, we have $\mathbb{P}(\gamma) = \mathbb{P}(\gamma')$ if and only if $\sigma = \sigma'$ and there exists $\lambda \in k_2^\times$ such that $\gamma = \lambda\gamma'$.

Example 2.5. The only automorphisms of the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Q}_p$ are the respective identities. The proof of this fact is elementary and carried out in [Con]. In the case of the field of real numbers, the proof uses the fact that $\mathbb{R}$ is an ordered field. For algebraically closed fields, the situation is more complicated. For example, the existence of a square root of -1 prohibits algebraically closed fields from being ordered. The only automorphisms of $\mathbb{C}$ that are continuous as maps of $\mathbb{R}^2$, are the identity and complex conjugation. However, these are not the only automorphisms of the field of complex numbers. In fact, for an arbitrary algebraically closed field $\overline{k}$, one can show that the cardinality of the set of automorphisms of $\overline{k}$ is $2^{\mathrm{card}(\overline{k})}$ (For details, see [Cha70]).

Remark 2.6. In the proof of Theorem 1.1, we will be able to determine whether a line in $\mathbb{P}(V_1)$ is mapped to a line in $\mathbb{P}(V_2)$ only for a dense open subset of lines in $\mathbb{P}(V_1)$. Hence, we need a variant of the Fundamental Theorem of Projective Geometry, taking into account this notion of genericity.

In order to make precise what we mean by a dense open subset of lines on a projective space, we recall that the Grassmannian variety $\mathrm{Gr}(1, \mathbb{P}(V))$ classifies the one-dimensional subspaces on the projective space $\mathbb{P}(V)$. (This is sometimes denoted $\mathrm{Gr}(2, V)$ as lines in $\mathbb{P}(V)$ correspond to two-dimensional subspaces of V.) For a discussion of the Grassmannian as a classical variety, the reader is referred to [Har92, Ch. 6]. The scheme-theoretic point of view is discussed in [EH00, Ch. III.2.7].

Definition 2.7 ([KLOS21, Def. 2.1.1])**.** A *definable projective space* is a triple (k, V, U) consisting of an infinite field k, a finite-dimensional k-vector space V and a subset $U \subseteq \mathrm{Gr}(1, \mathbb{P}(V))(k)$ containing the k-points of a dense open subset of the space $\mathrm{Gr}(1, \mathbb{P}(V))$ of lines in the projective space $\mathbb{P}(V)$. The elements of U are called *definable lines*. The dimension of (k, V, U) is defined to be

$$\dim(k, V, U) := \dim_k V - 1.$$

Definition 2.8 ([KLOS21, Def. 2.1.2])**.** Let k be a field and V a k-vector space. We define the *sweep* of a subset $U \subseteq \mathrm{Gr}(1, \mathbb{P}(V))$ to be the set $S_U(\mathbb{P}(V))$ of all points $P \in \mathbb{P}(V)$ contained in a line $L \in U^\circ$ where $U^\circ \subseteq U$ is the maximal open subset of $U \subseteq \mathrm{Gr}(1, \mathbb{P}(V))(k)$.

Theorem 2.9 (Variant Fundamental Theorem of Projective Geometry, [KLOS21, Thm. 2.1.5])**.** *Let (k_1, V_1, U_1) and (k_2, V_2, U_2) be definable projective spaces of the same dimension $n \geq 2$. If there is a bijection*

$$\varphi\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2)$$

such that for all definable lines $L \subseteq \mathbb{P}(V_1)$ (i.e., $L \in U_1$) the image $\varphi(L) \subseteq \mathbb{P}(V_2)$ is again a definable line, then there is an isomorphism of fields $\sigma\colon k_1 \xrightarrow{\sim} k_2$ and a σ-linear isomorphism

$$\gamma\colon V_1 \xrightarrow{\sim} V_2$$

such that $\mathbb{P}(\gamma)$ agrees with φ on a dense open subset of $\mathbb{P}(V_1)$ containing the sweep of (k_1, V_1, U_1).

Proof. We follow the proof given in [KLOS21, Thm. 2.1.5], which is very similar to the classical one (cf. [Art57, Thm. 2.26]).

As the definition of the sweep of (k_1, V_1, U_1) just depends on the maximal open subset $U_1^0 \subseteq U_1$, we can without loss of generality assume that $U_1 = U_1^\circ \subseteq \mathrm{Gr}(1, \mathbb{P}(V_1))(k_1)$ is an open subset.

For simplicity, we assume that k_1 and k_2 are algebraically closed. In particular, this allows us to work with the Grassmannian as a classical variety in the sense of [Har77, Ch. I.].

Let us first fix some notation. Let V be a finite-dimensional vector space over a field k. A point $P \in \mathbb{P}(V)$ corresponds to a one-dimensional subspace $l_P \subseteq V$. For $0 \neq v \in l_P$ we write $P = [v]$. Two distinct points $P, Q \in \mathbb{P}(V)$ span a unique line $L_{P,Q} \subseteq \mathbb{P}(V)$. If P (resp. Q) is represented by $v_P \in l_P$ (resp. $v_Q \in l_Q$), then v_P and v_Q are linearly independent in V and $L_{P,Q}$ corresponds to the two-dimensional subspace spanned by v_P and v_Q. Note that if P, Q and R are three distinct points on a line $L \subseteq \mathbb{P}(V)$, then, for a chosen vector $v_P \in l_P \setminus \{0\}$, there is a unique vector $v_R \in l_R \setminus \{0\}$ such that $Q = [v_P + v_R]$. Moreover, the map

$$\begin{aligned} \varepsilon^{P,Q,R}\colon k &\to L \setminus \{R\} \\ a &\mapsto [v_P + a v_R] \end{aligned}$$

is a bijection that maps 0 to P and 1 to Q. One easily verifies that $\varepsilon^{P,Q,R}$ is independent of the chosen representative v_P.

The first step of the proof of Theorem 2.9 is to construct an isomorphism of fields $\sigma\colon k_1 \to k_2$. For the remainder of the proof, fix a definable line $L_1 \subseteq \mathbb{P}(V_1)$ together with three distinct points $P, Q, R \in L_1$. As the image of L_1 under φ is again a definable line, we obtain a bijection

$$\sigma^{(L_1,P,Q,R)}\colon k_1 \xrightarrow{\varepsilon^{P,Q,R}} L_1 \setminus \{R\} \xrightarrow{\varphi} \varphi(L_1) \setminus \{\varphi(R)\} \xrightarrow{(\varepsilon^{\varphi(P),\varphi(Q),\varphi(R)})^{-1}} k_2.$$

Since $\varepsilon^{P,Q,R}(0) = P$ and $\varepsilon^{P,Q,R}(1) = Q$, we have $\sigma^{(L_1,P,Q,R)}(0) = 0$ and $\sigma^{(L_1,P,Q,R)}(1) = 1$. Our goal is to show that σ is an isomorphism of fields that is independent of the chosen line and points. The proof will revolve around the following configuration of lines:

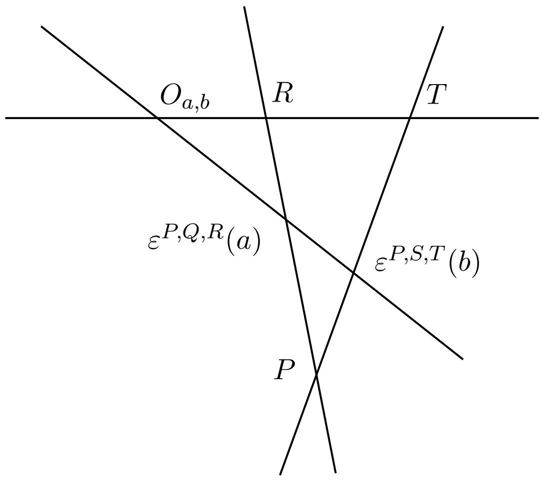

Figure 1: The configuration of lines considered in Construction 2.10.

Construction 2.10. Let $L_2 \subseteq \mathbb{P}(V_1)$ be an additional line intersecting L_1 precisely in the point P and choose two distinct points $S, T \in L_2 \setminus \{P\}$. Fix $a, b \in k_1 \setminus \{0\}$. Choose a representative $v_P \in l_P \setminus \{0\}$ of P. Then there are unique $v_R \in l_R \setminus \{0\}, v_T \in l_T \setminus \{0\}$ such that $Q = [v_P + v_R]$ and $S = [v_P + v_T]$. Moreover, we have $\varepsilon^{P,Q,R}(a) = [v_P + av_R] \neq P$ and $\varepsilon^{P,S,T}(b) = [v_P + bv_T] \neq P$. The point of intersection

$$\{O_{a,b}\} \coloneqq L_{T,R} \cap L_{\varepsilon^{P,Q,R}(a), \varepsilon^{P,S,T}(b)}$$

corresponds to the intersection of linear subspaces

$$\langle v_T, v_R \rangle \cap \langle v_P + av_R, v_P + bv_T \rangle = \langle v_T - \frac{a}{b} v_R \rangle \subseteq V_1.$$

In particular, we have

$$O_{a,b} = \varepsilon^{T,[v_T+v_R],R}(-a/b).$$

To compare $\sigma^{(L_1,P,Q,R)}$ and $\sigma^{(L_2,P,S,T)}$, we would like to apply φ to the lines constructed above and then carry out the same calculation in $\mathbb{P}(V_2)$. However, this is only possible if all the considered lines are definable. Otherwise, the images of the lines under φ may not be lines in $\mathbb{P}(V_2)$ anymore. In the following, we show that for a general choice of (L_2, S, T), all lines considered in Construction 2.10 are definable. Let us first introduce notions to make this statement precise.

The quotient map $V_1 \to V_1/l_P$ induces a closed embedding

$$\mathbb{P}(V_1/l_P) \cong \mathrm{Gr}(0, \mathbb{P}(V_1/l_P)) \hookrightarrow \mathrm{Gr}(1, \mathbb{P}(V_1)),$$

identifying $\mathbb{P}(V_1/l_P)$ with the closed subvariety of $\mathrm{Gr}(1, \mathbb{P}(V_1))$ corresponding to the set of lines going through the point P (see [Har92, Ch. 6.6]). Let

$$\mathscr{L} \to \mathbb{P}(V_1/l_P)$$

denote the *universal line through* P. In other words, $\mathscr{L} \subseteq \mathbb{P}(V_1) \times_k \mathbb{P}(V_1/l_P)$ can be described as the closed subvariety containing pairs (Q, L), where Q is a point on a line L through P. The morphism $\mathscr{L} \to \mathbb{P}(V_1/l_P)$ is the restriction of the second projection. Note that the restriction of the first projection $\mathscr{L} \to \mathbb{P}(V_1)$ is (isomorphic to) the blow-up of $\mathbb{P}(V_1)$ in the point P.

Definition 2.11. We define the scheme M_P to be the fiber product of the universal line through P, i.e., $\mathscr{L} \to \mathbb{P}(V_1/l_P)$ with itself.

$$M_P \coloneqq \mathscr{L} \times_{\mathbb{P}(V_1/l_P)} \mathscr{L}$$

Note that the closed points of M_P correspond to triples (L, S, T), where $L \subseteq \mathbb{P}(V_1)$ is a line through P and $S, T \in \mathbb{P}(V_1)$ are points contained in the line L.

Lemma 2.12. *The scheme M_P is integral.*

Proof. Being isomorphic to the blow-up of $\mathbb{P}(V_1)$ in the point P, the variety $\mathscr{L}$ is irreducible and smooth over k_1. Moreover, the fibers of $\mathscr{L} \to \mathbb{P}(V_1/l_P)$ are isomorphic to $\mathbb{P}^1_{k_1}$ and in particular irreducible of dimension $1 = \dim \mathscr{L} - \dim \mathbb{P}(V_1/l_P)$. The 'miracle flatness theorem' (cf. [Mat87, Thm. 23.1]) thus implies that $\mathscr{L} \to \mathbb{P}(V_1/l_P)$ is flat. As flatness is stable under base change, the projection $\pi\colon M_P = \mathscr{L} \times_{\mathbb{P}(V_1/l_P)} \mathscr{L} \to \mathscr{L}$ is flat as well. Note that the fibers of π are again isomorphic to projective lines. Hence, π is a flat (and thus open) morphism with irreducible fibers and irreducible target. A general fact from topology (cf. [Stacks, 004Z]) implies that the source of such a map is irreducible.

It remains to show that M_P is reduced. This follows from the fact that $\pi\colon M_P \to \mathscr{L}$ is flat and $\mathscr{L}$ is reduced (cf. [DG67, IV. Prop. 11.3.13]). ☐

Note that, in general, the fiber product of (morphisms between) integral varieties may not be irreducible.

Example 2.13. Fix $n > 2$. Consider the blow-up $\pi\colon \mathrm{Bl}_P(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ of $\mathbb{P}^n_k$ in a point $P \in \mathbb{P}^n_k$. Let X denote the fiber product of $\mathrm{Bl}_P(\mathbb{P}^n_k) \to \mathbb{P}^n_k$ with itself.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Bl}_P(\mathbb{P}^n_k) \arrow[d] \\ \mathrm{Bl}_P(\mathbb{P}^n_k) \arrow[r] & \mathbb{P}^n_k \end{tikzcd}$$

The morphism π restricts to an isomorphism $\pi^{-1}(\mathbb{P}^n_k \setminus \{P\}) \xrightarrow{\sim} \mathbb{P}^n_k \setminus \{P\}$. Hence, X has an open subset of dimension n. On the other hand, the fiber of $X \to \mathbb{P}^n_k$ over the point P is isomorphic to $\mathbb{P}^{n-1}_k \times_k \mathbb{P}^{n-1}_k$ and thus of dimension $2n - 2 \neq n$. In particular, this implies that X has irreducible components of different dimensions. Note however, that $\mathbb{P}^n_k$ is irreducible and π has irreducible fibers.

With these preparations out of the way, we can now show that we can always find triples (L, S, T) such that all of the lines occurring in Construction 2.10 are definable.

Lemma 2.14 ([KLOS21, Lem. 2.1.9])**.** *Fix $a, b \in k_1^\times$. Then there exists a non-empty open subset $U_{P,a,b} \subseteq M_P$ such that for all points (L, S, T) of $U_{P,a,b}$, the lines*

$$L_{P,T}, L_{T,R}, L_{\varepsilon^{P,Q,R}(a), \varepsilon^{P,S,T}(b)}$$

occurring in Construction 2.10 are definable.

Proof. Let $W \subseteq M_P$ be the open subset containing all points (L, S, T) that satisfy

$$S \neq T, P \neq T, R \neq T \text{ and } \varepsilon^{P,Q,R}(a) \neq \varepsilon^{P,S,T}(b).$$

We define the following morphisms of varieties sending points of W to the corresponding lines occuring in Construction 2.10.

$$\begin{aligned} q_1 &: W \to \mathrm{Gr}(1, \mathbb{P}(V_1)),\ (L, S, T) \mapsto L_{P,T} \\ q_2 &: W \to \mathrm{Gr}(1, \mathbb{P}(V_1)),\ (L, S, T) \mapsto L_{T,R} \\ q_3 &: W \to \mathrm{Gr}(1, \mathbb{P}(V_1)),\ (L, S, T) \mapsto L_{\varepsilon^{P,Q,R}(a),\varepsilon^{P,S,T}(b)} \end{aligned}$$

Note that by definition of W, each of the q_i is well-defined. As W is open and k_1 is infinite, we may choose points S, T on the fixed line L_1 such that $(L_1, S, T) \in W$. Then we have $q_i((L_1, S, T)) = L_1 \in U_1$ and thus $q_i^{-1}(U_1) \neq \emptyset$ for $i = 1, 2, 3$. As M_P is irreducible, the intersection of the non-empty open subsets $U_{P,a,b} := \bigcap_{i=1}^{3} q_i^{-1}(U_1)$ is again a non-empty open subset. By construction, the closed points in this set correspond precisely to the triples (L, S, T) for which all lines in Construction 2.10 are definable. $\square$

Lemma 2.15 ([KLOS21, Lem. 2.1.10])**.** *Let $P, Q \in \mathbb{P}(V_1)$ be two points in the sweep of U_1. Then there exists a definable line L_P through P and a definable line L_Q through Q such that L_P and L_Q intersect in a single point.*

Proof. The map $s\colon \mathbb{P}(V_1/l_P) \to \mathscr{L}$ sending a line $L \in \mathbb{P}(V_1/l_P)$ to the pair $(L, P) \in \mathscr{L}$ is a morphism between projective varieties, hence closed. Consider the restriction of the two projection maps to the open subvariety $\mathscr{L} \setminus \mathrm{im}(s) \subseteq \mathscr{L}$:

$$\begin{tikzcd} & \mathscr{L} \setminus \mathrm{im}(s) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}(V_1) \setminus \{P\} & & \mathbb{P}(V_1/l_P) \end{tikzcd}$$

The projection π_1 is an isomorphism. Indeed, the inverse morphism is given by

$$\begin{aligned} \mathbb{P}(V_1) \setminus \{P\} &\to \mathscr{L} \setminus \mathrm{im}(s) \\ Q &\mapsto (Q, L_{P,Q}) \end{aligned}$$

mapping a point $Q \neq P$ to the unique line through P and Q.

It follows that the set $S_P := \pi_1(\pi_2^{-1}(U_1)) \subseteq \mathbb{P}(V_1) \setminus \{P\}$, which corresponds to the set of points connected to P via a definable line, is open. Moreover, S_P is non-empty, since P is contained in the sweep of U_1. As $\mathbb{P}(V_1)$ is irreducible, the intersection of the non-empty open subsets $S_P \cap S_Q \subseteq \mathbb{P}(V_1)$ is non-empty. Hence, there is a point $R \in S_P \cap S_Q$ connected to both P and Q via definable lines. This finishes the proof. $\square$

We can finally show that σ is independent of any choices.

Claim 2.16. *The map $\sigma^{(L_1,P,Q,R)}$ does not dependent on the quadruple (L_1, P, Q, R).*

Proof. Let (L', P', Q', R') be a definable line on $\mathbb{P}(V_1)$ containing three marked points. Our aim is to show

$$\sigma^{(L_1,P,Q,R)}(a) = \sigma^{(L',P',Q',R')}(a)$$

for all $a \in k_1$. First consider the case $P = P'$. Fix an element $0 \neq a \in k_1$. By Lemma 2.14, there is a line $L \subseteq \mathbb{P}(V_1)$ containing two distinct points $S, T \in L$ such that $L' \neq L \neq L_1$ (Note that we do not exclude the case $L' = L_1$) and the lines

$$L_{P,T}, L_{T,R}, L_{\varepsilon^{P,Q,R}(a),\varepsilon^{P,S,T}(a)}, L_{\varepsilon^{P,Q,R}(1),\varepsilon^{P,S,T}(1)},$$

and

$$L_{T,R'}, L_{\varepsilon^{P,Q',R'}(a),\varepsilon^{P,S,T}(a)}, L_{\varepsilon^{P,Q',R'}(1),\varepsilon^{P,S,T}(1)}$$

are definable. In other words, all the lines appearing in Construction 2.10 for a, b replaced by a and a, b replaced by 1 are definable and thus mapped to lines in $\mathbb{P}(V_2)$ by φ. We claim that

$$\sigma^{(L_1,P,Q,R)}(a) = \sigma^{(L,P,S,T)}(a). \tag{2.1}$$

The calculation proceeds in the same way as in Construction 2.10. To simplify notation, we temporarily write $\sigma := \sigma^{(L_1,P,Q,R)}$ and $\sigma' := \sigma^{(L,P,S,T)}$. Choosing $v_P \in V_1$ and $w_P \in V_2$ such that $P = [v_P]$ and $\varphi(P) = [w_P]$ yields unique vectors $v_R, v_T \in V_1$ and $w_R, w_T \in V_2$ such that

$$R = [v_R],\ Q = [v_P + v_R],\ \varphi(R) = [w_R] \text{ and } \varphi(Q) = [w_P + w_R],$$

and

$$T = [v_T],\ S = [v_T + v_P],\ \varphi(T) = [w_T] \text{ and } \varphi(S) = [w_T + w_P].$$

By the definition of $\sigma^{(-)}$, we have

$$\varphi(\varepsilon^{P,Q,R}(a)) = \varepsilon^{\varphi(P),\varphi(Q),\varphi(R)}(\sigma(a)) = [w_P + \sigma(a)w_R]$$

and

$$\varphi(\varepsilon^{P,S,T}(a)) = \varepsilon^{\varphi(P),\varphi(S),\varphi(T)}(\sigma'(a)) = [w_P + \sigma'(a)w_T].$$

Hence,

$$\varphi(O_{a,a}) = [\sigma(a)w_R - \sigma'(a)w_T].$$

Note that $O_{a,a} = [v_R - v_T]$ is independent of the choice of $a \in k_1^\times$. As $\sigma(1) = \sigma'(1) = 1$, we conclude

$$[w_R - w_T] = \varphi(O_{1,1}) = \varphi(O_{a,a}) = [\sigma(a)w_R - \sigma'(a)w_T]$$

and thus $\sigma(a) = \sigma'(a)$. By symmetry, it follows that

$$\sigma^{(L_1,P,Q,R)}(a) = \sigma^{(L,P,S,T)}(a) = \sigma^{(L',P,Q',R')}(a).$$

In particular, the map $\sigma^{(L,P,Q,R)}$ is independent of the choice of Q and R. Moreover, using the notation above, we calculate

$$[w_P + \sigma^{(L_1,P,Q,R)}(a)w_R] = \varphi([v_P + av_R]) = \varphi([v_R + a^{-1}v_P]) = [w_R + \sigma^{(L_1,R,Q,P)}(a^{-1})w_P]$$

and obtain

$$\sigma^{(L_1,P,Q,R)}(a) = \left(\sigma^{(L_1,R,Q,P)}(a^{-1})\right)^{-1}. \tag{2.2}$$

Together with the conclusion above, this shows that $\sigma^{(L_1,P,Q,R)}$ is independent of the choice of points P, Q and R. Temporarily call this map σ^{L_1}.

Now consider the case $L_1 \neq L'$. If L_1 and L' intersect in a point P, then we can choose $Q, R \in L_1$ and $Q', R' \in L'$ as above and conclude

$$\sigma^{L_1} = \sigma^{(L_1,P,Q,R)} = \sigma^{(L',P,Q',R')} = \sigma^{L'}.$$

In the case that L_1 and L' do not intersect, Lemma 2.15 implies that there is a definable line L'' intersecting both L_1 and L'. Then we have

$$\sigma^{L_1} = \sigma^{L''} = \sigma^{L'}.$$

This finishes the proof. □

Set $\sigma := \sigma^{(L_1,P,Q,R)}$.

Claim 2.17. *For all $a \in k_1^\times$, we have*

$$(\sigma(a^{-1}))^{-1} = \sigma(a).$$

Proof. As calculated in the proof of the previous claim, we have

$$\sigma^{(L_1,P,Q,R)}(a) = \left(\sigma^{(L_1,R,Q,P)}(a^{-1})\right)^{-1}.$$

Since σ is independent of the chosen definable line and points, the claim follows. $\square$

Claim 2.18. *For all $a, b \in k_1$, we have*

$$\sigma(a \cdot b) = \sigma(a) \cdot \sigma(b).$$

Proof. It suffices to show that there is $\lambda \in k_2^\times$ such that for all $a, b \in k_1^\times$, we have

$$\sigma(-a/b) = -\lambda\sigma(a)/\sigma(b). \tag{2.3}$$

Indeed, plugging in $b = 1$, we obtain $\sigma(-a) = -\lambda\sigma(a)$ for all $a \in k_1$. In particular, we have

$$1 = \sigma(1) = -\lambda\sigma(-1) = \lambda^2\sigma(-(-1)) = \lambda^2.$$

We obtain

$$\sigma(a/b) = -\lambda\sigma(-a/b) = \lambda^2\sigma(a)/\sigma(b) = \sigma(a)/\sigma(b).$$

By Claim 2.17, we conclude

$$\sigma(a \cdot b) = \sigma(a/b^{-1}) = \sigma(a)/\sigma(b^{-1}) = \sigma(a)/\sigma(b)^{-1} = \sigma(a) \cdot \sigma(b).$$

Let us now give a proof of (2.3). The idea is again to inspect the point of intersection in Construction 2.10. Fix $a, b \in k_1^\times$. By Lemma 2.14 we find a definable line L and points $S, T \in L$ such that all the lines considered in Construction 2.10 for the fixed elements a, b as well as for a, b replaced by 1 are definable. In other words, (L, S, T) corresponds to a point in the intersection $U_{P,a,b} \cap U_{P,1,1} \subseteq M_P$. Fix representatives $v_P \in l_P \setminus \{0\}$ and $w_P \in l_{\varphi(P)} \setminus \{0\}$. Then there are unique $v_R, v_T \in V_1$ and $w_R, w_T \in V_2$ such that

$$R = [v_R],\ Q = [v_P + v_R],\ \varphi(R) = [w_R] \text{ and } \varphi(Q) = [w_P + w_R],$$

and

$$T = [v_T],\ S = [v_T + v_P],\ \varphi(T) = [w_T] \text{ and } \varphi(S) = [w_T + w_P].$$

As $\varphi([v_R + v_T])$ is contained in the definable line spanned by $\varphi(R)$ and $\varphi(T)$, there is $\lambda \in k_1^\times$ such that

$$\varphi([v_T + v_R]) = [w_T + \lambda^{-1}w_R].$$

Note that a priori, we do not know whether λ is independent of a and b. We have $O_{a,b} = [v_T - a/bv_R]$ and $\varphi(O_{a,b}) = [w_T - \sigma(a)/\sigma(b)w_R]$ by the calculation in Construction 2.10. This implies

$$\varepsilon^{T,[v_T+v_R],R}(-a/b) = [v_T - a/bv_R] = O_{a,b}$$

and

$$\varepsilon^{\varphi(T),[w_T+\lambda^{-1}w_R],\varphi(R)}(-\lambda\sigma(a)/\sigma(b)) = [w_T - \sigma(a)/\sigma(b)w_R] = \varphi(O_{a,b}).$$

We conclude $\sigma(-a/b) = -\lambda\sigma(a)/\sigma(b)$. Applying the same arguments with a and b replaced by 1 without changing (L, S, T) shows that λ is independent of a and b. $\square$

To see that σ is an isomorphism of fields, it remains to show that the map is compatible with addition. As in the case of multiplication, the strategy is to look at the points of intersection in a certain configuration of lines.

Construction 2.19. Consider a marked line (L_1, P, Q, R) and let S be a point not contained in L_1. Let T be a third point on the line $L_{S,R}$. Choose $v_R \in V_1$ such that $R = [v_R]$. This choice again yields unique $v_P, v_S \in V_1$ such that $P = [v_P]$, $S = [v_S]$, $Q = [v_P + v_R]$ and $T = [v_S + v_R]$. The lines $L_{S,Q}$ and $L_{P,T}$ intersect in a point corresponding to the subspace

$$\langle v_S, v_R + v_R\rangle \cap \langle v_P, v_S + v_R\rangle = \langle v_P + v_S + v_R\rangle \subseteq V_1.$$

Call this point $V := [v_P + v_S + v_R]$. The intersection of the lines $L_{R,V}$ and $L_{P,S}$ corresponds to the subspace

$$\langle v_R, v_P + v_S + v_R\rangle \cap \langle v_P, v_S\rangle = \langle v_P + v_S\rangle \subseteq V_1.$$

Let $W := [v_P + v_S]$ denote the point of intersection. Note that V and W are independent of the chosen vector $v_R \in V_1$.

Fix $a, b \in k_1$. Using the above notation, we have $\varepsilon^{P,Q,R}(a) = [v_P + av_R]$ and $\varepsilon^{S,T,R}(b) = [v_S + bv_R]$. The lines $L_{R,W}$ and $L_{\varepsilon^{P,Q,R}(a),\varepsilon^{S,T,R}(b)}$ thus intersect in a point corresponding to the linear subspace

$$\langle v_R, v_P + v_S\rangle \cap \langle v_P + av_R, v_S + bv_R\rangle = \langle v_P + v_S + (a + b)v_R\rangle.$$

Call this point $O := [v_P + v_S + (a + b)v_R] = \varepsilon^{W,V,R}(a + b)$.

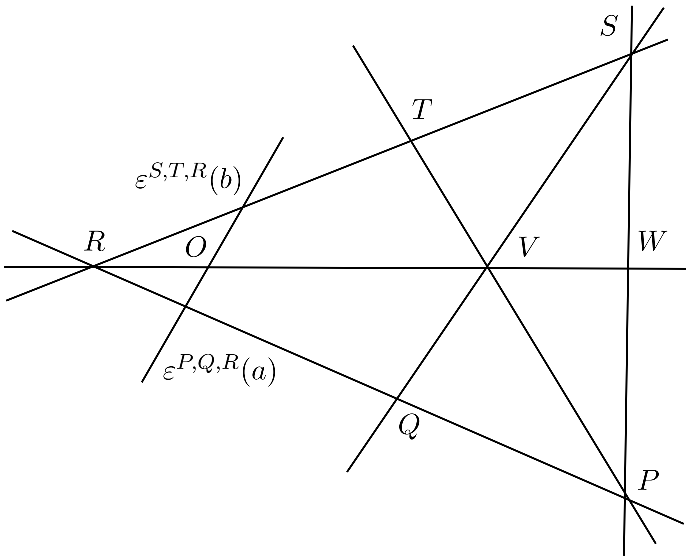

Figure 2: The configuration of lines considered in Construction 2.19

To use this construction in our favor, we have to ensure once again that all the considered lines are definable.

Lemma 2.20. *For $a, b \in k_1$, there exists a pointed line (L, P, Q, R) and points S, T such that all the lines in Figure 2 are definable.*

Proof. The proof of this lemma is very similar to the proof of Lemma 2.14 (see [KLOS21, Lem. 2.1.13]) and thus omitted for brevity. □

Claim 2.21. *For all $a, b \in k_1$, we have*

$$\sigma(a + b) = \sigma(a) + \sigma(b).$$

Proof. This follows from the calculation carried out in Construction 2.19 combined with Lemma 2.20. Indeed, for a choice of (L, P, Q, R) and S, T such that all the lines occurring in Construction 2.19 are definable, we have

$$O = \varepsilon^{W,V,R}(a+b) \text{ and } \varphi(O) = \varepsilon^{\varphi(W),\varphi(V),\varphi(R)}(\sigma(a)+\sigma(b)).$$

Hence, we conclude $\sigma(a+b) = \sigma(a) + \sigma(b)$. $\square$

Combining the claims above, we arrive at the desired conclusion.

Corollary 2.22. *The map $\sigma\colon k_1 \xrightarrow{\sim} k_2$ is an isomorphism of fields.*

The second step of the proof is to construct an injective σ-linear map $\gamma\colon V_1 \to V_2$.

Using the fact that k is infinite and that $U_1 \subseteq \mathrm{Gr}(1, \mathbb{P}(V_1))$ is open, we may choose a basis $(v_i) \in V_1^n$ of V_1 such that the lines $L_{[v_i],[v_j]} \subseteq \mathbb{P}(V_1)$ are definable for $1 \le i < j \le n$. In order to define a map $V_1 \to V_2$, we construct a sequence $(w_i) \in V_2^n$ of elements of V_2:

Begin by choosing a representative $w_1 \in l_{\varphi([v_1])} \setminus \{0\}$. For $i \ge 2$, the line $L_i := L_{[v_1],[v_i]}$ is definable. Therefore, its image $\varphi(L_i) \subseteq \mathbb{P}(V_2)$ is a definable line as well. Furthermore, $\varphi(L_i)$ contains the points $\varphi([v_1]), \varphi([v_i])$ and $\varphi([v_1+v_i])$. The choice of w_1 hence determines a unique $w_i \in l_{\varphi([v_i])} \setminus \{0\}$ such that

$$\varphi([v_1+v_i]) = [w_1+w_i].$$

Extending the map $v_i \mapsto w_i$ σ-linearly, we obtain a σ-linear map $\gamma\colon V_1 \to V_2$ satisfying

$$\gamma\left(\sum_{i=1}^n a_i v_i\right) = \sum_{i=1}^n \sigma(a_i) w_i.$$

Claim 2.23. *There is a non-empty open subset $W \subseteq k_1^n = \mathbb{A}^n_{k_1}(k_1)$ such that for all points $(a_1, \dots, a_n) \in W$, we have*

$$\varphi\left(\left[\sum_{i=1}^n a_i v_i\right]\right) = \left[\gamma\left(\sum_{i=1}^n a_i v_i\right)\right].$$

Proof. The idea of the proof is very similar to what we have seen above. First, we show that a certain set of lines is definable. Then, we compute points of intersection of these lines in $\mathbb{P}(V_1)$ and, after applying φ, in $\mathbb{P}(V_2)$.

For $1 \le i < j \le n$, we define morphisms of varieties

$$F_{i,j}\colon W^F_{i,j} \to \mathrm{Gr}(1, \mathbb{P}(V_1)),\ (a_k) \mapsto \mathbb{P}(\langle a_i v_i, a_{i+1} v_{i+1} + \dots + a_j v_j \rangle)$$

and

$$G_{i,j}\colon W^G_{i,j} \to \mathrm{Gr}(1, \mathbb{P}(V_1)),\ (a_k) \mapsto \mathbb{P}(\langle a_i v_i + \dots + a_{j-1} v_{j-1}, a_j v_j \rangle),$$

where $W^F_{i,j}, W^G_{i,j} \subseteq k_1^n$ are the maximal open subsets on which the maps are well-defined. Note that the definable line spanned by v_i and v_j is contained in the images of $F_{i,j}$ and $G_{i,j}$. Hence, the open set

$$W := \left(\bigcap_{1 \le i < j \le n} F^{-1}_{i,j}(U_1)\right) \cap \left(\bigcap_{1 \le i < j \le n} G^{-1}_{i,j}(U_1)\right) \subseteq k_1^n$$

is non-empty.

Take $a = (a_1, \dots, a_n) \in W$. Note that $\varphi([a_i v_i]) = \varphi([v_i]) = [w_i] = [\sigma(a_i) w_i] = [\gamma(a_i v_i)]$. We proceed by induction on $0 \le j < n$. For all $1 \le i \le n - j$, assume

$$\varphi([a_i v_i + \dots + a_{i+j} v_{i+j}]) = [\gamma(a_i v_i + \dots + a_{i+j} v_{i+j})].$$

Fix $1 \le i \le n-j-1$. Consider the lines $L := F_{i,i+j+1}(a)$ and $L' := G_{i,i+j+1}(a)$. They intersect in the point
$$L \cap L' = \{[a_i v_i + \cdots + a_{i+j+1} v_{i+j+1}]\}.$$
As $a \in W$, both L and L' are definable lines. Hence, their images $\varphi(L)$ and $\varphi(L')$ are definable lines as well. Note that $\varphi(L)$ is spanned by

$$\sigma([a_i v_i]) = [\gamma(a_i v_i)] \text{ and } \sigma([a_{i+1} v_{i+1} + \cdots + a_{i+j+1} v_{i+j+1}]) = [\gamma(a_{i+1} v_{i+1} + \cdots + a_{i+j+1} v_{i+j+1})],$$

while $\varphi(L')$ is spanned by

$$\sigma([a_i v_i + \cdots + a_{i+j} v_{i+j}]) = [\gamma(a_i v_i + \cdots + a_{i+j} v_{i+j})] \text{ and } \sigma([a_{i+j+1} v_{i+j+1}]) = [\gamma(a_{i+j+1} v_{i+j+1})].$$

In particular, we have

$$\varphi(L \cap L') = \varphi(L) \cap \varphi(L') = \{[\gamma(a_i v_i + \cdots + a_{i+j+1} v_{i+j+1}]\}$$

and conclude

$$\varphi([a_i v_i + \cdots + a_{i+j+1} v_{i+j+1}]) = [\gamma(a_i v_i + \cdots + a_{i+j+1} v_{i+j+1})].$$

By induction, we obtain $\varphi\left([\sum_{i=1}^n a_i v_i]\right) = [\gamma\left(\sum_{i=1}^n a_i v_i\right)]$. $\square$

Claim 2.24. *The map $\gamma\colon V_1 \to V_2$ is an isomorphism*

Proof. As V_1 and V_2 are vector spaces of the same dimension, it suffices to show that γ is injective. Suppose that $0 \neq \ker(\gamma)$. The previous claim shows that there is a non-empty open subset $W \subseteq V_1$ such that for all $v \in W$ one has $\sigma([v]) = [\gamma(v)]$. In particular, we have $\ker(\gamma) \cap W = \emptyset$. Fix $v \in W$. As W is open and $\ker(\gamma) \neq 0$, there is $w \in (v + \ker(\gamma)) \cap W$ such that v and w are linearly independent. But then we have $\varphi([v]) = [\gamma(v)] = [\gamma(w)] = \varphi([w])$, in contradiction to the injectivity of φ. $\square$

In particular, γ induces a bijection $\mathbb{P}(\gamma)\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2)$.

Claim 2.25. *The map $\mathbb{P}(\gamma)\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2)$ is independent of the chosen basis $v_1, \ldots, v_n \in V_1$.*

Proof. This follows from Claim 2.23 and the fact that for two injective σ-linear maps $\gamma, \gamma'\colon V_1 \to V_2$ such that the projectivizations $\mathbb{P}(\gamma), \mathbb{P}(\gamma')\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2)$ agree on a non-empty open (and thus dense) subset of $\mathbb{P}(V_1)$, we already have $\mathbb{P}(\gamma) = \mathbb{P}(\gamma')$. $\square$

To finish the proof of Theorem 2.9, it remains to show that $\mathbb{P}(\gamma)$ and φ agree on the sweep of the set of definable lines.

Claim 2.26. *The map $\mathbb{P}(\gamma)$ agrees with φ on the sweep of U_1.*

Proof. Let P be a point contained in the sweep of U_1. As $U_1 \subseteq \mathrm{Gr}(1, \mathbb{P}(V_1))$ is open, k is infinite and P is contained in the sweep of definable lines, we can choose a basis $v_1', \ldots, v_n'$ of V_1 such that v_1' represents the point P and for $i \neq j$, the line spanned by $[v_i']$ and $[v_j']$ is definable.

As the map $\mathbb{P}(\gamma)$ is independent of the chosen basis, we may assume $v_i = v_i'$. The construction of γ then yields

$$\mathbb{P}(\gamma)(P) = \mathbb{P}(\gamma)([v_1]) = [\gamma(v_1)] = [w_1] = \sigma([v_1]) = \sigma(P).$$

Therefore, $\mathbb{P}(\gamma)$ and φ agree on the sweep of U_1. $\square$

Remark 2.27. For an axiomatic approach to projective geometry, the reader is referred to [Mih72].

3 Divisorial structure and linear systems

In the following, all schemes are assumed to be Noetherian.

3.1 Divisors and reflexive sheaves

Recall from [Har77, Rem. II.6.11.2] that on a normal scheme X, the set of Cartier divisors may be identified with the set of locally principal Weil divisors. If X is locally-factorial, then every Weil divisor is Cartier (cf. [Har77, Prop. 6.11]) and thus corresponds to an invertible sheaf on X. For arbitrary normal varieties, this correspondence fails, leading to the notion of reflexive sheaves. In the following, we will briefly review the properties of reflexive sheaves needed in this thesis. For proofs and more information, the reader is referred to [Har80], [CLS12, Ch. 8] and [Sch].

Definition 3.1. Let D be a Weil divisor on a normal integral scheme X. Then

$$U \subseteq X \mapsto \{f \in K(X)^\times \mid (\mathrm{div} f + D)_{|U} \geq 0\} \cup \{0\}$$

yields a coherent sheaf $\mathcal{O}_X(D)$ on X. The divisor D is *Cartier* if and only if $\mathcal{O}_X(D)$ is invertible.

Definition 3.2. A coherent sheaf $\mathcal{F}$ is called *reflexive* if the natural map

$$\mathcal{F} \to \mathcal{F}^{\vee\vee}$$

is an isomorphism.

Proposition 3.3 ([Har80, Prop. 1.6])**.** *Let $\mathcal{F}$ be a coherent sheaf on a normal integral scheme X. Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is reflexive;*

(ii) *$\mathcal{F}$ is torsion-free and for every open set U and every closed subset $Z \subseteq U$ of codimension at least two, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U \setminus Z)$ is bijective.*

Proof. See [Har80, Prop. 1.6]. □

The following proposition relates Weil divisors and reflexive sheaves of rank one, generalizing the correspondence between Cartier divisors and invertible sheaves.

Proposition 3.4. *Let $\mathcal{F}$ be a coherent sheaf on a normal integral scheme X. Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is reflexive of rank one;*

(ii) *$\mathcal{F} \cong \mathcal{O}_X(D)$ for some Weil divisor D on X.*

Proof. See [CLS12, Prop. 8.0.7]. □

Lemma 3.5. *Let D and E be divisors on a normal integral scheme X. Then D and E are linearly equivalent if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$.*

Proof. In the case of X being locally-factorial (i.e., D and E Cartier), the proof is given in [Har77, Prop. II.7.7]. For the general case, see [Sch, Prop. 3.12]. □

We immediately obtain the following corollary.

Corollary 3.6. *Let D be an effective Cartier divisor on a normal integral scheme X. Then every effective divisor that is linearly equivalent to D, is Cartier as well.*

3.2 Divisorial structure

Let X be a normal integral scheme and let $\mathrm{Eff}(X) \subseteq \mathrm{Div}(X)$ denote the set of effective Weil divisors on X. Observe that $\mathrm{Eff}(X)$ is the free abelian monoid on the set of codimension one points of $|X|$, which we will denote as $X^{(1)}$. In particular, the datum of $\mathrm{Eff}(X)$ consists of a map $X^{(1)} \to \mathrm{Eff}(X)$. Similary, $\mathrm{Div}(X)$ is the free abelian group on the set $X^{(1)}$ and comes with a map $X^{(1)} \to \mathrm{Div}(X)$.

In order to give a more concise reformulation of Theorem 1.1, we introduce the notion of *divisorial structures.*

Definition 3.7 ([KLOS21, Def. 3.1.10])**.** The *divisorial structure* associated to a normal integral scheme X is the pair

$$\tau(X) := (|X|,\ c_X \colon X^{(1)} \to \mathrm{Cl}(X)),$$

where $|X|$ is the underlying topological space of X and

$$c_X \colon X^{(1)} \to \mathrm{Cl}(X)$$

is the map sending a point of codimension one to the rational equivalence class of the corresponding prime divisor in the class group. Note that by the universal property of free abelian groups (resp. monoids), the map c_X factors uniquely as

$$X^{(1)} \to \mathrm{Eff}(X) \to \mathrm{Div}(X) \to \mathrm{Cl}(X),$$

where the first map is given by the datum of $\mathrm{Eff}(X)$, the second one is the natural inclusion $\mathrm{Eff}(X) \subseteq \mathrm{Div}(X)$ and $\mathrm{Div}(X) \to \mathrm{Cl}(X)$ is the quotient map given by the rational equivalence relation on divisors.

Let X and Y be normal schemes and assume that there is a homeomorphism $f \colon |X| \xrightarrow{\sim} |Y|$. Note that f restricts to a bijection

$$f^{(1)} := f_{|X^{(1)}} \colon X^{(1)} \to Y^{(1)}$$

between the sets of points of codimension one. As $\mathrm{Eff}(X)$ is the free monoid on the set $X^{(1)}$, the bijection $f^{(1)}$ induces isomorphisms

$$\mathrm{Eff}(f) \colon \mathrm{Eff}(X) \xrightarrow{\sim} \mathrm{Eff}(Y)$$

and

$$\mathrm{Div}(f) \colon \mathrm{Div}(X) \xrightarrow{\sim} \mathrm{Div}(Y)$$

of monoids (respectively groups). However, in general, f does not induce an isomorphism between $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$.

Example 3.8. Let $X := \mathbb{A}^1_k$ and $Y := \mathbb{P}^1_k$ be the affine and projective line over an algebraically closed field k. Any bijection

$$\mathbb{A}^1(k) = k \xrightarrow{\sim} k \cup \{\infty\} = \mathbb{P}^1(k)$$

induces a homeomorphism $f \colon |X| \xrightarrow{\sim} |Y|$ between the underlying Zariski topologies. However,

$$c_X \colon X^{(1)} \to \mathrm{Cl}(X) = 0$$

is the zero map, while

$$c_Y \colon Y^{(1)} \to \mathrm{Cl}(Y) \cong \mathbb{Z}$$

is the map sending every point to $1 \in \mathbb{Z}$.

Definition 3.9 ([KLOS21, Def. 3.1.8]). Let X and Y be normal separated schemes. An *isomorphism of divisorial structures* is given by a homeomorphism

$$f\colon |X| \xrightarrow{\sim} |Y|$$

and an isomorphism of groups

$$c_f\colon \operatorname{Cl}(X) \xrightarrow{\sim} \operatorname{Cl}(Y)$$

such that the diagram

$$\begin{tikzcd} X^{(1)} \arrow[r, "f^{(1)}"] \arrow[d, "c_X"] & Y^{(1)} \arrow[d, "c_Y"] \\ \operatorname{Cl}(X) \arrow[r, "c_f"] & \operatorname{Cl}(Y) \end{tikzcd}$$

of maps of sets commutes.

Remark 3.10. Observe the group isomorphism c_f is uniquely determined by f, if it exists. In particular, the diagram above factors as

$$\begin{tikzcd} X^{(1)} \arrow[r] \arrow[d, "f^{(1)}"] & \operatorname{Eff}(X) \arrow[r] \arrow[d, "\operatorname{Eff}(f)"] & \operatorname{Div}(X) \arrow[r] \arrow[d, "\operatorname{Div}(f)"] & \operatorname{Cl}(X) \arrow[d, "c_f"] \\ Y^{(1)} \arrow[r] & \operatorname{Eff}(Y) \arrow[r] & \operatorname{Div}(Y) \arrow[r] & \operatorname{Cl}(Y) \end{tikzcd}$$

by the universal property of free abelian groups and monoids. From now on, we will drop c_f from the notation and say that $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$ is an isomorphism of divisorial structures.

We can now formulate a slightly stronger version of Theorem 1.1:

Theorem 3.11. *Let X and Y be proper normal irreducible schemes of dimension at least two over algebraically closed fields k_X and k_Y. If there is an isomorphism of divisorial structures $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$, then there is an isomorphism of schemes $\varphi\colon X \xrightarrow{\sim} Y$ such that the underlying map of topological spaces $|\varphi|\colon |X| \to |Y|$ agrees with f.*

The following example shows that even when $k_X = k_Y$, one can only expect the isomorphism $\varphi\colon X \xrightarrow{\sim} Y$ occurring in the previous theorem to be an isomorphism of schemes but not an isomorphism over k_X. In other words, the divisorial structure determines the isomorphism class of X as a scheme but not as a k_X-scheme.

Example 3.12. Let $\sigma\colon \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ be a non-trivial automorphism of the field complex numbers. Then σ induces an isomorphism of schemes $\varphi_\sigma\colon \mathbb{P}^n_{\mathbb{C}} \xrightarrow{\sim} \mathbb{P}^n_{\mathbb{C}}$ fitting into the diagram

$$\begin{tikzcd} \mathbb{P}^n_{\mathbb{C}} \arrow[r, "\varphi_\sigma"] \arrow[d] & \mathbb{P}^n_{\mathbb{C}} \arrow[d] \\ \operatorname{Spec}(\mathbb{C}) \arrow[r, "\sigma"] & \operatorname{Spec}(\mathbb{C}). \end{tikzcd}$$

Note that φ_σ sends a hypersurface $V_+(\sum a_I x^I)$ to the hypersurface $V_+(\sum \sigma(a_I)x^I)$. In particular, φ_σ preserves degrees of divisors and thus induces an isomorphism of divisorial structures. However, φ_σ is not a morphism of schemes over $\mathbb{C}$, just a morphism in the category of schemes.

We can already show a simple case of Theorem 3.11.

Proposition 3.13. *Fix $n, m \geq 2$ and let K and L be algebraically closed fields. If*

$$f\colon \tau(\mathbb{P}^n_K) \to \tau(\mathbb{P}^m_L)$$

is an isomorphism of divisorial structures, then we have $n = m$, $K \cong L$ and f is the underlying map on topological spaces of an isomorphism of schemes $\mathbb{P}^n_K \xrightarrow{\sim} \mathbb{P}^m_L$.

Proof. Note that n (resp. m) is the Krull dimension (i.e., the longest chain of irreducible closed subsets) of the underlying topological space $|\mathbb{P}^n_K|$. As f is a homeomorphism, this implies $n = m$. Since hyperplanes in $\mathbb{P}^n_K$ are precisely the effective divisors of degree one, we see that the isomorphism of divisorial structures f maps hyperplanes to hyperplanes. Choosing an isomorphism between K^{n+1} and its dual, we may interpret the action of f on the closed points of $\mathbb{P}^n_K$ as a bijection

$$\mathbb{P}(K^{n+1}) \to \mathbb{P}(L^{n+1})$$

that maps lines to lines. By the classical Fundamental Theorem of Projective Geometry 2.3, there is an isomorphism of fields $\sigma\colon K \xrightarrow{\sim} L$ and a σ-linear isomorphism $\psi\colon K^{n+1} \xrightarrow{\sim} L^{n+1}$ such that $f = \mathbb{P}(\psi)$. We conclude by observing that the isomorphism ψ naturally yields an isomorphism of schemes $\mathbb{P}^n_K \xrightarrow{\sim} \mathbb{P}^n_L$, which agrees with $\mathbb{P}(\psi)$ on the closed points $\mathbb{P}^n_K(K) = \mathbb{P}(K^{n+1})$. $\square$

A central theme occurring in the following sections is the observation that certain properties of X are determined by the divisorial structure $\tau(X)$. Recall that the linear system $|D|$ associated with a divisor is the set

$$|D| = \{E \in \operatorname{Eff}(X) \mid E \sim D\}$$

of effective divisors linearly equivalent to D. Note that by Corollary 3.6, the divisor D is Cartier if and only if every divisor contained in the linear system $|D|$ is Cartier. The following proposition shows that the divisorial structure determines the linear system $|D|$.

Proposition 3.14. *Let X and Y be normal separated schemes and $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$ an isomorphism of divisorial structures. Let D be an effective divisor on X. Then*

$$\operatorname{Eff}(f)\colon \operatorname{Eff}(X) \xrightarrow{\sim} \operatorname{Eff}(Y)$$

restricts to a bijection of sets $f_D\colon |D| \xrightarrow{\sim} |\operatorname{Eff}(f)(D)|$.

Proof. The divisorial structure determines a commutative diagram

$$\begin{tikzcd} \operatorname{Eff}(X) \arrow[r, "\operatorname{Eff}(f)"] \arrow[d, "q_X"] & \operatorname{Eff}(Y) \arrow[d, "q_Y"] \\ \operatorname{Cl}(X) \arrow[r, "c_f"] & \operatorname{Cl}(Y), \end{tikzcd}$$

where $\operatorname{Eff}(f)$ is an isomorphism of monoids and c_f is an isomorphism of groups. Note that $|D| = q_X^{-1}(q_X(D))$ and $|\operatorname{Eff}(f)(D)| = q_Y^{-1}(q_Y(\operatorname{Eff}(f)(D)))$. By commutativity of the diagram above, we have

$$|\operatorname{Eff}(f)(D)| = q_Y^{-1}(c_f(q_X(D))) = \operatorname{Eff}(f)(|D|)$$

and the claim follows. $\square$

Recall from the discussion in [Har77, II.7] that linear systems on a non-singular projective variety over an algebraically closed field are naturally endowed with the structure of a finite-dimensional projective space. Ultimately, these are the projective spaces that we would like to apply the Fundamental Theorem of Projective Geometry to. In the following, we characterize a broader class of schemes on which linear systems naturally possess the structure of projective spaces.

Definition 3.15. A normal integral scheme X is called *definable* if the following conditions hold:

(i) The global sections $k_X := \Gamma(X, \mathcal{O}_X)$ form an algebraically closed field.

(ii) The natural morphism $X \to \operatorname{Spec}(k_X)$ is separated and of finite type.

(iii) For every reflexive sheaf $\mathcal{F}$ of rank one, $\Gamma(X, \mathcal{F})$ is a finite-dimensional k_X-vector space.

Proposition 3.16. *Every proper normal irreducible scheme X over an algebraically closed field k_X is definable.*

Proof. As X is integral and proper, we have $\Gamma(X, \mathcal{O}_X) \cong k_X$. The second condition is satisfied by the definition of proper morphisms. Finally, note that every reflexive sheaf of rank one is coherent, and thus Serre's theorem (cf. [DG67, III. Thm. 3.2.1]) shows that its global sections form a finite-dimensional vector space over the field k_X. □

Remark 3.17. On the other hand, observe that positive-dimensional affine varieties are not definable as $\Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) \cong A$.

Proposition 3.18. *Let D be a divisor on a definable scheme X. Then there is a natural bijection between the linear system $|D|$ and the projective space $\mathbb{P}_{k_X}(\Gamma(X, \mathcal{O}_X(D)))$.*

Proof. First note that the definition of definable schemes ensures that $\mathbb{P}_{k_X}(\Gamma(X, \mathcal{O}_X(D))$ is a well-defined finite-dimensional projective space. By the definition of linear equivalence, an effective Divisor E is contained in $|D|$ if and only if there is an element $f \in K(X)^\times$ such that $E = \operatorname{div}(f) + D$. As E is effective, this already implies $f \in \Gamma(X, \mathcal{O}_X(D))$. Conversely, every non-zero section $0 \neq f \in \Gamma(X, \mathcal{O}_X(D))$ gives rise to the effective divisor $\operatorname{div}(f) + D \geq 0$ which is contained in $|D|$.

To show the claim, it suffices to observe that two sections $f, g \in \Gamma(X, \mathcal{O}_X(D))$ give rise to the same divisor if and only if there is a unit $u \in \Gamma(X, \mathcal{O}_X)^\times = k_X^\times$ such that $f = u \cdot g$. For Cartier divisors on locally-factorial schemes, this is shown in [Har77, Prop. II.7.7]. The general case is discussed in [Sch, Prop. 3.12]. □

We will eventually reduce the proof of Theorem 4.3 to the case of locally-factorial schemes on which every divisor is Cartier. Note that for a Cartier divisor D on X, the invertible sheaf $\mathcal{O}_X(D)$ is — by definition — a subsheaf of the constant sheaf with value $K(X)$. The following remark generalizes the notion of linear systems to arbitrary invertible sheaves.

Remark 3.19. Suppose that $\mathcal{L}$ is an invertible sheaf on a definable scheme X. By Lemma 3.4, there is a Cartier divisor D on X such that $\mathcal{L} \cong \mathcal{O}_X(D)$. This isomorphism induces a k_X-linear isomorphism $\varphi\colon \Gamma(X, \mathcal{L}) \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(D))$ which in turn yields an isomorphism

$$\mathbb{P}(\varphi)\colon \mathbb{P}(\Gamma(X, \mathcal{L})) \xrightarrow{\sim} \mathbb{P}(\Gamma(X, \mathcal{O}(D)) = |D|.$$

Recall from Lemma 3.6, that every divisor in $|D|$ is effective and Cartier. Furthermore, note that for a non-zero section $0 \neq s \in \Gamma(X, \mathcal{L})$, the zero locus of s is precisely the effective Cartier divisor $\mathbb{P}(\varphi)([s]) \in |D|$. We call

$$|\mathcal{L}| := \mathbb{P}_{k_X}(\Gamma(X, \mathcal{L}))$$

the *linear system* associated to $\mathcal{L}$.

Let X and Y be definable schemes and $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$ an isomorphism of divisorial structures. Let D be a divisor on X. The combination of Proposition 3.14 and 3.18 shows that f induces a bijection

$$\mathbb{P}(\Gamma(X, \mathcal{O}(D))) \xrightarrow{\sim} \mathbb{P}(\Gamma(Y, \mathcal{O}(f(D)))),$$

where we write $f(D)$ as an abbreviaton for $\operatorname{Eff}(f)(D)$. A priori, we only know that this is a bijection between sets. The goal of Section 3 is to show that this bijection satisfies the assumptions of the variant Fundamental Theorem of Projective Geometry 2.9. Before this is achieved in Section 3.7, we give some arguments to show that it suffices to consider the case of locally-factorial quasi-projective definable schemes.

3.3 Reduction to open subsets with complements in codimension two

In this section, we establish general arguments needed in order to reduce the proof of Theorem 1.1 to the case of locally-factorial quasi-projective definable schemes. The first step is to observe that definable schemes and isomorphisms of divisorial structures are compatible with restriction to open subsets containing all points of codimension one.

Lemma 3.20. *Let X and Y be normal separated schemes and $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$ an isomorphism of divisorial structures. If $U \subseteq X$ is an open subscheme containing all points of codimension one, then f restricts to an isomorphism of divisorial structures $f_{|U}\colon \tau(U) \xrightarrow{\sim} \tau(f(U))$.*

Proof. Note that as f is a homeomorphism, the set $f(U) \subseteq Y$ is an open subset containing all points of codimension one. As U contains all points of codimension one, we have $X^{(1)} = U^{(1)}$ and $\mathrm{Cl}(X) = \mathrm{Cl}(U)$ since these sets only depend on the points of codimension one. The same holds for $f(U)$ and thus the claim follows. □

Lemma 3.21. *Let X be a definable scheme. If $U \subseteq X$ is an open subscheme containing all points of codimension one, then U is a definable scheme as well.*

Proof. Note that by Lemma 3.3, we have $\Gamma(X, \mathcal{F}) = \Gamma(U, \mathcal{F})$ for all reflexive sheaves $\mathcal{F}$ of rank one on X. Hence, it suffices to show that for every reflexive sheaf $\mathcal{F}$ of rank one on U, there is a reflexive sheaf $\mathcal{F}'$ on X such that $\mathcal{F} \cong \mathcal{F}'_{|U}$. As in the previous proof, we have $X^{(1)} = U^{(1)}$ and $\mathrm{Cl}(X) = \mathrm{Cl}(U)$. By Lemma 3.4, this implies that there is a divisor D on X such that

$$\mathcal{F} \cong \mathcal{O}_U(D_{|U}) \cong \mathcal{O}_X(D)_{|U}.$$

This finishes the proof. □

The second part of this section is concerned with extending isomorphisms of schemes defined on open subsets containing all points of codimension one. More precisely, we will show the following proposition.

Proposition 3.22 ([KLOS21, Lem. 4.1.3])**.** *Assume that X and Y are normal separated schemes, and $U \subseteq X$ and $V \subseteq Y$ are dense open subschemes with complements of codimension at least two. If $f\colon |X| \xrightarrow{\sim} |Y|$ is a homeomorphism of Zariski topological spaces such that $f(U) = V$ and $f_{|U}$ is the underlying map of an isomorphism $\tilde{f}_U\colon U \xrightarrow{\sim} V$ of schemes, then $\tilde{f}_U$ extends to a unique isomorphism of schemes $\tilde{f}\colon X \xrightarrow{\sim} Y$ whose underlying map on topological spaces is f.*

Remark 3.23. Note that we do not assume X and Y to be schemes over a field k. However, a close look at the arguments below shows that if X and Y are k-schemes and $\tilde{f}_U$ is a morphism over k, then the lift $\tilde{f}$ is a morphism over k as well.

Before giving the proof, we need to make a few preliminary observations about separated schemes.

Lemma 3.24 ([KLOS21, Lem. 4.1.1])**.** *If X is an integral separated scheme, then we have*

$$\overline{\{x\}} = \bigcap_{y \in X^{(1)},\, x \in \overline{\{y\}}} \overline{\{y\}}$$

for all points $x \in X$.

Proof. The inclusion '$\subseteq$' is clear. For the inclusion '$\supseteq$', let $z \in \bigcap_{y\in X^{(1)}, x\in\overline{\{y\}}} \overline{\{y\}}$ be an element of the right-hand side. We claim that every open affine neighborhood of z contains x. Let $z \in U$ be an open affine neighborhood. As X is separated, the complement $X \setminus U$ has pure codimension one (cf. [Stacks, 0BCQ]). If x is not contained in U, then there exists $y \in X^{(1)}$ with $x \in \overline{\{y\}}$ such that $\overline{\{y\}} \subseteq X \setminus U$. But then $z \in \overline{\{y\}} \subseteq X \setminus U$. Contradiction. Hence, $x \in U$.

Therefore, it suffices to consider the affine case $X = \operatorname{Spec}(A)$ for some Noetherian integral ring A. Let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to the point $x \in X = \operatorname{Spec}(A)$. Write $\mathfrak{p} = (f_1, \dots, f_m)$. By primary decomposition (cf. [AM69, Ch. 4]), we have

$$V(f_i) = \bigcup_j V(\mathfrak{q}_{i,j}),$$

where the $\mathfrak{q}_{i,j} \subseteq A$ are the minimal prime ideals containing f_i. By Krull's principal ideal theorem (cf. [AM69, Cor. 11.17]), the prime ideals $\mathfrak{q}_{i,j}$ are of height one. In other words, the sets $V(\mathfrak{q}_{i,j}) \subseteq \operatorname{Spec}(A)$ are irreducible closed subsets of codimension one. We conclude

$$\overline{\{x\}} = V(\mathfrak{p}) = \bigcap_i V(f_i) = \bigcap_i \bigcup_j V(\mathfrak{q}_{i,j}) \supseteq \bigcap_{\mathfrak{q}_{i,j}\subseteq\mathfrak{p}} V(\mathfrak{q}_{i,j}) \supseteq \bigcap_{y\in X^{(1)}, x\in\overline{\{y\}}} \overline{\{y\}}.$$

This finishes the proof. □

Lemma 3.25 ([KLOS21, Lem. 4.1.2])**.** *Assume that $f_1, f_2 \colon |X| \to |Y|$ are homeomorphisms of the underlying spaces of two separated integral schemes. Let $U \subseteq |X|$ be an open subset containing all points of codimension one. If $f_{1|U} = f_{2|U}$, then $f_1 = f_2$.*

Proof. Take $x \in X$. As f_i is a homeomorphism for $i = 1, 2$, we have

$$f_i\left(\overline{\{x\}}\right) = \overline{\{f_i(x)\}}.$$

Hence $f_i(x)$ is the unique generic point of $f_i\left(\overline{\{x\}}\right)$ and therefore it suffices to show

$$f_1\left(\overline{\{x\}}\right) = f_2\left(\overline{\{x\}}\right).$$

In fact, Lemma 3.24 yields

$$f_i\left(\overline{\{x\}}\right) = f_i\left(\bigcap_{y\in X^{(1)},\, x\in\overline{\{y\}}} \overline{\{y\}}\right) = \bigcap_{y\in X^{(1)},\, x\in\overline{\{y\}}} f_i\left(\overline{\{y\}}\right) = \bigcap_{y\in X^{(1)},\, x\in\overline{\{y\}}} \overline{\{f_i(y)\}}.$$

As $f_1(y) = f_2(y)$ for all points $y \in X$ of codimension one, this finishes the proof. □

With these preparations out of the way, we can finally prove Proposition 3.22.

Proof of Proposition 3.22. Let us first show how to extend the morphism $\tilde{f}_U$ to a morphism on the whole of X. Since X is separated, it suffices to extend the morphism locally, and thus we may assume that $Y = \operatorname{Spec}(A)$ is affine. The morphism $\tilde{f}_U$ then corresponds to a morphisms of rings $A \to \Gamma(U, \mathcal{O}_X)$. As U contains all points of codimension one, Lemma 3.3 implies that $\Gamma(U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. Hence, we obtain a unique morphism of rings $A \to \Gamma(X, \mathcal{O}_X)$ such that the corresponding morphism $\tilde{f} \colon X \to Y = \operatorname{Spec}(A)$ of schemes satisfies $\tilde{f}_{|U} = \tilde{f}_U$. If we additionally assume that $\tilde{f}_U \colon U \to Y$ is a morphism in the category of k-schemes, then the morphism $A \to \Gamma(U, \mathcal{O}_X)$ is not only a morphism of rings but a morphism of k-algebras.

Applying the same argument to the homeomorphism f^{-1} and the isomorphism of schemes $\tilde{f}_U^{-1} \colon V \xrightarrow{\sim} U$, we obtain a morphism $\tilde{g} \colon Y \to X$ such that $(\tilde{f} \circ \tilde{g})_{|V} = \mathrm{id}_{|U}$ and $(\tilde{g} \circ \tilde{f})_{|U} = \mathrm{id}_{|V}$.

As X and Y are separated, this already implies that $\tilde{f}$ and $\tilde{g}$ are inverse to each other on all of X (resp. Y). In other words, $\tilde{f}$ is an isomorphism. In particular, the underlying map on topological spaces is a homeomorphism and agrees with f on $|U|$. By Lemma 3.25, this implies that $|\tilde{f}| = f$. □

3.4 Reduction to the locally-factorial case

Recall that a scheme X is called *locally-factorial* if all of the local rings $\mathcal{O}_{X,x}$ are unique factorization domains. As shown in [Har77, II.6.11], every divisor on a locally-factorial scheme is Cartier.

Lemma 3.26. *Every definable scheme X contains a locally-factorial definable open subscheme $U \subseteq X$ containing all points of codimension one.*

Proof. Every local regular ring is a unique factorization domain (cf. [AB59]). Hence, the claim follows from Lemma 3.21 and the fact that the regular locus $U \subseteq X$ on a normal scheme is an open subscheme containing all points of codimension one. □

Example 3.27. The regular locus can be strictly smaller than the locus of points $x \in X$ for which the local ring $\mathcal{O}_{X,x}$ is a unique factorization domain. Let k be a field of $\operatorname{char} k \neq 2$,

$$A \coloneqq k[x_1, \ldots, x_5]/(x_1^2 + \cdots + x_5^2)$$

and $X \coloneqq \operatorname{Spec}(A)$. The ring A is factorial by a theorem of Klein and Nagata, see [Sam68, 4.]. Therefore, X is locally-factorial. On the other hand, the localization of A at the origin $(x_1, \ldots, x_n) \in \operatorname{Spec}(A)$ is not regular.

Before combining the arguments of this section and the previous one into an actual reduction argument, we will give a short interlude on the detection of ample sheaves using divisorial structures. Afterwards, in Section 3.6, we will then see how the reduce the proof of Theorem 1.1 to the case of locally-factorial quasi-projective definable schemes.

3.5 Detecting ample sheaves

Linear systems associated with very ample invertible sheaves determine the ways of embedding X into projective space. In this section, we will investigate how to use the divisorial structure to determine whether a given invertible sheaf is (very) ample.

Definition 3.28 ([DG67, II. Thm. 4.5.2])**.** Let X be a k_X-scheme. Recall that an invertible sheaf $\mathcal{L}$ is *ample* if it satisfies the following equivalent conditions:

(i) For $n \gg 0$, $\mathcal{L}^{\otimes n}$ is very ample, i.e., global sections of $\mathcal{L}^{\otimes n}$ determine a locally-closed embedding of X into projective space.

(ii) For all quasicoherent sheaves $\mathcal{F}$, there is $n_0 \geq 0$ such that for $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated.

(iii) As s runs over all global sections of $\mathcal{L}^{\otimes n}$ for $n > 0$, the sets

$$X_s \coloneqq \{x \in X \mid s_x \notin \mathfrak{m}_x \cdot \mathcal{L}_x^{\otimes n}\}$$

form a base of the topology on X.

A Cartier divisor D is *(very) ample* if the associated invertible sheaf $\mathcal{O}_X(D)$ is (very) ample.

Recall that the *support* $\mathrm{Supp}(D) \subseteq |X|$ of an effective Weil divisor $D = \sum n_i Y_i$ is the union

$$\mathrm{Supp}(D) \coloneqq \bigcup \overline{\{Y_i\}}$$

of the irreducible closed subsets $\overline{\{Y_i\}} \subseteq |X|$.

Plugging in the definitions, we see that the support of an effective divisor D is determined by the divisorial structure $\tau(X)$:

Proposition 3.29. *Let X and Y be normal separated schemes. If there is an isomorphism $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$ of divisorial structures, then we have*

$$f(\mathrm{Supp}(D)) = \mathrm{Supp}(\mathrm{Eff}(f)(D)).$$

Proof. Let $D = \sum n_i Y_i$ be a divisor on X with $Y_i \in X^{(1)}$. Then we have $\mathrm{Eff}(f)(D) = \sum n_i f(Y_i)$. In particular, this implies

$$\mathrm{Supp}(\mathrm{Eff}(f)(D)) = \bigcup \overline{\{f(Y_i)\}} = f\left(\bigcup \overline{\{Y_i\}}\right) = f(\mathrm{Supp}(D)),$$

where the second equality holds as f is a homeomorphism. $\square$

We can now show that the divisorial structure $\tau(X)$ determines whether a Cartier divisor is ample.

Proposition 3.30 ([KLOS21, Prop. 3.2.7])**.** *Let X and Y be normal separated schemes and $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$ be an isomorphism of divisorial structures. If D and $\mathrm{Eff}(f)(D)$ are effective Cartier divisors, then D is ample if and only if $\mathrm{Eff}(f)(D)$ is ample.*

Proof. First note that if E is an effective divisor linearly equivalent to D given by a section $s \in \Gamma(X, \mathcal{O}_X(D))$, then E is Cartier and a computation on affine open subschemes shows that

$$\mathrm{Supp}(E) = \{x \in X \mid s_x \in \mathfrak{m}_x \cdot \mathcal{O}_X(D)_x\} = X \setminus X_s.$$

By Definition 3.28 and the observations above, D is ample if and only if the sets

$$X_E \coloneqq X \setminus \mathrm{Supp}(E)$$

for all $E \in |mD|$ and $m \geq 1$ form a basis of the topology on X. As f is a homeomorphism, this is the case if and only if the sets $f(X_E)$ form a basis of the topology on Y. By Lemma 3.29, we have

$$f(X_E) = Y \setminus f(\mathrm{Supp}(E)) = Y \setminus \mathrm{Supp}(\mathrm{Eff}(f)(E)) = Y_{\mathrm{Eff}(f)(E)}.$$

As f induces a bijection between $|mD|$ and $|\,\mathrm{Eff}(f)(mD)|$ by Lemma 3.14, we conclude that D is ample if and only if the same holds for $\mathrm{Eff}(f)(D)$. $\square$

3.6 Reduction to the quasi-projective case

Recall that a scheme X over a field k is called *quasi-projective* if and only if there is an embedding $X \hookrightarrow \mathbb{P}^n_k$ of k-schemes. Equivalently, X is quasi-projective if and only if X admits a (very) ample invertible sheaf. The results of the previous section show that isomorphisms of divisorial structures preserve quasi-projectivity. In this section, we will apply Chow's lemma to show that every definable scheme contains a quasi-projective definable open subscheme with a complement in codimension two. This is the final reduction step, allowing us to reduce the proof of Theorem 1.1 to the locally-factorial quasi-projective case.

Theorem 3.31 (Chow's lemma)**.** *Let X be a separated scheme of finite type over k. Then there exists a proper birational morphism $X' \to X$, where X' is a quasi-projective scheme over k.*

Proof. See [Stacks, 02O2]. $\square$

Lemma 3.32. *Every definable scheme X contains a quasi-projective definable open subscheme $U \subseteq X$ with $\operatorname{codim}(X \setminus U \subseteq X) \geq 2$.*

Proof. By definition, the definable scheme X is separated and of finite type over the field $k_X = \Gamma(X, \mathcal{O}_X)$. By Chow's lemma, there thus exists a quasi-projective k_X-scheme X' and a proper birational morphism $\pi \colon X' \to X$. As X is normal, there is an open subset $U \subseteq X$ containing all points of codimension one such that the restriction $\pi_{|\pi^{-1}(U)} \colon \pi^{-1}(U) \to U$ is an isomorphism. The claim follows. $\square$

Remark 3.33. Combining the results of Sections 3.3, 3.4 and 3.6, we see that it suffices to show the claim of Theorem 1.1 for locally-factorial quasi-projective definable schemes of dimension at least two.

Indeed, let X and Y be definable schemes of dimension at least two and $f \colon \tau(X) \xrightarrow{\sim} \tau(Y)$ an isomorphism of divisorial structures. By Lemma 3.26 and Lemma 3.32, there is a quasi-projective, locally-factorial open subscheme $U \subseteq X$ containing all points of codimension one. We apply the same arguments to the definable scheme $f(U)$ and, after possibly shrinking U even further, may assume that both U and $f(U)$ are quasi-projective, locally-factorial definable schemes of dimension at least two, containing all points in codimension one of X (resp. Y).

By Lemma 3.20, the isomorphism f restricts to an isomorphism of divisorial structures

$$f_{|U} \colon \tau(U) \xrightarrow{\sim} \tau(f(U)).$$

Suppose that we can show that there is an isomorphism of schemes

$$\varphi \colon U \xrightarrow{\sim} f(U)$$

such that $|\varphi| = f_{|U}$. Lemma 3.22 then implies that there is an isomorphism of schemes

$$\varphi' \colon X \xrightarrow{\sim} Y$$

whose underlying map on topological spaces is equal to f.

Hence, it suffices to prove Theorem 1.1 in the case of locally-factorial, quasi-projective definable schemes of dimension at least two.

3.7 Definable subspaces in linear systems

Recall that the linear system associated with an effective divisor D on a definable scheme is naturally endowed with the structure of a projective space by Lemma 3.18. As mentioned before, these are the projective spaces that we would like to apply the Fundamental Theorem of Projective Geometry to. In this section, we lay the necessary foundations. In particular, we will introduce a notion of definable lines on linear systems. Afterwards, we will show that these are preserved by isomorphisms of divisorial structures and satisfy the conditions of the variant Fundamental Theorem of Projective Geometry.

Definition 3.34 ([KLOS21, Def. 3.3.1])**.** Let D be an effective divisor on a definable scheme X. A subset $V \subseteq |D|$ is called a *definable subspace* if there is a subset $Z \subseteq X$ such that

$$V = V_X(Z) := V(Z) := \{E \in |D| \mid Z \subseteq \operatorname{Supp}(E)\} \subseteq |D|.$$

A definable subspace of dimension one is called a *definable line.*

Note that via the observations in Remark 3.19, this definition also applies to linear systems associated with arbitrary invertible sheaves. Observe that if $\overline{Z}$ is the closure of Z in X, then $V(Z) = V(\overline{Z})$, since $\mathrm{Supp}(E)$ is closed. Hence, it suffices to work with closed sets Z. The following lemma justifies calling $V(Z)$ a definable sub*space*.

Lemma 3.35. *Let Z be a closed subset of a definable scheme X. Let Z_{red} denote be the associated closed subscheme with reduced structure (cf. [Vak18, 8.3.9]). Let D be an effective Cartier divisor on X. Then we have*

$$V(Z) = \mathbb{P}(\ker(\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Z, \mathcal{O}_X(D)_{|Z_{\mathrm{red}}}))) \subseteq |D|.$$

In particular, a definable subspace is a subspace in the sense of Definition 2.1.

Proof. Let $\mathcal{L} := \mathcal{O}_X(D)$ denote the associated invertible sheaf. For $E \in |D|$ there is a section $0 \neq s \in \Gamma(X, \mathcal{L})$ such that $E = [s] \in \mathbb{P}(\Gamma(X, \mathcal{L}) = |D|$. First assume $Z \subseteq \mathrm{Supp}(E)$. Then we have $s_z \in \mathfrak{m}_z\mathcal{L}_z$ for all $z \in Z$. As $(\mathcal{L}_{|Z_{\mathrm{red}}})_z \cong \mathcal{L}_z/\mathfrak{m}_z\mathcal{L}_z$, this yields $(s_{|Z_{\mathrm{red}}})_z = 0$ for all $z \in Z$. As $\mathcal{L}_{|Z_{\mathrm{red}}}$ is a sheaf, we conclude $s_{|Z_{\mathrm{red}}} = 0$. Conversely, assume $s_{|Z_{\mathrm{red}}} = 0$. Then we have $(s_{|Z_{\mathrm{red}}})_z = 0$ in $(\mathcal{L}_{|Z_{\mathrm{red}}})_z \cong \mathcal{L}_z/\mathfrak{m}_z\mathcal{L}_z$ and thus $s_z \in \mathfrak{m}_z\mathcal{L}_z$ for all $z \in Z$. Hence, $Z \subseteq \mathrm{Supp}(E)$. □

In order to get a feeling for definable subspaces, let us inspect an easy example.

Example 3.36. Let $X = \mathbb{P}^2_{\mathbb{C}}$ and consider the standard very ample invertible sheaf $\mathcal{L} := \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$. One can show that every line on $|\mathcal{L}|$ is definable: A line $l \subseteq |\mathcal{L}|$ is given by two linearly independent elements $F_1, F_2 \in \Gamma(X, \mathcal{L}) = \langle X_0, X_1, X_2 \rangle_{\mathbb{C}}$ of the space of linear polynomials in three variables. As F_1 and F_2 are linearly independent, the hyperplanes $V_+(F_1)$ and $V_+(F_2)$ intersect in a single closed point $x = [x_0 : x_1 : x_2] \in \mathbb{P}^2_{\mathbb{C}}$.
We claim $V(\{x\}) = l$. Let $Z_{\mathrm{red}} \subseteq X$ be the closed subscheme with reduced structure on $\{x\}$. Transforming coordinates, we may assume $x = [1 : 0 : 0]$ and thus $F_1, F_2 \in \langle X_1, X_2 \rangle_{\mathbb{C}}$. The restriction map

$$\varphi\colon \Gamma(X, \mathcal{L}) \to \Gamma(Z, \mathcal{L}_{|Z_{\mathrm{red}}})$$

is given by

$$\langle X_0, X_1, X_2 \rangle \to \mathbb{C},\ aX_0 + bX_1 + cX_2 \mapsto a.$$

Hence, $\ker(\varphi) = \langle X_1, X_2 \rangle_{\mathbb{C}} = \langle F_1, F_2 \rangle_{\mathbb{C}}$ and we conclude $l = V(\{x\})$ by Lemma 3.35.

The following proposition shows that definable subspaces are determined by the divisorial structure.

Proposition 3.37. *Let X and Y be normal separated schemes and $f\colon X \xrightarrow{\sim} Y$ an isomorphism of divisorial structures. If $Z \subseteq |X|$ is a closed subset, then*

$$\mathrm{Eff}(f)(V_X(Z)) = V_Y(f(Z)).$$

Proof. Observe that for a divisor D on X, we have $Z \subseteq \mathrm{Supp}(D)$ if and only if

$$f(Z) \subseteq f(\mathrm{Supp}(D)) = \mathrm{Supp}(\mathrm{Eff}(f)(D)),$$

where the second equality holds by Lemma 3.29. As $\mathrm{Eff}(f)$ moreover induces a bijection between the linear systems $|D|$ and $|\,\mathrm{Eff}(f)(D)|$ by Lemma 3.14, the result follows. □

The following lemma is the crucial step in showing that the set of definable lines on a linear system is determined by the divisorial structure.

Lemma 3.38 ([KLOS21, Lem. 3.3.5]). *Let D be an effective Cartier divisor on a definable scheme X. Let $V(Z)$ be a non-empty definable subspace of $|D|$. Then there is an ascending chain of closed subsets*

$$Z = Z_1 \subsetneq \cdots \subsetneq Z_n$$

such that

$$V(Z) = V(Z_1) \supsetneq \cdots \supsetneq V(Z_n)$$

is a full flag of linear subspaces ending in a point, i.e., $\dim V(Z_{i+1}) + 1 = \dim V(Z_i)$ *and* $\dim V(Z_n) = 0$.

Proof. If $x \in X$ is a closed point, then $V(Z \cup \{x\}) = V(Z)$ or $V(Z \cup \{x\})$ has codimension 1 in $V(Z)$. Indeed, as k_X is algebraically closed, the residue field of every closed point $x \in X$ is k_X. Observe that the dimension of the kernel of the restriction map

$$\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(\{x\}, \mathcal{O}_X(D)_{|\{x\}_{\mathrm{red}}}) \cong \mathcal{O}_X(D)(x) \cong k_X$$

is at least $\dim \Gamma(X, \mathcal{O}_X(D)) - 1$. Thus, the vanishing of x imposes a codimension one condition on the linear system $|D|$.

We have $V(Z \cup \{x\}) = V(Z)$ if and only if $x \in \mathrm{Supp}(E)$ for all $E \in V(Z)$. Hence, it suffices to find $x \in X$ and $E \in V(Z)$ such that $x \notin \mathrm{Supp}(E)$. Pick $E \in |\mathcal{O}_X(D)|$ arbitrarily. As E corresponds to a non-zero section of $\mathcal{O}_X(D)$, there is a closed point $x \in X$ not contained in $\mathrm{Supp}(E)$. Applying this argument inductively, we obtain the desired chain of closed subsets. □

Corollary 3.39 ([KLOS21, Cor. 3.3.6]). *Let X and Y be definable schemes and*

$$f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$$

an isomorphism of divisorial structures. If D and $\mathrm{Eff}(f)(D)$ are effective Cartier divisors, then we have

$$\dim_{k_X} |D| = \dim_{k_Y} |\mathrm{Eff}(f)(D)|,$$

where $|D|$ and $|\mathrm{Eff}(f)(D)|$ are viewed as projective spaces via Lemma 3.18.

Proof. Setting $Z = \emptyset$ in Lemma 3.38, we obtain a full flag of linear subspaces

$$|D| = V_X(\emptyset) \supsetneq \cdots \supsetneq V_X(Z_n)$$

of length $\dim_{k_X} |D|$. By Lemma 3.37, the isomorphism of divisorial structures f takes this chain of linear subspaces to a proper descending chain of linear subspaces

$$|\mathrm{Eff}(f(D))| = V_Y(\emptyset) \supsetneq \cdots \supsetneq V_Y(f(Z_n)).$$

This implies $\dim_{k_X} |D| \leq \dim_{k_Y} |\mathrm{Eff}(f(D))|$ and equality follows by symmetry. □

We can finally show that definable lines are preserved by isomorphisms of divisorial structures.

Corollary 3.40 ([KLOS21, Cor. 3.3.7]). *Let X and Y be definable schemes and*

$$f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$$

an isomorphism of divisorial structures. If D and $\mathrm{Eff}(f)(D)$ are effective Cartier divisors, then the induced bijection

$$f_D\colon |D| \xrightarrow{\sim} |\mathrm{Eff}(f)(D)|$$

sends definable lines to definable lines.

Proof. We have already seen in Lemma 3.37 that f_D maps definable subspaces to definable subspaces. By symmetry, the same holds for the inverse map f_D^{-1}. Lemma 3.38 shows that the definable lines in $|D|$ are precisely the minimal (with respect to inclusion) definable subspaces containing more than one point. The claim follows. $\square$

The proof of Lemma 3.38 is one of the few parts of this thesis, where we use the fact that we are working over an algebraically closed field. The following example shows the statement of the lemma may fail over fields that are not algebraically closed.

Example 3.41 ([KLOS21, Ex. 3.3.8])**.** Consider the $\mathbb{R}$-scheme $X \coloneqq V_+(X_0^2 + X_1^2 + X_2^2) \subseteq \mathbb{P}^2_{\mathbb{R}}$ and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Note that for $X_0, X_1, X_2 \in \mathbb{R}$, the equation $X_0^2 + X_1^2 + X_2^2 = 0$ implies $X_0 = X_1 = X_2 = 0$. Hence, there are no $\mathbb{R}$-valued closed points on X. Therefore, every closed point of X has residue field $\mathbb{C}$ and imposes a codimension two condition on $|\mathcal{O}_X(1)|$. This shows that there is no definable line on $|\mathcal{O}_X(1)|$ and that Lemma 3.38 may fail if one does not assume the field k_X to be algebraically closed.

As an explicit example, let $\mathcal{O}_X(1)$ be the very ample invertible sheaf associated to the given embedding $V_+(X_0^2+X_1^2+X_2^2) \subseteq \mathbb{P}^2_{\mathbb{R}}$ and consider the closed point $x \in X$ given by the maximal ideal

$$(X_0, X_1 - 1, X_2^2 + 1) \subseteq \mathbb{R}[X_0, X_1, X_2].$$

Then the restriction map

$$\varphi\colon \Gamma(X, \mathcal{O}_X(1)) \to \Gamma(X, \mathcal{O}_X(1)_{|Z_{\mathrm{red}}})$$

for $Z = \{x\}$ can be written explicitly as

$$\begin{aligned} \varphi\colon \langle X_0, X_1, X_2\rangle_{\mathbb{R}} &\to \mathbb{R} \oplus \mathbb{R} \\ aX_0 + bX_1 + cX_2 &\mapsto (b, c). \end{aligned}$$

We conclude $\ker(\varphi) = \langle X_0\rangle_{\mathbb{R}}$ and thus $V(\{x\}) = \{[X_0]\} \subseteq V(\emptyset) = |\mathcal{O}_X(1)| \cong \mathbb{P}^2_{\mathbb{R}}$.

In view of the variant Fundamental Theorem of Projective Geometry, it remains to show that the set of definable lines contains a dense subset of the Grassmannian $\mathrm{Gr}(1, |\mathcal{O}_X(1)|)$. The proof will essentially be an application of Bertini's theorem. The following two lemmas give characterizations of definable lines, making the notion tangible to statements of Bertini type.

Lemma 3.42 ([KLOS21, Lem. 3.3.10])**.** *Let D be an effective Cartier divisor on a definable scheme X of dimension at least two and $l \subseteq |D|$ be a line. Let $Z \coloneqq \bigcap_{E\in l} \mathrm{Supp}(E) \subseteq |X|$ and Z_{red} denote the associated reduced closed subscheme. Then l is a definable line if and only if the dimension of the kernel*

$$K \coloneqq \ker(\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Z, \mathcal{O}_X(D)_{|Z_{\mathrm{red}}}))$$

is equal to two.

Proof. Let $T \subseteq \Gamma(X, \mathcal{O}_X(D))$ denote the two-dimensional subspace corresponding to the line $l \subseteq |D|$. Observe that T is contained in K, since Z is contained in the zero locus of every element of T. If the dimension of K is equal to two, then this inclusion is an equality as T is a two-dimensional vector space. Lemma 3.35 then implies that $l = V(Z)$ is a definable line.

Conversely, suppose that l is a definable line. Then there is a closed subset $Z' \subseteq |X|$ such that $l = V(Z')$. Let Z'_{red} denote the associated closed subscheme with reduced structure. By Lemma 3.35, l is the projective space associated to the kernel of the restriction map

$$\varphi\colon \Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Z', \mathcal{O}_X(D)_{|Z'_{\mathrm{red}}}).$$

In particular, $T = \ker(\varphi)$. Every section $s \in T$ vanishes on Z'. Hence, Z' is contained in the set-theoretic intersection of the zero-loci of elements of T and thus $Z' \subseteq Z$. Therefore, the restriction map

$$\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Z', \mathcal{O}_X(D)_{|Z'_{\mathrm{red}}})$$

factors as

$$\Gamma(X, \mathcal{O}_X(D)) \to \Gamma(Z, \mathcal{O}_X(D)_{|Z_{\mathrm{red}}}) \to \Gamma(Z', \mathcal{O}_X(D)_{|Z'_{\mathrm{red}}}).$$

Thus $K \subseteq \ker(\varphi)$. All in all, we have $T \subseteq K \subseteq \ker(\varphi) = T$ and thus $T = K$. □

Lemma 3.43 ([KLOS21, Lem. 3.3.11])**.** *Let $\mathcal{L}$ be an invertible sheaf on a definable scheme of dimension at least two. Let $F_1, F_2 \in \Gamma(X, \mathcal{L})$ be two linearly independent sections with zero loci Z_1 and Z_2. Assume that*

(i) *Z_1 is reduced;*

(ii) *the global sections $\Gamma(Z_1, \mathcal{O}_{Z_1})$ form a one-dimensional k_X-vector space;*

(iii) *the intersection $Z := Z_1 \cap Z_2$ is reduced and does not contain any irreducible component of Z_1.*

Then the line spanned by F_1 and F_2 in $|\mathcal{L}|$ is definable.

Proof. As Z_1 is the zero locus of F_1, the section $F_1 \colon \mathcal{O}_X \to \mathcal{L}$ yields the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{F_1} \mathcal{L} \xrightarrow{p_1} \mathcal{L}_{|Z_1} \to 0. \tag{3.1}$$

As Z_1 is reduced and Z does not contain any components of Z_1, the restriction of F_2 to Z_1 yields another short exact sequence

$$0 \to \mathcal{O}_{Z_1} \xrightarrow{F_2} \mathcal{L}_{|Z_1} \xrightarrow{p_2} \mathcal{L}_{|Z} \to 0. \tag{3.2}$$

Let $\mathcal{K}$ be the kernel of the morphism $p_2 \circ p_1 \colon \mathcal{L} \to \mathcal{L}_{|Z}$. Using the exactness of (3.1) and (3.2) and the universal property of the kernel, we obtain the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{F_1} \mathcal{K} \to \mathcal{O}_{Z_1} \to 0$$

of quasicoherent sheaves on X. Taking global sections then yields the following exact sequence of k_X-vector spaces:

$$0 \to k_X \cdot F_1 \to \Gamma(X, \mathcal{K}) \to \Gamma(Z_1, \mathcal{O}_{Z_1}).$$

As $\dim_{k_X} \Gamma(Z_1, \mathcal{O}_{Z_1}) = 1$ and F_1, F_2 both vanish on Z, we conclude

$$\dim_{k_X} \ker(\Gamma(X, \mathcal{L}) \to \Gamma(Z, \mathcal{L}_{|Z})) = 2.$$

Hence, the line spanned by F_1 and F_2 is definable by Lemma 3.42. □

Using this characterization, we can finally show that the definable lines considered in this section are definable lines in the sense of Section 2.

Proposition 3.44 ([KLOS21, Prop. 3.3.12])**.** *Let X be a definable scheme of dimension at least two and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Then the set of definable lines in $|\mathcal{O}_X(1)|$ contains a dense open subset of* $\mathrm{Gr}(1, |\mathcal{O}_X(1)|)$.

Remark 3.45. Note that we do not claim that the set of definable lines is open in $\mathrm{Gr}(1, |\mathcal{O}_X(1)|)$. An example where this set is not open will be given after the proof of the proposition.

The proof uses the following variant of Bertini's theorem:

Theorem 3.46 (Bertini). *Let k be an infinite field. Let X be a quasi-projective k-scheme and $X \subseteq \mathbb{P}^n_k$ an embedding. Let $Z \subseteq X$ be a reduced finite (possibly empty) subscheme of $\mathbb{P}^n_k$ contained in the regular locus of X.*

(i) *If X is reduced, then for a general hyperplane $H \subseteq \mathbb{P}^n_k$ containing Z, the intersection $H \cap X$ is reduced.*

(ii) *If X is integral of dimension ≥ 2, then for a general hyperplane $H \subseteq \mathbb{P}^n_k$ containing Z, the intersection $X \cap H$ is integral.*

Proof. A proof of this theorem is given in [GK20, Sec. 3]. Similar results may be found in [FOV99, 3.4.10, 3.4.14]. □

Remark 3.47. In the proof of Proposition 3.44, we will only apply Theorem 3.46 with $Z = \emptyset$. However, in a later part of this thesis we will need a version of Bertini's Theorem with basepoints, i.e., Z non-empty.

Proof of Proposition 3.44. Let us first consider the case that X is projective.

By the definition of definable schemes, X is integral, separated and of finite type over the algebraically closed field $k_X = \Gamma(X, \mathcal{O}_X)$. In particular, k_X is infinite.

Let $U \subseteq \mathrm{Gr}(1, |\mathcal{O}_X(1)|)$ be the set of lines spanned by linearly independent elements $F_1, F_2 \in \Gamma(X, \mathcal{O}_X(1))$ with zero loci Z_1 and Z_2 such that

(i) Z_1 is integral;

(ii) Z_2 is reduced;

(iii) $Z_1 \cap Z_2$ is reduced and does not contain any components of Z_1.

Note that the scheme Z_1 is integral and projective over the algebraically closed field k_X. Hence $\Gamma(X, \mathcal{O}_X) \cong k_X \cong \Gamma(Z_1, \mathcal{O}_{Z_1})$ (cf. [Stacks, 0BUG]). Therefore, F_1 and F_2 satisfy the assumptions of Lemma 3.43 and thus all lines in U are definable.

It remains to show that U contains the closed points of a dense open subset of $\mathrm{Gr}(1, |\mathcal{O}_X(1)|)$. Let $X \subseteq \mathrm{Proj}(\mathrm{Sym}^\bullet(\Gamma(X, \mathcal{O}_X(1)))) \cong \mathbb{P}^n_{k_X}$ be the closed embedding given by the very ample invertible sheaf $\mathcal{O}_X(1)$. By Bertini's Theorem 3.46, a general choice of sections $F_1, F_2 \in \Gamma(X, \mathcal{O}_X(1))$ will yield hypersurfaces $H_1, H_2 \subseteq \mathbb{P}^n$ such that the intersections $Z_1 = H_1 \cap X$ and $Z_2 = H_2 \cap X$ satisfy the conditions listed above. Therefore, there is a non-empty open subset of $\mathrm{Gr}(1, |\mathcal{O}_X(1)|)$ containing the closed points corresponding to lines in U. As $\mathrm{Gr}(1, |\mathcal{O}_X(1)|)$ is irreducible, the subset is dense. This finishes the proof in case X is projective.

Now consider the general case. Let $\overline{X}$ denote the scheme-theoretic closure of X in the embedding determined by the very ample invertible sheaf $\mathcal{O}_X(1)$. As the non-zero sections of the finite-dimensional vectorspace $\Gamma(X, \mathcal{O}_X(1))$ determine the embedding

$$i \colon X \to \overline{X} \to |\mathcal{O}_X(1)|^\vee \cong \mathbb{P}^n_{k_X},$$

we have $|\mathcal{O}_{\overline{X}}(1)| = |\mathcal{O}_X(1)|$, where $\mathcal{O}_{\overline{X}}(1)$ is the very ample invertible sheaf corresponding to the embedding $\overline{X} \to |\mathcal{O}_X(1)|^\vee$. Additionally, the fact that X is scheme-theoretically dense in $\overline{X}$ implies $V_X(Z \cap X) = V_{\overline{X}}(Z)$. Hence, the arguments above applied to the projective scheme $\overline{X}$ already yield the desired result on X. □

Example 3.48 ([KLOS21, Ex. 3.3.14])**.** In the Examples 3.36 and 3.41 the sets of definable lines (trivially) form the k_X-points of an open subscheme of $\mathrm{Gr}(1, |\mathcal{L}|)$. As hinted at in Remark 3.45, this is not the case in general:

Consider the projective plane $X \coloneqq \mathbb{P}^2_k$. Let $[X_0 : X_1 : X_2]$ denote the coordinates on X. For $\alpha, \beta \in k_X$ consider the line $L \subseteq X$ cut out by the equation

$$\alpha X_0 + \beta X_1 = 0$$

and the two-dimensional linear subspace

$$T_L = \{aX_0X_1 + b(\alpha X_0 + \beta X_1)X_2 \mid a, b \in k\} \subseteq \Gamma(\mathbb{P}^2, \mathcal{O}_X(2)).$$

In particular, T_L gives rise to a line in $|\mathcal{O}_{\mathbb{P}^2}(2)|$.

If α and β are non-zero, then the set theoretic intersection of the zero loci of sections in T_L is the set

$$Z \coloneqq \{[0 : 0 : 1], [0 : 1 : 0], [0 : 0 : 1]\}.$$

The space of homogenous quadratic polynomials over k_X vanishing on Z is three-dimensional. Hence, T_L is not definable by Lemma 3.42.

If $\alpha = 0$ or $\beta = 0$, then the intersection of zero loci Z is given by the union of the line L and the point $[0 : 1 : 0]$ (resp. $[1 : 0 : 0]$). In particular, we see that the space of quadratic homogenous polynomials over k_X vanishing on Z is precisely T_L. Therefore, the line in $|\mathcal{O}_{\mathbb{P}^2_k}(2)|$ given by T_L is definable.

The image of the morphism $\mathbb{P}^1_k \to \mathrm{Gr}(1, |\mathcal{O}_{\mathbb{P}^2}(2)|)$, sending $[\alpha : \beta] \in \mathbb{P}^2_k$ to the line $\mathbb{P}(T_L) \subseteq |\mathcal{O}_{\mathbb{P}^2_k}(2)|$, is closed and irreducible. As only two points in the image correspond to definable lines by the computation above, we see that the set of definable lines on $|\mathcal{O}_{\mathbb{P}^2_k}(2)|$ is not open in $\mathrm{Gr}(1, |\mathcal{O}_{\mathbb{P}^2}(2)|)$.

Remark 3.49. Combining the propositions given in this section, we see that for a very ample invertible sheaf $\mathcal{O}_X(1)$ on a definable scheme X of dimension at least two, the set of definable lines on $|\mathcal{O}_X(1)|$ satisfies the assumptions of the variant Fundamental Theorem of Projective Geometry. In particular, note that the existence of an embedding $X \hookrightarrow |\mathcal{O}_X(1)|^\vee$ implies

$$\dim_{k_X} |\mathcal{O}_X(1)| \geq \dim X \geq 2.$$

Recall that the sweep of the subset of definable lines is the set of all points contained in a line in the maximal open subset of the set of definable lines. We conclude this section by showing that certain kinds of divisors are contained in this set.

Corollary 3.50. *Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on a definable scheme X of dimension at least two. Then every irreducible divisor $D \in |\mathcal{O}_X(1)|$ is contained in the sweep of the set of definable lines in* $\mathrm{Gr}(1, |\mathcal{O}_X(1)|)$.

Proof. As in the proof of Proposition 3.44, it suffices to consider the case of X being projective. Being an irreducible divisor, D can be viewed as an integral closed subscheme Z_1 of X. By Bertini's Theorem 3.46, a general section $F_2 \in \Gamma(X, \mathcal{O}_X(1))$ yields a hyperplane $H \subseteq \mathbb{P}^n_{k_X}$ such that Z_1 and the intersection $Z_2 \coloneqq H \cap X \subseteq \mathbb{P}^n_{k_X}$ satisfy the conditions (i) - (iii) in the proof of Proposition 3.44 and thus, the line spanned by D and $[F_2] \in |\mathcal{O}_X(1)|$ is definable.

In particular, D is contained in this line and thus in the sweep of definable lines. □

Corollary 3.51 ([KLOS21, Lem. 4.2.2])**.** *Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on a definable scheme X of dimension at least two. Then for a general point $(D_1, \ldots, D_m) \in |\mathcal{O}_X(1)|^m$, the sum $\sum_{i=1}^m D_i \in |\mathcal{O}_X(m)|$ is contained in the sweep of the set of definable lines in* $\mathrm{Gr}(1, |\mathcal{O}_X(m)|)$.

Proof. Once again, it suffices to consider the case of X being projective. The idea is the same as in the previous proof. By Bertini's Theorem 3.46, for a general point $(D_1, \ldots, D_m) \in |\mathcal{O}_X(1)|^m$, each of the D_i is reduced. The sum $\sum_{i=1}^m D_i \in |\mathcal{O}_X(m)|$ is reduced if and only there is no pair D_i, D_j $(i \neq j)$ such that D_i and D_j share a common irreducible component. As this is an open condition, we conclude that a general sum $\sum_{i=1}^m D_i \in |\mathcal{O}_X(m)|$ is reduced. Set $Z_2 \coloneqq \sum_{i=1}^m D_i \in |\mathcal{O}_X(m)|$. Applying Bertini's Theorem once more, we see that a general section $F_1 \in \Gamma(X, \mathcal{O}_X(1))$ yields a hyperplane $H \subseteq \mathbb{P}^n_{k_X}$ such that $Z_1 \coloneqq H \cap X \subseteq \mathbb{P}^n_{k_X}$ and Z_2 satisfy the conditions (i) - (iii) in the proof of Proposition 3.44.

Therefore, a general sum $\sum_{i=1}^m D_i$ of divisors in $|\mathcal{O}_X(1)|$ is contained in the sweep of definable lines in $\mathrm{Gr}(1, |\mathcal{O}_X(m)|)$. $\square$

4 The reconstruction theorem

In this section, we combine the results of Sections 2 and 3 to prove Theorem 1.1.

4.1 Final preparations

First, we need some further preparations.

Lemma 4.1 ([KLOS21, Lem. 4.2.4])**.** *Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on a definable scheme of dimension at least two. For $m \geq 1$ and a regular closed point $x \in X$, we have*

$$\{x\} = \bigcap_{D \in S} \operatorname{Supp}(D) \subseteq |X|,$$

where S is the set of irreducible divisors in $|\mathcal{O}_X(m)|$ containing x.

Proof. The very ample invertible sheaf $\mathcal{O}_X(m)$ determines an embedding $X \subseteq \mathbb{P}^n_{k_X}$. As x is a regular closed point, Bertini's Theorem (i.e., Theorem 3.46) shows that intersecting a general hyperplane $H \subseteq \mathbb{P}^n_{k_X}$ containing x with $X \subseteq \mathbb{P}^n_{k_X}$ yields an irreducible divisor $D \in |\mathcal{O}_X(m)|$ with $x \in \operatorname{Supp}(D)$. Observe that the intersection of n general hyperplanes in $\mathbb{P}^n_{k_X}$ is zero-dimensional. Hence, the intersection of $n+1$ general hyperplanes containing x in $\mathbb{P}^n_{k_X}$ is just $\{x\}$. This finishes the proof. $\square$

The claim of Lemma 4.1 may fail if we do not assume x to be a regular point. The crucial difference to Lemma 3.24, which implies that a (possibly non-regular) closed point is the unique point in the intersection of all codimension one subvarieties of X containing x, is that we are only taking the intersection over the set of *irreducible* divisors in a fixed linear system on X.

Example 4.2. Consider the quadric cone $X \coloneqq V_+(X_0^2 + X_1^2 + X_2^2) \subseteq \mathbb{P}^3_{\mathbb{C}}$. The point $x \coloneqq [0 : 0 : 0 : 1] \in X$ is a non-regular closed point of X. One can check that every divisor containing x which is given by a hyperplane section $s \in |\mathcal{O}_X(1)|$ is reducible. In fact, such a divisor is always given as the union of two distinct lines going through the point x. Hence, the claim of the previous lemma does not hold in this situation.

4.2 The proof

We finally give the proof of Theorem 1.1. Note that by the reduction steps discussed in Section 3.6, it suffices to prove the following theorem:

Theorem 4.3. *Let X and Y be locally-factorial quasi-projective definable schemes of dimension at least two. If there is an isomorphism of divisorial structures $f\colon \tau(X) \xrightarrow{\sim} \tau(Y)$, then there is an isomorphism of schemes $\varphi\colon X \xrightarrow{\sim} Y$ such that the underlying map on topological spaces $|\varphi|\colon |X| \to |Y|$ agrees with f.*

Proof. We follow the proof given in [KLOS21, Prop. 4.2.5]. Let D be a very ample effective Cartier divisor on X. Let $\mathcal{O}_X(1) \coloneqq \mathcal{O}_X(D)$ denote the associated very ample invertible sheaf. As Y is locally-factorial, the divisor $\operatorname{Eff}(f)(D)$ is Cartier. By Proposition 3.30, the fact that D is ample implies that $\operatorname{Eff}(f)(D)$ is ample as well. Let $\mathcal{O}_Y(1) \coloneqq \mathcal{O}_Y(\operatorname{Eff}(f)(D))$ denote the associated invertible sheaf. Replacing D by nD for some $n \gg 0$, we may assume that $\mathcal{O}_Y(1)$ is very ample and that the multiplication maps

$$\Gamma(X, \mathcal{O}_X(1))^{\otimes m} \to \Gamma(X, \mathcal{O}_X(m))$$

and
$$\Gamma(Y,\mathcal{O}_Y(1))^{\otimes m} \to \Gamma(Y,\mathcal{O}_Y(m))$$
are surjective for all $m \geq 1$ (cf. [Laz04, 1.2.22]).

Let $S_X \coloneqq \mathrm{Sym}^\bullet(\Gamma(X,\mathcal{O}_X(1)))$ and $S_Y \coloneqq \mathrm{Sym}^\bullet(\Gamma(Y,\mathcal{O}_Y(1)))$ denote the symmetric algebras generated by the global sections of $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$. The very ample invertible sheaves give rise to embeddings
$$X \hookrightarrow \mathrm{Proj}(S_X) \text{ and } Y \hookrightarrow \mathrm{Proj}(S_Y).$$
Let $I_X \subseteq S_X$ and $I_Y \subseteq S_Y$ denote the homogenous ideals cutting out the scheme-theoretic closures $\overline{X}, \overline{Y}$ of the images of X and Y in $\mathrm{Proj}(S_X)$ and $\mathrm{Proj}(S_Y)$. Observe that the m-th graded part of I_X coincides with the kernel of the multiplication map given above:
$$I_{X,m} = \ker(\Gamma(X,\mathcal{O}_X(1))^{\otimes m} \twoheadrightarrow \Gamma(X,\mathcal{O}_X(m)).$$
By Lemma 3.14, the isomorphism of divisorial structures f induces bijections
$$f_m \colon |\mathcal{O}_X(m)| \to |\mathcal{O}_Y(m)|.$$
As discussed in Section 3.7, the definable lines on the linear system $|\mathcal{O}_X(m)|$ introduced in Definition 3.34 satisfy the conditions of the variant Fundamental Theorem of Projective Geometry stated in Section 2. Hence, there are isomorphisms of fields $\sigma_m \colon k_X \to k_Y$ and σ_m-linear isomorphisms $\gamma_m \colon \Gamma(X,\mathcal{O}_X(m)) \to \Gamma(Y,\mathcal{O}_Y(m))$ such that the projectivization
$$\mathbb{P}(\gamma_m) \colon |\mathcal{O}_X(m)| \to |\mathcal{O}_Y(m)|$$
agrees with f_m on a dense open subset containing the sweep of the set of definable lines in $|\mathcal{O}_X(m)|$.

Note that the σ-linear isomorphism $\gamma_1 \colon \Gamma(X,\mathcal{O}_X(1)) \xrightarrow{\sim} \Gamma(Y,\mathcal{O}_Y(1))$ induces an isomorphism
$$\gamma^\# \colon S_X \xrightarrow{\sim} S_Y$$
of algebras. Our aim is to show that $\gamma^\#$ induces an isomorphism between X and Y.

Let $+_{X,m} \colon |\mathcal{O}_X(1)|^{\times m} \to |\mathcal{O}_X(m)|$ (respectively $+_{Y,m}$) denote the addition map on divisors. Explicitely, this map is given as
$$\begin{gathered} |\mathcal{O}_X(1)|^{\times m} = \mathbb{P}(\Gamma(X,\mathcal{O}_X(1))^{\times m} \to \mathbb{P}(\Gamma(X,\mathcal{O}_X(m)) = |\mathcal{O}_X(m)| \\ ([s_1],\dots,[s_m]) \mapsto [s_1 \cdots s_m]. \end{gathered}$$
Observe that the diagram
$$\begin{tikzcd} |\mathcal{O}_X(1)|^{\times m} \arrow[r, "f_1^{\times m}"] \arrow[d, "+_{X,m}"] & |\mathcal{O}_Y(1)|^{\times m} \arrow[d, "+_{Y,m}"] \\ |\mathcal{O}_X(m)| \arrow[r, "f_m"] & |\mathcal{O}_Y(m)| \end{tikzcd}$$
of maps of sets commutes since addition of divisors is compatible with f. As a general sum of divisors in $|\mathcal{O}_X(1)|$ is contained in the set on which $\mathbb{P}(\gamma_m)$ and f_m agree by Lemma 3.51, the diagram
$$\begin{tikzcd} |\mathcal{O}_X(1)|^{\times m} \arrow[r, "\mathbb{P}(\gamma_1)^{\times m}"] \arrow[d, "+_{X,m}"] & |\mathcal{O}_Y(1)|^{\times m} \arrow[d, "+_{Y,m}"] \\ |\mathcal{O}_X(m)| \arrow[r, "\mathbb{P}(\gamma_m)"] & |\mathcal{O}_Y(m)| \end{tikzcd} \tag{4.1}$$
commutes on a dense open subset of $|\mathcal{O}_X(1)|^m$.

Claim 4.4. *The two isomorphisms of fields $\sigma_1, \sigma_m \colon k_X \xrightarrow{\sim} k_Y$ are equal.*

The proof of this claim is elementary but tedious. Therefore, we postpone it to the end of the section.

Set $\sigma := \sigma_1 = \sigma_m$. The maps $\mathbb{P}(\gamma_m) \circ (+_{X,m})$ and $(+_{Y,m}) \circ \mathbb{P}(\gamma_1)^{\times m}$ are projectivizations of σ-linear maps. As σ-linear maps over infinite fields are determined by their evaluation on non-empty open subsets, we see that diagram (4.1) commutes not only on a dense open subset but the whole of $|\mathcal{O}_X(1)|^{\times m}$. This then implies that the associated diagram of σ-linear maps

$$\begin{tikzcd} I_{X,m} \arrow[d] & I_{Y,m} \arrow[d] \\ \Gamma(X, \mathcal{O}_X(1))^{\otimes m} \arrow[r, "\gamma_1^{\otimes m}"] \arrow[d] & \Gamma(Y, \mathcal{O}_Y(1))^{\otimes m} \arrow[d] \\ \Gamma(X, \mathcal{O}_X(m)) \arrow[r, "\gamma_m"] & \Gamma(Y, \mathcal{O}_Y(m)) \end{tikzcd}$$

commutes up to a scalar. In particular, we get $\gamma_m^{\#}(I_{X,m}) = I_{Y,m}$ and thus $\gamma^{\#}(I_X) = I_Y$. Hence, we obtain an isomorphism $\varphi \colon \overline{X} \xrightarrow{\sim} \overline{Y}$ of schemes fitting into the diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "f"] \arrow[d, "\wr"] & \overline{Y} \arrow[d, "\wr"] \\ \operatorname{Proj}\left(\bigoplus_{m \geq 0} \Gamma(X, \mathcal{O}_X(m))\right) \arrow[r] \arrow[d, hook] & \operatorname{Proj}\left(\bigoplus_{m \geq 0} \Gamma(Y, \mathcal{O}_Y(m))\right) \arrow[d, hook] \\ \operatorname{Proj}(S_X) \arrow[r, "\gamma^{\#}"] & \operatorname{Proj}(S_Y) \end{tikzcd} \tag{4.2}$$

of morphisms of schemes.

However, a priori, it is not clear whether φ restricts to an isomorphism $\varphi_{|X} \colon X \xrightarrow{\sim} Y$. In the remaining part of the proof, we fix this by possibly modifying $\varphi_{|X}$ on a closed subset of codimension two.

Claim 4.5. *The continuous maps $|\varphi|$ and f agree on the regular locus of X.*

Proof. Let $D \subseteq \overline{X}$ be an irreducible divisor in $|\mathcal{O}_{\overline{X}}(m)|$. As diagram (4.2) commutes, it follows that the restriction $\operatorname{Eff}(\varphi)(D)_{|Y} \in |\mathcal{O}_Y(m)|$ of the image of D under f is given by $\mathbb{P}(\gamma_m)(D)$. The irreducible divisor D is contained in the sweep of the set of definable lines by Lemma 3.50. Hence,

$$\operatorname{Eff}(\varphi)(D)_{|Y} = \mathbb{P}(\gamma_m)(D) = f_m(D).$$

By the characterization of regular closed points as intersections of irreducible divisors in $|\mathcal{O}_X(m)|$ given in Lemma 4.1, we conclude that $|\varphi|$ and f agree on the underlying topological space of the regular locus of X. □

Let $U \subseteq |X|$ denote the regular locus of X. We have $\operatorname{codim}(X \setminus U) \geq 2$, as X is normal. Additionally, U is open, non-empty and thus dense in the integral scheme X. As $|\varphi|$ agrees with f on U, Proposition 3.22 implies that there is a unique isomorphism $\varphi' \colon X \xrightarrow{\sim} Y$ such that $|\varphi'| = f$. This finishes the proof of Theorem 4.3, modulo the proof of Claim 4.4. □

Proof of Claim 4.4. We closely follow the proof given in [KLOS21, Lem. 4.2.6]. Let $U_m \subseteq |\mathcal{O}_X(m)|$ be the sweep of the maximal open subset in the set of definable lines. Then $U_1^{\times m} \subseteq$

$|\mathcal{O}_X(1)|^{\times m}$ is a nonempty open subset, and therefore

$$V \coloneqq +_X^{-1}(U_m) \cap U_1^{\times m} \subseteq |\mathcal{O}_X(1)|^{\times m}$$

is non-empty and open as well. In particular, there are points

$$P, Q, R, P_2, \dots, P_m \in |\mathcal{O}_X(1)|$$

such that the points

$$(P, P_2, \dots, P_m), (Q, P_2, \dots, P_m), (R, P_2, \dots, P_m) \in |\mathcal{O}_X(1)|^{\times m}$$

are contained in V and $P, Q, R \in |\mathcal{O}_X(m)|$ lie on a line $L \subseteq |\mathcal{O}_X(1)|$. Since $\mathbb{P}(\gamma_1)$ and $\mathbb{P}(\gamma_m)$ agree with the maps induced by f on U_1 and U_m, we have

$$\overline{P} \coloneqq (\mathbb{P}(\gamma_m) \circ +_X)(P, P_2, \dots, P_m) = (+_Y \circ \mathbb{P}(\gamma_1)^{\times m})(P, P_2, \dots, P_m) \in |\mathcal{O}_Y(m)|$$

$$\overline{Q} \coloneqq (\mathbb{P}(\gamma_m) \circ +_X)(Q, P_2, \dots, P_m) = (+_Y \circ \mathbb{P}(\gamma_1)^{\times m})(Q, P_2, \dots, P_m) \in |\mathcal{O}_Y(m)|$$

$$\overline{R} \coloneqq (\mathbb{P}(\gamma_m) \circ +_X)(R, P_2, \dots, P_m) = (+_Y \circ \mathbb{P}(\gamma_1)^{\times m})(R, P_2, \dots, P_m) \in |\mathcal{O}_Y(m)|$$

Let $\overline{L} \subseteq |\mathcal{O}_Y(m)|$ denote the line through $\overline{P}$ and $\overline{Q}$. Observe that $+_X(L \times \{P_2\} \times \cdots \times \{P_m\})$ is the line in $|\mathcal{O}_X(m)|$ through the two points

$$+_X(P, P_2, \dots, P_m) \text{ and } +_X(Q, P_2, \dots, P_m).$$

As γ_m is a σ_m-linear map, $\mathbb{P}(\gamma_m)$ sends lines to lines and thus the above implies that

$$(\mathbb{P}(\gamma_m) \circ +_X)(L \times \{P_2\} \times \cdots \times \{P_m\}) = \overline{L}.$$

Similarly, $\mathbb{P}(\gamma_1)$ sends lines to lines and agrees on U_1 with the map defined by f. It follows that

$$(+_Y \circ \mathbb{P}(\gamma_1)^{\times m})(L \times \{P_2\} \times \cdots \times \{P_m\}) = \overline{L}.$$

Via the identification

$$L \cong L \times \{P_2\} \times \cdots \times \{P_m\},$$

we can view L as a subset of $|\mathcal{O}_X(1)|^{\times m}$. Recall that $\mathbb{P}(\gamma_m) \circ +_X$ and $+_Y \circ \mathbb{P}(\gamma_1)^{\times m}$ agree on a dense open subset of $|\mathcal{O}_X(1)|^{\times m}$ containing the point $(P_1, \dots, P_m)$. Hence, the two compositions

$$k_X \xrightarrow{\alpha} L \subseteq |\mathcal{O}_X(1)|^{\times m} \xrightarrow{\mathbb{P}(\gamma_m) \circ +_X} \overline{L} \xrightarrow{\beta^{-1}} k_Y$$

and

$$k_X \xrightarrow{\overline{\alpha}} L \subseteq |\mathcal{O}_X(1)|^{\times m} \xrightarrow{+_Y \circ \mathbb{P}(\gamma_1)^{\times m}} \overline{L} \xrightarrow{\overline{\beta}^{-1}} k_Y$$

agree on all but finitely many elements of k_X. Here, $\alpha \colon k_X \xrightarrow{\sim} L$, $\beta \colon k_Y \xrightarrow{\sim} \overline{L}$ (resp. $\overline{a}$, $\overline{\beta}$) are the bijections obtained as in the proof of Theorem 2.9 using the three points P, Q, R (resp. $\overline{P}, \overline{Q}, \overline{R}$). Observe that the first map is σ_m and the second one is σ_1. All in all, we have seen that $\sigma_1(a) = \sigma_m(a)$ for all but finitely many elements $a \in k_X$. As k_X is an infinite field, this implies $\sigma_1 = \sigma_m$. $\square$

This finally completes the proof of Theorem 1.1. Let us conclude the section with a few remarks.

Remark 4.6. As alluded to in Example 3.12, we do not recover the isomorphism class of X as a k_X-scheme but only as an abstract scheme over $\mathbb{Z}$. Note however, that the construction above yields an isomorphism of fields $\sigma\colon k_X \xrightarrow{\sim} k_Y$ fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec}(k_X) \arrow[r, "\sigma"] & \operatorname{Spec}(k_Y). \end{tikzcd}$$

A posterio, this isomorphism of fields may also be recovered using the isomorphisms

$$k_X \cong \Gamma(X, \mathcal{O}_X) \xrightarrow{\sim} \Gamma(Y, \mathcal{O}_Y) \cong k_Y.$$

Remark 4.7. The assumption that k_X is algebraically closed may be dropped if one instead assumes X to be geometrically integral over an infinite field k_X. We will not prove this but give a rough sketch of the changes which have to be made to obtain the more general result. Recall the parts of this thesis where we used the fact that the ground field is algebraically closed:

- In the proof of Proposition 3.44, i.e., the proof of the claim that the set of definable lines on a linear system contains a dense open subset, we utilized the fact that the field k_X is algebraically closed in two ways. On the one hand, k_X being infinite allowed us to apply Bertini's theorem. On the other hand, we used the fact that X is integral over the algebraically closed field k_X to deduce that $\Gamma(X, \mathcal{O}_X) \cong k_X$. This then allowed us to apply the characterization of definable lines given in Lemma 3.43. Note that this is still possible if we just assume X to be geometrically integral over an infinite field.

- In the proof of the fact that the set of definable lines is determined by the divisorial structure, i.e., Lemma 3.38 and Corollary 3.40, we used the fact that k_X is algebraically closed to deduce that one can always find full flags of linear subspaces on a given linear system. This is not possible over arbitrary fields (cf. Example 3.41). Hence, for arbitrary infinite fields one must apply different methods to conclude that the divisorial structure determines the set of definable lines. In [KLOS21], this issue is resolved by proving the following lemma, using methods similar to the ones we have seen in Section 2.

  Lemma 4.8 ([KLOS21, Thm. 2.2.1]). *Let (k_1, V_1, U_1) and (k_2, V_2, U_2) be definable projective spaces. Assume that*
  $$\varphi\colon \mathbb{P}(V_1) \to \mathbb{P}(V_2)$$
  is a bijection such that for every definable line $L \in U_1$, the image $\varphi(L) \subseteq \mathbb{P}(V_2)$ is a linear subspace and for every definable line $L' \in U_2$, $\varphi^{-1}(L') \subseteq \mathbb{P}(V_1)$ is a linear subspace. Then φ maps definable lines to lines in $\mathbb{P}(V_2)$.

  Note that by Proposition 3.37, which does not critically rely on the fact that k_X is algebraically closed, the bijections between linear systems induced by an isomorphism of divisorial structures satisfy the conditions of Lemma 4.8.

Applying the changes listed in the remark above, one obtains the following generalization of Theorem 1.1:

Theorem ([KLOS21, Main Theorem]). *Let X be a proper normal geometrically integral variety of dimension at least 2 over an infinite field. Then X is determined as a scheme by the pair*

$$(|X|,\ c\colon X^{(1)} \to \mathrm{Cl}(X)),$$

where $X^{(1)}$ is the set of codimension one points of X and c is the map sending a codimension one point of X to its divisor class.

5 Examples and stronger results

In this section, we discuss examples and stronger results related to Theorem 1.1. In particular, we are interested in the following question.

Question 5.1. *Given a scheme X as in the statement of Theorem 1.1, is $\tau(X)$ already determined by the topological space $|X|$ alone?*

One is immediately tempted to formulate the following corollary to Theorem 1.1:

Corollary 5.2. *Let X and Y be proper normal integral schemes of dimension at least 2 over algebraically closed fields k_X and k_Y. If $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$ are trivial, then X and Y are isomorphic as schemes if and only if the underlying topological spaces $|X|$ and $|Y|$ are homeomorphic.*

However, it is easy to see that this corollary is somewhat tautological.

Proposition 5.3. *Every definable projective scheme X with trivial class group is zero-dimensional.*

Proof. Since X is projective, there is a very ample invertible sheaf $\mathcal{O}_X(1)$ on X. As $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X) = 0$, we have $\mathcal{O}_X(1) \cong \mathcal{O}_X$. Therefore

$$\dim X \leq \dim_{k_X} \Gamma(X, \mathcal{O}_X(1)) - 1 = \dim_{k_X} \Gamma(X, \mathcal{O}_X) - 1 = 0$$

and hence, X is zero-dimensional. $\square$

Remark 5.4. If we drop the condition on X to be projective, then the claim of the previous proposition may fail. One construction of a proper normal surface with trivial Picard group is given in [Sch99].

5.1 Curves

In this section, a *curve over a field k* is a proper geometrically integral one-dimensional k-scheme. We will present a counterexample to Theorem 1.1 in the case of curves over non-algebraically closed fields (cf. 4.7). Afterwards, we apply a Theorem of Torelli type proved by B. Zilber to obtain a variant of Theorem 1.1 in the case of curves over algebraically closed field.

Franchetta's conjecture [Sch03, Thm. 5.1] states that the class group of the *generic curve of genus g over k* is an infinite cyclic group generated by the canonical divisor. We exploit this fact to give an example of non-isomorphic curves with isomorphic divisorial structures. Let us first recall the definition of the generic curve, following the exposition given in [Sch03, p. 3].

Fix an algebraically closed field k. Let $g \geq 3$ be an integer. Let M_g be the *coarse moduli space* of smooth curves of genus g over k. Deligne and Mumford [DM69] have shown that M_g is an irreducible scheme of finite type over k. The closed points of M_g correspond to isomorphism classes of smooth curves over k. Moreover, there is a coarse moduli space $M_{g,1}$ of pointed smooth curves of genus g over k. Forgetting the pointed structure yields a morphism $M_{g,1} \to M_g$. The fiber of this morphism over the generic point $\eta_g \in M_g$ is denoted C_g and called the *generic curve of genus g over k*. It turns out that C_g is again a smooth curve of genus g over the function field $k(\eta_g)$ (cf. [Sch03, p. 3]). Note that, while k is assumed to be algebraically closed, the function field $k(\eta_g)$ is not algebraically closed.

Proposition 5.5 ([KLOS21, Prop. 4.5.1.])**.** *For any pair of integers $g, h \geq 3$, there is an isomorphism of divisorial structures*

$$\tau(C_g) \xrightarrow{\sim} \tau(C_h).$$

Proof. As observed multiple times before, a homeomorphism between C_g and C_h is nothing but a bijection between the sets of closed points. The difficult part is finding such a bijection that respects the class group. By Franchetta's conjecture [Sch03, Thm. 5.1], we have

$$\mathrm{Cl}(C_g) = \mathbb{Z} \cdot [K_{C_g}],$$

where K_{C_g} is the canonical divisor on the curve C_g. Let $c_g\colon C_g^{(1)} \to \mathrm{Cl}(C_g) = \mathbb{Z} \cdot [K_{C_g}]$ denote the map sending a closed point to its class in the divisor class group. As $\deg K_{C_g} = 2g - 2 > 0$, we have $\mathrm{im}(c_g) \subseteq \mathbb{N} \cdot [K_{C_g}]$. For $n \in \mathbb{N}$, we define

$$D_g(n) \coloneqq \{x \in C_g^{(1)} \mid c_g(x) = n \cdot [K_{C_g}]\}.$$

We claim that there is a bijection $D_g(n) \leftrightarrow k(\eta_g)$. Consider the linear system $|nK_{C_g}|$ for $n \geq 1$. We want to show that $D_g(n) \subseteq |nK_{C_g}|$ is a non-empty open subset. The claim then follows from the fact that, over an infinite field, a non-empty open subset of a positive-dimensional projective space has the same cardinality as the ground field.

Using the Riemann–Roch formula (cf. [Har77, Thm. IV.1.3]), we compute

$$\begin{aligned} \dim |nK_{C_g}| &= \dim |(1-n)K_{C_g}| + n \deg(K_{C_g}) + 1 - g \\ &= \dim |(1-n)K_{C_g}| + (2n-1)g + 1 - 2n. \end{aligned}$$

As $\deg(1-n)K_{C_g} = (1-n)(2g-2) < 0$ for $n > 1$, the above simplifies to

$$\dim |nK_{C_g}| = \begin{cases} g - 1 & \text{if } n = 1 \\ (2n-1)g - 2n & \text{if } n > 1. \end{cases}$$

In particular, we have $\dim |nK_{C_g}| + \dim |mK_{C_g}| < \dim |(n+m)K_{C_g}|$ for all $n, m \geq 1$. Hence, the image of the natural map

$$+_{n,m}\colon |nK_{C_g}| \times |mK_{C_g}| \to |(n+m)K_{C_g}|,$$

given by addition of divisors, is a proper closed subset. Observe that $D_g(n) \subseteq |nK_{C_g}|$ is precisely the set of points not contained in the images of any of the addition maps. We conclude that $D_g(n) \subseteq |nK_{C_g}|$ is a non-empty open subset.

Hence, we have established the existence of bijections $D_g(n) \leftrightarrow k(\eta_g)$. Furthermore, as $k(\eta_g)/k$ is a field extension of finite transcendence degree $3g - 3$ and k is infinite, we observe that $k(\eta_g)$ and k have the same cardinality. We conclude that there is a bijection $D_g(n) \leftrightarrow k$.

Applying the arguments above to C_h, we see that for each $n \geq 1$, there is a bijection

$$D_g(n) \leftrightarrow k \leftrightarrow D_h(n).$$

As $C_g^{(1)} = \bigsqcup_{n \geq 1} D_g(n)$, these bijections compose to a bijection $C_g^{(1)} \leftrightarrow C_h^{(1)}$ that respects the divisorial structure. This finishes the proof. $\square$

For $g \neq h$, the curves C_g and C_h are not isomorphic as schemes, since they are geometrically integral smooth curves of different genus. Hence, Proposition 5.5 shows that Theorem 1.1 may fail for schemes of dimension one over non-algebraically closed fields (cf. Remark 4.7).

On the other hand, it turns out that a variant of Theorem 1.1 holds for curves over algebraically closed fields.

Let k be an algebraically closed field and C a smooth projective curve over k of genus g greater than one. Recall that the group $\mathrm{Pic}^0(C) \subseteq \mathrm{Cl}(X)$ is the subgroup of divisors of degree zero modulo rational equivalence. Fixing a point P on C, we obtain an injective map

$$i_P\colon C^{(1)} \to \mathrm{Pic}^0(C),\ x \mapsto [x] - [P].$$

The following theorem of Torelli type proved by B. Zilber in [Zil12] using model-theoretic methods states that this datum is enough to recover C as a scheme:

Theorem 5.6 ([Zil12])**.** *Let k be an algebraically closed field. Let C and D be smooth projective curves over k of genus at least two. Assume that there are points $P \in C^{(1)}$, $Q \in D^{(1)}$, a bijection*

$$f\colon C^{(1)} \to D^{(1)}$$

between the sets of closed points, and an isomorphism of groups

$$\tilde{f}\colon \operatorname{Pic}^0(C) \xrightarrow{\sim} \operatorname{Pic}^0(D)$$

such that the diagram

$$\begin{tikzcd} C^{(1)} \arrow[r, "f"] \arrow[d, "i_P"] & D^{(1)} \arrow[d, "i_Q"] \\ \operatorname{Pic}^0(C) \arrow[r, "\tilde{f}"] & \operatorname{Pic}^0(D) \end{tikzcd}$$

commutes. Then C and D are isomorphic as schemes.

Observing that the divisorial structure already determines the datum considered in the above theorem, we obtain a variant of Theorem 1.1 for smooth projective curves of genus at least two:

Corollary 5.7. *Let k be an algebraically closed field. Let C and D be smooth projective curves over k of genus at least two over k. If there is an isomorphism of divisorial structures*

$$f\colon \tau(C) \xrightarrow{\sim} \tau(D),$$

then C and D are isomorphic as schemes.

Proof. Let $c_f\colon \operatorname{Cl}(C) \xrightarrow{\sim} \operatorname{Cl}(D)$ denote the group isomorphism given by the isomorphism of divisorial structures. Let us first show that c_f restricts to an isomorphism $\operatorname{Pic}^0(C) \to \operatorname{Pic}^0(D)$. By definition, we have

$$\operatorname{Pic}^0(C) = \ker(\deg\colon \operatorname{Cl}(C) \to \mathbb{Z}).$$

Hence, it suffices to show that the degree map $\deg\colon \operatorname{Cl}(C) \to \mathbb{Z}$ is determined by the divisorial structure. Indeed, for every $x \in C^{(1)}$ one has $\deg([x]) = 1 = \deg([f(x)]) = \deg(c_f([x]))$ as k is algebraically closed. As the divisor class group is generated by the classes of closed points, the claim follows.

Fix a point $P \in C^{(1)}$. Set $Q := f(P) \in D^{(1)}$. Then we have

$$c_f \circ i_P = c_f([x] - [P]) = c_f([x]) - c_f([P]) = [f(x)] - [f(P)] = [f(x)] - [Q] = i_Q \circ f$$

for all $x \in C^{(1)}$. All in all, we see that the assumptions of Theorem 5.6 are satisfied and thus C and D are isomorphic as schemes. $\square$

5.2 Positive characteristic

Recall the question posed at the beginning of this section.

Question. *Given a scheme X as in the statement of Theorem 1.1, is $\tau(X)$ already determined by the topological space $|X|$ alone?*

In general, the answer to this question is negative:

Proposition 5.8 ([Wie81, Cor. 1]). *Let $p, q \in \mathbb{Z}$ be prime numbers. Then the underlying topological spaces of the projective planes*

$$\mathbb{P}^2_{\overline{\mathbb{F}}_p} \quad \text{and} \quad \mathbb{P}^2_{\overline{\mathbb{F}}_q}$$

over the algebraic closures of the respective finite fields are homeomorphic.

Proof. See [Wie81, Cor. 1]. In fact, the proof shows that every homeomorphism

$$|C| \xrightarrow{\sim} |C'|$$

between curves $C \subseteq \mathbb{P}^2_{\overline{\mathbb{F}}_p}$ and $C' \subseteq \mathbb{P}^2_{\overline{\mathbb{F}}_q}$ can be extended to a homeomorphism $|\mathbb{P}^2_{\overline{\mathbb{F}}_p}| \xrightarrow{\sim} |\mathbb{P}^2_{\overline{\mathbb{F}}_q}|$. $\square$

Note that for $p \neq q$, the schemes $\mathbb{P}^2_{\overline{\mathbb{F}}_p}$ and $\mathbb{P}^2_{\overline{\mathbb{F}}_q}$ are not isomorphic as the rings of global regular functions are of different characteristic. In case $p = q$, the proof of the proposition allows us to construct homeomorphisms between isomorphic definable schemes that do not arise as underlying continuous maps of morphisms of schemes:

Example 5.9. Let $k := \overline{\mathbb{F}}_p$ and $H \subseteq \mathbb{P}^2$ be a hyperplane, i.e., a line. Pick two distinct closed points $x_1, x_2 \in H$. The bijection $f\colon |H| \to |H|$, given by swapping the points x_1 and x_2, and the identity everywhere else, is a self-homeomorphism of the Zariski topology on H. On the other hand, one easily verifies that f is not the underlying map of a morphism of schemes.

By [Wie81, Cor. 1], the homeomorphism between H and itself can be extended to a homeomorphism

$$|\mathbb{P}^2_k| \xrightarrow{\sim} |\mathbb{P}^2_k|.$$

Restriction to $H \subseteq \mathbb{P}^2$ then shows that this homeomorphism is not the underlying map of a morphism of schemes.

Similarly, we obtain a counterexample to Theorem 1.1 in the affine case.

Proposition 5.10. *There is an isomorphism*

$$f\colon \tau(\mathbb{A}^2_{\overline{\mathbb{F}}_p}) \xrightarrow{\sim} \tau(\mathbb{A}^2_{\overline{\mathbb{F}}_q})$$

between the divisorial structures of affine planes over the algebraic closures of finite fields.

Proof. To simplify notation, set $K := \overline{\mathbb{F}}_p$ and $L := \overline{\mathbb{F}}_q$. As $K[x, y]$ and $L[x, y]$ are unique factorization domains, we have $\mathrm{Cl}(\mathbb{A}^2_K) = 0 = \mathrm{Cl}(\mathbb{A}^2_L)$. Hence, it suffices to construct a homeomorphism $f\colon \mathbb{A}^2_K \xrightarrow{\sim} \mathbb{A}^2_L$.

Let $H_p \subseteq \mathbb{P}^2_K$, $H_q \subseteq \mathbb{P}^2_L$ be hyperplanes (i.e., lines). Note that H_q and H_p are both countable as sets. Any bijection $H_p \leftrightarrow H_q$ that maps the generic point of H_p to the generic point of H_q is a homeomorphism of Zariski topologies. The proof of Proposition 5.10 ([Wie81, Cor. 1]) shows that one can extend this homeomorphism to a homeomorphism $f\colon |\mathbb{P}^2_K| \to |\mathbb{P}^2_L|$. Restricting f to the complements of H_p and H_q then yields a homeomorphism

$$\mathbb{A}^2_K \cong \mathbb{P}^2_K \setminus H_p \xrightarrow{f} \mathbb{P}^2_L \setminus H_q \cong \mathbb{A}^2_L$$

between the underlying topological spaces of the affine planes. $\square$

Note that for $p \neq q$, the schemes $\mathbb{A}^2_K$ and $\mathbb{A}^2_L$ are not isomorphic, since the corresponding rings $K[x, y]$ and $L[x, y]$ are of different characteristic.

Another class of examples of homeomorphisms, which are not induced by isomorphisms of schemes, arises from purely inseparable morphisms, i.e., morphisms that induce purely inseparable extensions on function fields. A prototypical example is the so-called Frobenius morphism.

From now on, fix a perfect field k of characteristic $p > 0$.

Definition 5.11. Let X be a scheme over k. The *absolute Frobenius* is the morphism

$$F_X \colon X \to X$$

given as the identity on the underlying topological spaces and

$$F_X^{\#} \colon \mathcal{O}_X \to \mathcal{O}_X,\ f \mapsto f^p$$

on structure sheaves.

Note that F_X is not a morphism in the category of k-schemes in general.

Definition 5.12. The *Frobenius twist* $X^{(p)}$ of a k-scheme X is the fiber product

$$\begin{tikzcd} X^{(p)} := X \times_{\mathrm{Spec}(k), F_{\mathrm{Spec}(k)}} \mathrm{Spec}(k) \arrow[r] \arrow[d, "\varphi"] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "F_{\mathrm{Spec}(k)}"] & \mathrm{Spec}(k), \end{tikzcd}$$

where the projection $\varphi \colon X^{(p)} \to \mathrm{Spec}(k)$ realizes $X^{(p)}$ as a k-scheme. Via the universal property of the fiber product applied to $(X \to \mathrm{Spec}(k),\ F_X \colon X \to X)$, we obtain a morphism

$$F_{X/k} \colon X \to X^{(p)},$$

which is called the *relative Frobenius*. By construction, $F_{X/k}$ is a morphism of k-schemes.

Note that as k is perfect, $F_{\mathrm{Spec}(k)}$ is an isomorphism of schemes, and thus, $X^{(p)}$ and X are isomorphic as abstract schemes. However, we will later see examples where X and $X^{(p)}$ are not isomorphic over k.

Example 5.13. For $n \geq 1$, we have canonical isomorphisms

$$(\mathbb{P}^n_k)^{(p)} \cong \mathbb{P}^n_k \times_{F_k, k} \mathrm{Spec}(k) \cong (\mathbb{P}^n_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathrm{Spec}(k)) \times_{k, F_k} \mathrm{Spec}(k) \cong \mathbb{P}^n_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathrm{Spec}(k) \cong \mathbb{P}^n_k.$$

Under this identification, the relative Frobenius $F_{\mathbb{P}^n/k} \colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ is given by $[x, y] \mapsto [x^p, y^p]$.

We recall some properties of the Frobenius twist and relative Frobenius.

Lemma 5.14. *Let X be a geometrically integral, proper k-scheme. Then $X^{(p)}$ is a geometrically integral, proper k-scheme as well, and the relative Frobenius $F_{X/k}$ is a finite dominant morphism of degree $p^{\dim X}$. Furthermore, $F_{X/k}$ is a universal homeomorphism in the sense of [Stacks, 04DC].*

Proof. See [Stacks, 0CC6]. ☐

Lemma 5.15. *Let $f \colon X \to Y$ be a morphism of k-schemes. Then the diagram*

$$\begin{tikzcd} X \arrow[r, "F_{X/k}"] \arrow[d, "f"] & X^{(p)} \arrow[d, "f^{(p)}"] \\ Y \arrow[r, "F_{Y/k}"] & Y^{(p)} \end{tikzcd}$$

commutes, where $f^{(p)} := f \times_{k, F_k} \mathrm{id}_k$ is the base change of f along $F_{\mathrm{Spec}(k)}$.

Proof. See [Stacks, 0CCA]. ☐

Lemma 5.16. *If C is a smooth curve (cf. Section 5.1) over k, then $C^{(p)}$ is smooth as well and*

$$g(C) = g(C^{(p)}).$$

Proof. Since smoothness is stable under base change (cf. [Stacks, 01VB]), smoothness of C over k directly implies smoothness of $C^{(p)}$ over k. The claim about the genus of $C^{(p)}$ follows from the fact that $g(C) = \dim_k H^1(C, \mathcal{O}_C)$ and

$$H^1(C^{(p)}, \mathcal{O}_{C^{(p)}}) = H^1(C, \mathcal{O}_C) \otimes_{k,F} k$$

by flat base change (cf. [Stacks, 02KH]). $\square$

Remark 5.17. As stated initially, we are interested in homeomorphisms that are not induced by isomorphisms of schemes. Lemma 5.14 shows that, for $\dim X > 0$, the relative Frobenius $F_{X/k}$ is not an isomorphism of schemes for degree reasons. However, the underlying map on topological spaces is induced by an isomorphism of schemes.

To see this, let $G\colon X^{(p)} \to X$ denote the projection of the Frobenius twist onto X. Being the pullback of the isomorphism $F_{\mathrm{Spec}(k)}$ along $X \to \mathrm{Spec}(k)$, the morphism G is an isomorphism. By construction, we have $G \circ F_{X/k} = F_X$. As the absolute Frobenius F_X is the identity on topological spaces, this implies that the underlying continuous maps of $F_{X/k}$ and G^{-1} agree.

However, we can use the relative Frobenius to construct examples of the desired form in the following way:

Proposition 5.18. *Let X be an integral proper variety over an algebraically closed field k of characteristic p. If X and $X^{(p)}$ are not isomorphic as k-schemes, then*

$$|F_{X/k} \times_k \mathrm{id}_X|\colon |X \times_k X| \to |X^{(p)} \times_k X|$$

is a homeomorphism that is not induced by an isomorphism of schemes.

Proof. This is a generalization of [KLOS21, Ex. 5.5.1]. Note that $F_{X/k} \times_k \mathrm{id}_X$ is a well-defined morphism as both the relative Frobenius and the identity are morphisms over k. Let us first show that $|F_{X/k} \times_k \mathrm{id}_X|$ is a homeomorphism. As $|F_{X/k}|$ and $|\mathrm{id}_X|$ are homeomorphisms and X is a variety over an algebraically closed field, we see that the continuous map $|F_{X/k} \times_k \mathrm{id}_X|$ is a bijection. Since $X \times_k X$ and $X^{(p)} \times_k X$ are proper over k, the continuous bijection $|F_{X/k} \times_k \mathrm{id}_X|$ is closed, hence a homeomorphism.

It remains to show that the map is not induced by an isomorphism of schemes. On the contrary, suppose

$$G\colon X \times_k X \to X^{(p)} \times_k X$$

were an isomorphism of schemes such that $|G| = |F_{E/k} \times_k \mathrm{id}_X|$. Then, as X and $X^{(p)}$ are integral proper schemes over the algebraically closed field k, the action of G on global sections induces an isomorphism

$$s_G\colon k = \Gamma(X \times_k X, \mathcal{O}_{X \times_k X}) \xrightarrow{\sim} \Gamma(X^{(p)} \times_k X, \mathcal{O}_{X^{(p)} \times_k X}) = k$$

of fields, fitting into the commutative diagram

$$\begin{tikzcd} X \times_k X \arrow[r, "G"] \arrow[d] & X^{(p)} \times_k X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "s_G"] & \mathrm{Spec}(k). \end{tikzcd}$$

As G agrees with $|F_{X/k} \times_k \mathrm{id}_X|$ on the level of topological spaces, the restriction to fibers of the two projections over closed points yields isomorphisms of abstract schemes

$$G_1 \colon X \xrightarrow{\sim} X^{(p)} \text{ and } G_2 \colon X \xrightarrow{\sim} X,$$

which fit into the commutative diagrams

$$\begin{tikzcd} X \arrow[r, "G_1"] \arrow[d] & X^{(p)} \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "s_G"] & \mathrm{Spec}(k) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "G_2"] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "s_G"] & \mathrm{Spec}(k). \end{tikzcd}$$

Hence, $G_1 \circ G_2^{-1} \colon X \xrightarrow{\sim} X^{(p)}$ is an isomorphism in the category of k-schemes. Contradiction. □

Note that elliptic curves over k are geometrically integral and proper. In the following, we will give examples of elliptic curves E/k for which E and $E^{(p)}$ are not isomorphic over k. For the theory of elliptic curves over fields of positive characteristic, we refer to [Sil09, III.]. In particular, recall that to every elliptic curve E over k, we can associate a quantity $j(E) \in k$ called the *j-invariant of E*. Furthermore, two elliptic curves $E, E'/k$ are isomorphic over the algebraic closure of k if and only if $j(E) = j(E')$, see [Sil09, III.1.4.c].

Example 5.19. Let $k := \overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ for some prime $p > 3$. Choose $j_0 \in k \setminus \mathbb{F}_p$. Consider the elliptic curve given by the Weierstrass equation

$$E : y^2 + xy = x^3 - \frac{36}{j_0 - 1728}x - \frac{1}{j_0 - 1728}.$$

A straightforward calculation shows that the j-invariant $j(E)$ of E is equal to j_0. Moreover, the Frobenius twist of E is given by the Weierstrass equation

$$E^{(p)} : y^2 + xy = x^3 - \frac{36}{j_0^p - 1728}x - \frac{1}{j_0^p - 1728},$$

see [Sil09, Ex. III.4.6]. Hence, $j(E^{(p)}) = j_0^p$ and, as $j_0 \notin \mathbb{F}_p$, we conclude $j(E) \neq j(E^{(p)})$. As desired, this shows that E and $E^{(p)}$ are not isomorphic over k. Combining the above with Proposition 5.18, we conclude that the homeomorphism

$$|F_{E/k} \times_k \mathrm{id}_E| \colon |E \times_k E| \to |E^{(p)} \times_k E|$$

is not induced by an isomorphism of schemes.

Example 5.20. Let $k := \overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$ for some prime $p > 3$ and set $X := \mathbb{P}^n_k \times_k E$. Then the Albanese morphism (cf. [Bad01, Ch. 5])

$$X \to E \cong \mathrm{Alb}(X)$$

is given by the projection onto the second factor, as the Albanese is compatible with taking products and the Albanese of $\mathbb{P}^n_k$ is trivial. One can show that there is an isomorphism

$$\mathrm{Alb}(X^{(p)}) \cong \mathrm{Alb}(X)^{(p)}$$

of k-schemes (cf. [Moc12, Prop. A.3]). Furthermore, an isomorphism $X \xrightarrow{\sim} X^{(p)}$ over k induces an isomorphism

$$E \cong \mathrm{Alb}(X) \xrightarrow{\sim} \mathrm{Alb}(X^{(p)}) \cong \mathrm{Alb}(X)^{(p)} \cong E^{(p)}$$

over k. We conclude, that X and $X^{(p)}$ are isomorphic over k if and only if the same holds for E. For elliptic curves E as in Example 5.19, Proposition 5.18 then implies that the homeomorphism

$$|F_{X/k} \times_k \mathrm{id}_X| \colon |X \times_k X| \to |X^{(p)} \times_k X|$$

is not induced by an isomorphism of schemes.

Recall that a smooth curve C over k is called *hyperelliptic* if there is a finite k-morphism $f\colon C \to \mathbb{P}^1_k$ of degree two. In the following, we generalize Example 5.19 to hyperelliptic curves of arbitrary genus. From now on, fix an algebraically closed field k of characteristic different from two.

Example 5.21. Every curve C of genus two is hyperelliptic. The linear system $|\omega_X|$ associated with the canonical sheaf has dimension one and degree two by Riemann–Roch. Moreover, $|\omega_X|$ is basepoint-free (cf. [Har77, Lem. V.5.1]) and therefore determines a finite morphism $C \to \mathbb{P}^1$ of degree two. For genera strictly greater than two, one can always find non-hyperelliptic curves. For example, plane quartic curves are not hyperelliptic.

Lemma 5.22. *If C is a hyperelliptic curve of genus $g \geq 2$, then there is a unique finite k-morphism $f\colon C \to \mathbb{P}^1$ of degree two, up to automorphism of $\mathbb{P}^1_k$.*

Proof. See [Har77, Prop. IV.5.3]. □

Lemma 5.23. *Let C be smooth curve over k. A finite k-morphism $f\colon C \to \mathbb{P}^1$ of degree two is ramified over exactly $2g(C) + 2$ distinct points.*

Proof. This directly follows from the Hurwitz formula (cf. [Har77, IV.2.4]). □

Lemma 5.24. *For every set of points $R = \{x_1, \dots, x_{2g+2}\}$, there is a unique hyperelliptic curve C over k and a finite k-morphism $f\colon C \to \mathbb{P}^1_k$, ramifying precisely over R.*

Proof. For a proof, see [Vak18, Prop. 19.5.2]. □

All in all, we conclude that, up to isomorphism, hyperelliptic curves of genus at least two over k correspond precisely to unordered sets of $2g + 2$ distinct points on $\mathbb{P}^1$, modulo the action of $\mathrm{Aut}(\mathbb{P}^1_k)$. The following lemma shows that the Frobenius twist of a hyperelliptic curve is again a hyperelliptic curve.

Lemma 5.25. *Let C be a hyperelliptic curve of genus $g \geq 2$ such that the unique morphism $f\colon C \to X$ ramifies precisely over the points $x_1, \dots, x_{2g+2} \in \mathbb{P}^1$. Then $C^{(p)}$ is a hyperelliptic curve over k, and the morphism $f^{(p)}\colon C^{(p)} \to (\mathbb{P}^1)^{(p)} = \mathbb{P}^1$ is finite of degree two and ramifies precisely over the points $x_1^p, \dots, x_{2g+2}^p$.*

Proof. As being finite is stable under base change (cf. [Stacks, 01WL]), the morphism $f^{(p)}$ is finite. Moreover, by Lemma 5.15, the diagram

$$\begin{tikzcd} C \arrow[r, "F_{C/k}"] \arrow[d, "f"] & C^{(p)} \arrow[d, "f^{(p)}"] \\ \mathbb{P}^1_k \arrow[r, "F_{\mathbb{P}^1/k}"] & \mathbb{P}^1_k \end{tikzcd}$$

commutes. Applying Lemma 5.14, we obtain

$$2p = \deg(F_{\mathbb{P}^1/k} \circ f) = \deg(f^{(p)} \circ F_{C/k}) = p \deg(f^{(p)})$$

and conclude $\deg f^{(p)} = 2$. Hence, the curve $C^{(p)}$ is hyperelliptic. As k is algebraically closed, the points over which $f^{(p)}$ is ramified are precisely the points $x \in \mathbb{P}^1$ whose preimage under $f^{(p)}$ consists of a single point. As $F_{C/k}$ and $F_{\mathbb{P}^1/k}$ are bijective on the underlying sets, the morphism $f^{(p)}$ is therefore ramified precisely over the points $F_{\mathbb{P}^1/k}(x_i) = x_i^p \in \mathbb{P}^1_k$ (cf. Example 5.13). □

Example 5.26. In this example, we construct hyperelliptic curves C/k of genus ≥ 2 such that C and $C^{(p)}$ are not isomorphic over k. By Proposition 5.18, it then follows that the homeomorphism

$$|F_{C/k} \times_k \mathrm{id}_C|\colon |C \times_k C| \to |C \times_k C^{(p)}|$$

is not induced by an isomorphism of schemes.

Fix an integer $g \geq 2$ and a prime $p \geq 2g$. Set $q := p^{(p+1)!}$. Recall that there is a unique subfield $\mathbb{F}_q \subseteq \overline{\mathbb{F}}_p$ with q elements. As $\overline{\mathbb{F}}_p$ is infinite, we may choose $x \in \overline{\mathbb{F}}_p \setminus \mathbb{F}_q$. In particular, this implies $F(x) \neq 0$ for every non-zero polynomial $0 \neq F \in \mathbb{F}_p[X]$ of degree at most $p+1$.

Consider the hyperelliptic curve $f\colon C \to \mathbb{P}^1$ over $\overline{\mathbb{F}}_p$ of genus g ramified over the $2g+2$ points

$$R := \{x, 0, 1, \infty, 2, 3, \ldots, 2g-1\} \subseteq \mathbb{P}^1,$$

where we identify $z \mathrel{\widehat{=}} [z : 1]$ and $\infty \mathrel{\widehat{=}} [1 : 0]$ for $z \in \overline{\mathbb{F}}_p$. By Lemma 5.25, the morphism $f^{(p)}\colon C^{(p)} \to \mathbb{P}^1$ endows $C^{(p)}$ with the structure of a hyperelliptic curve ramified over the points

$$R^{(p)} := \{x^p, 0, 1, \infty, 2, 3, \ldots, 2g-1\} \subseteq \mathbb{P}^1.$$

We claim that C and $C^{(p)}$ are not isomorphic over $\overline{\mathbb{F}}_p$. Assume the contrary. Then, by the discussion above, there is an automorphism $\Psi \in \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$ such that $\Psi(R^{(p)}) = R$. Set

$$[\alpha_0 : \beta_0] := \Psi([0 : 1]),\ [\alpha_1 : \beta_1] := \Psi([1 : 1]) \text{ and } [\alpha_\infty : \beta_\infty] := \Psi([1 : 0]).$$

As the $[\alpha_i\colon \beta_i]$ are pairwise distinct, the morphism

$$\begin{aligned} \Phi\colon \mathbb{P}^1 &\to \mathbb{P}^1 \\ [a : b] &\mapsto [(\beta_0 a - \alpha_0 b)(\alpha_1\beta_\infty - \beta_1\alpha_\infty) : (\beta_\infty a - \alpha_\infty b)(\beta_0\alpha_1 - \beta_1\alpha_0)] \end{aligned}$$

is an automorphism of $\mathbb{P}^1$. Moreover, Φ satisfies

$$\Phi([\alpha_0 : \beta_0]) = [0 : 1], \Phi([\alpha_1\colon \beta_1]) = [1 : 1] \text{ and } \Phi([\alpha_\infty : \beta_\infty]) = [1 : 0].$$

As every element of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$ is uniquely determined by its image over three distinct points, we see that Φ and Ψ are mutual inverses. Therefore, $\Phi(R) = R^{(p)}$.

Claim 5.27. *For $i = 0, 1, \infty$, we have $[\alpha_i : \beta_i] \neq [x\colon 1]$.*

Proof. First assume $[\alpha_0 : \beta_0] = [x : 1]$. Possibly multiplying by elements in $\overline{\mathbb{F}}_p^\times$, we may then assume $\alpha_1, \beta_1, \alpha_\infty, \beta_\infty \in \mathbb{F}_p$. Since $|R^{(p)}| = 2g + 2 > 4$, we can pick $[a : b] \in R$ with $a, b \in \mathbb{F}_p$ such that $\Phi([a : b]) \in R^{(p)} \setminus \{0, 1, \infty, x^p\}$. In particular, this implies $\Phi([a : b])^p = \Phi([a : b])$. On the other hand, we compute

$$\Phi([a : b]) = [(a - xb)(\alpha_1\beta_\infty - \beta_1\alpha_\infty) : (\beta_\infty a - \alpha_\infty b)(\alpha_1 - \beta_1 x)]$$

and

$$\Phi([a : b])^p = [(a - x^p b)(\alpha_1\beta_\infty - \beta_1\alpha_\infty) : (\beta_\infty a - \alpha_\infty b)(\alpha_1 - \beta_1 x^p)],$$

as every occurring term except x is contained in $\mathbb{F}_p$. Since $\Phi([a : b])^p = \Phi([a : b])$ and $[a, b] \neq [\alpha_i : \beta_i]$, we obtain

$$(a - xb)(\alpha_1 - \beta_1 x^p) = (a - x^p b)(\alpha_1 - \beta_1 x).$$

Simplifying both sides of this equation yields

$$\alpha_1 bx + a\beta_1 x^p = \alpha_1 bx^p + a\beta_1 x.$$

As x and x^p are linearly independent over $\mathbb{F}_p$, we conclude $\alpha_1 b = \beta_1 a$ and thus $[a : b] = [\alpha_1\colon \beta_1]$. Contradiction.

Therefore, $[\alpha_0 : \beta_0] \neq [x : 1]$. Very similar arguments show $[\alpha_1 : \beta_1] \neq [x : 1] \neq [\alpha_\infty : \beta_\infty]$. □

Hence, we have $[\alpha_i, \beta_i] \in R \setminus \{[x:1]\} \subseteq \mathbb{F}_p \cup \{\infty\}$ and may assume $\alpha_i, \beta_i \in \mathbb{F}_p$. Observe that this implies $\Phi(a) = \Phi(a)^p$ for all $a \in R \setminus \{[x:1]\}$. Since $x \neq x^p$ implies $x^p \neq x^{p^2}$, we thus have $\Phi([x:1]) = [x^p:1]$. Explicitely,

$$[(\beta_0 x - \alpha_0)(\alpha_1\beta_\infty - \beta_1\alpha_\infty) : (\beta_\infty x - \alpha_\infty)(\beta_0\alpha_1 - \beta_1\alpha_0)] = [x^p:1],$$

i.e.,

$$(\beta_0 x - \alpha_0)(\alpha_1\beta_\infty - \beta_1\alpha_\infty) = x^p(\beta_\infty x - \alpha_\infty)(\beta_0\alpha_1 - \beta_1\alpha_0).$$

In other words, there is a non-zero polynomial $0 \neq F \in \mathbb{F}_p[X]$ of degree at most $p+1$ satisfying $F(x) = 0$. Contradiction.

All in all, we conclude that C and $C^{(p)}$ are not isomorphic over $\overline{\mathbb{F}}_p$. Hence, the homeomorphism

$$|F_{C/\overline{\mathbb{F}}_p} \times \mathrm{id}_C| \colon |C \times_{\overline{\mathbb{F}}_p} C| \to |C^{(p)} \times_{\overline{\mathbb{F}}_p} C|$$

is not induced by an isomorphism of schemes by Proposition 5.18.

5.3 Characteristic zero

Note that the examples in the previous section heavily relied on working over a field of positive characteristic. The following proposition states that any morphism between normal varieties over an algebraically closed field of characteristic zero, which induces a homeomorphism on the underlying topological spaces, already is an isomorphism. This is an obstruction to the existence of counterexamples of the form of Proposition 5.18 in characteristic zero.

Proposition 5.28. *Let X and Y be irreducible varieties of dimension at least two over an algebraically closed field k of characteristic zero and let $f\colon X \to Y$ be a morphism over k. If Y is normal and f induces a homeomorphism on the underlying topological spaces, then f is an isomorphism of k-varieties.*

Proof. This is an application of Zariski's Main Theorem. For a proof, see [Vit87, Thm. 3.8]. □

Example 5.29. If we drop the condition on Y to be normal, then the claim of the previous proposition may fail. For example, consider the *pinched plane* (cf. [Vak18, Ex. 12.5.I])

$$Y := \mathrm{Spec}(A) \text{ with } A := \mathbb{C}[x^2, x^3, xy, y] \subseteq \mathbb{C}[x,y].$$

Observe that $A_{x^2} \cong \mathbb{C}[x,y]_x$ and $A_y \cong \mathbb{C}[x,y]_y$. Hence, the normalization

$$f\colon \mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y]) \to \mathrm{Spec}(\mathbb{C}[x^2,x^3,xy,y]) = Y$$

is an isomorphism on $\mathbb{A}^2_{\mathbb{C}} \setminus \{(0,0)\}$. In particular, Y is regular in codimension one and f is injective. Being a normalization, f is also surjective and closed, hence a homeomorphism of the underlying Zariski topologies. But since A is not normal, f is not an isomorphism of varieties.

In fact, for fields of characteristic zero, it turns out that the answer to Question 5.1 is positive. Building upon Theorem 1.1, J. Kollár proved the following theorem:

Theorem 5.30 ([Kol20; KLOS21])**.** *Let K, L be fields of characteristic zero and X_K, Y_L normal, geometrically integral projective varieties over K (resp. L) satisfying one the following two conditions:*

(i) *X_K and Y_L are of dimension at least four.*

(ii) *K and L are uncountable, algebraically closed fields and X_K and Y_L are of dimension at least two.*

If $\Phi\colon |X_K| \xrightarrow{\sim} |Y_L|$ is a homeomorphism of the underlying Zariski topologies, then Φ is the composition of a field isomorphism $\phi\colon K \xrightarrow{\sim} L$ and an isomorphism of L-varieties $X_K^\phi \xrightarrow{\sim} Y_L$.

Proof. By Theorem 1.1, it suffices to show that the underlying topological space determines the linear equivalence relation on divisors. See [Kol20] for a proof of the theorem assuming (i). A proof of the theorem under the assumption (ii) is given in [KLOS21]. □

Note that Theorem 5.30 has slightly stronger assumptions than Theorem 1.1 (also note Remark 4.7). However, this not due to the existence of counterexamples, but limitations of the techniques applied in the proof. In fact, in [Kol20], J. Kollár formulates the following conjecture:

Conjecture 5.31 ([Kol20])**.** *Let K, L be fields of characteristic zero and X_K, Y_L normal, geometrically integral proper varieties over K (resp. L) of dimension at least two . If*

$$\Phi\colon |X_K| \xrightarrow{\sim} |Y_L|$$

is a homeomorphism, then Φ is the composition of a field isomorphism $\phi\colon K \xrightarrow{\sim} L$ and an isomorphism of L-varieties $X_K^\phi \xrightarrow{\sim} Y_L$.

Recall that for a smooth variety X over $\mathbb{C}$, the set of closed points $X(\mathbb{C})$ can be endowed with the structure of a complex manifold (cf. [Har77, App. B]). The corresponding topology on $X(\mathbb{C})$ is called the *analytic topology* on X.

Example 5.32. As stated before in Example 2.5, the only automorphisms of the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Q}_p$ are the respective identities. On the other hand, there are infinitely many automorphisms of the field of complex numbers. Moreover, one can even find examples of automorphisms $\sigma\colon \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ and projective varieties X over $\mathbb{C}$ such that the analytic topologies on X and its conjugate $X^\sigma := X \times_{\mathbb{C},\sigma} \operatorname{Spec}(\mathbb{C})$ are not homeomorphic. For example, in [Ser64], J.-P. Serre constructed a projective $\mathbb{C}$-variety X and an automorphism $\sigma\colon \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ such that

$$\pi_1(X(\mathbb{C})) \ncong \pi_1(X^\sigma(\mathbb{C})).$$

Conversely, as X and X^σ are isomorphic as schemes, the underlying Zariski topologies $|X|$ and $|X^\sigma|$ are homeomorphic.

On the other hand, one can also find examples of varieties over $\mathbb{C}$ with homeomorphic analytic topologies but non-homeomorphic Zariski topologies.

For $n \geq 0$, consider the n-th *Hirzebruch surface* $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_\mathbb{C}} \oplus \mathcal{O}_{\mathbb{P}^1_\mathbb{C}}(n)) \to \mathbb{P}^1_\mathbb{C}$ (cf. [Bad01, 12.5]). In his thesis [Hir51], F. Hirzebruch showed that, as real manifolds, $\mathbb{F}_n$ and $\mathbb{F}_m$ are diffeomorphic if and only if $n \equiv m \mod 2$, while as complex manifolds, they are isomorphic if and only if $n = m$. Using the well-known fact that $\mathbb{F}_n$ and $\mathbb{F}_m$ are isomorphic as abstract schemes if and only if $n = m$, we show that the underlying Zariski topologies of distinct Hirzebruch surfaces are not homeomorphic:

Proposition 5.33. *For $n \neq m \geq 0$, the underlying Zariski topologies of the Hirzebruch surfaces $\mathbb{F}_n$ and $\mathbb{F}_m$ are not homeomorphic.*

Proof. Observe that $\mathbb{C}$ is an uncountable algebraically closed field and $\mathbb{F}_n$ is a two-dimensional integral smooth projective $\mathbb{C}$-variety. By Theorem 5.30, it thus suffices to see that $\mathbb{F}_n$ and $\mathbb{F}_m$ are not isomorphic as schemes. One way of proving this fact is the following: Using basic intersection theory (cf. [Har77, Ch. V.1]), one deduces that for $n \geq 1$, there is a unique irreducible curve $C \subseteq \mathbb{F}_n$ with negative self-intersection (cf. [Bad01, 12.5]). Furthermore, one computes $C^2 = -n$. In the case of $n = 0$, observe that every curve $C \subseteq \mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ has non-negative self-intersection.

Assume that there is an isomorphism $F\colon \mathbb{F}_n \xrightarrow{\sim} \mathbb{F}_m$ of schemes. Then F carries irreducible curves $C \subseteq \mathbb{F}_n$ to irreducible curves on $\mathbb{F}_m$ and preserves self-intersection numbers. By the observations above, we conclude $n = m$. This finishes the proof. $\square$

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