Compact Hyperkähler Manifolds

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Introduction

Compact hyperkähler manifolds, or irreducible symplectic manifolds as they will be frequently called here, are higher-dimensional analogues of K3 surfaces. That they indeed share many of the well-known properties of K3 surfaces this paper intends to show.

To any K3 surface X one naturally associates its period consisting of the weight-two Hodge structure on $H^2(X, \mathbb{Z})$ together with the intersection pairing. The period describes and determines a K3 surface in the following sense:

- Firstly, the period encodes the algebraic (and to some extent also the differential) geometry of the K3 surface. E.g. the ample (Kähler) cone can be described purely in terms of the period.
- Secondly, any point in the period domain is realized as the period of a K3 surface.
- Thirdly, and this is certainly the culmination of the theory, two K3 surfaces with the same period are isomorphic (Global Torelli Theorem).

The second cohomology $H^2(X, \mathbb{Z})$ of a compact irreducible hyperkähler (or irreducible symplectic) manifold also carries a natural weight-two Hodge structure. Moreover, due to a result of Beauville $H^2(X, \mathbb{Z})$ can be endowed with a natural non-degenerate quadratic form q_X generalizing the intersection pairing of a K3 surface. For higher-dimensional varieties, in contrast to varieties of dimension two, the knowledge of the second cohomology usually reveals only a small portion of the full geometry. However, for irreducible symplectic manifolds the situation is quite different and it is not completely unreasonable to expect $(H^2(X, \mathbb{Z}), q_X)$ to encode all 'essential' information about the geometry of X. In this paper we present several results underpinning this expectation. But to temper the enthusiasm a little at this point we mention that Debarre has constructed examples showing that the Global Torelli Theorem as formulated for K3 surfaces fails in higher dimensions.

The first basic result showing that $(H^2(X, \mathbb{Z}), q_X)$ controls the geometry of X, at least to a certain extent, is the following projectivity criterion (Theorem 9.11) which is also of importance for many of the other results in the later chapters.

- An irreducible symplectic manifold X is projective if and only if there exists a line bundle L on X with positive square $q_X(c_1(L)) > 0$.

Another instance of the significance of the second cohomology for the geometry of X is a description of the ample cone. Due to a missing concept of (-2)-classes in higher dimensions the result is less explicit than the known one for K3 surfaces. It can roughly be formulated as

- A class in $H^2(X, \mathbb{Z})$ is ample if and only if it has positive square with respect to q_X and cannot be separated from the Kähler cone by any (integral) hyperplane.

For the precise statement see Theorem 13.5. A similar description is obtained for the Kähler cone (Theorem 13.6). Again, the result does not reach the explicitness of the result for K3 surfaces, but it suffices to describe the Kähler cone of certain hyperkähler manifolds completely. One of the consequences is the following result (Corollary 12.7):

- For a general irreducible symplectic manifold X the Kähler cone equals the positive cone, i.e. the Kähler cone is a connected component of $\{\alpha \in H^{1,1}(X, \mathbb{R}) | q_X(\alpha) > 0\}$.

An irreducible symplectic manifold is general if the corresponding point in the moduli space is contained in the complement of countably many nowhere dense closed subspaces (see 7.10). In fact, the description of the Kähler cone is based on the higher dimensional analogue of the transitivity of the Weyl-action for K3 surfaces. The latter says that any general $(1,1)$-class with positive selfintersection on a K3 surface can be mapped to a Kähler class by reflecting it in a finite number of hyperplanes orthogonal to (-2)-curves. Proposition 12.1 and Theorem 12.9 are in this spirit, although in higher dimensions there does not exist a Weyl-group and the resulting class is only Kähler on an irreducible symplectic manifold birational to the given one.

The information about the Kähler cone allows us to attack the question of which point in the period domain really occurs as the period of an irreducible symplectic manifold:

- The period map surjects any non-empty connected component of the moduli space of marked irreducible symplectic manifolds onto the period domain $\{x | q_\Gamma(x) = 0,\ q_\Gamma(x + \bar{x}) > 0\} \subset \mathbb{P}(\Gamma_\mathbb{C})$.

This is Theorem 14.1 (cf. Section 7 for the notation). The last result we wish to mention explicitly in the introduction deals with birational irreducible symplectic manifolds. It has been known for some time that seemingly different examples of irreducible symplectic manifolds are often related to each other either by deformation or by birational correspondences. In [31] we could show that certain moduli spaces of stable rank two sheaves on a K3 surface are deformation equivalent (and hence diffeomorphic) to the Hilbert scheme of points on this K3 surface. In fact, we showed more generally that under a certain assumption on the codimension of the exceptional sets any two birational irreducible symplectic manifolds are diffeomorphic. Using the techniques developed in this paper we are now able to prove this result in full generality. As a consequence of the main Theorem 10.12 we mention:

- Two birational projective irreducible symplectic manifolds are deformation equivalent and, hence, diffeomorphic.

This replaces the fact that birational K3 surfaces are isomorphic. The result also compares nicely with a recent theorem of Batyrev and Kontsevich saying in particular that birational projective manifolds with trivial canonical bundle have the same Betti numbers (see the discussion in Section 10).

I decided not just to write up the proofs of the results mentioned above but instead to give an introduction to the theory of compact hyperkähler manifolds as such. I hope that this makes the paper accessible to everyone with a basic knowledge in complex algebraic geometry. In particular, I present complete proofs of the main results of Beauville, Bogomolov, and Fujiki on the cohomology of compact hyperkähler manifolds. The only important fact I will not discuss is a recent result of Verbitsky on the structure of the part of the cohomology which is generated by classes of degree two.

Thus, Sections 1 - 8 contain essentially no new results, although some of the statements (5.12, 5.13, 5.15, 6.8) I could not find in the literature or at least not in this generality. The new results are contained in Sections 9 - 17. Section 10 is in fact a sequel to [31]. In order to keep the exposition self-contained I included Theorem 10.8 and Proposition 10.11 which correspond to 3.4 respectively 4.3 in [31].

Let us now give a detailed plan of the paper: Section 1 is devoted to the classification of compact Ricci-flat manifolds. Due to Beauville's decomposition theorem and Berger's classification of irreducible holonomies there exist essentially two types of compact Ricci-flat manifolds: Calabi-Yau and irreducible hyperkähler manifolds. This section also recalls basic notions from deformation theory and the unobstructedness result of Tian and Todorov.

A hyperkähler structure on a Riemannian manifold consists of three parallel complex structures satisfying the usual quaternion relations. The twistor space is a convenient way to encode this information in holomorphic terms. These concepts will be discussed in Section 2.

Section 3 illuminates the relation between simply connected compact Kähler manifolds with a unique non-degenerate holomorphic two-form, so called irreducible symplectic manifolds, and compact irreducible hyperkähler manifolds.

In Section 4 we collect all known examples of irreducible symplectic manifolds. No proofs are given in this section as it would take us too far afield. However, we try to indicate the relations between the various constructions.

In Section 5 we state and prove Beauville's Local Torelli Theorem (Theorem 5.6). It allows one to define a quadratic form q_X on the second cohomology $H^2(X,\mathbb{Z})$ naturally generalizing the intersection pairing on a K3 surface (Theorem 5.11). This quadratic form also controls intersection numbers with respect to certain classes of higher degree (Theorem 5.12). As a consequence, the Hirzebruch-Riemann-Roch formula for a line bundle L is shown to be a universal expression in $q_X(c_1(L))$. The Kähler cone and the positive cone are introduced in 5.14.

Section 6 contains hyperkähler and holomorphic versions of the standard Lefschetz theorems. These results are mainly due to Fujiki. Beauville's quadratic form is used to establish a link between curves and divisors on an irreducible symplectic manifold via the holomorphic symplectic form (6.8) similarly to the Hard Lefschetz Theorem with respect to a Hodge class.

Section 7 discusses various local and global aspects of the moduli space of marked irreducible symplectic manifolds. The period map and the period domain are defined in 7.6 and 7.9. Deformations of irreducible symplectic manifolds respecting the type of certain classes are studied in 7.16. These facts are indispensable for the proofs of the results in the later chapters.

The beautiful theory of K3 surfaces is sketched in its rough outlines in Section 8. I hope that it may serve as a motivation for the considerations in higher dimensions in the other chapters.

In Section 9 we prove the projectivity criterion mentioned above (Theorem 9.11). It is based on results of Demailly, Bonavero and Ji-Shiffman. We also include a result due to Fujiki (9.5) showing that algebraic manifolds are dense in the moduli space.

The topic of Section 10 is the relation between different but birational irreducible symplectic manifolds. We begin with the easy fact that two such manifolds cannot be distinguished by their periods (10.1). Then (10.2) we recall Mukai's construction of a birational surgery, called elementary transformation, which leads to interesting examples. Two important ones, due to Beauville and Debarre, are sketched in 10.4. Theorem 10.7, a generalization of the 'Main Lemma' of Burns and Rapoport [13], shows that manifolds corresponding two non-separated points in the moduli space are always birational. The converse of this statement takes up the rest of this section. We first recall a result from [31] that the converse holds true if the birational correspondence is given in terms of elementary transformations (Theorem 10.8). The general case is more subtle and we have to add a projectivity assumption in order to make the proof go through (Theorem 10.12).

Consequences of 10.12 are presented in 10.13 and Section 11.

The Weyl-group of a K3 surface, which is generated by reflections in hyperplanes orthogonal to (-2)-classes, acts transitively on the set of chambers. In particular, the orbit of any general positive class meets the Kähler cone. In Section 12 we prove an analogue of this fact for manifolds of higher dimension (Proposition 12.1). It allows one to describe the Kähler cone of very general hyperkähler manifolds (12.6, 12.7).

Section 13 aims at a description of the ample and the Kähler cone similar to the known one for K3 surfaces. Although the results (13.5, 13.6) are far less explicit in higher dimensions, they still give a complete answer for certain hyperkähler manifolds not quite as general as those in Section 12.

In Section 14 we prove that the period map surjects any non-empty component of the moduli space of marked irreducible symplectic manifolds onto the period domain. The proof deviates from the standard ones for K3 surfaces.

In the final two chapters we discuss results and questions related to the automorphism group of irreducible symplectic manifolds and to the moduli space of these manifolds. We add some speculations on a possible Global Torelli Theorem.

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Contents

1 Compact Kähler Manifolds with $\mathbf{c_1 = 0}$

Compact complex Kähler manifolds with vanishing first Chern class occupy a distinguished place in the classification of complex manifolds. In dimension two these are Enriques and K3 surfaces, tori, and hyperelliptic surfaces. Their geometry has been intensively studied and a good deal is known about these surfaces today. The techniques to approach them include algebraic and complex geometry, differential geometry, arithmetic, automorphic forms, lattice theory, etc. The rich geometry and the many interesting features encountered already in dimension two, promise a rewarding study of higher dimensional manifolds with $c_1 = 0$.

The first step towards a classification of such manifolds relies on Berger's description of irreducible holonomies. For details we refer to [5]. It roughly says that an irreducible (i.e. the holonomy is irreducible) Riemannian manifold (M, g) is either locally symmetric or its holonomy occurs in a list of eight groups. In this list there are 'general' holonomies like $\mathrm{SO}(m)$ or $\mathrm{U}(m)$, corresponding to 'general' oriented respectively 'general' Kähler manifolds, and there are special ones like $\mathrm{SU}(m)$ or $\mathrm{Sp}(n)$. A classification theory aims at an understanding of manifolds with special holonomy.

Definition 1.1 — *A compact Riemannian manifold (M, g) of dimension $2m$ is called Calabi-Yau if its holonomy equals $\mathrm{SU}(m)$.*

A compact Riemannian manifold (M, g) of dimension $4n$ is called (irreducible) hyperkähler if its holonomy is contained in (equals) $\mathrm{Sp}(n)$.

The holonomy principle tells us that the mere fact that $\mathrm{SU}(m)$ and $\mathrm{Sp}(n)$ are contained in $\mathrm{U}(m)$ respectively $\mathrm{U}(2n)$ implies that these manifolds are Kähler. More precisely, there exists a complex structure I on M such that g is a Ricci-flat Kähler metric on the complex manifold $X := (M, I)$. The holonomy principle also shows that the canonical bundle K_X of X is trivial, i.e. $K_X \cong \mathcal{O}_X$. In particular, $c_1(X) = 0$.

These two classes of Ricci-flat manifolds are in fact the building-blocks of all Ricci-flat Kähler manifolds. This becomes apparent by the following decomposition theorem (cf. [2], [7]).

Theorem 1.2 — *If X is a compact complex manifold endowed with a Ricci-flat Kähler metric, then its universal cover $\tilde{X}$ decomposes (as a Kähler manifold):*

$$\tilde{X} = \mathbb{C}^m \times \Pi X_i \times \Pi Y_i.$$

Here, the X_i's and Y_i's are endowed with Kähler metrics g_i and h_i such that the underlying Riemannian manifolds (M_i, g_i) and (N_i, h_i) are compact (!) simply connected irreducible hyperkähler respectively Calabi-Yau manifolds. □

For any compact complex manifold X with $c_1(X) = 0$ there exists a unique Ricci-flat Kähler metric (the Calabi-Yau metric) realizing a prescribed Kähler class. This is due to Yau's celebrated result [58]. Thus, the above theorem can also be regarded as a decomposition result for compact Kähler manifolds with vanishing first Chern class.

Using the Calabi-Yau metric one can in fact reformulate Definition 1.1 in purely holomorphic data. In the following proposition this is done for Calabi-Yau manifolds (for a proof see [2]). The analogous result for hyperkähler manifolds will be discussed in more detail in Sections 2 and 3.

Proposition 1.3 — *Let Y be a compact Kähler manifold of dimension m. Then the following conditions are equivalent:*

- *Y admits a Kähler metric with holonomy* $\mathrm{SU}(m)$.
- *The canonical bundle K_Y is trivial and $H^0(Y', \Omega^p_{Y'}) = 0$ for $0 < p < m$ and any finite unramified cover Y' of Y.* □

Example 1.4 — Using 1.3 it is rather easy to construct examples of Calabi-Yau manifolds. E.g. smooth complete intersections of type $(a_1, \ldots, a_{N-m})$ in $\mathbb{P}^N$, where $\sum a_i = N + 1$, are Calabi-Yau manifolds.

Compact irreducible hyperkähler manifolds are much harder to come by. A list of the known examples will be provided in Section 4. Here we only note that any K3 surface is irreducible symplectic and that there exist examples of compact irreducible hyperkähler manifolds in all possible dimensions.

Many results in the theory of compact Kähler manifolds with vanishing first Chern class are based on or are related to the particularly nice behaviour of these manifolds under deformation. Here are the main facts:

A *deformation* of a compact manifold X is a smooth proper holomorphic map $\mathcal{X} \to S$, where S is an analytic space and the fibre over a distinguished point $0 \in S$ is isomorphic to X.

Let X be a compact Kähler manifold. Then there exists a *semi-universal* deformation denoted by $\mathcal{X} \to Def(X)$, where $Def(X)$ is a germ of an analytic space and the fibre $\mathcal{X}_0$ over $0 \in Def(X)$ is isomorphic to X. The *Zariski tangent space* of $Def(X)$ is naturally isomorphic to $H^1(X, \mathcal{T}_X)$. If $H^0(X, \mathcal{T}_X) = 0$, i.e. if X does not allow infinitesimal automorphisms, then $\mathcal{X} \to Def(X)$ is *universal*, i.e. for any deformation $\mathcal{X}_S \to S$ of X there exists locally a uniquely determined holomorphic map $S \to Def(X)$ such that $\mathcal{X}_S \cong \mathcal{X} \times_{Def(X)} S$.

For Ricci-flat manifolds one has the following important result originally due to Tian [52] and Todorov [53]. A beautiful algebraic proof of it was given by Kawamata [34] and Ran [48]. The special case of hyperkähler manifolds was earlier treated by Bogomolov [8].

Theorem 1.5 — *Any compact Kähler manifold X with $c_1(X) = 0$ has unobstructed deformations, i.e. $Def(X)$ is smooth.* □

The main reason for the unobstructedness of Ricci-flat Kähler manifolds is the fact that the dimension of $H^1(X, \mathcal{T}_X)$ turns out to be a Hodge number, which is constant in a family of Kähler manifolds. This also implies that the universal family $\mathcal{X} \to Def(X)$ is universal at any point $t \in Def(X)$.

2 Hyperkähler Manifolds

Recall that by definition a hyperkähler manifold is a Riemannian manifold (M, g) of dimension $4n$ such that its holonomy is contained in $\mathrm{Sp}(n)$. By the holonomy principle, the quaternions $\mathbb{H}$ act as parallel endomorphisms on the tangent bundle of M. This gives rise to three different complex structures:

Proposition 2.1 — *If (M, g) is a hyperkähler manifold, then there exist complex structures I, J, and K on M satisfying $K = IJ = -JI$ such that g is a Kähler metric with respect to all of them.* □

Remark 2.2 — In fact, any $\lambda = aI + bJ + cK$ with $a^2 + b^2 + c^2 = 1$ defines a complex structure on M with respect to which g is a Kähler metric. The associated Kähler form $g(\lambda\,.\,,\,.\,)$ is denoted ω_λ.

Thus, a hyperkähler metric g on a manifold M defines a family of complex Kähler manifolds $(M, \lambda, \omega_\lambda)$, where $\lambda \in S^2 \cong \mathbb{P}^1$.

The fact that on any hyperkähler manifold there exists a two-sphere of complex structures can be turned into the concept of the twistor space of a hyperkähler manifold: Consider the product $M \times \mathbb{P}^1$ and define

$$\mathbb{I} : T_m M \oplus T_\lambda \mathbb{P}^1 \to T_m M \oplus T_\lambda \mathbb{P}^1 \quad , \quad (v, w) \mapsto (\lambda(v), I_{\mathbb{P}^1}(w)),$$

where $\lambda \in \mathbb{P}^1 \cong S^2$ as above. Obviously, $\mathbb{I}$ is an almost complex structure. But in fact:

Theorem 2.3 — *$\mathbb{I}$ is a complex structure on $M \times \mathbb{P}^1$.*

Proof. See [30, 49]. □

Definition 2.4 — *The complex manifold $\mathcal{M} := (M \times \mathbb{P}^1, \mathbb{I})$ is called the twistor space associated with the hyperkähler metric g.*

By construction, the projection $\pi : \mathcal{M} \to \mathbb{P}^1$ is holomorphic and the fibre over $\lambda \in \mathbb{P}^1$ is isomorphic to (M, λ). Although any fibre of $\pi : \mathcal{M} \to \mathbb{P}^1$ has a natural Kähler structure, the twistor space $\mathcal{M}$ is in general not Kähler. Any point $m \in M$ defines a holomorphic section $\{m\} \times \mathbb{P}^1 \subset \mathcal{M}$ of π. The normal bundle of $\{m\} \times \mathbb{P}^1$ in $\mathcal{M}$ is isomorphic to $\mathcal{O}(1)^{\oplus 2n}$. Another feature of the twistor space is that the antipodal map on S^2 defines an anti-holomorphic involution $\iota : \mathcal{M} \to \mathcal{M}$. As will be discussed in the next section, the two-form $\omega_J + i\omega_K$ defines a non-degenerate holomorphic two-form on (M, λ), i.e. an element of $H^0((M, \lambda), \Omega^2_{(M,\lambda)})$. This can be globalized to give a section of $\Omega^2_\pi \otimes \pi^*\mathcal{O}(2)$ on $\mathcal{M}$, which is non-degenerate on any fibre of $\pi : \mathcal{M} \to \mathbb{P}^1$.

Theorem 2.5 — *Let (M, g) be a hyperkähler manifold. Then there exists a complex manifold $\mathcal{M}$ and a smooth holomorphic map $\pi : \mathcal{M} \to \mathbb{P}^1$ such that:*

i) There exists a covering family of holomorphic sections of π with normal bundle $\mathcal{O}(1)^{\oplus 2n}$.

ii) There exists an element in $H^0(\mathcal{M}, \Omega^2_\pi \otimes \pi^*\mathcal{O}(2))$ restricting to a non-degenerate holomorphic two-form on any fibre of π.

iii) The antipodal map on $S^2 = \mathbb{P}^1$ admits a lift ι to a real structure on $\mathcal{M}$ that is compatible with the structures in i) and ii).

Conversely, the hyperkähler structure on M is uniquely determined by $\pi : \mathcal{M} \to \mathbb{P}^1$ satisfying i), ii), and iii).

Proof. See [30]. □

In fact, the manifold M can be recovered as (a component of) the parameter space of the sections of $\pi : \mathcal{M} \to \mathbb{P}^1$ invariant under the involution.

It is an interesting but hard problem to construct explicit examples of hyperkähler metrics especially if the underlying manifold is compact. The twistor space allows one to approach this problem by means of complex geometry. This has successfully be carried out by Topiwala in the case of K3 surfaces [54].

3 Symplectic Manifolds

If not stated otherwise, symplectic manifolds, symplectic structures, etc., will always mean the holomorphic versions of the usual concepts. In particular, one defines

Definition 3.1 — *A symplectic structure on a complex manifold X is an everywhere non-degenerate holomorphic two-form $\sigma \in H^0(X, \Omega^2_X)$.*

Any $\sigma \in H^0(X, \Omega_X^2)$ defines a homomorphism $\sigma : \mathcal{T}_X \to \Omega_X$. The two-form σ is everywhere non-degenerate if and only if $\sigma : \mathcal{T}_X \to \Omega_X$ is an isomorphism. A symplectic manifold in this context would just mean a complex manifold endowed with a symplectic structure. We will usually only work with the following much stronger concept:

Definition 3.2 — *A compact Kähler manifold X is called irreducible symplectic if*

i) X is simply connected and

ii) $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate two-form σ.

Throughout, we will fix the complex dimension of X to be $2n$.

Here are some immediate consequences of the definition.

Lemma 3.3 — *If X is an irreducible symplectic manifold then $K_X \cong \mathcal{O}_X$ and, therefore, $c_1(X) = 0$. Moreover, the complex dimension of X is even and $h^{0,2}(X) = h^{2,0}(X) = 1$.* □

As it turns out, irreducible symplectic manifolds with a fixed Kähler class and compact irreducible hyperkähler manifolds are the holomorphic respectively metric incarnation of the same object. More precisely, one has the following result due to Beauville [2].

Theorem 3.4 — *If X is an irreducible symplectic manifold and $\alpha \in H^2(X, \mathbb{R})$ is a Kähler class, then there exists a unique Kähler metric g with Kähler class α such that (M, g) is irreducible hyperkähler, where M denotes the real manifold underlying X.*

Conversely, if (M, g) is a compact irreducible hyperkähler manifold, then there exists a complex structure I on M as in 2.1 such that $X := (M, I)$ is irreducible symplectic.

Proof. Let us give an outline of the proof. For the details we refer to [2].

First, let X be irreducible symplectic and let α be a Kähler class on X. Then take the uniquely determined Ricci-flat metric g on X with Kähler class α. By the decomposition theorem and since X is simply connected, X endowed with g splits in a product $\Pi_i^\ell X_i$, where the X_i's are irreducible hyperkähler. Since σ is unique (up to scalars), one has $\ell = 1$, i.e. the underlying Riemannian manifold (M, g) is irreducible hyperkähler.

Conversely, if (M, g) is hyperkähler and I, J, K are complex structures as in 2.1, then $\sigma := \omega_J + i\omega_K$ is a holomorphic everywhere non-degenerate two-form on $X = (M, I)$. If M is compact and g is irreducible hyperkähler, i.e. the holonomy is $\mathrm{Sp}(n)$, then the holonomy principle shows that σ is unique up to scalars. It remains to show that X is simply connected. To this end, evoke the holonomy principle again and compare the holomorphic Euler characteristics of X and its universal cover, which by the decomposition theorem is a product of irreducible symplectic manifolds. □

It is worth emphasizing that the hyperkähler metric on an irreducible symplectic manifold is just known to exist and cannot be constructed explicitly. This is due to the fact that Yau's solution of the Calabi conjecture is a pure existence result. The holomorphic version of this problem would be the quest for the twistor space of an irreducible symplectic manifold with respect to a fixed Kähler class.

Let X be an irreducible symplectic manifold. We introduce the following notation. If $\alpha \in H^{1,1}(X)_{\mathbb{R}} := H^{1,1}(X) \cap H^2(X,\mathbb{R})$, then

$$P(X) := (\sigma \cdot \mathbb{C} \oplus \bar{\sigma} \cdot \mathbb{C}) \cap H^2(X,\mathbb{R}) \qquad \text{and} \qquad F(\alpha) := P(X) \oplus \alpha \cdot \mathbb{R},$$

where σ is a holomorphic two-form spanning $H^0(X,\Omega_X^2)$. If α is a Kähler class and the induced hyperkähler structure is (M,g,I,J,K), where $X = (M,I)$, then $\sigma = \omega_J + i\omega_K$ (up to scalar factors) and, therefore, $P(X) = [\omega_J]\cdot\mathbb{R}\oplus[\omega_K]\cdot\mathbb{R}$ and $F(\alpha) = [\omega_I]\cdot\mathbb{R}\oplus[\omega_J]\cdot\mathbb{R}\oplus[\omega_K]\cdot\mathbb{R}$. Hence, $F(\alpha) \subset H^2(X,\mathbb{R}) = H^2(M,\mathbb{R})$ is independent of the complex structure and depends only on the metric g on M. Sometimes, $F(\alpha)$ is called the *HK 3-space* associated with the hyperkähler metric g.

4 Examples

For the reader's convenience we collect the known examples of irreducible symplectic manifolds. In all cases the verification is reduced to the conditions of Definition 3.2 and not to an explicit construction of an irreducible hyperkähler metric. Explicit examples of irreducible hyperkähler metrics on compact manifolds would be highly desirable.

4.1. K3 surfaces. *A complex manifold of dimension two is irreducible symplectic if and only if it is a K3 surface.*

By definition a K3 surface is a compact connected surface with trivial canonical bundle and vanishing first Betti number. One can show that any K3 surface is deformation equivalent to a smooth quartic hypersurface in $\mathbb{P}^3$ and, therefore, simply connected. That a K3 surface is Kähler is due to Siu [51] (see also [1]). Examples of Guan [26] show that in higher dimensions not every simply connected holomorph-symplectic manifold is Kähler.

4.2. Hilbert schemes of K3 surfaces. *If S is a K3 surface, then* $\mathrm{Hilb}^n(S)$ *is irreducible symplectic (cf. [2])*

By the Hilbert scheme $\mathrm{Hilb}^n(S)$ we mean the Douady space of zero-dimensional subspaces $(Z,\mathcal{O}_Z)$ of S of length $\dim_{\mathbb{C}}\mathcal{O}_Z = n$. Strictly speaking, $\mathrm{Hilb}^n(S)$ is a scheme only if S is algebraic. In general, it is just a complex space. Using that S is smooth, compact, connected, and of dimension two, one shows that $\mathrm{Hilb}^n(S)$ is a smooth compact connected manifold of dimension $2n$. By results of Varouchas [55] the Hilbert scheme is Kähler if the underlying surface is Kähler which is the case for K3 surfaces. Beauville then concluded that for any K3 surface S the Hilbert scheme $\mathrm{Hilb}^n(S)$ is irreducible symplectic by showing that $\mathrm{Hilb}^n(S)$ admits a unique (up to scalars) everywhere non-degenerate holomorphic two-form and that it is simply connected. For $n = 2$ this result was also obtained by Fujiki. It is interesting to note that for $n > 1$ one has $b_2(\mathrm{Hilb}^n(S)) = 23$. Moreover, the second cohomology $H^2(\mathrm{Hilb}^n(S), \mathbb{Z})$ endowed with the natural quadratic form q_X (cf. Section 5) is isomorphic to the lattice $H^2(S, \mathbb{Z}) \oplus (-2(n-1) \cdot \mathbb{Z})$.

4.3. Generalized Kummer varieties. *If A is a two-dimensional torus, then* $\mathrm{K}^{n+1}(A)$ *is irreducible symplectic (cf. [2]).*

The generalized Kummer variety $\mathrm{K}^{n+1}(A)$ is by definition the fibre over $0 \in A$ of the natural morphism $\mathrm{Hilb}^{n+1}(A) \to S^{n+1}(A) \xrightarrow{\Sigma} A$, where Σ is the summation and $0 \in A$ is the zero-point of the torus. $\mathrm{Hilb}^{n+1}(A)$ itself also admits an everywhere non-degenerate two-form, but neither is this two-form unique nor is $\mathrm{Hilb}^{n+1}(A)$ simply connected. That both conditions are satisfied for $\mathrm{K}^{n+1}(A)$ was shown by Beauville. That $\mathrm{K}^{n+1}(A)$ is Kähler follows again from the results in [55]. The second Betti number of $\mathrm{K}^{n+1}(A)$ is 7 (cf. [2]).

I usually refer to the examples provided by the Hilbert schemes of K3 surfaces and by the generalized Kummer varieties as the two standard series of examples of irreducible symplectic manifolds. Note that by means of these examples we have in any real dimension $4n$ at least two different compact real manifolds admitting irreducible hyperkähler metrics. That they are not diffeomorphic (in fact, not even homeomorphic) follows from a comparison of their second Betti numbers.

4.4. New examples by deformation. *Any small deformation of an irreducible symplectic manifold is irreducible symplectic (cf. [2]).*

The stability results in [38] say that any small deformation of a compact Kähler manifold is again Kähler. Since the Hodge number $h^{2,0}$ is constant in families of compact Kähler manifolds, any small deformation of an irreducible symplectic manifold admits a unique non-trivial non-degenerate two-form which is everywhere non-degenerate. In fact, using the splitting theorem one can show that any Kähler deformation of an irreducible symplectic manifold is again irreducible symplectic. For the details see [2]. It seems to be an open problem if any deformation of an irreducible symplectic manifold is Kähler and, therefore, irreducible symplectic. Deforming the Hilbert scheme $\mathrm{Hilb}^n(S)$ of a K3 surface or the generalized Kummer variety $\mathrm{K}^{n+1}(A)$ provides new examples of irreducible symplectic manifolds. Indeed, for $n > 1$ one has $\dim Def(\mathrm{Hilb}^n(S)) = h^{1,1}(\mathrm{Hilb}^n(S)) = h^{1,1}(S) + 1 = \dim Def(S) + 1 (= 21)$ and $\dim Def(\mathrm{K}^{n+1}(A)) = h^{1,1}(\mathrm{K}^{n+1}(A)) = h^{1,1}(A) + 1 = \dim Def(A) + 1 (= 5)$. Thus one can think of the deformations of $\mathrm{Hilb}^n(S)$ that remain Hilbert schemes as a hypersurface in the full deformation space of $\mathrm{Hilb}^n(S)$. An analogous result holds true for Kummer varieties.

4.5. New examples by birational transformation. *If a compact Kähler manifold admits an everywhere non-degenerate two-form and is birational to an irreducible symplectic manifold, then it is irreducible symplectic as well.*

This follows easily from the observation that $\pi_1(X) = \pi_1(Y)$ and $h^{2,0}(X) = h^{2,0}(Y)$ for two birational compact manifolds X and Y. Can one drop the assumption that the manifold is Kähler? Or equivalently, is any compact manifold that is birational to an irreducible symplectic manifold and that admits an everywhere non-degenerate two-form, automatically Kähler? In Section 10 we will explain how to construct new irreducible symplectic manifolds starting from a given one by doing some kind of surgery along projective bundles embedded in the manifold. Also note that the above fact was used in many instances to show that a symplectic manifold is indeed irreducible symplectic (cf. 4.8).

The following examples are related to one of the standard series by deformation or birational transformation as described above. In particular, the proof that they are irreducible symplectic is reduced to the proof of this fact for either a Hilbert scheme or a Kummer variety.

4.6. Fano varieties of cubics [4]. Let $Y \subset \mathbb{P}^5$ be a smooth cubic hypersurface and let $X := F(Y)$ be the Fano variety of lines on Y. Then X is smooth and projective of dimension 4. All cubics are deformation equivalent and, hence, so are the corresponding Fano varieties. Beauville and Donagi showed that for a special cubic Y the Fano variety $X = F(Y)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^2(S)$ of a special K3 surface of degree 14 in $\mathbb{P}^8$. Hence, for an arbitrary cubic Y the Fano variety $X = F(Y)$ is a deformation of $\mathrm{Hilb}^2(S)$ and, therefore, irreducible symplectic. The map $Y \mapsto F(Y)$ identifies $\mathit{Def}(Y)$ with a hypersurface of $\mathit{Def}(X)$ parametrizing algebraic deformations of $\mathrm{Hilb}^2(S)$. It would be interesting to understand the space of algebraic deformations of $\mathrm{Hilb}^n(S)$ for other K3 surfaces.

4.7. Relative Jacobians [41]. Markushevich constructed an explicit example of a projective irreducible symplectic manifold which is completely integrable: Let $\pi : S \to \mathbb{P}^2$ be a generic double cover ramified along a sextic. Then S is a K3 surface. The dual space $\mathbb{P}^{2\vee}$ can be regarded as the base space of the family of hyperelliptic curves (of genus 2) of the form $\pi^{-1}(\ell)$, where $\ell \subset \mathbb{P}^2$ is a line. Then the compactified relative Jacobian $X \to \mathbb{P}^{2\vee}$ of this family of curves is smooth projective and admits an everywhere non-degenerate two-form. Since the map that sends $[Z] \in \mathrm{Hilb}^2(S)$ to the divisor (Z) on the curve $\pi^{-1}(\ell_{\pi(Z)})$, where $\ell_{\pi(Z)}$ is the line through $\pi(Z)$, defines a birational map $\mathrm{Hilb}^2(S) - - \to X$, the manifold X is irreducible symplectic. In fact, X can be seen as an example of a moduli space of simple sheaves (or rather stable sheaves with pure support as treated below). Similar examples were considered by Beauville and Mukai. But, as it is not surprising, in all cases the resulting spaces are birational to some Hilbert scheme of the underlying K3 surfaces.

4.8. Moduli spaces of sheaves on K3 surfaces. Let S be a K3 surface. Consider $M_H(v)$ – the moduli space of sheaves with primitive Mukai vector v which are stable with respect to a generic polarization H (see [46] or [32] for the notation). This space is a projective variety and by the general smoothness criterion it is non-singular. Note that also $\mathrm{Hilb}^n(S)$ can be considered as such a moduli space, namely the moduli space of stable rank one sheaves. Mukai

[43] constructed on $M_H(v)$ an everywhere non-degenerate two-form. Later it was shown that $M_H(v)$ is indeed irreducible symplectic (in [25] for the rank two case and in [46] for arbitrary rank). The proof in both cases goes roughly as follows: Any smooth deformation of (S,H) induces a deformation of $M_H(v)$ which is smooth as long as H stays v-generic. Using this one reduces the proof to the case of a very special K3 surface (e.g. an elliptic surface in [46]), where one can show that the moduli space is birational to the Hilbert scheme of the same dimension. Hence we have: *The moduli spaces $M_H(v)$ are irreducible symplectic manifolds whenever v is primitive and H is v-generic.*

5 The Quadratic Form

This section gives a self-contained account of some of the results of Beauville [2], Fujiki [23], and Bogomolov [8, 9].

Let X be an irreducible symplectic manifold of complex dimension $2n$. Fix $\sigma \in H^0(X,\Omega_X^2)$ such that $\int(\sigma\bar\sigma)^n = 1$.

Definition 5.1 — *The real quadratic form f_X on $H^2(X,\mathbb{R})$ is defined by*

$$f_X(\alpha) := \frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\alpha^2 + (1-n)\left(\int\sigma^{n-1}\bar\sigma^n\alpha\right)\cdot\left(\int\sigma^n\bar\sigma^{n-1}\alpha\right).$$

Of course, f_X can naturally be extended to a quadratic form on $H^2(X,\mathbb{C})$. It does not depend on σ as long as $\int(\sigma\bar\sigma)^n = 1$. If we write $\alpha = \lambda\sigma + \beta + \mu\bar\sigma$ according to the Hodge decomposition on X, i.e. $\lambda,\mu \in \mathbb{C}$ and $\beta \in H^{1,1}(X)$, then

$$f_X(\alpha) = \lambda\mu + \frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\beta^2. \tag{1}$$

Thus, $H^{1,1}(X)$ is orthogonal (with respect to f_X) to $(H^{2,0}\oplus H^{0,2})(X)$.

The aim of this section is to show that a scalar multiple of f_X defines an integral quadratic form q_X that shares many of the properties of the intersection pairing on a K3 surface. The proof of this result begins with a detour: The Local Torelli Theorem.

Let $\pi : \mathcal{X} \to S$ be a deformation of $X = \mathcal{X}_0$ over a simply connected smooth base S (recall that $Def(X)$ is smooth, cf. 1.5). Trivializing the local systems $R^i\pi_*\mathbb{Z}$ provides isomorphisms $H^i(\mathcal{X}_t,\mathbb{Z}) \cong H^i(X,\mathbb{Z})$ $(t \in S)$ which respect the ring structure of $H^*(X,\mathbb{Z})$ and $H^*(\mathcal{X}_t,\mathbb{Z})$, respectively. Via these isomorphisms a class on a fibre $\mathcal{X}_t$ can also be considered as an element in $H^*(X,\mathbb{C})$. In particular, a holomorphic two-form σ_t on the fibre $\mathcal{X}_t$ can be viewed as a class in $H^2(X,\mathbb{C})$. (We usually assume that $\sigma_t \in H^2(X,\mathbb{C})$ depends continuously on t.)

Formula (1) shows that $f_X(\sigma) = 0$ and $f_X(\sigma+\bar\sigma) = 1 > 0$. In fact, one has

Lemma 5.2 — *If $\sigma_t \in H^2(X,\mathbb{C})$ is the class of a holomorphic two-form on a nearby fibre $\mathcal{X}_t$, then $f_X(\sigma_t) = 0$ and $f_X(\sigma_t+\bar\sigma_t) > 0$.*

Proof. The inequality is obvious for t close to 0. Since σ_t is a $(2,0)$-form on $\mathcal{X}_t$ and $\mathcal{X}_t$ has complex dimension $2n$, the class $\sigma_t^{n+1} \in H^{2(n+1)}(X,\mathbb{C})$ is trivial. Hence, $\int_X \sigma_t^{n+1}\bar{\sigma}^{n-1} = 0$. Let $\sigma_t = \lambda\sigma + \alpha + \mu\bar{\sigma}$ be the Hodge decomposition of σ_t as a class on X. Then the $(2n,2)$-component of σ_t^{n+1} on X is

$$(\sigma_t^{n+1})_{(2n,2)} = (n+1)\lambda^n\mu\sigma^n\bar{\sigma} + \binom{n+1}{2}\lambda^{n-1}\sigma^{n-1}\alpha^2.$$

Hence,

$$\begin{aligned} 0 &= \int_X \sigma_t^{n+1}\bar{\sigma}^{n-1} = \int_X (\sigma_t^{n+1})_{(2n,2)}\bar{\sigma}^{n-1} \\ &= (n+1)\lambda^n\mu\int_X(\sigma\bar{\sigma})^n + \binom{n+1}{2}\lambda^{n-1}\int_X(\sigma\bar{\sigma})^{n-1}\alpha^2 \\ &= (n+1)\lambda^{n-1}(\lambda\mu + \frac{n}{2}\int_X(\sigma\bar{\sigma})^{n-1}\alpha^2) \\ &= (n+1)\lambda^{n-1}f_X(\sigma_t). \end{aligned}$$

Since $\lambda \neq 0$ for t close to 0, this yields $f_X(\sigma_t) = 0$ □

Definition 5.3 — *Let $Q_X \subset \mathbb{P}(H^2(X,\mathbb{C}))$ be the set $\{\alpha| \quad f_X(\alpha) = 0, f_X(\alpha+\bar{\alpha}) > 0\}$.*

Thus, Q_X is an open (in the classical topology) subset of the quadric (f_X) defined by f_X.

Lemma 5.4 — *The rank of f_X is at least three. In particular, (f_X) is irreducible.*

Proof. Since the decomposition $(H^{2,0} \oplus H^{0,2})(X) \oplus H^{1,1}(X)$ is orthogonal with respect to f_X and f_X is non-degenerate on $(H^{2,0} \oplus H^{0,2})(X) = P(X)_{\mathbb{C}}$, it suffices to show that $\int(\sigma\bar{\sigma})^{n-1}\alpha^2 \neq 0$ for some $\alpha \in H^{1,1}(X)$. If $\alpha = \omega_I$ is a Kähler class and (I,J,K) is the associated hyperkähler structure (cf. 2.1, 3.4), then $\sigma\bar{\sigma} = \omega_J^2 + \omega_K^2$ and hence $\int(\sigma\bar{\sigma})^{n-1}\alpha^2 = \int(\omega_J^2+\omega_K^2)^{n-1}\omega_I^2$, which is certainly positive. □

In Section 7 we will see that Q_X is a connected open subset of the irreducible quadric (f_X).

Definition 5.5 — *If $\mathcal{X} \to S$ is a deformation of $X = \mathcal{X}_0$ as above then the map*

$$\mathcal{P}_X : S \longrightarrow \mathbb{P}(H^2(X,\mathbb{C})), \quad t \mapsto [\sigma_t].$$

is called the period map.

The period map $\mathcal{P}_X$ is holomorphic. This is essentially due to the fact that $\pi_*\Omega^2_{\mathcal{X}/S}$ is a holomorphic subbundle of $R^2\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_S$. By 5.2 the image of $\mathcal{P}_X$ is contained in Q_X.

Theorem 5.6 *(Local Torelli Theorem [2])* — *Consider the universal family $\mathcal{X} \to Def(X)$. The induced period map $\mathcal{P}_X : Def(X) \to Q_X$ is a local isomorphism.*

Proof. The differential $d\mathcal{P}_X$ of the period map at the point $0 \in Def(X)$ is the map given by the contraction

$$T_0 Def(X) = H^1(X, \mathcal{T}_X) \to \mathrm{Hom}(H^{2,0}(X), H^2(X,\mathbb{C})/H^{2,0}(X)).$$

Since $H^{2,0}(X) = \sigma \cdot \mathbb{C}$ and the contraction $\sigma : H^1(X, \mathcal{T}_X) \to H^1(X, \Omega_X) = H^{1,1}(X) \subset H^2(X,\mathbb{C})/H^{2,0}(X)$ is induced by the isomorphism $\sigma : \mathcal{T}_X \cong \Omega_X$, this map is injective. Both spaces $Def(X)$ and Q_X are of dimension $h^{1,1}(X)$ and the image of $\mathcal{P}_X$ is contained in Q_X. Hence, $\mathcal{P}_X : Def(X) \to Q_X$ is a local isomorphism in $0 \in Def(X)$. □

Corollary 5.7 — *Up to scalar factors the quadratic form f_X remains unchanged under small deformations of X.*

Proof. Let $t \in Def(X)$. Then $\mathcal{X} \to Def(X)$ can also be considered as a deformation of $\mathcal{X}_t$. By definition of the period map the diagram

$$\begin{array}{cccccc} \mathcal{P}_X : & Def(X) & \longrightarrow & Q_X & \subset & \mathbb{P}(H^2(X,\mathbb{C})) \\ & \downarrow = & & & & \downarrow \cong \\ \mathcal{P}_{\mathcal{X}_t} : & Def(X) & \longrightarrow & Q_{\mathcal{X}_t} & \subset & \mathbb{P}(H^2(\mathcal{X}_t,\mathbb{C})) \end{array}$$

commutes. Hence, a non-empty open subset of Q_X maps to a non-empty open subset of $Q_{\mathcal{X}_t}$. Since Q_X and $Q_{\mathcal{X}_t}$ are open subsets of the irreducible quadrics (f_X) respectively $(f_{\mathcal{X}_t})$, this shows that $f_X = c \cdot f_{\mathcal{X}_t}$ for some $c \in \mathbb{R}^*$. □

To shorten the notation let us introduce $v(\alpha) := \int_X \alpha^{2n}$. This is a purely topological invariant of X. The following formula turns out to be extremely useful in the further investigation of the quadratic form f_X.

Lemma 5.8 — *For all $\alpha, \beta \in H^2(X,\mathbb{C})$ one has*

$$v(\alpha)^2 f_X(\beta) = f_X(\alpha)\left((2n-1)v(\alpha)\int \alpha^{2n-2}\beta^2 - (2n-2)\left(\int \alpha^{2n-1}\beta\right)^2\right). \tag{2}$$

Proof. Let us first prove (2) for $\alpha = \sigma + s\bar{\sigma}$, where $s \in \mathbb{C}$. If α is of this form then

$$\begin{aligned} \int \alpha^{2n} &= \binom{2n}{n} s^n \qquad \text{and} \qquad f_X(\alpha) = s \\ \alpha^{2n-1} &= \binom{2n-1}{n}(s^{n-1}\sigma^n\bar{\sigma}^{n-1} + s^n\sigma^{n-1}\bar{\sigma}^n) \\ \alpha^{2n-2} &= \binom{2n-2}{n}\sigma^n(s\bar{\sigma})^{n-2} + \binom{2n-2}{n-1}\sigma^{n-1}(s\bar{\sigma})^{n-1} + \binom{2n-2}{n-2}\sigma^{n-2}(s\bar{\sigma})^n. \end{aligned}$$

If $\beta = \lambda\sigma + \gamma + \mu\bar{\sigma}$, with $\gamma \in H^{1,1}(X)$, then a simple calculation shows

$$\int \alpha^{2n-2}\beta^2 = s^n\binom{2n-2}{n}(\lambda^2 + s^{-2}\mu^2) + s^{n-1}\binom{2n-2}{n-1}(2\lambda\mu + \int(\sigma\bar{\sigma})^{n-1}\gamma^2)$$

and
$$\int \alpha^{2n-1}\beta = s^n \binom{2n-1}{n}(\lambda + s^{-1}\mu).$$
Inserting these two expressions on the right hand side of (2) immediately gives the desired equality.

Since f_X and $f_{\mathcal{X}_t}$ only differ by a scalar, (2) holds also for all α of the form $\sigma_t + s\bar{\sigma}_t$. But these classes cover a small open subset of $H^2(X,\mathbb{C})$. Indeed, σ_t varies freely in an open subset of the hypersurface defined by f_X and $s\bar{\sigma}$ provides another direction. Thus, (2) holds for all α in an open subset of $H^2(X,\mathbb{C})$ and, hence, for all α. □

In Section 3 we defined for a Kähler class α the three-dimensional subspace $F := F(\alpha) \subset H^2(X,\mathbb{R})$. We denote by $H^2(X,\mathbb{R})_F$ the real primitive $(1,1)$-cohomology with respect to α, i.e. $H^2(X,\mathbb{R})_F := \{\beta \in H^{1,1}(X)_{\mathbb{R}} | \int \alpha^{2n-1}\beta = 0\}$. The reason for indexing this space with F (instead of α) will become clear in Section 6.

Lemma 5.9 — *The quadratic form f_X is positive definite on F and negative definite on $H^2(X,\mathbb{R})_F$. Moreover, the direct sum decomposition $H^2(X,\mathbb{R}) = F \oplus H^2(X,\mathbb{R})_F$ is orthogonal with respect to f_X.*

Proof. In the course of the proof of Lemma 5.4 we already saw that $f_X|_F$ is positive definite. On the other hand, (2) shows that for $\beta \in H^2(X,\mathbb{R})_F$ one has
$$f_X(\beta) = (2n-1)f_X(\alpha)v(\alpha)\int \alpha^{2n-2}\beta^2.$$
Thus, f_X restricted to $H^2(X,\mathbb{R})_F$ is a positive multiple of the standard Hodge-Riemann bilinear form, which is negative definite. That the sum is indeed direct follows from Hodge theory:
$$\begin{aligned} H^2(X,\mathbb{C}) &= (H^{2,0} \oplus H^{0,2})(X) \oplus \alpha \cdot \mathbb{C} \oplus H^2(X,\mathbb{C})_F \\ &= (F \oplus H^2(X,\mathbb{R})_F)_{\mathbb{C}}. \end{aligned}$$
In order to prove the orthogonality of $F \oplus H^2(X,\mathbb{R})_F$ with respect to f_X one argues as follows: First, recall that $H^{1,1}(X)$ is orthogonal to $(H^{2,0} \oplus H^{0,2})(X)$ due to (1). Second, the decomposition $F \oplus H^2(X,\mathbb{R})_F$ and the quadratic form f_X do not change when X moves in the twistor family $\mathcal{X}(\alpha) \to T(\alpha)$ (this notation will be introduced in 7.13) associated with the hyperkähler structure given by α. On the other hand, $(H^{2,0} \oplus H^{0,2})(X) \subset F$ does change. Hence, all of F is orthogonal to $H^2(X,\mathbb{R})_F$. □

Corollary 5.10 — *f_X on $H^2(X,\mathbb{R})$ is non-degenerate of index $(3, b_2(X)-3)$.* □

Now choose $\alpha \in H^2(X,\mathbb{Q})$ close to $\sigma + \bar{\sigma}$. In particular, $f_X(\alpha) > 0$. Then by (2)
$$f_X(\beta)/f_X(\alpha) = \frac{1}{v(\alpha)^2}\left((2n-1)v(\alpha)\int \alpha^{2n-2}\beta^2 - (2n-2)\left(\int \alpha^{2n-1}\beta\right)^2\right).$$

If $\beta \in H^2(X,\mathbb{Q})$, then the right hand side is certainly rational. Summarizing the results we obtain Beauville's

Theorem 5.11 — *There exists a positive constant $c \in \mathbb{R}$ such that $q_X := cf_X$ defines a primitive integral form on $H^2(X,\mathbb{Z})$ of index $(3, b_2(X)-3)$ such that $H^2(X,\mathbb{R}) = F(\alpha) \oplus H^2(X,\mathbb{R})_{F(\alpha)}$ is orthogonal for all Kähler classes α. Moreover, $q_X(\sigma) = 0$ and $q_X(\sigma+\bar\sigma) > 0$.* □

It is one of the very interesting facts in the theory of compact hyperkähler manifolds that the quadratic form not only carries information about the second cohomology but also about the intersection behaviour of higher degree cohomology classes. The following result, although not quite in this generality, was proved by Fujiki [23] and more recently by Lunts and Looijenga [39]. The proof we will give is based on an observation of Bogomolov [9].

Theorem 5.12 — *Assume $\alpha \in H^{4j}(X,\mathbb{C})$ is of type $(2j,2j)$ on all small deformations of X (in particular on X itself). Then there exists a constant $c \in \mathbb{C}$ depending on α such that*

$$\int \alpha\beta^{2(n-j)} = cq_X(\beta)^{n-j}$$

for all $\beta \in H^2(X,\mathbb{C})$. In particular, if $\alpha = [X] \in H^0(X,\mathbb{C})$, then $v(\beta) = \int \beta^{2n} = cq_X(\beta)^n$ with c a positive real constant.

Proof. By the Local Torelli Theorem the period map defines an isomorphism $\mathcal{P}_X : Def(X) \cong U$, where U is an open subset of the quadric $(q_X) \subset \mathbb{P}(H^2(X,\mathbb{C}))$ defined by the quadratic form q_X on $H^2(X,\mathbb{C})$. If $\beta = \mathcal{P}_X(t)$, then β represents a $(2,0)$-form on the fibre $\mathcal{X}_t$. Since α is of type $(2j,2j)$ on $\mathcal{X}_t$ by assumption and $\mathcal{X}_t$ is of complex dimension $2n$, the classes $\alpha\beta^i \in H^{4j+2i}(X,\mathbb{C})$ vanish for $2i+2j > 2n$, i.e. for $i > n-j$. Thus, for $i > n-j$ the complete intersection defined by $\alpha(\,.\,)^i = 0$ contains U and hence (q_X).

Let us now come to the proof of the assertion. The integral $\int \alpha\beta^{2(n-j)}$ defines a homogeneous polynomial g_α of degree $2(n-j)$ on the projective space $\mathbb{P}(H^2(X,\mathbb{C}))$. The associated hypersurface is denoted (g_α). If $g_\alpha = 0$, then set $c = 0$. If $g_\alpha \neq 0$ and $\mathrm{supp}(g_\alpha) \subset (q_X)$, then there exists a constant c such that $g_\alpha = cq_X^{n-j}$, i.e. $\int \alpha\beta^{2(n-j)} = cq_X(\beta)^{n-j}$. Let now $g_\alpha \neq 0$ and $\mathrm{supp}(g_\alpha) \not\subset (q_X)$. Then pick $\beta \in (g_\alpha)$ such that $\beta \notin (q_X)$. For the generic $\gamma \in \mathbb{P}(H^2(X,\mathbb{C})) \setminus (g_\alpha)$ the line $\overline{\beta\gamma}$ meets q_X in two distinct points δ_0, δ_1. Denoting representatives of all the classes in $H^2(X,\mathbb{C})$ by the same symbols, one can write $\gamma = a_0\delta_0 + a_1\delta_1$ and $\beta = b_0\delta_0 + b_1\delta_1$. Since $\beta \notin (q_X)$, both coefficients b_0 and b_1 are non-zero. Now use $\gamma \notin (g_\alpha)$

and $\alpha\delta_0^i = \alpha\delta_1^i = 0$ for $i > n-j$ to prove

$$\begin{aligned} 0 &\neq \int \alpha\gamma^{2(n-j)} = \int \alpha(a_0\delta_0 + a_1\delta_1)^{2(n-j)} \\ &= \binom{2(n-j)}{n-j}(a_0a_1)^{n-j}\int \alpha(\delta_0\delta_1)^{n-j} \\ &= \binom{2(n-j)}{n-j}(\tfrac{a_0a_1}{b_1})^{n-j}\int \alpha\delta_0^{n-j}(\beta - b_0\delta_0)^{n-j} \\ &= \binom{2(n-j)}{n-j}(\tfrac{a_0a_1}{b_1})^{n-j}\int \alpha(\delta_0\beta)^{n-j}. \end{aligned}$$

Hence, $\int \alpha(\delta_0\beta)^{n-j} \neq 0$. On the other hand, $\beta \in (g_\alpha)$ yields

$$\begin{aligned} 0 &= \int \alpha\beta^{2(n-j)} = \int \alpha(b_0\delta_0 + b_1\delta_1)^{2(n-j)} \\ &= \binom{2(n-j)}{n-j}(b_0b_1)^{n-j}\int \alpha(\delta_0\delta_1)^{n-j} \\ &= \binom{2(n-j)}{n-j}b_0^{n-j}\int \alpha\delta_0^{n-j}(\beta - b_0\delta_0)^{n-j} \\ &= \binom{2(n-j)}{n-j}b_0^{n-j}\int \alpha(\delta_0\beta)^{n-j}. \end{aligned}$$

Since $b_0 \neq 0$, this contradicts $\int \alpha(\delta_0\beta)^{n-j} \neq 0$. □

A typical example of a cohomology class α that is of type $(2j, 2j)$ on all deformations of X is the Chern class $c_{2j}(X)$ or, more generally, any class contained in the subring generated by the Chern classes of X. As the full cohomology ring of a compact hyperkähler manifold is still a mystery, it is not quite clear whether any α satisfying the assumption of the theorem is in fact a polynomial in the Chern classes of X.

Corollary 5.13 — *Let X be an irreducible symplectic manifold. Then there exist constants $a_i \in \mathbb{Q}$ such that for any line bundle L on X one has:*

$$\chi(L) = \sum \frac{a_i}{(2i)!}q_X(c_1(L))^i.$$

Proof. Indeed, by the Hirzebruch-Riemann-Roch formula $\chi(L) = \sum\int ch_i(L)td_{2n-i}(X)$, where $(2i)!ch_{2i}(L) = c_1(L)^{2i}$. The i-th Todd polynomial $td_i(X)$ can be expressed as a polynomial in the Chern classes $c_j(X)$. Since $c_j(X) = c_j(\mathcal{T}_X) = c_j(\Omega_X) = (-1)^j c_j(\mathcal{T}_X)$, the odd Chern classes of X are trivial. Hence, $td_j(X) = 0$ for $j \equiv 1(2)$. For the even Todd polynomials one applies 5.12 in order to get $c_1(L)^{2i}td_{2n-2i}(X) = a_iq_X(c_1(L))^i$, where a_i is the constant c in the notation of Theorem 5.12 corresponding to the class $td_{2n-2i}(X)$. □

To conclude this section we introduce the notion of the *Kähler cone* $\mathcal{K}_X$ and the *positive cone* $\mathcal{C}_X$.

Definition 5.14 — *The Kähler cone* $\mathcal{K}_X \subset H^{1,1}(X)_{\mathbb{R}} := H^{1,1}(X) \cap H^2(X, \mathbb{R})$ *is the open convex cone of all Kähler classes on* X. *The positive cone* $\mathcal{C}_X$ *is by definition the component of* $\{\alpha \in H^{1,1}(X)_{\mathbb{R}} | q_X(\alpha) > 0\}$ *that contains the Kähler cone* $\mathcal{K}_X$.

Note that one has a Hodge Index Theorem with respect to q_X: A class $\beta \in H^{1,1}(X)_{\mathbb{R}}$ which is orthogonal to a Kähler class (with respect to q_X) satisfies $q_X(\beta) < 0$ or is zero. In particular, if $\alpha \in \mathcal{K}_X$ then $q_X(\alpha, \,.\,)$ is positive on $\mathcal{C}_X$. Moreover, if $\alpha \in \mathcal{K}_X$ then $\int \alpha^{2n-1}\beta > 0$ for $\beta \in \mathcal{C}_X$. Indeed, if not then one could find $\beta \in \mathcal{C}_X$ with $\int \alpha^{2n-1}\beta = 0$, i.e. $\beta \in H^2(X, \mathbb{R})_{F(\alpha)}$, and hence $q_X(\beta) < 0$. This is absurd.

Using the relation between $q_X(\alpha)$ and $v(\alpha)$, the fact that $q(\alpha, \,.\,)$ is positive on $\mathcal{C}_X$ for $\alpha \in \mathcal{K}_X$ can be generalized to the following statement:

Lemma 5.15 — *Let* α *be a Kähler class and let* $Y \subset X$ *be an effective divisor, then* $q_X(\alpha, [Y]) > 0$.

(Later we will see that any integral class in $\mathcal{C}_X$ is effective (9.10, 9.12).)

Proof. One possible proof goes as follows: If $[Y] \in \mathcal{C}_X$ then $q_X(\alpha, [Y]) > 0$ as we have just seen. Also, since $\int_Y \alpha^{2n-1} > 0$ and hence $\int \alpha^{2n-1}(-[Y]) < 0$, the case $-[Y] \in \mathcal{C}_X$ can be excluded. Therefore, it remains to verify the claim for $q_X([Y]) \leq 0$. Of course, $q_X(\alpha, [Y]) > 0$ if and only if $q_X(\alpha + [Y]) > q_X(\alpha) + q_X([Y])$. Thus, it suffices to show $q_X(\alpha + [Y]) > q_X(\alpha)$. Replacing α by $k\beta$ with $k \gg 0$ and β fixed, this is equivalent to $\int (k\beta + [Y])^{2n} = cq_X(k\beta + [Y])^n > cq_X(k\beta)^n = \int (k\beta)^{2n}$, where c is a positive constant. Since $\int_Y \beta^{2n-1} > 0$, this follows immediately. □

6 Cohomology

Let X be an irreducible symplectic manifold. The following identities are immediate consequences of the definition:

$$\begin{aligned} H^1(X, \mathcal{O}_X) &= 0, \quad H^2(X, \mathcal{O}_X) \cong \mathbb{C} \\ H^0(X, \mathcal{T}_X) &\cong H^0(X, \Omega_X) = 0 \\ H^1(X, \mathcal{T}_X) &\cong H^1(X, \Omega_X). \end{aligned}$$

In this section more results concerning the cohomological structure of irreducible symplectic manifolds are discussed. Most of them are due to Fujiki. For details and other results in this direction we refer to Enoki's survey article [18], to the original paper of Fujiki [23] and to the more recent articles by Looijenga and Lunts [39] and Verbitsky [56].

Let (M, g) be a compact irreducible hyperkähler manifold of real dimension $4n$. Let F denote the associated HK 3-space spanned by the three Kähler forms $[\omega_I], [\omega_J], [\omega_K]$. The

Lefschetz operator $L_\lambda := L_{\omega_\lambda}$ on the Kähler manifold $(M, \lambda, \omega_\lambda)$ for $\lambda \in S^2$ acts on the cohomology $H^*(M, \mathbb{R})$ and allows one to define the primitive cohomology $H^k((M, \lambda), \mathbb{R})_{\rm pr} := \ker(L_\lambda^{2n-k+1} : H^k(M, \mathbb{R}) \to H^{4n-k+2}(M, \mathbb{R}))$ $(k \leq 2n)$. Recall, the *Lefschetz decomposition* on the compact Kähler manifold $(M, \lambda, \omega_\lambda)$ is the direct sum decomposition

$$H^k(M, \mathbb{R}) = H^k((M, \lambda), \mathbb{R}) = \bigoplus_{(k-\ell) \geq (k-2n)^+} L_\lambda^{k-\ell}(H^{2\ell-k}((M, \lambda), \mathbb{R})_{\rm pr})$$

and the *Hard Lefschetz Theorem* asserts that for $k \leq 2n$

$$L_\lambda^{2n-k} : H^k(M, \mathbb{R}) \cong H^{4n-k}(M, \mathbb{R}).$$

In particular,

$$L_\lambda^{k-\ell} : H^{2\ell-k}((M, \lambda), \mathbb{R})_{\rm pr} \cong L_\lambda^{k-\ell}(H^{2\ell-k}((M, \lambda), \mathbb{R})_{\rm pr}).$$

Recall, that both Lefschetz theorems are formal consequences of the representation theory of sl(2), where the sl(2)-action is constructed by means of the Lefschetz operator and its adjoint. Combining the statements for all $\lambda \in S^2$ one obtains the following results [23, 18]:

Theorem 6.1 — *Let $N^* \subset H^*(M, \mathbb{R})$ be the subalgebra generated by F and denote by $H^*(M, \mathbb{R})_F$ the subspace $\bigoplus H^k(M, \mathbb{R})_F = \bigoplus \{\alpha \in H^k((M, \lambda), \mathbb{R})_{\rm pr} |$ for all $\lambda \in S^2\}$. Then*

$$H^k(M, \mathbb{R}) = \bigoplus_{2(k-l) \geq (k-2n)^+} N^{k-\ell} H^\ell(M, \mathbb{R})_F$$

and for $k \leq n$

$$N^{k-\ell} \otimes_{\mathbb{R}} H^\ell(M, \mathbb{R})_F \cong N^{k-\ell} H^\ell(M, \mathbb{R})_F.$$

The first statement in particular says $H^2(M, \mathbb{R}) = F \oplus H^2(M, \mathbb{R})_F$. Note that this decomposition is exactly the one used in Section 5. Indeed, for any $\lambda \in S^2$ the space $H^2(M, \mathbb{R})_F$ is contained in $H^{1,1}((M, \lambda), \mathbb{R})$.

Let us now come to the holomorphic counterpart. Using the holonomy principle, Beauville [2] proved the following:

Lemma 6.2 — *If X is an irreducible symplectic manifold and $0 \neq \sigma \in H^0(X, \Omega_X^2)$, then*

$$H^0(X, \Omega_X^p) \cong \begin{cases} 0 & p \equiv 1(2) \\ \Lambda^{p/2}\sigma \cdot \mathbb{C} & p \equiv 0(2). \end{cases}$$

□

This was substantially generalized by Fujiki who proved holomorphic versions of the Lefschetz decomposition and the Hard Lefschetz Theorem. To formulate his results we need to introduce the following notation. Let $L_\sigma : H^q(X, \Omega_X^p) \to H^q(X, \Omega_X^{p+2})$ and $L_{\bar\sigma} : H^q(X, \Omega_X^p) \to H^{q+2}(X, \Omega_X^p)$ be the map given by the cup-product with σ and $\bar\sigma$, respectively, and let $H^q(X, \Omega_X^p)_\sigma := \ker(L_\sigma^{n-p+1})$ and $H^q(X, \Omega_X^p)_{\bar\sigma} := \ker(L_{\bar\sigma}^{n-q+1})$.

Theorem 6.3 — *There is a holomorphic respectively antiholomorphic Lefschetz decomposition*

$$H^q(X,\Omega_X^p) = \bigoplus_{(p-\ell)\geq(p-n)^+} L_\sigma^{p-\ell} H^q(X,\Omega_X^{2\ell-p})_\sigma = \bigoplus_{(q-\ell)\geq(q-n)^+} L_{\bar\sigma}^{q-\ell} H^{2\ell-q}(X,\Omega_X^p)_{\bar\sigma}$$

and a holomorphic Hard Lefschetz Theorem

$$\begin{array}{rlcll} L_\sigma^{n-p}: & H^q(X,\Omega_X^p) & \cong & H^q(X,\Omega_X^{2n-p}) & \text{for } p\leq n \\ L_{\bar\sigma}^{n-q}: & H^q(X,\Omega_X^p) & \cong & H^{2n-q}(X,\Omega_X^p) & \text{for } q\leq n. \end{array}$$

Proof. Let us first give an ad-hoc argument for the holomorphic Hard Lefschetz Theorem (which could also be proved by using the sl(2)-action on the cohomology constructed below). It is in fact a consequence of the elementary fact that the exterior product with σ^{n-p} defines an isomorphism $\Omega_X^p \cong \Omega_X^{2n-p}$. In order to see this we argue locally. Let V be a $2n$-dimensional vector space with a symplectic structure σ on V^*. Fix a basis e_i of V such that $\sigma = \sum_{i=1}^n e_i \wedge e_{i+n}$. Any basis vector v of $\Lambda^p V$ can uniquely be written as

$$(e_{i_1}\wedge\ldots\wedge e_{i_a})\wedge(e_{j_1+n}\wedge\ldots\wedge e_{j_b+n})\wedge(e_{k_1}\wedge e_{k_1+n}\wedge\ldots\wedge e_{k_c}\wedge e_{k_c+n}),$$

where the numbers $1\leq i_m, j_m, k_m\leq n$ are pairwise distinct and $a+b+2c=p$. Then

$$\begin{array}{rcl} v\wedge\sigma^{n-p} & = & \sum(e_{i_1}\wedge\ldots\wedge e_{i_a})\wedge(e_{j_1+n}\wedge\ldots\wedge e_{j_b+n}) \\ & & \wedge(e_{k_1}\wedge e_{k_1+n}\wedge\ldots\wedge e_{k_c}\wedge e_{k_c+n}) \\ & & \wedge(e_{\ell_1}\wedge e_{\ell_1+n}\wedge\ldots\wedge e_{\ell_{n-p}}\wedge e_{\ell_{n-p}+n}), \end{array}$$

where $\{\ell_1,\ldots,\ell_{n-p}\}\cap\{i_1,\ldots,i_a,j_1,\ldots,j_b,k_1,\ldots,k_c\}=\emptyset$. But given $v\wedge\sigma^{n-p}$ of this shape and knowing that v is a basis vector, one can easily read off all the indices $i_1,\ldots,k_c$ necessary to identify v. Hence, $\sigma^{n-p}:\Lambda^p\to\Lambda^{2n-p}$ is injective. Thus, $\sigma^{n-p}:\Omega_X^p\to\Omega_X^{2n-p}$ is injective at every point. Since both vector bundles are of the same rank, it is an isomorphism. Taking q-th cohomology proves the first isomorphism of the Hard Lefschetz Theorem. The second one is obtained from the first by complex conjugation.

The proof of the Lefschetz decomposition follows closely the original argument for the standard Lefschetz decomposition. Fix a Ricci-flat Kähler metric on X and let $*$ be the associated Hodge operator. The hyperkähler structure associated with the metric provides three complex structures I, J, and K as in Section 3, where I is the one giving X. By rescaling σ, we may assume that $\sigma=\omega_J+i\omega_K$ (see 3.4). Then define $\Lambda_\sigma := *^{-1}\circ L_{\bar\sigma}\circ *$, which is adjoint to L_σ with respect to the Hodge metric. Indeed,

$$\begin{array}{rcl} (\alpha,\Lambda_\sigma\beta) & = & \int\alpha\wedge*\overline{(*^{-1}L_{\bar\sigma}(*\beta))} = \int\alpha\wedge*(*^{-1}L_\sigma(*\bar\beta)) \\ & = & \int\alpha\wedge L_\sigma(*\bar\beta) = \int L_\sigma(\alpha)\wedge*\bar\beta \\ & = & (L_\sigma(\alpha),\beta), \end{array}$$

where we use that $\overline{L_{\bar\sigma}} = L_\sigma$ and that $*$ is $\mathbb{C}$-linear. Moreover, since $\Delta(\bar\sigma) = 0$ and $*\Delta = \Delta*$ (see e.g. [57]), Λ_σ commutes with Δ and is thus well-defined on the level of cohomology. Let now Π be the operator on $H^*(X, \Omega_X^*)$ given by multiplication with $(p-n)$ on $H^*(X, \Omega_X^p)$. Then $[\Lambda_\sigma, \Pi] = 2\Lambda_\sigma$, $[\Pi, L_\sigma] = 2L_\sigma$, and $[L_\sigma, \Lambda_\sigma] = 4\Pi$. The first two equations are trivial. The last one follows from a calculation similar to the one for the usual Hodge operators. As it is a completely local statement, it is enough to consider a vector space $V = \mathbb{H}^n$ with an orthonormal basis (e_{ij}), $i = 1, \ldots, n$, $j = 1, \ldots, 4$, where $(e_{i1}, e_{i2}, e_{i3}, e_{i4})$ is a standard basis for the i-th factor $\mathbb{H}$ of V, i.e. $e_{i2} = Ie_{i1}$, $e_{i3} = Je_{i1}$, $e_{i4} = Ke_{i1}$. Then $\sigma = \sum_i e_{i1} \wedge e_{i3} + e_{i4} \wedge e_{i2} + i(e_{i1} \wedge e_{i4} + e_{i2} \wedge e_{i3})$. Instead of giving the details of this unpleasant calculation, which is very similar to the standard one, let us consider two specific examples that give evidence for the scalar factor $4(p-n)$. First, let $p = 0$. Then we only have to compute $[L_\sigma, \Lambda_\sigma](1) = -\Lambda_\sigma(L_\sigma(1)) = -\Lambda_\sigma(\sigma) = -(\sigma, \bar\sigma) = -4n$. Next, consider the case $n = 1$ and $p = 2$. We have to compute $[L_\sigma, \Lambda_\sigma](\sigma)$. As we have just seen $\Lambda(\sigma) = 4$. Hence, $[L_\sigma, \Lambda_\sigma](\sigma) = 4\sigma = 4(2-1)\sigma$.

Thus, $(1/2)L_\sigma$, $(1/2)\Lambda_\sigma$, and Π define an sl(2)-action on $H^*(X, \Omega_X^*)$. The first decomposition claimed in the theorem is then a formal consequence of the representation theory of sl(2). The second is obtained by complex conjugation. $\square$

There exists also the natural non-degenerate pairing $\Omega_X^p \otimes \Omega_X^{2n-p} \to K_X \cong \mathcal{O}_X$ which combined with $\Lambda^{2n-p}\sigma : \Lambda^{2n-p}(\mathcal{T}_X) \cong \Omega_X^{2n-p}$ induces an isomorphism $\Omega_X^p \cong \Omega_X^{2n-p}$. For $p \neq 1$ it is different from the isomorphism given by the exterior product with σ^{n-p}. However, if $p = 1$ both maps coincide. We thus obtain an alternative proof of the theorem for $p = 1$. As this is the result we will make use of later on, let us formulate it again as

Corollary 6.4 — *If X is an irreducible symplectic manifold of dimension $2n$ and L_σ and $L_{\bar\sigma}$ are defined as above, then*

$$\begin{array}{llcll} L_\sigma^{n-1}: & H^q(X, \Omega_X) & \cong & H^q(X, \Omega_X^{2n-1}) & \text{for all } q \\ L_{\bar\sigma}^{n-1}: & H^1(X, \Omega_X^p) & \cong & H^{2n-1}(X, \Omega_X^p) & \text{for all } p. \end{array}$$

$\square$

In Section 9 we will also use a version of this isomorphism on the level of smooth forms.

A consequence of Theorem 6.3 is the following proposition which will be crucial at several places in the exposition.

Proposition 6.5 — *The following natural pairings are perfect:*

$$\begin{array}{lclcl} H^1(X, \mathcal{T}_X) & \otimes & H^{2n-1}(X, \Omega_X^{2n-1}) & \to & H^{2n}(X, \Omega_X^{2n-2}) \cong (\bar\sigma^n \sigma^{n-1}) \cdot \mathbb{C} \\ H^1(X, \mathcal{T}_X) & \otimes & H^1(X, \Omega_X) & \to & H^2(X, \mathcal{O}_X) \cong \bar\sigma \cdot \mathbb{C} \end{array}$$

Proof. By Serre duality it suffices to show that the induced homomorphisms $H^1(X,\mathcal{T}_X) \to \mathrm{Hom}(H^{2n-1}(X,\Omega_X^{2n-1}), H^{2n}(X,\Omega_X^{2n-2})) \cong \mathrm{Hom}(H^0(X,\Omega_X^2), H^1(X,\Omega_X))$ and $H^1(X,\mathcal{T}_X) \to \mathrm{Hom}(H^1(X,\Omega_X), H^2(X,\mathcal{O}_X)) \cong \mathrm{Hom}(H^{2n-2}(X,\Omega_X^{2n}), H^{2n-1}(X,\Omega_X^{2n-1}))$ are bijective. Using the general form of the holomorphic Hard Lefschetz Theorem above one finds that these maps are also given by the cup-product. Then the perfectness of the first pairing follows from the fact that $\sigma : \mathcal{T}_X \to \Omega_X$ is bijective and the proof of the second uses in addition that the composition of $L_\sigma^n : H^1(X,\mathcal{T}_X) \to H^1(X,\Omega_X^{2n-1})$ followed by $L_{\bar\sigma}^{n-1} : H^1(X,\Omega_X^{2n-1}) \to H^{2n-1}(X,\Omega_X^{2n-1})$ is bijective. □

Corollary 6.6 — *For any $0 \neq \alpha \in H^1(X,\Omega_X)$ (or $\in H^{2n-1}(X,\Omega_X^{2n-1})$) the natural map $\alpha : H^1(X,\mathcal{T}_X) \to H^2(X,\mathcal{O}_X)$ (respectively $H^1(X,\mathcal{T}_X) \to H^{2n}(X,\Omega_X^{2n-2})$) is surjective.* □

Remark 6.7 — We will also use the following fact: Let $\beta \in H^{2n-1}(X,\Omega_X^{2n-1})$, then the cup-product with β sends $v \in H^1(X,\mathcal{T}_X)$ to $\beta \cdot v \in H^{2n}(X,\Omega_X^{2n-2})$. Hence, $(\beta\cdot v)\sigma$ can be integrated over X. On the other hand, β can be regarded as a linear form on $H^1(X,\Omega_X)$, and thus can be evaluated on the image α of v under the isomorphism $H^1(X,\mathcal{T}_X) \cong H^1(X,\Omega_X)$. Clearly, the two expressions $\int_X(\beta\cdot v)\sigma$ and $\beta(\alpha) = \int\beta\alpha$ agree. A similar statement holds for $\beta \in H^1(X,\Omega_X)$. Here we have $\int(\beta\cdot v)\sigma^n\bar\sigma^{n-1} = \int(\beta\alpha)(\sigma\bar\sigma)^{n-1} = cq_X(\beta,\alpha)$, where c is a positive constant (for the definition of q_X see Section 5).

By means of the integral quadratic form q_X one can establish a close link between rational classes of dimension one and those of codimension one. Namely, one has

Proposition 6.8 — *If c is the positive constant such that $c\cdot f_X = q_X$, then $cL_{\sigma\bar\sigma}^{n-1}$ defines an isomorphism $H^{1,1}(X)_{\mathbb{Q}} \cong H^{2n-1,2n-1}(X)_{\mathbb{Q}}$.*

(Note that this is quite similar to the Hard Lefschetz Theorem with respect to a Hodge class.)

Proof. Indeed, if $\alpha \in H^{1,1}(X)$ and $\beta := cL_{\sigma\bar\sigma}^{n-1}(\alpha)$, then for any $\gamma \in H^2(X,\mathbb{Q})$ one has

$$\int\beta\gamma = \int\beta\gamma^{1,1} = c\int L_{\sigma\bar\sigma}^{n-1}(\alpha)\gamma^{1,1} = \frac{2}{n}q_X(\alpha,\gamma^{1,1}) = \frac{2}{n}q_X(\alpha,\gamma).$$

Now, if α is rational then $q_X(\alpha,\gamma) \in \mathbb{Q}$ and hence $\int\beta\gamma \in \mathbb{Q}$ for all $\gamma \in H^2(X,\mathbb{Q})$. Thus $\beta \in H^{4n-2}(X,\mathbb{Q})$. Conversely, if β is rational, then $\int\beta\gamma \in \mathbb{Q}$ and therefore $q_X(\alpha,\gamma) \in \mathbb{Q}$. Since q_X is non-degenerate and integral, this shows $\alpha \in H^2(X,\mathbb{Q})$. □

7 Period Map and Moduli Space

In this Section we first introduce the notion of marked irreducible symplectic manifolds. In contrast to the space of irreducible symplectic manifolds the moduli space of marked irreducible symplectic manifolds can be endowed with the structure of a smooth, although non-separated, complex space. It also allows one to define a global period map with values in the period domain which is independent of any specific X. The second part deals with local properties of the moduli space. Special deformations arising from the twistor construction and those respecting the type of a given cohomology class will be studied in some detail.

Recall that for any irreducible symplectic manifold X the second cohomology $H^2(X,\mathbb{Z})$ comes with an integral non-degenerate quadratic form q_X of index $(3, b_2(X)-3)$. Let Γ be a lattice of index $(3, b-3)$. The quadratic form on Γ is usually denoted q_Γ. Analogously to 5.3 we define

Definition 7.1 — *The period domain $Q \subset \mathbb{P}(\Gamma_\mathbb{C})$ is the set*

$$Q := \{x \in \mathbb{P}(\Gamma_\mathbb{C}) | \;\; q_\Gamma(x) = 0, \;\; q_\Gamma(x+\bar{x}) > 0\}.$$

Clearly, Q is an open subset of the smooth irreducible quadric defined by q_Γ. The following two lemmas are taken from [1]. The proofs given there for the intersection lattice of a K3 surface translate literally. The results will be needed in Section 14 to show that the period map is surjective.

Lemma 7.2 — *The map sending a point $x \in Q \subset \mathbb{P}(\Gamma_\mathbb{C})$ to $P(x) := (x \cdot \mathbb{C} \oplus \bar{x} \cdot \mathbb{C}) \cap \Gamma_\mathbb{R}$ defines a natural isomorphism between Q and the Grassmannian $\tilde{Q}$ of positive oriented planes in $\Gamma_\mathbb{R}$.*

Proof. The orientation of $P(x)$ is defined by $(\mathrm{Re}(x), \mathrm{Im}(x))$. Conversely, if a positive plane P is given then there exist exactly two isotropic lines in $P_\mathbb{C}$. An orientation of P distinguishes one of them. This line gives rise to a point $x \in Q$. Clearly, the two constructions are inverse to each other. □

Corollary 7.3 — *The period domain Q is connected.*

Proof. Here one uses the fact that q_Γ has at least three positive eigenvalues. Any positive plane P admits an orthogonal basis a_1, a_2 with $q_\Gamma(a_i) > 0$. For fixed a_1 the second vector a_2 is allowed to vary in the open subset of the orthogonal complement of a_1 of all vectors with positive square. The assumption ensures that this is a connected open subset, for details see [1]. □

Definition 7.4 — *Two points $x, y \in Q$ are called equivalent if there exists a sequence $x = x_1, x_2, \ldots, x_k = y \in Q$ such that the subspaces $\langle P(x_i), P(x_{i+1})\rangle \subset \Gamma_\mathbb{R}$ are of dimension three and such that $q_\Gamma|_{\langle P(x_i), P(x_{i+1})\rangle}$ is positive definite. (A subspace of dimension three with this property is called a positive 3-space.)*

Lemma 7.5 — *Any two points in Q are equivalent.*

Proof. Since Q is connected, it suffices to show that the equivalence relation is open, i.e. that any equivalence class is open. If a, b, c is a basis of a positive 3-space, then for any (a', b') in a neighbourhood of (a, b) the spaces $\langle a, b', c\rangle$ and $\langle a', b', c\rangle$ are positive 3-spaces as well. Thus, $\langle a, b\rangle \sim \langle a, c\rangle \sim \langle b', c\rangle \sim \langle a', b'\rangle$. □

Let us now come to the concept of marked manifolds and the moduli space of those.

Definition 7.6 — *A marking of an irreducible symplectic manifold X is an isomorphism $\varphi : (H^2(X, \mathbb{Z}), q_X) \cong (\Gamma, q_\Gamma)$. The pair (X, φ) is called a marked manifold.*

$\mathfrak{M}_\Gamma$ is the set $\{(X, \varphi)\}/\sim$, where (X, φ) is a marked irreducible symplectic manifold of dimension $2n$ and $(X, \varphi) \sim (X', \varphi')$ if and only if there exists an isomorphism $f : X \cong X'$ such that $f^* = \pm(\varphi^{-1} \circ \varphi')$.

$\mathfrak{M}_\Gamma$ is called the moduli space of marked irreducible symplectic manifolds, but a priori it is just a set. The composite of the period map $\mathcal{P}_X$ and the isomorphism $\mathbb{P}(H^2(X, \mathbb{C})) \cong \mathbb{P}(\Gamma_\mathbb{C})$ induced by a marking φ gives rise to the period map $\mathcal{P} : Def(X) \to \mathbb{P}(\Gamma_\mathbb{C})$ for the marked manifold (X, φ). The image is contained in Q and by the Local Torelli Theorem 5.6 the map $\mathcal{P} : Def(X) \to Q$ is a local isomorphism. This allows one to prove

Proposition 7.7 — *Glueing the deformation spaces $Def(X)$ defines on $\mathfrak{M}_\Gamma$ the structure of a smooth non-separated (i.e. non-Hausdorff) complex space.*

Proof. For any irreducible symplectic manifold X we consider its universal deformation $\mathcal{X} \to Def(X)$. We can think of $Def(X)$ as a small disk. A marking φ of X naturally induces markings φ_t of all fibres $\mathcal{X}_t$. By the Local Torelli Theorem 5.6 the period map $\mathcal{P} : Def(X) \to Q \subset \mathbb{P}(\Gamma_\mathbb{C})$ defined by φ is a local isomorphism. Hence, two different fibres $\mathcal{X}_{t_0}$ and $\mathcal{X}_{t_1}$ are non-isomorphic as marked manifolds (with the markings φ_{t_0} and φ_{t_1}, respectively). Thus, the base space $Def(X)$ can be regarded as a subset of $\mathfrak{M}_\Gamma$ containing (X, φ). If $(X, \varphi), (X', \varphi') \in \mathfrak{M}_\Gamma$, then the intersection $Def(X) \cap Def(X')$ is an open subset of $Def(X)$ (which might be empty), for $\mathcal{X} \to Def(X)$ is a universal deformation for any $t \in Def(X)$. Therefore, the complex structures of the deformation spaces $Def(X)$ glue to a complex structure on $\mathfrak{M}_\Gamma$. Since $Def(X)$ is smooth, also $\mathfrak{M}_\Gamma$ is smooth. □

Remark 7.8 — At this point, a warning remark is in order. $\mathfrak{M}_\Gamma$ is in fact the moduli space of marked irreducible symplectic manifolds in the following sense: For any family $(\mathcal{X}, \varphi) \to S$ of marked irreducible symplectic manifolds there exists a uniquely defined holomorphic map $S \to \mathfrak{M}_\Gamma$ such that $t \in S$ is mapped to $(\mathcal{X}_t, \varphi_t) \in \mathfrak{M}_\Gamma$ and such that locally $\mathcal{X}_S \to S$ is the pull-back of the universal family $\mathcal{X} \to Def(X)$ under the holomorphic map $S \to Def(X) \subset \mathfrak{M}_\Gamma$, where $Def(X) \subset \mathfrak{M}_\Gamma$ is defined by φ_0. But, there does not exist a universal family over $\mathfrak{M}_\Gamma$! This is due to the fact that an irreducible symplectic manifold might have automorphisms that act trivially on the second cohomology. An example for such a phenomenon was given by Beauville in [3]. Note that in dimension two, i.e. for K3 surfaces, $\mathfrak{M}_\Gamma$ is fine, i.e. there exists a universal family (the Global Torelli holds!).

Definition 7.9 — *The global period map is the holomorphic map*

$$\mathcal{P} : \mathfrak{M}_\Gamma \to Q \subset \mathbb{P}(\Gamma_\mathbb{C})$$

obtained by glueing the local period maps $Def(X) \to Q_X \cong Q_\Gamma \subset \mathbb{P}(\Gamma_\mathbb{C})$.

One of the main problems in the theory of irreducible symplectic manifolds is to understand the fibres and the image of this map. E.g. the Global Torelli Theorem for K3 surfaces asserts that points in the same fibre of $\mathcal{P}$ correspond to just one K3 surface with different markings (for a more precise statement see Section 8). Moreover, for K3 surfaces the period map $\mathcal{P}$ is known to surject $\mathfrak{M}_\Gamma$ onto Q. The latter statement generalizes to higher dimensions as will be shown in Section 14.

We now come back to more local questions related to the deformation theory of irreducible symplectic manifolds. As a consequence of the general Tian-Todorov unobstructedness result 1.5 one knows that for an irreducible symplectic manifold X the deformation space $Def(X)$ is smooth of dimension $h^1(X, \mathcal{T}_X) = h^1(X, \Omega_X) = h^{1,1}(X)$ (which is positive, for X is Kähler). Also recall that any small deformation of an irreducible symplectic manifold is irreducible symplectic ([2], see also 4.4).

Convention: 7.10 — In these notes we will say that a certain property holds for the *generic* fibre, if for an open (in the analytic topology) dense set $U \subset S$ and all $t \in U$ the fibre $\mathcal{X}_t$ has this property. The property holds for the *general* fibre if such a set U exists which is the complement of the union of countably nowhere dense closed (in the analytic topology) subsets.

A very special deformation of an irreducible symplectic manifold X is associated with any Kähler class $\alpha \in H^{1,1}(X)_\mathbb{R}$. Namely, if g is the Calabi-Yau metric realizing α, then the underlying real manifold M endowed with g is irreducible hyperkähler. As we explained in Section 2 the sphere of complex structures $\{\lambda = aI + bJ + cK | a^2 + b^2 + c^2 = 1\}$ gives rise to a twistor space which in the present context will be denoted by $\mathcal{X}(\alpha) \to \mathbb{P}^1$, i.e. $\mathcal{X}(\alpha) \to \mathbb{P}^1$ is a smooth proper holomorphic map such that the fibre over $\lambda \in S^2 \cong \mathbb{P}^1$ is isomorphic to (M, λ). In particular, the fibre over I is X. Obviously, $\mathcal{X}(\alpha) \to \mathbb{P}^1$ can as well be considered as the twistor space of any of the fibres $X' := (M, \lambda)$ with respect to the Kähler class $[\omega_\lambda]$.

As a deformation of X the twistor space $\mathcal{X}(\alpha) \to \mathbb{P}^1$ induces a (local) map $\mathbb{P}^1 \to Def(X)$. The image of the associated map between the tangent spaces (the Kodaira-Spencer map) is described by the following lemma due to Fujiki [23].

Lemma 7.11 — *The image of the Kodaira-Spencer map induced by the twistor space of (X, α) is the one-dimensional subspace of $H^1(X, \Omega_X)$ spanned by α, where $H^1(X, \mathcal{T}_X)$ is identified with $H^1(X, \Omega_X)$ via the symplectic structure on X.*

Proof. By the very construction of the twistor space a $(2,0)$-form on any fibre is contained in the span of ω_I, ω_J, and ω_K, i.e. in $F(\alpha)_{\mathbb{C}} = \sigma\cdot\mathbb{C}\oplus\bar\sigma\cdot\mathbb{C}\oplus\alpha\cdot\mathbb{C}$. In particular, classes $\beta\in H^2(X,\mathbb{C})_F$ are of type $(1,1)$ on all the fibres of $\mathcal{X}(\alpha)\to\mathbb{P}^1$. Thus, if v is the Kodaira-Spencer class of $\mathcal{X}(\alpha)\to\mathbb{P}^1$ then the contraction $v: H^2(X,\mathbb{C})_F\subset H^{1,1}(X)\to H^{0,2}$ is zero. By 6.7 for any $\beta\in H^2(X,\mathbb{R})$ the class $v\beta\in H^{0,2}(X)$ is trivial if and only if $q_X(\sigma(v),\beta)=0$. Hence, $\sigma(v)\in F\cap H^{1,1}(X)$, i.e. $\sigma(v)\in\alpha\cdot\mathbb{C}$. □

Note that the unobstructedness of X can also be proved along this line: Any class in $\mathcal{K}_X\subset H^{1,1}(X)$ can be realized as a Kodaira-Spencer class of a smooth curve and $\mathcal{K}_X$ is open in the real part $H^{1,1}(X)_{\mathbb{R}}$ of $H^{1,1}(X)$ (see [23] for the details).

The twistor space can also be described globally. This is done by means of the period map as follows.

Lemma 7.12 — *If φ is a marking of X and $F(\alpha)_{\mathbb{C}} = \sigma\cdot\mathbb{C}\oplus\bar\sigma\cdot\mathbb{C}\oplus\alpha\cdot\mathbb{C}$, then the period map $\mathcal{P}:\mathbb{P}^1\to\mathbb{P}(\Gamma_{\mathbb{C}})$ defines an isomorphism $\mathbb{P}^1\cong Q\cap\mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}}))$. Thus, $\mathbb{P}^1$, as the base space of the twistor space, is naturally identified with the quadric $T(\alpha)\subset\mathbb{P}(F(\alpha)_{\mathbb{C}})$ defined by $q_X=0$.*

Proof. By definition the period $\mathcal{P}$ maps $\lambda\in\mathbb{P}^1$ to $[\varphi(\sigma_\lambda)]$, where σ_λ is a holomorphic two-form on the fibre $\mathcal{X}_\lambda$. In particular, $\sigma_\lambda\in F(\alpha)_{\mathbb{C}}$ and $[\varphi(\sigma_\lambda)]\in\mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}}))$. By the period relations: $\mathrm{im}(\mathcal{P})\subset Q\cap\mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}}))$. Since q_X is positive definite on $F(\alpha)$ by Lemma 5.9, the quadric $Q\cap\mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}}))$ is irreducible. On the other hand, the Kodaira-Spencer class of $\mathcal{X}(\alpha)\to\mathbb{P}^1$ is non-trivial at any $\lambda\in\mathbb{P}^1$ and hence $\mathbb{P}^1\to Q$ is unramified. Therefore, $\mathcal{P}:\mathbb{P}^1\cong Q\cap\mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}}))$. □

Remark 7.13 — In the sequel, we will denote the twistor space of $(\mathcal{X},\alpha)$ by $\mathcal{X}(\alpha)\to T(\alpha)$. This can be slightly generalized as follows. If α is just any class of type $(1,1)$ on X we define $T(\alpha) := Def(X)\cap\mathcal{P}^{-1}(\mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}})))$ and $\mathcal{X}(\alpha)$ as the restriction of the universal family to $T(\alpha)$. Note that for a Kähler class α the space $T(\alpha)$ means the complete base $\mathbb{P}^1$ of the twistor space, but in general it is only defined as a closed subset of $Def(X)$.

At several points in these notes we will have to deal with joint deformations of an irreducible symplectic manifold X together with a line bundle L on X. Again, there exists a universal deformation $(\mathcal{X},\mathcal{L})\to Def(X,L)$, i.e. a deformation $\mathcal{X}\to Def(X,L)$ of $\mathcal{X}_0=X$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ with $\mathcal{L}_0:=\mathcal{L}|_{\mathcal{X}_0}\cong L$ such that any other deformation $(\mathcal{X}_S,\mathcal{L}_S)\to S$ of this kind is isomorphic to the pull-back of $(\mathcal{X},\mathcal{L})$ via a uniquely determined map $S\to Def(X,L)$. The Zariski tangent space of $Def(X,L)$ is canonically isomorphic to $H^1(X,\mathcal{D}(L))$, where $\mathcal{D}(L)$ is the sheaf of differential operators on L of order ≤ 1.

Lemma 7.14 — *If X is irreducible symplectic and L is a non-trivial line bundle on X, then $Def(X,L)$ is a smooth hypersurface in $Def(X)$ and its Zariski tangent space $H^1(X,\mathcal{D}(L))$ is the kernel of the surjection $H^1(X,\mathcal{T}_X)\to H^2(X,\mathcal{O}_X)$ given by the contraction with $c_1(L)$.*

Proof. Let us begin with the last part of the assertion. By 6.6 the contraction $c_1(L) : H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X)$ is surjective whenever $c_1(L) \neq 0$, i.e. when L is non-trivial (X is simply connected). This contraction is in fact the boundary map in the long exact cohomology sequence of the symbol sequence $0 \to \mathcal{O}_X \to \mathcal{D}(L) \to \mathcal{T}_X \to 0$. Since $H^1(X, \mathcal{O}_X) = 0$, the kernel of it is exactly $H^1(X, \mathcal{D}(L))$. As $h^2(X, \mathcal{O}_X) = 1$, it is one-codimensional. The obstructions to deform (X, L) are contained in $H^2(X, \mathcal{D}(L))$ which, again by the symbol sequence, injects into $H^2(X, \mathcal{T}_X)$. The latter contains the obstruction to deform X alone, but those are trivial by the result of Tian and Todorov. Thus, the space $Def(X, L)$ is a smooth hypersurface of $Def(X)$. $\square$

Proposition 6.5 also shows that for two line bundles L and M with $c_1(L)$ and $c_1(M)$ linearly independent the two hypersurfaces $Def(X, L)$ and $Def(X, M)$ intersect transversally.

Analogously, one defines hypersurfaces for cohomology classes as follows.

Definition 7.15 — *Let (X, φ) be a marked irreducible symplectic manifold. For any $\alpha \in H^2(X, \mathbb{R})$ let*

$$S_\alpha := \{t \in Def(X) | q_\Gamma(\varphi(\alpha), \mathcal{P}(t)) = 0\},$$

i.e. S_α is the pull-back of the hyperplane defined by $q_\Gamma(\varphi(\alpha), .)$. Analogously, for a class $\alpha \in H^{4n-2}(X, \mathbb{R})$ considered as a linear form on $H^2(X, \mathbb{R})$ we define

$$S_\alpha := \{t \in Def(X) | \alpha(\varphi^{-1}(\mathcal{P}(t))) = 0\}.$$

By the properties of q_X, the set S_α for $\alpha \in H^2(X, \mathbb{R})$ is easily identified as the set of points $t \in Def(X)$ such that α is of type $(1,1)$ on $\mathcal{X}_t$. Using the holomorphic Lefschetz theorem 6.3 one finds that S_α for $\alpha \in H^{4n-2}(X, \mathbb{R})$ is the set of points $t \in Def(X)$ such that α is of type $(2n-1, 2n-1)$ on $\mathcal{X}_t$.

Lemma 7.16 — *Let X be an irreducible symplectic manifold of dimension $2n$.*

i) For $0 \neq \alpha \in H^2(X, \mathbb{R})$ or $0 \neq \alpha \in H^{4n-2}(X, \mathbb{R})$ the set S_α is a possibly empty smooth hypersurface of $Def(X)$.

ii) If $0 \in S_\alpha$, i.e. $\alpha \in H^{1,1}(X)_\mathbb{R}$ respectively $\alpha \in H^{2n-1,2n-1}(X)_\mathbb{R}$, the tangent space $T_0 S_\alpha$ of S_α is the kernel of the surjection $\alpha : H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X)$ respectively $\alpha : H^1(X, \mathcal{T}_X) \to H^{2n}(X, \Omega_X^{2n-2})$.

iii) If $\alpha, \alpha' \in H^{1,1}(X)$ are linearly independent, then S_α and $S_{\alpha'}$ intersect transversally in $0 \in Def(X)$. The analogous statement holds true for linearly independent classes $\alpha, \alpha' \in H^{2n-1,2n-1}(X)$.

iv) If $\alpha \in H^2(X, \mathbb{Z})$, then there exists a line bundle $\mathcal{L}$ on $\mathcal{X}|_{S_\alpha}$ such that $c_1(\mathcal{L}_t) = \alpha$ for all $t \in S_\alpha$. Moreover, if $\alpha = c_1(L)$, then $S_\alpha = Def(X, L)$.

Proof. By the very definition the sets S_α are isomorphic to the intersection of Q with the hyperplanes defined by $q_\Gamma(\varphi(\alpha),\,.\,)$ or $\int \alpha\varphi^{-1}(\,.\,)$, respectively. If $q_X(\alpha) \neq 0$ respectively $q_X((L_{\sigma\bar\sigma}^{n-1})^{-1}(\alpha)) \neq 0$, then this quadric is non-degenerate, hence smooth. If $q_X(\alpha) = 0$, then this quadric is degenerate, but its only singular point would be $\varphi(\alpha)$ which is not contained in the period domain, for $q_X(\alpha+\bar\alpha) = 4q_X(\alpha) = 0$. The same argument can be applied to the case $q_X((L_{\sigma\bar\sigma}^{n-1})^{-1}(\alpha)) = 0$.

This proves *i)*. The second statement follows from the standard fact that an (i,i)-class α keeps being of type (i,i) under an infinitesimal change $v \in H^1(X, \mathcal{T}_X)$ if and only if the contraction $v\alpha \in H^{i-1,i+1}(X)$ is trivial. The surjectivity in *ii)* and assertion *iii)* follow from 6.5. *iv)* is a consequence of 7.14. □

A consequence of the lemma is the following very useful result due to Fujiki [22]:

Corollary 7.17 — *If α is a Kähler class and $\mathcal{X}(\alpha) \to T(\alpha)$ is the associated twistor space, then the general fibre contains neither effective divisors nor curves.*

Proof. Indeed, the tangent space of the twistor space $\mathcal{X}(\alpha) \to T(\alpha)$ is spanned by the image v of α under the isomorphism $H^1(X, \Omega_X) \cong H^1(X, \mathcal{T}_X)$. On the other hand, if $D \subset X$ is an effective divisor and $C \subset X$ is a curve, then the tangent space of the hypersurface $S_{[D]}$ (respectively $S_{[C]}$) is the kernel of the map $H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X)$ (respectively $H^1(X, \mathcal{T}_X) \to H^{2n}(X, \Omega_X^{2n-2})$). Since up to scalar factors $\int_X([D]\cdot v)\sigma^n\bar\sigma^{n-2} = q_X(\alpha, [D]) > 0$ and $\int_X([C]\cdot v)\sigma = \int_C \alpha \neq 0$ (use 5.12 resp. 6.5 and the assumption that α is Kähler), the hypersurfaces $S_{[D]}$ and $S_{[C]}$ meet $T(\alpha)$ locally only in 0. Or equivalently, neither D nor C deforms in the twistor space. Since there are only countably many such classes $[D]$ and $[C]$, this proves the claim. □

8 K3 Surfaces (Survey)

The theory of K3 surfaces is a rewarding topic for everyone studying geometry nowadays. It combines techniques from differential, complex, and algebraic geometry. This section surveys those parts of this beautiful theory which motivated the present work. Thus, some important aspects will not be touched upon, e.g. the arithmetic of K3 surfaces or Nikulin's work on Picard lattices.

The theory of K3 surfaces rapidly developed in the late seventies and early eighties with main contributions by Shafarevich and its school, Burns and Rapoport, Siu, Todorov, and many others. I learned this subject from [1], but there are other highly recommendable expositions as well, e.g. [11, 40]. For precise statements, proofs, and references the reader may profitably consult any of these. We will put special emphasize on the logical order of arguments which is quite intriguing at times. It may be noteworthy that for certain generalizations to higher dimension this order has to be modified.

Definition 8.1 — *A K3 surface is a compact connected complex surface X with vanishing $c_1(X) \in H^2(X,\mathbb{Z})$ and $b_1(X) = 0$.*

Applying the exponential sequence, one finds that the canonical bundle of a K3 surface is trivial. Equivalently, any K3 surface admits a unique non-degenerate holomorphic two-form. As it turns out at a later stage, K3 surfaces are simply connected. This is an easy consequence of the Local Torelli Theorem. K3 surfaces are also Kähler, but this is a rather deep result due to Siu. Thus, K3 surfaces are the two-dimensional irreducible symplectic manifolds.

Clearly, if X is a K3 surface then $h^1(X,\mathcal{O}_X) = 0$ and $h^2(X,\mathcal{O}_X) = h^0(X,\mathcal{O}_X) = 1$. By Noether's formula $2 = \chi(\mathcal{O}_X) = c_2(X)/12$. Hence, $c_2(X) = 24$ and, therefore, $b_2(X) = 22$. The intersection pairing defines a non-degenerate quadratic form q_X on $H^2(X,\mathbb{Z})$. It is unimodular, as for any four-manifold, and even, for $c_1(X) = 0$. Its index can be computed by Hirzebruch's signature formula; it is $(19,3)$. This yields

Proposition 8.2 — *$(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to $2E_8 \oplus 3H$.*

Thus, a marking (cf. 7.6) of a K3 surface is an isomorphism of lattices $\varphi : (H^2(X,\mathbb{Z}), q_X) \cong 2E_8 \oplus 3H$. Two marked K3 surfaces (X,φ), (X',φ') are called equivalent if there exists an isomorphism $f : X \cong X'$ with $\varphi^{-1} \circ \varphi = \pm f^*$. Note that this definition slightly deviates from the standard definition by allowing the negative sign.

A K3 surface is projective if and only if there exists an integral class $\alpha \in H^{1,1}(X)$ with $q_X(\alpha) > 0$ (cf. 9.4). In order to identify ample classes on a projective K3 surface one has to pay special attention to so called (-2)-classes.

Definition 8.3 — *An irreducible curve $C \subset X$ is called a (-2)-curve, if $C^2 = -2$. A class $\alpha \in H^2(X,\mathbb{Z})$ with $q_X(\alpha) = -2$ is called a (-2)-class.*

By adjunction $C \subset X$ is a (-2)-curve if and only if C is a smooth irreducible rational curve. If $\alpha \in H^2(X,\mathbb{Z})$ is a (-2)-class and α is of type $(1,1)$, then by the Hirzebruch-Riemann-Roch formula α or $-\alpha$ is effective and thus $\alpha = \pm[C]$, where C is a (-2)-curve if irreducible. The Nakai-Moishezon criterion determines the ample cone explicitly:

Proposition 8.4 — *A class $\alpha \in H^2(X,\mathbb{Z})$ is ample if and only if α is contained in the positive cone $\mathcal{C}_X$ and $\alpha.C > 0$ for all (-2)-curves C.*

Recall that the positive cone $\mathcal{C}_X$ is the component of $\{\alpha \in H^{1,1}(X)_{\mathbb{R}} | q_X(\alpha) > 0\}$ that contains a Kähler class. Analogously to the description of the ample cone, one defines (!!) the Kähler cone $\mathcal{K}_X$ to be the part of $\mathcal{C}_X$ that is positive on all (-2)-curves. Certainly, any Kähler class is contained in $\mathcal{K}_X$, but to verify that $\mathcal{K}_X$ really is the Kähler cone in the sense of 5.14, i.e. the cone of all Kähler classes, one needs to prove the Global Torelli Theorem first.

Examples 8.5 — Any smooth complete intersection in $\mathbb{P}^N$ of type $(a_1, \ldots, a_{N-2})$ with $\sum a_i = N+1$ is a K3 surface. In particular, a smooth quartic in $\mathbb{P}^3$ is a K3 surface. It is a little more difficult to write down non-projective K3 surfaces. In fact, some of the Kummer surfaces defined below are non-projective.

Also note the following

Proposition 8.6 — *If X is a K3 surface such that* $\mathrm{Pic}(X) = L \cdot \mathbb{Z}$ *with* $q_X(c_1(L)) = 4$*, then X is isomorphic to a quartic in* $\mathbb{P}^3$.

The reflection in a hyperplane orthogonal to a (-2)-class defines an orthogonal transformation of $H^2(X, \mathbb{Z})$. The Weyl-group W_X is by definition the group generated by reflections in hyperplanes orthogonal to (-2)-classes of type $(1,1)$, or, equivalently, by all classes of (-2)-curves. W_X leaves invariant the space $H^{1,1}(X)_{\mathbb{R}}$ and the positive cone $\mathcal{C}_X$. The latter is endowed with a chamber structure, where the chambers are the connected components of the complement of all hyperplanes orthogonal to (-2)-classes. The two important facts are:

Proposition 8.7 — *W_X operates transitively on the set of chambers. Each chamber is an open subset of $\mathcal{C}_X$.*

In particular, for any class $\alpha \in \mathcal{C}_X$ not orthogonal to any (-2)-class there exists an element of the Weyl-group that maps α to an element in the Kähler cone $\mathcal{K}_X$.

If (X, φ) is a marked K3 surface, then the period $\mathcal{P}(X, \varphi)$ is the point in $\mathbb{P}((2E_8 \oplus 3H)_{\mathbb{C}})$ given by $\varphi(H^{2,0}(X))$. It is in the period domain Q, defined by the two conditions $q(x) = 0$ and $q(x + \bar{x}) > 0$ (cf. 7.1). The Local Torelli Theorem (due to Andreotti and Weil) states (cf. 5.6):

Theorem 8.8 — *The period map $\mathcal{P} : Def(X) \to Q$ is a local isomorphism.*

Since the Picard group of a K3 surface can be read off its period, one can use 8.6 to show

Proposition 8.9 — *Any K3 surface is simply connected.*

Indeed, one shows that the points $x \in Q$ such that 'the Picard group' $(x\mathbb{C} + \bar{x}\mathbb{C})^{\perp} \cap (2E_8 \oplus 3H)$ is spanned by an element with square 4, is dense in Q. Then, the Local Torelli proves that any K3 surface is a deformation of a quartic hypersurface in $\mathbb{P}^3$ which is certainly simply connected. In fact, the argument also shows that any two K3 surfaces are deformation equivalent and, hence, diffeomorphic.

There is another distinguished class of K3 surfaces the periods of which are dense in Q: Kummer surfaces. Let T be a complex torus of dimension two and let $\iota : T \to T$ be the involution $\iota \mapsto -\iota$. The sixteen fix-points of this action correspond to the sixteen singular points (all being ordinary double points) of the quotient T/ι. The blow-up of T/ι in these points is a K3 surface. Surfaces of this type are called Kummer surfaces. The exceptional curves are smooth rational curves, i.e. (-2)-curves. As for quartics the period remembers whether a given K3 surface is Kummer:

Proposition 8.10 — *If a K3 surface X contains sixteen disjoint (-2)-curves the sum of which is divisible by two in $H^2(X,\mathbb{Z})$, then X is a Kummer surface.*

The period not only recognizes whether a K3 surface is Kummer, but it is also able to distinguish among Kummer surfaces. This is the Global Torelli Theorem for Kummer surfaces due to Piateckii-Shapiro and Shafarevich. (Note that there is no direct argument showing that the period distinguishes different quartics.)

Theorem 8.11 — *Assume that X and X' are two Kummer surfaces. Let ψ be an isomorphism $(H^2(X,\mathbb{Z}),q_X) \cong (H^2(X',\mathbb{Z}),q_{X'})$ such that $\psi(H^{2,0}(X)) = H^{2,0}(X')$ and $\psi(\mathcal{K}_X) = \mathcal{K}_{X'}$. Then there exists a unique isomorphism $f: X \to X'$ with $f^* = \psi$.*

The proof of this theorem is based on delicate arguments showing that an isometry of the second cohomology groups of the Kummer surfaces defines an isometry of the second cohomology groups, and eventually also of the first one, of the underlying tori. The latter identifies the two tori and, eventually, also the Kummer surfaces.

The next step towards the Global Torelli Theorem is the 'Main Lemma' of Burns and Rapoport:

Theorem 8.12 — *Let $\pi : \mathcal{X} \to S$ and $\pi' : \mathcal{X}' \to S$ be two smooth families of K3 surfaces and $\varphi : R^2\pi_*\mathbb{Z} \cong R^2\pi'_*\mathbb{Z}$ an isomorphism. If there exists a sequence $t_i \in S$ approaching $0 \in S$ and isomorphisms $f_i : \mathcal{X}'_{t_i} \cong \mathcal{X}_{t_i}$ with $f_i^* = \varphi_{t_i}$, then $\mathcal{X}_0$ and $\mathcal{X}'_0$ are isomorphic. Moreover, if $\varphi_0(\mathcal{K}_{\mathcal{X}_0}) = \mathcal{K}_{\mathcal{X}'_0}$, then there exists an isomorphism $f : \mathcal{X}'_0 \cong \mathcal{X}_0$ with $f^* = \varphi_0$.*

In fact, Burns and Rapoport proved this result for Kählerian K3 surfaces and later Siu in general. Together with the fact that the periods of Kummer surfaces are dense in the period domain Q this result is enough to prove the Global Torelli Theorem for all K3 surfaces:

Theorem 8.13 — *Assume that X and X' are two arbitrary K3 surfaces. Let ψ be an isomorphism $(H^2(X,\mathbb{Z}),q_X) \cong (H^2(X',\mathbb{Z}),q_{X'})$ such that $\psi(H^{2,0}(X)) = H^{2,0}(X')$ and $\psi(\mathcal{K}_X) = \mathcal{K}_{X'}$. Then there exists a unique isomorphism $f: X \to X'$ with $f^* = \psi$.*

There is also an essentially different proof of the Global Torelli Theorem due to Friedman [19], who first shows by a global argument that the Global Torelli holds for quartics and then uses those instead of Kummer surfaces. It might also be noteworthy that any automorphism of a K3 surface inducing the identity on the second cohomology is the identity.

The Global Torelli Theorem can be interpreted as a statement about the fibres of the period map $\mathcal{P} : \mathfrak{M} \to Q$, where $\mathfrak{M}$ is the moduli space of equivalence classes of marked K3 surfaces:

If two K3 surfaces occur in the same fibre of $\mathcal{P}$, then they are isomorphic. Moreover, if $\mathcal{P}(X,\varphi) = \mathcal{P}(X,\varphi')$ then there exists a unique element g in the Weyl-group W_X such that ${\varphi'}^{-1} \circ \varphi = \pm g$.

The Global Torelli Theorem certainly prompts the question about the image of the period map. It is answered by a result due to Siu, Todorov, and Looijenga:

Theorem 8.14 — *Any point in the period domain Q is the period of a Kählerian K3 surface.*

The proof relies on the Local Torelli Theorem, the Nakai-Moishezon criterion, the transitivity of the Weyl-action, and some facts about the associated hyperkähler structure.

The surjectivity of the period map has two interesting consequences. Firstly, combined with the Global Torelli Theorem it proves that $\mathcal{K}_X$ really is the Kähler cone:

Theorem 8.15 — *A class $\alpha \in \mathcal{C}_X$ is Kähler if and only if it is positive on all (-2)-curves.*

Secondly, Siu used the surjectivity of the period map to show

Theorem 8.16 — *Any K3 surface is Kähler.*

It is important to note that the analogous result in higher dimension fails.

9 Projectivity

Let X be an irreducible symplectic manifold. The goal of this section is to show that the second cohomology $H^2(X,\mathbb{Z})$ endowed with the natural weight-two Hodge structure and the quadratic form q_X (cf. Section 5) determines whether X is a projective variety. The following list collects known projectivity criteria that either motivate the main result (Theorem 9.11) or are essential for its proof.

9.1 — Let X be a compact complex manifold. Then X is projective if and only if X is Kähler and Moishezon (cf. [45], [47]).

9.2 — Let X be a compact complex manifold. Then X is projective if and only if X admits a Kähler form ω such that its cohomology class $[\omega] \in H^2(X,\mathbb{C})$ is integral (cf. [36], [24]).

9.3 — Let X be a compact complex surface. Then X is projective if and only if X is Moishezon (cf. [35], [11]).

9.4 — Let X be a compact complex surface. Then X is projective if and only if there exists a line bundle L on X such that $\int c_1^2(L) > 0$ (cf. [11]).

It is easy to see that a surface X that admits a line bundle L with $\int c_1^2(L) > 0$ is Moishezon. Indeed, the Hirzebruch-Riemann-Roch formula shows that either $h^0(X,L^m) \sim m^2$ or $h^0(X,L^{-m} \otimes K_X) \sim m^2$. For manifolds of dimension > 2 the existence of a big line bundle

L, i.e. a line bundle with $\int c_1(L)^{\dim(X)} > 0$, is not sufficient to conclude that X is Moishezon, for the Hirzebruch-Riemann-Roch formula only gives $\sum h^{2i}(X, L^m) \sim m^{\dim(X)}$.

Let us begin with a result due to Fujiki [22] (see also [14]), which in particular shows that projective irreducible symplectic manifolds are dense in the moduli space. Some of the techniques used in later chapters are based on the proof of this result. Thus, we decided to reproduce it here.

Theorem 9.5 [22]— *Let X be an irreducible symplectic manifold, let $0 \in S \subset Def(X)$ be a smooth analytic subset of positive dimension and let $\mathcal{X} \to S$ denote the restriction of the Kuranishi family to S. Then any open neighbourhood $U \subset S$ of $0 \in S$ contains a point $t \neq 0$ such that the fibre $\mathcal{X}_t$ over t is projective.*

Proof. We may assume that S is one-dimensional. Then, its tangent space at zero T_0S is a line in $T_0 Def(X) = H^1(X, \mathcal{T}_X)$ spanned by, say, $0 \neq v \in H^1(X, \mathcal{T}_X)$. By 6.5 the induced map $\tilde{v} : H^1(X, \Omega_X) \to H^2(X, \mathcal{O}_X) \cong \mathbb{C}$ is surjective. Since the Kähler cone $\mathcal{K}_X$ is an open subset of $H^{1,1}(X)_{\mathbb{R}} = H^{1,1}(X) \cap H^2(X, \mathbb{R})$, which spans $H^1(X, \Omega_X)$ as a complex vector space, there exists a Kähler form ω on X such that $\tilde{v}([\omega]) \neq 0$. Consider the corresponding hypersurface $S_{[\omega]} \subset Def(X)$ of deformations of X where $[\omega]$ stays of type $(1,1)$ (cf. 7.15). The tangent space of $S_{[\omega]}$ at 0 is the kernel of the map $H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{O}_X) \cong \mathbb{C}$ induced by the product with $[\omega] \in H^1(X, \Omega_X)$. Since $\tilde{v}([\omega]) \neq 0$, the tangent spaces $T_0 S_{[\omega]}$ and $T_0 S$ have zero intersection. Hence, $S_{[\omega]}$ and S meet transversally in $0 \in Def(X)$. Shrinking S we can also assume that $S_{[\omega]} \cap S = \{0\}$.

Next, pick classes $\alpha_i \in H^2(X, \mathbb{Q})$ converging to $[\omega]$ and consider the associated hypersurfaces $S_{\alpha_i} \subset Def(X)$. Then the hypersurfaces S_{α_i} converge to $S_{[\omega]}$ and therefore $S_{\alpha_i} \cap S \neq \emptyset$ for $i \gg 0$. Moreover, if we choose α_i such that they are not of type $(1,1)$ on X, then $0 \notin S_{\alpha_i} \cap S$. Hence, there exist points $t_i \in (S_{\alpha_i} \cap S) \setminus \{0\}$ converging to 0. Using the isomorphism $H^2(\mathcal{X}_t, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$, the classes α_i are considered as rational classes of type $(1,1)$ on $\mathcal{X}_{t_i}$. Intuitively, the classes α_i on $\mathcal{X}_{t_i}$ converge to the Kähler class $[\omega]$ on $X = \mathcal{X}_0$ and thus should be Kähler for $i \gg 0$. This can be made rigorous as follows: Fix a diffeomorphism $\mathcal{X} \cong X \times S$ (compatible with the projections to S). By a result of Kodaira and Spencer (Thm. 3.1 in [37], Thm. 15 in [38]) there exists a real two-form $\tilde{\omega}$ on $X \times S$ such that the restriction $\tilde{\omega}_t$ of $\tilde{\omega}$ to $\mathcal{X}_t = X \times \{t\}$ is a Kähler form for all $t \in S$ and $\tilde{\omega}_0 = \omega$. One also finds two-forms $\tilde{\omega}_i$ on $X \times S$ such that $(\tilde{\omega}_i)_t$ is harmonic with respect to $\tilde{\omega}_t$ and $[(\tilde{\omega}_i)_t] = \alpha_i$ (Section 2 in [37]). In particular, $(\tilde{\omega}_i)_{t_i}$ is a harmonic $(1,1)$-form on $\mathcal{X}_{t_i}$ representing α_i. Since $(\tilde{\omega}_i)_{t_i}$ converges to ω, the harmonic $(1,1)$-form $(\tilde{\omega}_i)_{t_i}$ is a Kähler form on $\mathcal{X}_{t_i}$ for $i \gg 0$. Hence, α_i is a Hodge class on $\mathcal{X}_{t_i}$ for $i \gg 0$. In particular, $\mathcal{X}_{t_i}$ is projective for $i \gg 0$. $\square$

For the proof of the main result of this section we need to recall some facts about positive forms and currents. Let X be an irreducible symplectic manifold and let ω be a fixed Kähler form on X. By $\mathcal{A}^{p,p}(X)$ (resp. $\mathcal{A}^{p,p}(X)_{\mathbb{R}}$) we denote the space of smooth (real) (p,p)-forms on X. A form $\varphi \in \mathcal{A}^{p,p}(X)_{\mathbb{R}}$ is called *positive* if locally it can be written as a linear combination with non-negative coefficients of forms of type $i\alpha_1 \wedge \bar\alpha_1 \wedge \ldots \wedge i\alpha_p \wedge \bar\alpha_p$, where the α_i's are smooth $(1,0)$-forms. For $p = 2n-1$ we introduce the convex cone $C_{pos} \subset \mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}}$ that is formed by the positive forms.

Remark 9.6 — Of course, ω itself is a positive $(1,1)$-form. Conversely, if φ is a d-closed positive $(1,1)$-form, then $\varphi + \varepsilon\omega$ is a Kähler form for any $\varepsilon > 0$.

For the next lemma recall that σ^{n-1} defines an isomorphism of complex bundles $T_X^{*\,1,0} \cong T_X^{*\,2n-1,0}$ (use the argument in 6.3). Analogously, $\bar\sigma^{n-1}$ defines an isomorphism $T_X^{*\,0,1} \cong T_X^{*\,0,2n-1}$. Passing to form valued sections of these bundles one obtains isomorphisms ($p \geq 0$):

$$\sigma^{n-1} : \mathcal{A}^{1,p}(X) \cong \mathcal{A}^{2n-1,p}(X) \qquad \text{and} \qquad \bar\sigma^{n-1} : \mathcal{A}^{p,1}(X) \cong \mathcal{A}^{p,2n-1}(X)$$

and, since $\sigma\bar\sigma$ is real, also $(\sigma\bar\sigma)^{n-1} : \mathcal{A}^{1,1}(X)_{\mathbb{R}} \cong \mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}}$.

Lemma 9.7 — *Let $\psi \in \mathcal{A}^{1,1}(X)_{\mathbb{R}}$. If $(\sigma\bar\sigma)^{n-1}\psi \in \mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}}$ is positive, then also ψ is positive.*

Proof. It is enough to prove the assertion for those ψ for which $(\sigma\bar\sigma)^{n-1}\psi$ can locally be written as

$$\begin{aligned} (\sigma\bar\sigma)^{n-1}\psi &= i\alpha_1 \wedge \bar\alpha_1 \wedge \ldots \wedge i\alpha_{2n-1} \wedge \bar\alpha_{2n-1} \\ &= i^{2n-1}(-1)^{(2n-1)(n-1)}\alpha_1 \wedge \ldots \wedge \alpha_{2n-1} \wedge \bar\alpha_1 \wedge \ldots \wedge \bar\alpha_{2n-1}. \end{aligned}$$

First, observe that $i^{2n-1}(-1)^{(2n-1)(n-1)} = i$. Second, there exists a $(1,0)$-form β such that $\sigma^{n-1} \wedge \beta = \alpha_1 \wedge \ldots \wedge \alpha_{2n-1}$. Hence, $(\sigma\bar\sigma)^{n-1}\psi = i(\sigma\bar\sigma)^{n-1}(\beta \wedge \bar\beta)$. Therefore, $\psi = i\beta \wedge \bar\beta$, i.e. ψ is positive. $\square$

The choice of the Kähler form ω induces a Hodge decomposition

$$\mathcal{A}^{p,p}(X) = \mathcal{H}^{p,p}(X) \oplus (\mathrm{im}(d) \oplus \mathrm{im}(d^*)) \cap \mathcal{A}^{p,p}(X).$$

The space $\mathcal{H}^{p,p}(X)$ of harmonic (p,p)-forms is naturally isomorphic to $H^{p,p}(X)$. Since $\sigma\bar\sigma = \omega_J^2 + \omega_K^2$ is harmonic, where (I,J,K) is the hyperkähler structure associated with (ω, I) (cf. 3.4), the isomorphism in 6.3 can be understood as the isomorphism given by

$$\begin{array}{ccc} \mathcal{H}^{1,1}(X)_{\mathbb{R}} & \cong & \mathcal{H}^{2n-1,2n-1}(X)_{\mathbb{R}} \\ \alpha & \mapsto & (\sigma\bar\sigma)^{n-1}\alpha \end{array}$$

By definition a $(1,1)$-*current* T is a continuous linear map $\mathcal{A}^{2n-1,2n-1}(X) \to \mathbb{C}$. The topology on $\mathcal{A}^{2n-1,2n-1}(X)$ is the one induced by the Hodge metric. T is *positive* if $T(\varphi) \geq 0$ for all $\varphi \in C_{pos}$. The current T is *closed* if $T(d\gamma) = 0$ for all γ or, equivalently, if T factorizes over $\mathcal{A}^{2n-1,2n-1}(X)/(\mathrm{im}(d) \cap \mathcal{A}^{2n-1,2n-1}(X))$. Any closed $(1,1)$-current T gives rise to a cohomology class $[T] \in H^{1,1}(X)$ which is given by restricting T to $\mathcal{H}^{2n-1,2n-1}(X)$ and identifying the dual space of $\mathcal{H}^{2n-1,2n-1}(X)$ with $\mathcal{H}^{1,1}(X)$. A current T is *real* if it is the natural extension of a continuous linear map $\mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}} \to \mathbb{R}$.

Proposition 9.8 — *Let X be an irreducible symplectic manifold and let $\alpha \in H^{1,1}(X)_{\mathbb{R}}$ such that $q_X(\alpha, .)$ is positive on the Kähler cone $\mathcal{K}_X$. Then there exists a closed positive real $(1,1)$-current T such that $[T] = \alpha$.*

Proof. The proof is inspired by an argument of Peternell in [47].

Let $A := \mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}}/(\mathrm{im}(d) \cap \mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}})$, let C be the image of C_{pos} under the projection $\mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}} \to A$, and let $B := \mathcal{H}^{2n-1,2n-1}(X)_{\mathbb{R}}$. Using Hodge decomposition one has $A = \mathcal{H}^{2n-1,2n-1}(X)_{\mathbb{R}} \oplus (\mathrm{im}(d^*) \cap \mathcal{A}^{2n-1,2n-1}(X)_{\mathbb{R}})$. Thus B can be considered as a (finite dimensional) subspace of A. The class α defines a continuous linear map $f_0 : B \to \mathbb{R}$. A closed positive real $(1,1)$-current T with $[T] = \alpha$ is a continuous linear extension $f : A \to \mathbb{R}$ of f_0 with $f|_C \geq 0$.

The existence of such an extension is due to the following general fact: If A is a topological vector space, $B \subset A$ a subspace, and $C \subset A$ a convex cone such that $C \cap (-C) = 0$ and $B \cap \overset{o}{C} \neq \emptyset$, then any continuous linear function $f_0 : B \to \mathbb{R}$ with $f_0|_{B\cap C} \geq 0$ can be extended to a continuous linear function $f : A \to \mathbb{R}$ with $f|_C \geq 0$. ([12], Ch.II, Sect. 3).

Let us now verify the assumptions in our situation: Clearly, C is a convex cone as it is the image of C_{pos}. If $\varphi, -\varphi' \in C_{pos}$ with $\varphi = \varphi' + d\gamma$ then the positivity of ω implies $0 \leq \int \varphi\omega = \int \varphi'\omega \leq 0$ and hence $\int \varphi\omega = \int \varphi'\omega = 0$. Since ω is Kähler, we get $\varphi = \varphi' = 0$. Therefore, $C \cap (-C) = 0$. The intersection $B \cap \overset{o}{C}$ is not empty as it contains the class ω^{2n-1} which is an inner point of C_{pos} (cf. [29]). Last but not least, one checks $f_0|_{B\cap C} \geq 0$. Indeed, if $\varphi \in \mathcal{H}^{2n-1,2n-1}(X)_{\mathbb{R}}$ is positive, then there exists $\psi \in \mathcal{H}^{1,1}(X)_{\mathbb{R}}$ with $(\sigma\bar{\sigma})^{n-1}\psi = \varphi$ and ψ is positive by Lemma 9.7. Hence, $\psi + \varepsilon\omega$ is Kähler for all $\varepsilon > 0$. The assumption on α then implies $q_X(\alpha, [\psi + \varepsilon\omega]) > 0$ for all $\varepsilon > 0$ and hence $q_X(\alpha, [\psi]) \geq 0$. Therefore, $\int \varphi\alpha = \int (\sigma\bar{\sigma})^{n-1}\psi\alpha = c q_X([\psi], \alpha) \geq 0$ for some positive constant c. Hence, $f_0(\varphi) \geq 0$. □

Remark 9.9 — *i)* Demailly [17] introduced the notion of pseudo-effective classes: A class $\alpha \in H^{1,1}(X)_{\mathbb{R}}$ is called pseudo-effective if it can be represented by a closed positive $(1,1)$-current. They span a closed cone $H^{1,1}_{psef}(X) \subset H^{1,1}(X)_{\mathbb{R}}$. The proposition says that the cone dual to the Kähler cone is contained in $H^{1,1}_{psef}(X)$.

ii) By Section 5 any class $\alpha \in \mathcal{C}_X$ is positive on $\mathcal{K}_X$. Hence $\mathcal{C}_X \subset H^{1,1}_{psef}(X)$. In fact, as $\mathcal{C}_X$ is an open cone, any class $\alpha \in \mathcal{C}_X$ can be represented by a closed positive real $(1,1)$-current T such that $T - \varepsilon\omega$ is still positive for some $\varepsilon > 0$.

As an immediate consequence of [17] we obtain

Corollary 9.10 — *If X is a projective irreducible symplectic manifold and L is a line bundle such that $c_1(L) \in \mathcal{C}_X$, then L has maximal Kodaira-dimension, i.e. $h^0(L^k)$ grows like $c \cdot k^{2n}$ for some $c > 0$. In particular, $\mathcal{C}_X \cap H^2(X, \mathbb{Z})$ is contained in the effective cone.* □

The main result of this section is the following

Theorem 9.11 — *Let X be an irreducible symplectic manifold. Then X is projective if and only if there exists a line bundle L on X with $q_X(c_1(L)) > 0$.*

Proof. If X is projective then $c_1(L)$ of an ample line bundle L is a Kähler class and, therefore, $q_X(c_1(L)) > 0$ (cf. Section 5).

Conversely, if X admits a line bundle L with $c_1(L) \in \mathcal{C}_X$, then there exists a closed $(1,1)$-current T representing $c_1(L)$ such that $T - \varepsilon\omega$ is positive for some $\varepsilon > 0$. By a result of Ji-Shiffman [33] and Bonavero [10] this implies that X is Moishezon and, since it is also Kähler, that it is projective. □

Remark 9.12 — *i)* The theorem only asserts that X is projective, but not that any line bundle L with positive square $q_X(c_1(L))$ is ample (this is not even true for surfaces). A criterion for the ampleness of a line bundle will be discussed in Section 13. However, if $\mathrm{Pic}(X) \cong L \cdot \mathbb{Z}$ with $q_X(c_1(L)) > 0$, then either L or $L^\vee$ is ample.

ii) In the original approach to prove the theorem I tried to avoid the result of Bonavero and Ji-Shiffman, which in turn relies on Demailly's very complicated holomorphic Morse inequalities. The argument should use the relatively easy Corollary 9.10 on projective deformations and some kind of semi-continuity argument. But there are still some technical problems. Also note that if one is willing to use the holomorphic Morse inequalities (or rather their singular version) then one can in fact see that Corollary 9.10 works without the projectivity assumption.

iii) The techniques above can also be used to prove a result for Kähler surfaces I was not aware of before and which I could not find in the literature. However, I believe that going through the classification of surfaces an easier proof, not using singular Morse inequalities, should exist. The result is: A compact Kähler surface is projective if and only if the dual of the Kähler cone, i.e. the elements with positive intersection with any Kähler class, contains an inner integral point.

10 Birational Manifolds

This section deals with the relation between birational irreducible symplectic manifolds and non-separated points in the moduli space $\mathfrak{M}_\Gamma$ (cf. 7.6). Already in dimension two, i.e. for K3 surfaces, the moduli space of marked irreducible symplectic manifolds is not separated (i.e. non-Hausdorff). But there, two non-separated points always correspond to just one K3 surface with two different markings. The situation is more subtle in higher dimensions: One easily generalizes the 'Main Lemma' of Burns and Rapoport [13], the algebraic version of which is [42], to the effect that the underlying manifolds X and X' of two non-separated points $(X,\varphi),(X',\varphi') \in \mathfrak{M}_\Gamma$ are birational (Theorem 10.7). But contrary to the two-dimensional case, this does not imply that X and X' are isomorphic. In fact, it is a standard procedure to construct via certain birational transformations out of one irreducible symplectic manifold new ones (cf. 4.5).

It is the goal of this section to prove that in general two birational irreducible symplectic manifolds correspond to non-separated points in $\mathfrak{M}_\Gamma$ (Theorem 10.12). As the method is completely algebraic the main result is limited to the case of projective manifolds. However, if the birational correspondence is given by an elementary transformation the result holds true without the projectivity assumption (10.8).

The result has strong consequences for the classification of examples. This will be explained in the next section. The relation to a conjectural 'Global Torelli Theorem' for higher dimensional irreducible symplectic will be discussed in Section 16.

Here is the plan for this section: We first prove an easy lemma saying that birational irreducible symplectic manifolds have isomorphic periods. Next we discuss the most explicit type of birational correspondences, called elementary transformations, which were introduced by Mukai. Interesting examples of elementary transformations were studied by Beauville and Debarre. In particular, Debarre found counterexamples to the Global Torelli Theorem (10.4). Then we turn to the generalization of the Main Lemma of Burns and Rapoport alluded to above (cf. 10.7). For elementary transformations there exists a rather simple proof of the main result 10.12 which works without any further assumption. This is Proposition 10.8. Last, we proof the non-separatedness of any two birational projective irreducible symplectic manifolds and discuss the relation of some immediate consequences to a result of Batyrev and Kontsevich.

Lemma 10.1 — *Let X and X' be two birational irreducible symplectic manifolds. Then there exist open subsets $U\subset X$ and $U'\subset X'$ such that $U\cong U'$ and* $\mathrm{codim}(X\setminus U), \mathrm{codim}(X'\setminus U')\geq 2$. *Moreover, there exists a natural isomorphism $H^2(X,\mathbb{Z})\cong H^2(X',\mathbb{Z})$ compatible with the Hodge structures and the quadratic forms q_X and $q_{X'}$ [18, 44, 46].*

Proof. The first assertion holds in fact for all manifolds with nef canonical bundle. In our situation it can be proved as follows. Let $U\subset X$ be the largest open subset where the birational map $f: X\to X'$ is regular. Then $\mathrm{codim}(X\setminus U)\geq 2$. Let σ and σ' be non-trivial two-forms on X and X', respectively. Since $\mathbb{C}=H^0(X,\Omega_X^2)=H^0(U,\Omega_U^2)$, the pull-back $f^*\sigma'$ is a non-trivial multiple of σ. Since σ is everywhere non-degenerate, f must be quasi-finite on U. Since it is generically one-to-one, it is an embedding. Thus, if $U\subset X$ and $U'\subset X'$ denote the maximal open subsets where f and f^{-1}, respectively, are regular, then $U\cong U'$ and $\mathrm{codim}(X\setminus U)$, $\mathrm{codim}(X'\setminus U')\geq 2$.

We provide three slightly different descriptions for the claimed isomorphism.

If U and U' are as above, then one defines the isomorphism as the composition $H^2(X,\mathbb{Z})\cong H^2(U,\mathbb{Z})\cong H^2(U',\mathbb{Z})\cong H^2(X',\mathbb{Z})$. Since these isomorphisms commute with $H^0(X,\Omega_X^2)\cong H^0(U,\Omega_U^2)\cong H^0(U',\Omega_{U'}^2)\cong H^0(X',\Omega_{X'}^2)$, the isomorphism $H^2(X,\mathbb{Z})\cong H^2(X',\mathbb{Z})$ is compatible with the Hodge structures.

The second description is in terms of the closure of the graph $Z\subset X\times X'$ of the birational map $X --\to X'$. Then $[Z]_*: H^2(X,\mathbb{Z})\to H^2(X',\mathbb{Z})$, defined by $\alpha\mapsto p'_*([Z].p^*\alpha)$, equals the above isomorphism. We also write $[Z]_*$ for the map $\beta \mapsto p_*([Z].p'^*\beta) \in H^2(X,\mathbb{Z})$. Using the first description, one finds $[Z]_* \circ [Z]_* = \mathrm{id}$. At one point in the discussion later on we will need $q'_X([Z]_*\alpha,\beta) = q_X(\alpha,[Z]_*\beta)$, which follows easily from the compatibility with the quadratic form, proved below, and $[Z]_* \circ [Z]_* = \mathrm{id}$.

Yet another description goes as follows: Let $\tilde{Z} \to Z$ be a desingularization. Then there are isomorphisms $H^2(\tilde{Z},\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \oplus \bigoplus_i [E_i]\cdot\mathbb{Z}$ and $H^2(\tilde{Z},\mathbb{Z}) \cong H^2(X',\mathbb{Z}) \oplus \bigoplus_i [E_i]\cdot\mathbb{Z}$, where the E_i's are the exceptional divisors of $\tilde{Z} \to X$ (or, equivalently, of $\tilde{Z} \to X'$). The isomorphism $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ is then given as the composition of the natural inclusion followed by the projection.

In either of the three descriptions one sees that the isomorphism maps the class of an effective divisor in X to the class of an effective divisor in X'.

To see the compatibility with the quadratic forms q_X and $q_{X'}$, let σ and σ' be two-forms on X and X', respectively, with $\int_X(\sigma\bar{\sigma})^n = 1 = \int_{X'}(\sigma'\bar{\sigma}')^n$. Their pull-backs to $\tilde{Z}$ coincide. Using the description of q_X and $q_{X'}$ given in Section 5, one finds that both can be defined on $\tilde{Z}$ and that via $[Z]_*$ they coincide if $\sigma^{n-1}|_{E_i} = 0$. The latter equality now follows from the observation that the fibers (which are of positive dimension) of the morphisms $E_i \to X$ and $E_i \to X'$ are different and, hence, the rank of σ (or σ') on E_i drops at least by two. $\square$

An explicit birational correspondence between two symplectic manifolds was introduced by Mukai [43]. We briefly recall his construction.

Let X be a complex manifold of dimension $2n$ with an everywhere non-degenerate two-form $\sigma \in H^0(X,\Omega_X^2)$. Furthermore, let $P \subset X$ be a closed submanifold which itself is a projective bundle $P = \mathbb{P}(F) \xrightarrow{\phi} Y$. Here, F is a vector bundle of rank $N+1$ on the manifold Y. Using the symplectic structure one defines the elementary transformation X' of X along P as follows: Since a projective space $\mathbb{P}^N$ does not admit any regular two-form, the restriction of σ to any fibre of ϕ is trivial. More is true, the relative tangent bundle $\mathcal{T}_\phi$ of ϕ is orthogonal to $\mathcal{T}_P$ with respect to the restriction of σ, i.e. $\sigma|_P : \mathcal{T}_\phi \times \mathcal{T}_P \to \mathcal{O}_P$ vanishes. Indeed, this follows from the isomorphism $H^0(Y,\Omega_Y^2) \cong H^0(P,\Omega_P^2)$, i.e. $\sigma|_P$ is the pull-back of a two-form on Y. Thus the composition of $\mathcal{T}_P \subset \mathcal{T}_X|_P$ with the isomorphism $\mathcal{T}_X|_P \cong \Omega_X|_P$ and the projection $\Omega_X|_P \twoheadrightarrow \Omega_P \twoheadrightarrow \Omega_\phi$ vanishes. Thus, σ induces a vector bundle homomorphism $\mathcal{N}_{P/X} \cong \mathcal{T}_X|_P/\mathcal{T}_P \to \Omega_\phi$.

Now let $\mathrm{codim}(P) = N$. Then both vector bundles $\mathcal{N}_{P/X}$ and Ω_ϕ are of rank N and, since σ is non-degenerate, $\mathcal{N}_{P/X} \to \Omega_\phi$ is an isomorphism.
Let $\tilde{X} \to X$ denote the blow-up of X along $P \subset X$ and let $D \subset \tilde{X}$ be the exceptional divisor. The projection $D \to P$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{N}_{P/X}) \cong \mathbb{P}(\Omega_\phi) \to P$.
The natural isomorphism of the incidence variety $\{(x,H)|x \in H\} \subset \mathbb{P}^N \times \mathbb{P}^{N\vee}$ as a projective bundle over $\mathbb{P}^N$ with the projective bundle $\mathbb{P}(\Omega_{\mathbb{P}^N}) \to \mathbb{P}^N$ can be generalized to the relative situation, i.e. there is a canonical embedding $D = \mathbb{P}(\Omega_\phi) \subset \mathbb{P}(F) \times_Y \mathbb{P}(F^\vee)$ compatible with the projection to $\mathbb{P}(F)$. The other projection $D \to \mathbb{P}(F^\vee)$ is a projective bundle as well.

The adjunction formula shows that $\mathcal{O}_{\tilde{X}}(D)$ restricts to $\mathcal{O}(-1)$ on every fibre of $D \to \mathbb{P}(F^\vee)$.

By a result of Nakano and Fujiki [20] there exists a blow-down $\tilde{X} \to X'$ to a smooth manifold X' such that $D \subset \tilde{X}$ is the exceptional divisor and $D \to X'$ is the projection $D \to \mathbb{P}(F^\vee) \subset X'$.

Definition 10.2 — *$X' = elm_P(X)$ is called the elementary transformation of the symplectic manifold X along the projective bundle P as well.*

The symplectic structure on X restricted to the complement of P and then extended again to X' defines a holomorphic two-form on $elm_P(X)$. It is certainly everywhere non-degenerate by a codimension argument. Hence, $elm_P(X)$ is again symplectic. Moreover, if σ is unique, i.e. $h^0(X, \Omega_X^2) = 1$, then the same holds true for X'. Furthermore, if X is simply connected then also X' is simply connected. This proves [44]:

Proposition 10.3 — *Let X be an irreducible symplectic manifold and let $X' := elm_P(X)$ be the elementary transformation of X along an embedded $\mathbb{P}^N$-bundle of codimension N. If X' is Kähler then it is an irreducible symplectic manifold.*

Is the elementary transformation of a Kähler manifold always Kähler?

Example 10.4 — First, let us recall an example of Beauville [3]: If $S \subset \mathbb{P}^3$ is a smooth quartic hypersurface, then the generic line $\ell \subset \mathbb{P}^3$ meets S in four distinct points. Thus one defines a rational map $\mathrm{Hilb}^2(S) - - \to \mathrm{Hilb}^2(S)$ by sending $[Z] \in \mathrm{Hilb}^2(S)$ to $(\ell_Z \cap S) \setminus \{Z\}$, where ℓ_Z is the uniquely defined line through Z. If S does not contain any line then this map extends to an automorphism of $\mathrm{Hilb}^2(S)$. On the other hand, if S contains k lines $\ell_1, \ldots, \ell_k$, then $\mathrm{Hilb}^2(S) - - \to \mathrm{Hilb}^2(S)$ is the elementary transformation along the naturally embedded planes $\mathbb{P}^2 \cong \mathrm{Hilb}^2(\ell_i) \subset \mathrm{Hilb}^2(S)$. But, not any elementary transformation of an irreducible symplectic manifold X is again isomorphic to X! One should rather think of Beauville's example as an exception. However, it is usually not easy to show that $X' = elm_P(X)$ is not isomorphic to X. One explicit example was given by Debarre in [16]:

Let S be a K3 surface such that $\mathrm{Pic}(S) = \mathcal{O}(C) \cdot \mathbb{Z}$, where C is a (-2)-curve. The Hilbert scheme $X := \mathrm{Hilb}^n(S)$ contains the projective space $\mathrm{Hilb}^n(C) \cong S^n(C) \cong \mathbb{P}^n$ as above. Let X' be the elementary transformation of X along this projective space. If X and X' were isomorphic, then this would yield a birational automorphism of X. Debarre then shows that, due to the special shape of the surface S, the induced map on the second cohomology would be trivial and that this implies that the birational automorphism can be extended to an isomorphism, which is absurd. Due to 10.1 the two manifolds X and X' have isomorphic periods, but in order to obtain an honest counterexample to the Global Torelli Theorem one has in addition to ensure that X' is Kähler. This is not clear in general, but if S is close to an algebraic K3 surface S_0 as in the example above, then X' is close to $elm_{\mathbb{P}^n}(\mathrm{Hilb}(S_0)) \cong \mathrm{Hilb}^n(S_0)$. But any small deformation of a Kähler manifold is Kähler; hence there are examples for which X' is Kähler. Note that in Debarre's example the manifolds are not projective. It would be interesting to find also a projective counterexample to the Global Torelli Theorem. Good candidates are the moduli space of stable sheaves on a K3 surface and the Hilbert scheme of the surface, which in some cases are birational, but most likely not isomorphic (cf. 4.8).

Let us now come to the phenomenon that the moduli space of marked irreducible symplectic manifolds is non-separated. Before going into the subject we say a few words about what 'non-separated' means in practice. To this end we formulate the following (almost tautological):

Lemma 10.5 — *Let X and X' be irreducible symplectic manifolds, let $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ be deformations of X and X', respectively, and let $V \subset S$ be an open (in the classical topology) non-empty subset such that i) S is one-dimensional and ii) $0 \in \partial V$, and iii) $\mathcal{X}|_V \cong \mathcal{X}'|_V$ compatible with the projections to S. Then there exist markings φ and φ' of X and X', respectively, such that $(X,\varphi),(X',\varphi') \in \mathfrak{M}_\Gamma$ are non-separated. Conversely, if $(X,\varphi),(X'\varphi') \in \mathfrak{M}_\Gamma$ are two non-separated points in $\mathfrak{M}_\Gamma$, then there exist deformations $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ and $V \subset S$ satisfying i), ii), and iii) such that for $t \in V$ the natural isomorphisms $H^2(X,\mathbb{Z}) \cong H^2(\mathcal{X}_t,\mathbb{Z}) \cong H^2(\mathcal{X}'_t,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ are compatible with φ and φ'.* □

Remark 10.6 — The arguments in the proof of Proposition 12.1 show that in part two of the lemma one can in fact arrange things such that $V = S \setminus \{0\}$. See Remark 12.3.

Let us now come to a straightforward generalization of the 'Main Lemma' in [13]. The proof follows closely Beauville's exposé in [1].

Theorem 10.7 — *If $(X,\varphi),(X',\varphi') \in \mathfrak{M}_\Gamma$ are non-separated points in the moduli space of marked irreducible symplectic manifolds, then X and X' are birational.*

Proof. Consider deformations $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ of $\mathcal{X}_0 = X$ and $\mathcal{X}'_0 = X'$, respectively, and $V \subset S$ as in 10.5. In particular, we have for $t \in V$ the canonical isomorphism $H^2(X,\mathbb{Z}) \cong H^2(\mathcal{X}_t,\mathbb{Z}) \cong H^2(\mathcal{X}'_t,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$ which is compatible with φ and φ'. Pick a sequence $t_i \in V$ converging to $0 \in S$ and consider the corresponding isomorphisms $f_i : \mathcal{X}_{t_i} \cong \mathcal{X}'_{t_i}$ and their graphs $\Gamma_i \subset \mathcal{X}_{t_i} \times \mathcal{X}'_{t_i}$. By [37] there exist Kähler forms ω_t and ω'_t on the fibres $\mathcal{X}_t$, respectively $\mathcal{X}'_t$, for all $t \in S$ depending continuously (this is enough) on t. The volume of Γ_{t_i} with respect to these Kähler forms, can be computed by

$$\begin{aligned} vol(\Gamma_i) &= \int_{\mathcal{X}_{t_i}} (\omega_{t_i} + f_i^*\omega'_{t_i})^{2n} \\ &= \int_{\mathcal{X}_{t_i}} ([\omega_{t_i}] + f_i^*[\omega'_{t_i}])^{2n} \\ &= \int_X ([\omega_{t_i}] + (\varphi'^{-1} \circ \varphi)[\omega'_{t_i}])^{2n}. \end{aligned}$$

Hence, $vol(\Gamma_i)$ converges to $\int_X([\omega_0] + (\varphi'^{-1} \circ \varphi)[\omega'_0])^{2n} < \infty$. By a result of Bishop [6] the boundedness of the volume of Γ_i is enough to conclude the existence of a limit cycle $\Gamma_\infty \subset X \times$

X' with the same cohomological properties as the Γ_i's. In particular, $[\Gamma_\infty] \in H^{4n}(X \times X', \mathbb{Z})$ satisfies $p_*[\Gamma_\infty] = [X] \in H^0(X, \mathbb{Z})$; $p'_*[\Gamma_\infty] = [X'] \in H^0(X', \mathbb{Z})$; and $p'_*([\Gamma_\infty].p^*\alpha) = ({\varphi'}^{-1} \circ \varphi)(\alpha)$ for all $\alpha \in H^2(X, \mathbb{Z})$. Here, p and p' denote the two projections from $X \times X'$. Splitting Γ_∞ into its irreducible components and using the first two properties yields either

- $\Gamma_\infty = Z + \sum Y_i$, where $p : Z \to X$ and $p' : Z \to X'$ are generically one-to-one, or
- $\Gamma_\infty = Z + Z' + \sum Y_i$, where $p : Z \to X$ and $p' : Z' \to X'$ are generically one-to-one, but neither $p' : Z \to X'$ nor $p : Z' \to X$ is generically finite.

In both cases, $p_*[Y_i] = 0$ and $p'_*[Y_i] = 0$.

The second possibility can be excluded: If σ and σ' are non-trivial holomorphic two-forms on X and X' respectively, then $({\varphi'}^{-1} \circ \varphi)([\sigma]) = \lambda[\sigma']$ for some $\lambda \neq 0$. Indeed, since the period domain, i.e. the quadric $Q \subset \mathbb{P}(\Gamma_{\mathbb{C}})$, is separated, ${\varphi'}^{-1} \circ \varphi$ is compatible with the natural Hodge structures on $H^2(X, \mathbb{Z})$ and $H^2(X', \mathbb{Z})$. If $\Gamma_\infty = Z + Z' + \sum Y_i$, then

$$
\begin{aligned}
0 &\neq \lambda \int_{X'} (\sigma'\bar{\sigma}')^n = \int_{X'} ({\varphi'}^{-1} \circ \varphi)[\sigma]({\sigma'}^{n-1}\bar{\sigma}'^n) \\
&= \int_{X'} p'_*([\Gamma_\infty].p^*[\sigma])({\sigma'}^{n-1}\bar{\sigma}'^n) = \int_{X \times X'} [\Gamma_\infty].p^*[\sigma].p'^*[{\sigma'}^{n-1}\bar{\sigma}'^n] \\
&= \int_Z p^*[\sigma].p'^*[{\sigma'}^{n-1}\bar{\sigma}'^n] + \int_{Z'} p^*[\sigma].p'^*[{\sigma'}^{n-1}\bar{\sigma}'^n] + \sum \int_{Y_i} p^*[\sigma].p'^*[{\sigma'}^{n-1}\bar{\sigma}'^n].
\end{aligned}
$$

The first and third term vanish, because $p'(Z)$ and $p'(Y_i)$ are of dimension $< 2n$, but $\bar{\sigma}'^n$ is a $(0, 2n)$-form on X'. For simplicity assume Z' smooth (otherwise pass to a desingularization). Since $Z' \to X'$ is birational, $H^0(Z', \Omega^2_{Z'}) = p'^*(\sigma')|_{Z'} \cdot \mathbb{C}$. On the other hand, $p^*(\sigma)|_{Z'} \in H^0(Z', \Omega^2_{Z'})$. Since $p^*(\sigma)|_{Z'}$ is everywhere degenerate, but $p'^*(\sigma')|_{Z'}$ is at least generically non-degenerate, we must have $p^*(\sigma)|_{Z'} = 0$. Thus, also the second term vanishes. Contradiction.

Therefore, only the decomposition $\Gamma_\infty = Z + \sum Y_i$ can occur. Since $Z \to X$ and $Z \to X'$ are generically one-to-one, X and X' are birational. $\square$

Note that the birational correspondence between X and X' constructed this way does not, in general, induce ${\varphi'}^{-1} \circ \varphi$ on H^2. The theorem (or rather its proof) has various interesting consequences and, with the proof still in mind, the reader may wish to have a look at Corollary 12.2 already at this point in the discussion.

Let us now come to the converse: Are two birational irreducible symplectic manifolds non-separated in their moduli space? Recall that for an appropriate choice of the markings the periods of two birational irreducible symplectic manifolds are equal (cf. 10.1). Therefore, their period points in $\mathbb{P}(\Gamma_{\mathbb{C}})$ coincide.

As a warm-up we treat the case that the birational correspondence between X and X' is given by an elementary transformation, i.e. $X' = elm_P(X)$ for some $\mathbb{P}^N$-bundle $P \subset X$. The following theorem was proved in [31].

Theorem 10.8 — *Let X be an irreducible symplectic manifold and let $P \subset X$ be a smooth $\mathbb{P}^N$-bundle of codimension N. Then there exist two smooth proper families $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ over a smooth and one-dimensional base S, such that $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic as families over $S \setminus \{0\}$ and the fibres over $0 \in S$ satisfy $\mathcal{X}_0 \cong X$ and $\mathcal{X}'_0 \cong X' := elm_P(X)$.*

Proof. Let P be the $\mathbb{P}^N$-bundle $\phi : P = \mathbb{P}(F) \to Y$, where Y is a compact manifold and F is a vector bundle of rank $N+1$. Consider a deformation $\mathcal{X} \to S$ of X and assume that S is smooth and one-dimensional. Let $v \in H^1(X, \mathcal{T}_X)$ be its Kodaira-Spencer class, i.e. $v \cdot \mathbb{C}$ is the image of the Kodaira-Spencer map $T_0 S \to H^1(X, \mathcal{T}_X)$. Furthermore, denote by $\bar{v} \in H^1(X, \Omega_X)$ the image of v under the isomorphism $H^1(X, \mathcal{T}_X) \cong H^1(X, \Omega_X)$ induced by the symplectic structure.

Claim — Assume that $\bar{v} \in H^1(X, \Omega_X)$ is a Kähler class. Then the normal bundle $\mathcal{N}_{P/\mathcal{X}}$ is isomorphic to $\phi^* F^\vee \otimes \mathcal{O}_\phi(-1)$.

Proof of the Claim. We certainly can assume that the base manifold Y is connected and hence $H^1(P, \mathcal{N}_{P/X}) \cong H^1(P, \Omega_\phi) \cong H^0(Y, \mathcal{O}_Y) \cong \mathbb{C}$ (use the Leray Spectral Sequence for $\phi : P \to Y$). By construction, the isomorphism $\mathcal{N}_{P/X} \cong \Omega_\phi$ commutes with the projections $\mathcal{T}_X \to \mathcal{N}_{P/X}$ and $\Omega_X \to \Omega_\phi$, and the symplectic structure $\mathcal{T}_X \cong \Omega_X$. In particular, the image ξ of v under $H^1(X, \mathcal{T}_X) \to H^1(P, \mathcal{N}_{P/X})$ is non-zero if and only if $\bar{v}$ maps to a non-zero class under $H^1(X, \Omega_X) \to H^1(P, \Omega_\phi)$. Since $\bar{v}$ is Kähler and thus its restriction to the fibres of ϕ non-trivial, one concludes that the extension corresponding to ξ is isomorphic to the unique (up to scalars) non-trivial extension of $\mathcal{O}_P$ by $\mathcal{N}_{P/X} \cong \Omega_\phi$, i.e. it is isomorphic to the relative Euler sequence

$$0 \longrightarrow \Omega_\phi \longrightarrow \phi^* F^\vee \otimes \mathcal{O}_\phi(-1) \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

In order to prove the claim it suffices to show that ξ is also the extension class of the canonical sequence

$$0 \longrightarrow \mathcal{N}_{P/X} \longrightarrow \mathcal{N}_{P/\mathcal{X}} \longrightarrow \mathcal{N}_{X/\mathcal{X}}|_P \longrightarrow 0,$$

where we use $\mathcal{N}_{X/\mathcal{X}} \cong \mathcal{O}_X$. This follows easily from the definition of the Kodaira-Spencer class v as the extension class of

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathcal{X}}|_X \longrightarrow \mathcal{N}_{X/\mathcal{X}} \longrightarrow 0.$$

Now pick a one-dimensional deformation $\mathcal{X} \to S$ of X such that $\bar{v}$ is Kähler. Denote the blow-up of $\mathcal{X}$ along P by $\tilde{\mathcal{X}} \to \mathcal{X}$. By the Claim above the exceptional divisor $\mathcal{D} \to P$ is isomorphic to the projective bundle $\mathbb{P}(\phi^* F^\vee) \to P$. Obviously, $\mathbb{P}(\phi^* F^\vee) \cong \mathbb{P}(F) \times_Y \mathbb{P}(F^\vee)$. Consider the second projection $\mathcal{D} \cong \mathbb{P}(\phi^* F^\vee) \to \mathbb{P}(F^\vee)$. As in the description of elementary transformations one checks that $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ restricts to $\mathcal{O}(-1)$ on every fibre of this projection, i.e. the condition of the Nakano-Fujiki criterion is satisfied. Thus, $\tilde{\mathcal{X}}$ can be blown-down to a smooth manifold $\mathcal{X}'$ such that the exceptional divisor $\mathcal{D}$ is contracted to $\mathbb{P}(F^\vee)$. By the very construction $\mathcal{X}' \leftarrow \tilde{\mathcal{X}} \to \mathcal{X}$ is compatible with $X' \leftarrow \tilde{X} \to X$, i.e. $\mathcal{X}' \to S$ is a smooth proper family, isomorphic to $\mathcal{X}$ over $S \setminus \{0\}$, and its special fibre $\mathcal{X}'_0$ is isomorphic to X'. □

Remark 10.9 — Note that the manifolds X and X' are not assumed to be projective. In fact, the proof works for any Kähler manifold X with a symplectic structure. X' might be even non-Kähler.

This theorem answers a question of Mukai, who asked whether the elementary transformation of a symplectic manifold X is deformation equivalent to X (Problem 4.6 in [44]).

Let us now turn to the case of arbitrary birational transformations. Consider the following situation:

10.10 — *i)* X and X' are irreducible symplectic manifolds. *ii)* X' is projective. *iii)* There exists a birational map $f : X - - \to X'$.

Let $U \subset X$ and $U' \subset X'$ be the maximal open subsets where f, respectively f^{-1}, are regular. Then $U \cong U'$ and $\mathrm{codim}(X \setminus U), \mathrm{codim}(X' \setminus U') \geq 2$ (see 10.1). Since X is Kähler and birational to the projective manifold X', also X is projective (cf. 9.1). For the algebraic Picard groups we have canonical isomorphisms $\mathrm{Pic}(X) \cong \mathrm{Pic}(U) \cong \mathrm{Pic}(U') \cong \mathrm{Pic}(X')$. If $L \in \mathrm{Pic}(X)$ we denote by L' the corresponding line bundle on X', i.e. L' is the line bundle on X' such that $L|_U \cong L'|_{U'}$. In the sequel, L' will usually be ample. If L is ample as well, then the isomorphism $U \cong U'$ can be extended to an isomorphism of X and X' and there is nothing to show.

The following proposition was proved in [31]:

Proposition 10.11 — *Under the assumptions 10.10, let L' be ample and let $\pi : (\mathcal{X}, \mathcal{L}) \to S$ be a deformation of $(\mathcal{X}_0, \mathcal{L}_0) = (X, L)$ over a smooth one-dimensional base S. If for any $n \gg 0$ the dimension $h^0(\mathcal{X}_t, \mathcal{L}_t^n)$ is constant in a neighbourhood of $0 \in S$, then, after shrinking S if necessary, there exists a deformation $(\mathcal{X}', \mathcal{L}') \to S$ of $(\mathcal{X}'_0, \mathcal{L}'_0) = (X', L')$ and an S-birational correspondence between $\mathcal{X}$ and $\mathcal{X}'$ respecting $\mathcal{L}$ and $\mathcal{L}'$.*

Proof. The idea of the proof is as follows: The linear system $|\mathcal{L}_t^m|$ $(m \gg 0)$ defines a rational map from $\mathcal{X}$ to $\mathbb{P}_S((\pi_* \mathcal{L}^m)^\vee)$ and we define $\mathcal{X}'$ as the closure of the image of this rational map. Then one shows that $\mathcal{X}'$ is S-flat with special fibre X'. The line bundle $\mathcal{L}'^m$ is the restriction of the relative $\mathcal{O}(1)$ on $\mathbb{P}_S((\pi_* \mathcal{L}^m)^\vee)$.

Here are the technical details: We may assume from the very beginning that L' is in fact very ample. First, shrink S to the open subset of points $t \in S$ such that $h^0(\mathcal{X}_t, \mathcal{L}_t)$ is constant on S. Hence $\pi_* \mathcal{L}$ is locally free on S with fibre $(\pi_* \mathcal{L})(t) = H^0(\mathcal{X}_t, \mathcal{L}_t)$.

By the very ampleness of L' the base locus $Bs(L)$ of L is contained in $X \setminus U$ and therefore of codimension at least 2. The set $\bigcup_{t \in S} Bs(\mathcal{L}_t)$ is a closed subset of $\mathcal{X}$ and hence $\mathrm{codim}_{\mathcal{X}_t} Bs(\mathcal{L}_t) \geq 2$ for t in an open neighbourhood of $t = 0$ (semicontinuity of the fibre dimension). Since $Bs(\mathcal{L}_t^n) \subset Bs(\mathcal{L}_t)$, we can assume that $\mathrm{codim}_{\mathcal{X}_t} Bs(\mathcal{L}_t^n) \geq 2$ for all $n > 0$ and $t \in S$.

The rational maps $\phi_{|\mathcal{L}_t|}: \mathcal{X}_t - - \to \mathbb{P}(H^0(\mathcal{X}_t, \mathcal{L}_t)^\vee)$, defined by the complete linear system $|\mathcal{L}_t|$, glue to a rational S-map $\phi : \mathcal{X} - - \to \mathbb{P}((\pi_*\mathcal{L})^\vee)$. Then ϕ is regular at all points of $\mathcal{X}_t \setminus Bs(\mathcal{L}_t)$ $(t \in S)$. Let $\mathcal{Z}$ be the scheme-theoretic closure of the graph Γ_ϕ of ϕ in $\mathcal{X} \times_S \mathbb{P}((\pi_*\mathcal{L})^\vee)$, i.e. the closure of Γ_ϕ with the reduced induced structure.

The projection $\varphi : \mathcal{Z} \to \mathcal{X}$ is an isomorphism over every point of $\mathcal{X}_t \setminus Bs(\mathcal{L}_t)$, $t \in S$. Note that a fibre $\mathcal{Z}_t$ of $\mathcal{Z}$ over $t \in S$ does not necessarily coincide with the closure of the graph of $\phi_{|\mathcal{L}_t|}$. However, since $\mathcal{X}$ has irreducible fibres and hence Γ_ϕ, the generic fibre of $\mathcal{Z} \to S$ is irreducible as well. Thus, shrinking to an open neighbourhood of $t = 0$, we can assume that $\mathcal{Z}_t$ is irreducible for $t \neq 0$. In particular, $\mathcal{Z}_{t\neq 0}$ equals the closure of the graph of $\phi_{|\mathcal{L}_t|}$ in $\mathcal{X}_t \times \mathbb{P}(H^0(\mathcal{X}_t, \mathcal{L}_t)^\vee)$ at least set-theoretically. Since $\mathcal{Z}$ is integral, i.e. irreducible and reduced, and S is smooth and one-dimensional, the dominant morphism $\mathcal{Z} \to S$ is flat ([27], III. 9.7.).

Now consider the other projection $\psi : \mathcal{Z} \to \mathbb{P}((\pi_*\mathcal{L})^\vee)$ and denote its image by $\mathcal{X}' \subset \mathbb{P}((\pi_*\mathcal{L})^\vee)$. Strictly speaking, $\mathcal{X}'$ is the scheme-theoretic image of ψ and since $\mathcal{Z}$ is reduced, this is the image with the reduced induced structure. Since $\mathcal{X}'$ then is integral and $\mathcal{X}' \to S$ is dominant, $\mathcal{X}'$ is flat over S.

Obviously, X' is contained in $\mathcal{X}'_0$. To conclude that $X' = \mathcal{X}'_0$ it is enough to show that $h^0(X', \mathcal{O}(n)|_{X'}) \geq h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0})$ for $n \gg 0$, where $\mathcal{O}(1)$ is the tautological ample line bundle on $\mathbb{P}(H^0(\mathcal{X}_0, L)^\vee)$. Since $\mathcal{O}(1)|_{X'} \cong L'$ and $h^0(X', L'^n) = h^0(X, L^n)$, this is equivalent to $h^0(X, L^n) \geq h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0})$ for $n \gg 0$. For any n there exists an open neighbourhood $S_n \subset S$ of $0 \in S$, such that $h^0(\mathcal{X}_t, \mathcal{L}_t^n)$ is constant for $t \in S_n$. On the intersection $\bigcap S_n \subset S$, which is the complement of countably many points, one has $h^0(\mathcal{X}_t, \mathcal{L}_t^n) = h^0(X, L^n)$. Using this and the flatness of $\mathcal{X}' \to S$, the inequality $h^0(X, L^n) \geq h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0})$ is equivalent to $h^0(\mathcal{X}_t, \mathcal{L}_t^n) \geq h^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t})$ for $n \gg 0$. But the latter can be derived using the composition

$$H^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t}) \xrightarrow{\psi^*} H^0(\mathcal{Z}_t, \psi^*\mathcal{O}(n)) \xrightarrow{i^*} H^0(\mathcal{X}_t \setminus Bs(\mathcal{L}_t), \mathcal{L}_t^n) \cong H^0(\mathcal{X}_t, \mathcal{L}_t^n).$$

Indeed, ψ^* is injective since $\mathcal{Z}_t \to \mathcal{X}'_t$ is surjective, and i^* is injective, since it is induced by the dense open embedding $\mathcal{X}_t \setminus Bs(\mathcal{L}_t) \subset \mathcal{Z}_t$ $(t \neq 0)$. The last isomorphism is a consequence of $\mathrm{codim}(\mathcal{X}_t \setminus Bs(\mathcal{L}_t)) \geq 2$. This shows that $\mathcal{X}'_0 = X'$. Shrinking S further we can also assume that $\mathcal{X}' \to S$ is smooth ([27], III. Ex. 10.2).

It remains to show the assertion on the birationality. Let $\mathcal{Z}^*$ and $\mathcal{X}'^*$ denote the fibre products $\mathcal{Z} \times_S (S \setminus \{0\})$ and $\mathcal{X}' \times_S (S \setminus \{0\})$, respectively. Stein factorization decomposes $\psi : \mathcal{Z}^* \to \mathcal{X}'^*$ into a finite morphism $f : \mathcal{Y} \to \mathcal{X}'^*$ and a morphism $\mathcal{Z}^* \to \mathcal{Y}$ with connected fibres. One first shows that $f : \mathcal{Y} \to \mathcal{X}'^*$ is in fact an isomorphism. Since $f_t : \mathcal{Y}_t \to \mathcal{X}'_t$ is finite, the line bundle $f_t^*\mathcal{O}(1)$ is ample. Thus $f_t^*\mathcal{O}(n)$ is very ample for $n \gg 0$. In order to prove that f is an isomorphism, it is therefore enough to show that $f_t^* : H^0(\mathcal{X}'_t, \mathcal{O}(n)) \to H^0(\mathcal{Y}_t, f_t^*\mathcal{O}(n))$ is surjective. We argue as above: Consider

$$H^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t}) \xrightarrow{f_t^*} H^0(\mathcal{Y}_t, f_t^*\mathcal{O}(n)) \hookrightarrow H^0(\mathcal{Z}_t, \psi^*\mathcal{O}(n)) \hookrightarrow H^0(\mathcal{X}_t, \mathcal{L}_t^n)$$

and use $h^0(\mathcal{X}'_t, \mathcal{O}(n)|_{\mathcal{X}'_t}) = h^0(\mathcal{X}'_0, \mathcal{O}(n)|_{\mathcal{X}'_0}) = h^0(X', L'^n) = h^0(X, L^n) = h^0(\mathcal{X}_t, \mathcal{L}_t^n)$ for all $t \in \bigcap S_n$. Hence f_t^* is bijective for t in the complement of countably many points and therefore $\mathcal{Y} \cong \mathcal{X}'^*$ after shrinking S. Thus $\mathcal{Z}^* \to \mathcal{X}'^*$ has connected fibres. On the other hand, $\dim \mathcal{Z}_t = \dim \mathcal{X}_t = \dim \mathcal{X}'_t$. Hence, $\mathcal{Z}_t \to \mathcal{X}'_t$ is birational for $t \neq 0$. □

The main technical problem that limited the results in [31] to the case that $\mathrm{codim}(X \setminus U), \mathrm{codim}(X' \setminus U') \geq 3$ was the assumption $h^0(\mathcal{X}_t, \mathcal{L}_t^n) \equiv const$, which is easy to establish under this additional assumption on the codimension. The projectivity criterion Theorem 9.11, which was not yet available in [31], can be used to arrange things such that the assumption on $h^0(\mathcal{X}_t, \mathcal{L}_t)$ holds.

Theorem 10.12 — *Let X and X' be birational projective irreducible symplectic manifolds. Then there exist deformations $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ of $\mathcal{X}_0 = X$ and $\mathcal{X}'_0 = X'$, respectively, such that*

- *S is smooth and one-dimensional.*
- *There exists an S-isomorphism $\mathcal{X}|_{S\setminus\{0\}} \cong \mathcal{X}'|_{S\setminus\{0\}}$.*

Proof. By 5.13 there exist constants (a_i) and (a'_i) $(i = 1, \ldots, n)$ such that for any line bundle L on X (resp. L' on X') the Hirzebruch-Riemann-Roch formula can be written as

$$\chi(X, L) = \sum_{i=1}^{n} \frac{a_i}{(2i)!} q_X(c_1(L))^i \qquad \text{resp.} \qquad \chi(X', L') = \sum_{i=1}^{n} \frac{a'_i}{(2i)!} q_{X'}(c_1(L'))^i.$$

Without loss of generality we may assume that in the lexicographic order $(a_i) \geq (a'_i)$, i.e. $\sum a_i t^i \geq \sum a'_i t^i$ for $t \gg 0$. If now $L' \in \mathrm{Pic}(X')$ is ample and L is the corresponding line bundle on X, then $q_X(c_1(L)) = q_{X'}(c_1(L')) > 0$. For the equality see 10.1 and for the inequality 5.9. Hence, $\chi(X, L^m) \geq \chi(X', L'^m)$ for $m \gg 0$.

Let $(\mathcal{X}, \mathcal{L}) \to S$ be a deformation of (X, L) over a smooth and one-dimensional base S such that $\rho(\mathcal{X}_t) = 1$ for general $t \in S$, e.g. take a curve $S \subset \mathit{Def}(X, L)$ not contained in any $\mathit{Def}(X, M)$ where M is a line bundle linearly independent of L (cf. 7.14, 7.15). By Theorem 9.11 the positivity of $q_{\mathcal{X}_t}(c_1(\mathcal{L}_t)) = q_X(c_1(L))$ implies that all fibres $\mathcal{X}_t$ are projective. Hence, for general $t \in S$ either $\mathcal{L}_t$ or its dual is ample. The non-vanishing of $H^0(X, L^m) = H^0(X', L'^m)$ for $m \gg 0$ and the semi-continuity of $h^0(\mathcal{X}_t, \mathcal{L}_t)$ excludes the latter possibility for t close to $0 \in S$. Thus, we can apply the Kodaira Vanishing Theorem to $\mathcal{L}_t$: For general $t \in S$ one has $h^0(\mathcal{X}_t, \mathcal{L}_t^m) = \chi(\mathcal{X}_t, \mathcal{L}_t^m) = \chi(X, L^m)$ for all $m > 0$. Using semi-continuity this yields $h^0(X, L^m) \geq h^0(\mathcal{X}_t, \mathcal{L}_t^m) = \chi(X, L^m)$ for $m > 0$. On the other hand, $h^0(X', L'^m) = h^0(X, L^m)$ (use $\mathrm{codim}(X \setminus U), \mathrm{codim}(X' \setminus U') \geq 2$) and $\chi(X', L'^m) = h^0(X', L'^m)$ (use L' ample on X'). Therefore

$$\chi(X', L'^m) = h^0(X, L^m) \geq h^0(\mathcal{X}_t, \mathcal{L}_t^m) = \chi(X, L^m).$$

Under the assumptions $(a_i) \geq (a'_i)$ and $q_X(c_1(L)) = q_{X'}(c_1(L')) > 0$ this implies $h^0(X, L^m) = h^0(\mathcal{X}_t, \mathcal{L}_t^m)$ for $m \gg 0$. Hence, for $m \gg 0$ the dimension $h^0(\mathcal{X}_t, \mathcal{L}_t^m)$ is constant in a neighbourhood of $0 \in S$.

Therefore, Proposition 10.11 can be applied and we find a deformation $(\mathcal{X}', \mathcal{L}') \to S$ of (X', L') which is S-birational to $\mathcal{X}$. The rest of the proof is as the one of the corresponding theorem in [31]: For $t \in S$ general and close to $0 \in S$ the birational correspondence $\mathcal{X}_t - - \to \mathcal{X}'_t$ is a birational correspondence between two projective manifolds with Picard number $\rho(\mathcal{X}_t) = \rho(\mathcal{X}'_t) = 1$, which must be an isomorphism. Hence, $\mathcal{X} - - \to \mathcal{X}'$ is an isomorphism on the general fibre. Restricting to an open neighbourhood of $0 \in S$ we can in fact achieve that $\mathcal{X}|_{S\setminus\{0\}} \cong \mathcal{X}'|_{S\setminus\{0\}}$ (cf. [31]). $\square$

More in the spirit of the Main Lemma of Burns and Rapoport (see 10.7) Theorem 10.12 can equivalently be formulated as

Theorem 10.12' — *Let X and X' be birational projective irreducible symplectic manifolds. Then there exist two markings $\varphi : H^2(X, \mathbb{Z}) \cong \Gamma$ and $\varphi' : H^2(X', \mathbb{Z}) \cong \Gamma$ such that $(X, \varphi), (X', \varphi') \in \mathfrak{M}_\Gamma$ are non-separated points.* $\square$

The fact that X and X' can be realized as the special fibres of the same family has strong consequences, which are not at all obvious just from the fact that they are birational. We only mention:

Corollary 10.13 — *If X and X' are birational projective irreducible symplectic manifolds, then:*

i) X and X' are diffeomorphic.

ii) For all k the weight-k Hodge structures of X and X' are isomorphic. $\square$

It is interesting to compare *ii)* in the corollary with a recent result of Batyrev and Kontsevich. They show that *ii)* holds true for all birational smooth projective manifolds with trivial canonical bundle; in particular for irreducible symplectic but also for Calabi-Yau manifolds. Assertion *i)* is not expected to hold for the more general class of projective manifolds with trivial canonical bundle.

11 Examples Revisited

We now wish to apply the results of the previous section to the list of examples in Section 4. As we tried to explain, there are essentially two types of manifolds for which one can verify directly that they are irreducible symplectic, namely for the Hilbert schemes $\mathrm{Hilb}^n(S)$ of a K3 surface S and for the generalized Kummer varieties $\mathrm{K}^{r+1}(A)$, where A is a torus (see [2]). The verification for all the other examples is reduced to these two by means of deformation and birational correspondences. As the birational correspondences used in Section 4 are either given in terms of Mukai's elementary transformations or are birational correspondences between projective manifolds, Theorems 10.8 and 10.12 suffice to prove:

Corollary 11.1 — *All examples of irreducible symplectic manifolds in Section 4 are deformation equivalent (and hence diffeomorphic) either to* $\mathrm{Hilb}^n(S)$ *or to* $\mathrm{K}^{n+1}(A)$*, where* S *is a K3 surface and* A *is a torus.* □

In particular, one has

Corollary 11.2 — *If* S *is a K3 surface and* $v = (v_0, v_1, v_2)$ *is a Mukai vector with* v_1 *primitive, then for any* v*-generic polarization* H *the moduli space* $M_H(v)$ *of semistable sheaves with Mukai vector* v *is an irreducible symplectic manifold that is deformation equivalent to* $\mathrm{Hilb}^n(S)$*, where* $n = (v, v) + 2$*. (For the notation see Section 4, [46] or [32].)* □

12 An Analogue of the Weyl-Action

The main obstacle to generalize the techniques in the theory of K3 surfaces to higher dimensions, besides the missing Global Torelli Theorem, is the absence of a Weyl-group, i.e. the group of automorphisms of the second cohomology generated by reflections in hyperplanes orthogonal to (-2)-classes. To a certain extent, Proposition 12.1 (and its corollary) is a good replacement for the fact that the Weyl-group acts transitively on the set of chambers (see also Remark 12.4). It has immediate consequences such as the description of the Kähler cone of very general irreducible symplectic manifolds (see 12.6 and 12.7). The question will be further pursued in Section 13. A posteriori the not very concrete 'generality' assumption in 12.1 can be made more specific (Theorem 12.9). This was pointed out to me by Deligne.

Proposition 12.1 — *Let* $(X, \varphi) \in \mathfrak{M}_\Gamma$ *be a marked irreducible symplectic manifold. Assume* $\alpha \in \mathcal{C}_X$ *is general, i.e.* α *is contained in the complement of countably many nowhere dense closed subsets. Then there exists a point* $(X', \varphi') \in \mathfrak{M}_\Gamma$*, which cannot be separated from* (X, φ) *such that* $({\varphi'}^{-1} \circ \varphi)(\alpha) \in H^2(X', \mathbb{R})$ *is a Kähler class.*

Together with Theorem 10.7 (or rather its proof) the proposition shows

Corollary 12.2 — *Let* X *be an irreducible symplectic manifold of dimension* $2n$ *and let* $\alpha \in \mathcal{C}_X$ *be general. Then there exists another irreducible symplectic manifold* X' *together with an effective cycle* $\Gamma := Z + \sum Y_i \subset X \times X'$ *of dimension* $2n$ *satisfying the following conditions:*

i) Z *defines a birational map* $X \dashrightarrow X'$*.*

ii) The projections $Y_i \to X$ *and* $Y_i \to X'$ *have positive fibre dimension.*

iii) $[\Gamma]_* : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ *defines an isomorphism of Hodge structures compatible with* q_X *and* $q_{X'}$*. Moreover,* $[\Gamma]_* \circ [\Gamma]_*$ *is the identity on* $H^2(X, \mathbb{Z})$*, resp.* $H^2(X', \mathbb{Z})$*.*

iv) $[\Gamma]_*(\alpha) \in \mathcal{K}_{X'}$*.* □

Proof of Proposition 12.1. First, let $\alpha \in \mathcal{C}_X$ be completely arbitrary. Choose a Kähler class $\gamma \in \mathcal{K}_X \subset \mathcal{C}_X$ and let $\beta := (1+\varepsilon)\alpha - \varepsilon\gamma \in \mathcal{C}_X$, where $0 < \varepsilon \ll 1$. Then, pick a sequence $\beta_i \in H^2(X, \mathbb{Q})$ converging to β such that $\beta_i \notin H^{1,1}(X)$ (or, equivalently, $\beta_i \notin \mathcal{C}_X$). The associated hypersurfaces $S_{\beta_i} \subset Def(X)$ (for the notation see 7.15) converge to $S_\beta \subset Def(X)$. Since $S_\beta \neq \emptyset$, also $S_{\beta_i} \neq \emptyset$ for $i \gg 0$. Let $U \subset \mathcal{C}_X$ be an open neighbourhood of α such that for all $\alpha' \in U$ the class $((1+\varepsilon)\alpha' - \beta)/\varepsilon$ is still Kähler. Such an open neighbourhood exists, because $\mathcal{K}_X \subset \mathcal{C}_X$ is open. For any $\alpha' \in U$ let $\mathcal{X}(\alpha') \to T(\alpha')$ be the deformation obtained by restricting the universal deformation to $T(\alpha') := Def(X) \cap \mathcal{P}^{-1}(\mathbb{P}(\varphi(F(\alpha')_\mathbb{C})))$ (see also 7.17). Since $\alpha', \beta \in \mathcal{C}_X$ and, hence, $q_X(\alpha', \beta) \neq 0$, the curve $T(\alpha')$ is not contained in S_β (cf. 6.5, 7.15). Therefore, $T(\alpha') \cap S_\beta$ is zero-dimensional and, moreover, non-empty, as 0 is contained in $T(\alpha') \cap S_\beta$. Hence, $T(\alpha') \cap S_{\beta_i}$ is zero-dimensional and non-empty for $i \gg 0$ as well. Thus, there exist points $t_i \in T(\alpha') \cap S_{\beta_i}$ approaching 0.

We need the following

Claim — For general $\alpha' \in U$ the points $t_i \in T(\alpha') \cap S_{\beta_i}$ are general points of S_{β_i} in the sense that $H^{1,1}(\mathcal{X}(\alpha')_{t_i})_\mathbb{Q}$ is spanned by β_i or, equivalently, that $\rho(\mathcal{X}(\alpha')_{t_i}) = 1$.

Proof of the Claim. Consider the map $\psi_i : U \to S_{\beta_i}$ that sends $\alpha' \in U$ to the intersection $\{t_i\} = T(\alpha') \cap S_{\beta_i}$, which, at least locally, consists of a single point. The map ψ_i is constant along the orbits of the natural $\mathbb{R}^*$-action on $U \subset H^{1,1}(X)_\mathbb{R}$. Thus $d\psi_i : T_{\alpha'}U \to T_{t_i}S_{\beta_i}$ factorizes through $T_{\alpha'}U = H^{1,1}(X)_\mathbb{R} \to H^{1,1}(X)_\mathbb{R}/\alpha' \cdot \mathbb{R} \to T_{t_i}S_{\beta_i} \subset H^{1,1}(\mathcal{X}(\alpha')_{t_i})$. Using the period map, which is holomorphic, it is easy to extend the map ψ_i to a holomorphic map $\tilde{\psi}_i : \tilde{U} \to S_{\beta_i}$, where $\tilde{U}$ is an open subset of the complex vector space $H^{1,1}(X)$ with $\tilde{U} \cap H^{1,1}(X)_\mathbb{R} = U$. Again, $\tilde{\psi}_i$ is constant along the orbits of the natural $\mathbb{C}^*$-action on $\tilde{U} \subset H^{1,1}(X)$ and, therefore, its tangent map factorizes as follows $T_{\alpha'}\tilde{U} \cong H^{1,1}(X) \to H^{1,1}(X)/\alpha' \cdot \mathbb{C} \to T_{t_i}S_{\beta_i} \subset H^{1,1}(\mathcal{X}(\alpha')_{t_i})$.

Next, one proves that the map $\tilde{\psi} : \tilde{U} \to S_{\beta_i}$ is injective modulo the $\mathbb{C}^*$-action. Indeed, if $H^{2,0} = \sigma \cdot \mathbb{C}$, then $F(\alpha')_\mathbb{C} = \sigma \cdot \mathbb{C} \oplus \bar{\sigma} \cdot \mathbb{C} \oplus \alpha' \cdot \mathbb{C}$. Therefore, for linearly independent $\alpha', \alpha'' \in \tilde{U} \subset H^{1,1}(X)$ the two planes $\mathbb{P}(\varphi(F(\alpha')_\mathbb{C})), \mathbb{P}(\varphi(F(\alpha'')_\mathbb{C})) \subset \mathbb{P}(\Gamma_\mathbb{C})$ intersect in $\mathbb{P}(\varphi(\sigma \cdot \mathbb{C} \oplus \bar{\sigma} \cdot \mathbb{C}))$. The latter space is a projective line which meets the period domain $Q \subset \mathbb{P}(\Gamma_\mathbb{C})$ in exactly two points, namely $\mathbb{P}(\varphi(\sigma \cdot \mathbb{C} \oplus \bar{\sigma} \cdot \mathbb{C})) \cap Q = \{\varphi[\sigma], \varphi[\bar{\sigma}]\}$. Shrinking $Def(X)$ such that $\varphi[\bar{\sigma}] \notin \mathcal{P}(Def(X))$, one has in particular $\varphi[\bar{\sigma}] \notin \mathcal{P}(S_{\beta_i})$ and, therefore, $T(\alpha') \cap S_{\beta_i} \neq T(\alpha'') \cap S_{\beta_i}$ for linearly independent α' and α''. Thus $\tilde{\psi} : \tilde{U} \to S_{\beta_i}$ is injective modulo $\mathbb{C}^*$.

But if $\tilde{\psi}_i : \tilde{U} \to S_{\beta_i}$ is injective modulo the $\mathbb{C}^*$-action, then its tangent map $d\tilde{\psi}_i : H^{1,1}(X)/\alpha' \cdot \mathbb{C} \to T_{t_i}S_{\beta_i}$ is injective for general $\alpha' \in \tilde{U}$. In fact, the set where $d\tilde{\psi}_i$ fails to be injective is an analytic set and, therefore, it cannot contain any open part of U. Thus, even for general $\alpha' \in U$ the tangent map can be assumed to be injective. Since both spaces $H^{1,1}(X)/\alpha' \cdot \mathbb{C}$ and $T_{t_i}S_{t_i}$ have dimension $h^{1,1}(X) - 1$, the tangent map $d\tilde{\psi}_i$ at such a point must be bijective. In particular, $\mathrm{im}(d\tilde{\psi}_i)$ is not contained in any $T_{t_i}S_\delta \cap T_{t_i}S_{\beta_i}$ for any $(1,1)$-class δ on $\mathcal{X}(\alpha')_{t_i}$ that is linearly independent of β_i. Since $d\tilde{\psi}_i$ is $\mathbb{C}$-linear and $H^{1,1}(X)_\mathbb{R}$ spans

$H^{1,1}(X)$, this also shows that $\mathrm{im}(d\psi_i)$ is not contained in any such $T_{t_i}S_\delta \cap T_{t_i}S_{\beta_i}$. Hence, the image of $\psi_i : U \to S_{\beta_i}$ is not contained in any hypersurface $S_\delta \cap S_{\beta_i}$ with δ linearly independent of β_i. As there are only countably many β_i's and δ's to be considered, one can assume that for the general $\alpha' \in U$ the intersection $T(\alpha') \cap S_{\beta_i}$ is not contained in any S_δ for δ linearly independent of β_i. In other words, if $\{t_i\} = T(\alpha') \cap S_{\beta_i}$ with α' general, then $\rho(\mathcal{X}(\alpha')_{t_i}) = 1$. This proves the claim.

Let us now replace α by a general (in the above sense) $\alpha' \in U$. Analogously, replace $\gamma = ((1+\varepsilon)\alpha - \beta)/\varepsilon$ by the Kähler class $\gamma' = ((1+\varepsilon)\alpha' - \beta)/\varepsilon$. In other words, from now on we are in the following situation: $\alpha \in \mathcal{C}_X$, $\gamma = ((1+\varepsilon)\alpha - \beta)/\varepsilon \in \mathcal{K}_X$, and there are points $t_i \in T(\alpha) \cap S_{\beta_i}$ converging to 0 such that $\rho(\mathcal{X}(\alpha)_{t_i}) = 1$.

Let $\mathcal{X} := \mathcal{X}(\alpha) \to T(\alpha)$ and denote by α_t a $(1,1)$-class on $\mathcal{X}_t$ that spans the orthogonal complement of $P(\mathcal{X}_t) := (\sigma_t \cdot \mathbb{C} + \bar\sigma_t \cdot \mathbb{C})_{\mathbb{R}}$ in $F(\alpha) \subset H^2(X,\mathbb{R}) \cong H^2(\mathcal{X}_t,\mathbb{R})$. We may choose α_t depending continuously on t such that $\alpha_0 = \alpha$. Next, let $\gamma_i := ((1+\varepsilon)\alpha_{t_i} - \beta)/\varepsilon$ which is considered as a class on X or as a class on $\mathcal{X}_{t_i}$ via the isomorphism $H^2(X,\mathbb{R}) \cong H^2(\mathcal{X}_{t_i},\mathbb{R})$. Since the union of the Kähler cones $\mathcal{K}_{\mathcal{X}_t}$ in $\bigcup_t H^{1,1}(\mathcal{X}_t)_{\mathbb{R}}$ is open, γ_i is a Kähler class on $\mathcal{X}_{t_i}$ for $i \gg 0$ (see also the arguments in the proof of Fujiki's Theorem 9.5). On the other hand, for $i \gg 0$ the class β_i on $\mathcal{X}_{t_i}$ is of type $(1,1)$ and $q_X(\beta_i) > 0$. Theorem 9.11 then asserts that $\mathcal{X}_{t_i}$ is projective. Since $H^{1,1}(\mathcal{X}_{t_i})_{\mathbb{Q}} = \beta_i \cdot \mathbb{Q}$ and $\beta_i \in \mathcal{C}_{\mathcal{X}_{t_i}}$ for $i \gg 0$, the class β_i is ample on $\mathcal{X}_{t_i}$. Thus, α_{t_i} is contained in the segment $[\gamma_i, \beta_i]$ joining the two Kähler classes γ_i and β_i on $\mathcal{X}_{t_i}$. Since the Kähler cone is convex, α_{t_i} is a Kähler class on $\mathcal{X}_{t_i}$ for $i \gg 0$.

All we need from the preceeding discussion is the following statement:

> *If X is an irreducible symplectic manifold and $\alpha \in \mathcal{C}_X$ is general, then there exists a point $t \in T(\alpha)$ such that α_t is Kähler on $\mathcal{X}_t = \mathcal{X}(\alpha)_t$.*

Let us fix this point t. We denote the induced marking $H^2(\mathcal{X}_t,\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \xrightarrow{\varphi} \Gamma$ of $\mathcal{X}_t$ by φ_t. Then $\varphi(F(\alpha)) = \varphi(P(X) \oplus \alpha \cdot \mathbb{R}) = \varphi_t(P(\mathcal{X}_t) \oplus \alpha_t \cdot \mathbb{R}) = \varphi_t(F(\alpha_t))$. This way one identifies $T(\alpha)$ with an open subset of the base of the twistor space associated to $(\mathcal{X}_t, \alpha_t)$. (Note that for an arbitrary $(1,1)$-class δ the space $T(\delta)$ is only locally defined but if δ is a Kähler class then $T(\delta)$ means the complete base of the twistor space, which is a $\mathbb{P}^1$, see 7.13) Hence, there are two families $\mathcal{X} \to T(\alpha)$ and $\mathcal{X}' \to T(\alpha)$ over the same base $T(\alpha)$, where the latter is (an open subset of) the twistor space of $\mathcal{X}_t \cong \mathcal{X}'_t$. Both deformations are endowed with the natural markings φ and φ' such that ${\varphi'_t}^{-1} \circ \varphi_t$ is induced by the isomorphism $\mathcal{X}_t \cong \mathcal{X}'_t$.

Moreover, $\alpha'_s := ({\varphi'_s}^{-1} \circ \varphi_s)(\alpha_s)$ is a Kähler class on $\mathcal{X}'_s$ for all $s \in T(\alpha)$. In particular, α'_0 is a Kähler class on $X' := \mathcal{X}'_0$. It remains to show that $(X', \varphi') := (\mathcal{X}'_0, \varphi'_0)$ cannot be separated from (X, φ).

Consider the maximal open subset $V \subset T(\alpha)$ containing t such that there exists an isomorphism $\mathcal{X}|_V \cong \mathcal{X}'|_V$ extending $\mathcal{X}_t \cong \mathcal{X}'_t$. (Use the Local Torelli Theorem 5.6 in order to show that V is open.) Denote by $\overline{V}$ the closure of V in $T(\alpha)$ and let $\partial V := \overline{V} \setminus V$ be its boundary. Let $s \in \partial V$. Then there exists an effective cycle $\Gamma = Z + \sum Y_i \subset \mathcal{X}_s \times \mathcal{X}'_s$ as in the proof of 10.7. In particular, $\mathcal{X}_s \leftarrow Z \rightarrow \mathcal{X}'_s$ is a birational correspondence, the projections $Y_i \rightarrow \mathcal{X}_s$ and $Y_i \rightarrow \mathcal{X}'_s$ have positive fibre dimension, and $[\Gamma]_* = {\varphi'_s}^{-1} \circ \varphi_s$. Let us assume that in addition $\mathcal{X}'_s$ neither contains non-trivial curves nor effective divisors, e.g. $H^{1,1}(\mathcal{X}'_s)_{\mathbb{Z}} = 0$ (cf. 6.8). Then, $\mathcal{X}_s \cong Z \cong \mathcal{X}'_s$ and $[Y_i]_* : H^2(\mathcal{X}_s, \mathbb{Z}) \rightarrow H^2(\mathcal{X}'_s, \mathbb{Z})$ is trivial (see Lemma 12.5 below). Since $\alpha'_s = ({\varphi'_s}^{-1} \circ \varphi)(\alpha_s) = [\Gamma]_*(\alpha) = [Z]_*(\alpha_s)$, this yields that α_s is a Kähler class on $\mathcal{X}_s$. Next we claim that in fact $\Gamma = Z$. Indeed, if we compute the volume with respect to the Kähler class $p_1^*\alpha_s + p_2^*\alpha'_s$, then

$$\begin{aligned} vol(\Gamma) &= vol(Z) + \sum vol(Y_i) \\ &= \int_Z (p_1^*\alpha_s + p_2^*\alpha'_s)^{2n} + \sum vol(Y_i) \\ &= \int_{\mathcal{X}'_s} ([Z]_*(\alpha_s) + \alpha'_s)^{2n} + \sum vol(Y_i) \\ &= \int_{\mathcal{X}'_s} ([\Gamma]_*(\alpha_s) + \alpha'_s)^{2n} + \sum vol(Y_i) \\ &= \int_{\mathcal{X}'_s} (2\alpha'_s)^{2n} + \sum vol(Y_i) \end{aligned}$$

On the other hand, if $s_i \in V$ converges to s, then the volume of the graph Γ_i of the isomorphism $\mathcal{X}_{s_i} \cong \mathcal{X}'_{s_i}$ converges to $vol(\Gamma)$. But $vol(\Gamma_i) = \int_{\Gamma_i} (p_1^*\alpha_{s_i} + p_2^*\alpha'_{s_i})^{2n} = \int_{\mathcal{X}'_{s_i}} (2\alpha'_{s_i})^{2n}$. Hence, $\sum vol(Y_i) = 0$, i.e. $\Gamma = Z$. But this would contradict the maximality of V. Thus, if $s \in \partial V$, then $\mathcal{X}'_s$ either contains non-trivial curves or divisors. By 7.17 the set of points $s \in T(\alpha)$ with this property is countable. Hence, ∂V is countable. But then ∂V could not separate two non-empty open subset V and $T(\alpha) \setminus \overline{V}$. Hence, $\overline{V} = T(\alpha)$, which proves that (X, φ) and (X', φ') cannot be separated. □

Remark 12.3 — In fact, we can modify $\mathcal{X}'$ appropriately such that it becomes isomorphic to $\mathcal{X}$ over $T(\alpha) \setminus \{0\}$. (However, it will not be a twistor space any longer.) Indeed, any countable closed subset in $T(\alpha)$ has an isolated point, but in the neighbourhood of an isolated point $\neq 0$ one can replace $\mathcal{X}'$ by $\mathcal{X}$.

Remark 12.4 — The transitivity of the action of the Weyl-group on the set of chambers of a K3 surface is equivalent to the following statement: If X is a K3 surface and $\alpha \in \mathcal{C}_X$ is general in the sense that it is not orthogonal to any (-2)-class, then there exists a cycle $\Gamma = \Delta + \sum C_i \times C_i \subset X \times X$, where Δ is the diagonal and the C_i's are (-2)-curves, such that $[\Gamma]_*(\alpha)$ is a Kähler class. In this light, the above proposition is a weak generalization of the transitivity of the action of the Weyl-group. Unfortunately, it seems to be hard to specify the assumption on α to be 'general'. In particular, one would like to replace it by an open condition. For a K3 surface this is granted by the fact that the union of all walls is closed (and hence its complement is open) in the positive cone. However, as we will see below (Corollary 12.8), a class is 'general' in the sense of the proposition if it is not orthogonal to any integral class.

The following lemma is rather elementary. It was used in the previous proof and will come up again in Section 13.

Lemma 12.5 — *Let X and X' be compact complex manifolds of dimension m and let $Y \subset X \times X'$ be an m-dimensional irreducible subvariety such that the projection $p' : Y \to X'$ has positive fibre dimension. If the induced homomorphism $[Y]_* := p'_*([Y].p^*(\,.\,)) : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is non-zero, then $p'(Y) \subset X'$ is a divisor. In this case $[Y]_*(\alpha) = (\int_C \alpha)[p'(Y)]$, where $C = {p'}^{-1}(x)$ is the general fibre curve of $p' : Y \to p'(Y)$.* □

Before approaching the Kähler cone of an arbitrary irreducible symplectic manifold in Section 13 let us deduce here some immediate consequences of Proposition 12.1.

Corollary 12.6 — *Let X be an irreducible symplectic manifold without effective divisors. Then for general $\alpha \in \mathcal{C}_X$ there exists a birational correspondence $Z \subset X \times X'$ between X and another irreducible symplectic manifold X' with $[Z]_*(\alpha) \in \mathcal{K}_{X'}$. If in addition X contains no rational curves, then $\mathcal{C}_X = \mathcal{K}_X$*

Proof. If X does not contain any divisor, then by the lemma the maps $[Y_i] : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ are trivial, where $\Gamma = Z + \sum Y_i$ is as in Corollary 12.2. Hence, if $\alpha' := [\Gamma]_*(\alpha)$, which is a Kähler class by Corollary 12.2, then $\alpha = [\Gamma]_*(\alpha') = [Z]_*(\alpha')$ and, therefore, $[Z]_*(\alpha) = \alpha' \in \mathcal{K}_{X'}$. If X does not contain any rational curves, then any birational correspondence extends to an isomorphism. Hence, α is Kähler. □

From the last assertion one easily obtains also the following

Corollary 12.7 — *If $H^{1,1}(X)_{\mathbb{Z}} = 0$, i.e. there exists no non-trivial line bundle on X, then $\mathcal{C}_X = \mathcal{K}_X$. In particular, if X is a general irreducible symplectic manifold then $\mathcal{C}_X = \mathcal{K}_X$.*

Proof. The assumption $H^{1,1}(X)_{\mathbb{Z}} = 0$ implies $H^{2n-1,2n-1}(X)_{\mathbb{Z}} = 0$ by Proposition 6.8. Hence, X contains neither curves nor effective divisors. Thus, we can apply the previous corollary. For the second assertion recall that the deformations of X that admit non-trivial line bundles form a countable union of hypersurfaces in *Def*(X) (Lemma 7.16). □

Once the Kähler cone of a general irreducible symplectic manifold is described (Corollary 12.7), Proposition 12.1 can be sharpened to yield:

Corollary 12.8 — *Let $(X,\varphi) \in \mathfrak{M}_\Gamma$ be a marked irreducible symplectic manifold. Assume that $\alpha \in \mathcal{C}_X$ is not orthogonal to any $0 \neq \beta \in H^{1,1}(X)_{\mathbb{Z}}$. Then there exists a point $(X',\varphi') \in \mathfrak{M}_\Gamma$ which cannot be separated from (X,φ) such that $({\varphi'}^{-1} \circ \varphi)(\alpha)$ is in the Kähler cone of X'.*

Proof. As in the proof of Proposition 12.1 we consider the 'twistor space' $\mathcal{X}(\alpha) \to T(\alpha)$. By the assumption on α, the general fibre $\mathcal{X}_t := \mathcal{X}(\alpha)_t$ satisfies $H^{1,1}(\mathcal{X}_t)_{\mathbb{Z}} = 0$. By Corollary 12.7 this implies $\mathcal{K}_{\mathcal{X}_t} = \mathcal{C}_{\mathcal{X}_t}$. If α_t is a $(1,1)$-class on $\mathcal{X}_t$ that spans the orthogonal complement of $P(\mathcal{X}_t) := (\sigma_t \cdot \mathbb{C} + \bar{\sigma}_t \cdot \mathbb{C})_{\mathbb{R}}$ in $F(\alpha) \subset H^2(X, \mathbb{R}) \cong H^2(\mathcal{X}_t, \mathbb{R})$ (as in the proof of 12.1), then the class α_t is Kähler on $\mathcal{X}_t$ for general t close to 0. Now proceed as in the final paragraph of the proof of 12.1 $\square$.

Once a point $(X', \varphi') \in \mathfrak{M}_\Gamma$ non-separated from (X, φ) with $({\varphi'}^{-1} \circ \varphi)(\alpha)$ in the Kähler cone $\mathcal{K}_{X'}$ has been shown to exist, one easily proves

Theorem 12.9 — *Let $\mathcal{X} \to Def(X)$ be the universal deformation of an irreducible symplectic manifold X and let $\alpha \in \mathcal{C}_X$ be a class not orthogonal to any $0 \neq \beta \in H^{1,1}(X)_{\mathbb{Z}}$. Then there exists another irreducible symplectic manifold X' and a smooth proper family $\mathcal{X}' \to Def(X)$ with $X' = \mathcal{X}'_0$ such that over an open subset containing the complement of the union of all hypersurfaces S_β with $\beta \in H^2(X, \mathbb{Z})$ both families $\mathcal{X}$ and $\mathcal{X}'$ are isomorphic and the induced isomorphism $H^2(X, \mathbb{R}) \cong H^2(X', \mathbb{R})$ maps α to a Kähler class on X'.*

Proof. Again, this is just a reformulation of the proof of 12.1 with the extra input 12.8, which in turn is based on 12.1 in its original form. Fix a marking φ of X and take (X', φ') as in 12.8. Consider $Def(X')$ as an open subset of $\mathfrak{M}_\Gamma$. Using the period map, both spaces $Def(X)$ and $Def(X')$ can be identified. Thus, we have two marked families $(\mathcal{X}, \varphi) \to Def(X)$ and $(\mathcal{X}', \varphi') \to Def(X)$, both universal for X respectively X'. Moreover, for some point $t \in Def(X)$ there is an isomorphism $f : \mathcal{X}_t \cong \mathcal{X}'_t$ with $f^* = {\varphi'}^{-1} \circ \varphi$. Let V be the maximal open subset such that there exists an isomorphism $\mathcal{X}|_V \cong \mathcal{X}'|_V$ extending f. As we have seen in the proof of 12.1, for a point s in the boundary $\partial V := \overline{V} \setminus V$ the group $H^{1,1}(\mathcal{X}_s)_{\mathbb{Z}}$ is non-trivial. Hence, ∂V is contained in the union of all hypersurfaces S_β with $\beta \in H^2(X, \mathbb{Z})$. Since a countable union of real codimension two subsets cannot separate two non-empty open subsets, one of the open sets V or $Def(X) \setminus \overline{V}$ must be empty. Hence, $\overline{V} = Def(X)$. $\square$

It seems likely that in the above theorem one can achieve that the open subset is in fact the complement of finitely many hypersurfaces S_{β_i} with $\beta_i \in H^{1,1}(X)_{\mathbb{Z}}$, but at the moment I do not know how to prove this.

13 The Ample and the Kähler Cone

After having established a projectivity criterion for irreducible symplectic manifolds in Section 9, we now strive for an understanding of the ample cone of a projective irreducible symplectic manifold. Recall the following well-known results:

13.1 — Let X be a projective variety. A line bundle L on X is ample if and only if $\int_Z c_1^i(L) > 0$ for any integral subscheme $Z \subset X$ of dimension $i \leq \dim(X)$ (cf. [28]).

13.2 — Let X be a K3 surface. A line bundle L on X is ample if and only if $c_1(L)$ is contained in the positive cone and $\int_C c_1(L) > 0$ for all (-2)-curves (i.e. irreducible smooth and rational) $C \subset X$ (cf. [11], [1]).

The second statement, a special case of the Nakai-Moishezon criterion, says that on a two-dimensional irreducible symplectic manifold the ample cone is the integral part of the positive cone that is positive on all (-2)-curves. The main result of this section (Corollary 13.5) is formulated in this spirit. However, the result is extremely weak compared to 13.2 as it only says that an integral class in the positive cone that cannot be separated from the Kähler cone by any *integral* wall is ample. The problem one has to face in higher dimensions is that the analogue of (-2)-curves is not (yet?) available.

Recall, the following notation: $\mathcal{K}_X \subset H^{1,1}(X)_{\mathbb{R}}$ is the cone of all Kähler classes and the positive cone $\mathcal{C}_X$ is the component of $\{\alpha \in H^{1,1}(X)_{\mathbb{R}} | q_X(\alpha) > 0\}$ that contains $\mathcal{K}_X$.

We begin with a variant of 13.1 in the case of projective manifolds with trivial canonical bundle. The following proposition, which is a straightforward consequence of the Basepoint-Free Theorem, says that in order to check whether a line bundle is ample it suffices to test it on subvarieties which are either X itself or of dimension one.

Proposition 13.3 — *Let X be a projective manifold with $K_X \cong \mathcal{O}_X$. Then a line bundle L on X is ample if and only if $\int c_1(L)^{\dim(X)} > 0$ and $\int_C c_1(L) > 0$ for all curves $C \subset X$.*

Proof. Obviously, if L is ample, then both inequalities hold. For the converse let us first recall the Basepoint-Free Theorem [15]: If X is any smooth projective variety and $L \in \mathrm{Pic}(X)$, then some high power of L is globally generated if L is nef and $L \otimes K_X{}^\vee$ is nef and big (i.e. $c_1(L \otimes K_X{}^\vee)^{\dim(X)} > 0$). Let us apply this to our situation. If L is a line bundle satisfying both inequalities, then L is nef and $L \otimes K_X^\vee \cong L$ is nef and big. Then, the Basepoint-Free Theorem shows that L^m is globally generated for $m \gg 0$. But any globally generated line bundle that is positive on all curves is ample (cf. [28]). □

Applied to irreducible symplectic manifolds this yields:

Corollary 13.4 — *Let X be a projective irreducible symplectic manifold of dimension $2n$ and let L be a line bundle on X. Then*

- *L is ample if and only if $c_1(L) \in \mathcal{C}_X$ and $\int_C c_1(L) > 0$ for all curves $C \subset X$.*
- *L is in the closure of the ample cone if and only if $c_1(L) \in \overline{\mathcal{C}}_X$ and L is nef.*

Proof. Observe that for the first assertion we do not need the projectivity of X, for $c_1(L) \in \mathcal{C}_X$ implies that X is projective (Theorem 9.11). Both assertions follow directly from the proposition: If L is ample, then certainly $\int_C c_1(L) > 0$, $q_X(c_1(L)) > 0$, and $q_X(c_1(L), \alpha) > 0$

for any Kähler class α (cf. Section 5) and, therefore, $c_1(L) \in \mathcal{C}_X$. Conversely, if $c_1(L) \in \mathcal{C}_X$ then $\int_X c_1^{2n}(L) > 0$ by 5.12. For the second assertion replace L by $L^m \otimes M$ with M an ample line bundle and $m \gg 0$. □

Using the isomorphism $cL_{\sigma\bar\sigma}^{n-1} : H^{1,1}(X)_{\mathbb{Q}} \cong H^{2n-1,2n-1}(X)_{\mathbb{Q}}$ (cf. 6.8) Corollary 13.4 allows us to formulate also the following

Corollary 13.5 — *Let X be an irreducible symplectic manifold and let L be a line bundle on X. Then L is ample if and only if $c_1(L)$ satisfies*

i) $q_X(c_1(L), .)$ *is positive on* $\mathcal{K}_X$.

ii) If $M \in \mathrm{Pic}(X)$ *such that* $q_X(c_1(M), .)$ *is positive on* $\mathcal{K}_X$, *then* $q_X(c_1(M), c_1(L)) > 0$.

Proof. If L is ample, then $c_1(L) \in \mathcal{K}_X$ and, therefore, *i)* and *ii)* follow. If L satisfies *i)* and *ii)*, then $c_1(L) \in \mathcal{C}_X$. Hence, X is projective (Theorem 9.11). Thus it remains to verify that L is positive on all curves (13.4). If C is a curve then $[C] \in H^{2n-1,2n-1}(X)_{\mathbb{Q}}$ and therefore there exists an $\alpha \in H^{1,1}(X)_{\mathbb{Q}}$ with $cL_{\sigma\bar\sigma}^{n-1}(\alpha) = [C]$. Since any Kähler class $[\omega]$ is positive on C, one has $q_X(\alpha, \omega) > 0$. Hence, $q_X(\alpha, c_1(L)) > 0$ by assumption, for α is a rational class. □

The statement seems rather weak, and indeed, it cannot be considered as a true generalization of the Nakai-Moishezon criterion for K3 surfaces. However, it is non-trivial. E.g., if X is a projective irreducible symplectic manifold such that its Picard group is spanned by two line bundles L_1 and L_2 with L_1 ample and L_2 non-ample, then there exists a ($\mathbb{Z}$-)linear combination of L_1 and L_2 that is negative (with respect to q_X) on L_2 and positive on the whole Kähler (and not only on the ample!) cone. Of course, if $h^{1,1}(X) = \rho(X)$, then the assertion of then 13.5 is void.

Note that the description of the ample cone given above does not make use of Proposition 12.1, but I expect that Corollary 13.5 together with Proposition 12.1 implies that the ample cone is locally finitely polyhedral away from $\partial\overline{\mathcal{C}}_X$. But at the moment I do not know how to prove this.

An analogous result can be proved for the Kähler cone of a not necessarily projective irreducible symplectic manifold. Again, the result is much weaker than the known ones for K3 surfaces and says that a class in the positive cone that cannot be separated by any integral wall from the Kähler cone is Kähler itself. It might be noteworthy that the description of the ample cone of a K3 surface is a rather easy consequence of the Nakai-Moishezon criterion, whereas the description of the Kähler cone of a K3 surface relies on the Global Torelli Theorem, which is in no form available in higher dimensions.

The result can most powerfully be applied to the case where there are no or only few integral classes. In particular it generalizes Corollary 12.7 to the case that X is projective and the Picard number is one (cf. 13.7).

Theorem 13.6 — *Let X be an irreducible symplectic manifold. Then a class α is contained in the closure of the Kähler cone $\overline{\mathcal{K}}_X$ if and only if*

i) $\alpha \in \overline{\mathcal{C}}_X$.

ii) If $q_X(c_1(M), \,.\,)$ is non-negative on the Kähler cone $\mathcal{K}_X$ for a line bundle $M \in \mathrm{Pic}(X)$, then $q_X(c_1(M), \alpha) \geq 0$.

Proof. The principal idea of the proof is modelled on Beauville's exposé in [1]. Arguments using the Weyl-group are replaced by Proposition 12.1.

It is obvious that *i)* and *ii)* are necessary for $\alpha \in \overline{\mathcal{K}}_X$. For the converse, assume that α satisfies both conditions. We first consider $\alpha + \varepsilon \cdot \gamma$, where $0 < \varepsilon \ll 1$ and γ is a general Kähler class. This class is contained in $\mathcal{C}_X$ and satisfies the strong inequality in *ii)* for all line bundles M with $q_X(c_1(M), \,.\,)$ positive on $\mathcal{K}_X$. If we can show that for general γ the class $\alpha + \varepsilon\gamma$ is a Kähler class, then $\alpha \in \overline{\mathcal{K}}_X$. Certainly, $\alpha + \varepsilon\gamma$ is general (in the sense of Proposition 12.1) if γ is a general Kähler class. Thus, we only have to deal with the following situation: α is a general class in $\mathcal{C}_X$ such that $q_X(c_1(M), \alpha) > 0$ whenever $q_X(c_1(M), \,.\,)$ is positive on $\mathcal{K}_X$, where M is an arbitrary line bundle on X. In this situation we show that α is a Kähler class.

Indeed, by Proposition 12.1 and Corollary 12.2 there exists another irreducible symplectic manifold X' together with an effective cycle $\Gamma = Z + \sum Y_i \subset X \times X'$ satisfying *i)-iv)* of Corollary 12.2. In particular, there exists a Kähler class α' on X' with $[\Gamma]_*(\alpha) = \alpha'$. The latter condition is equivalent to $[\Gamma]_*(\alpha') = \alpha$. Since $[\Gamma]_*$ respects the quadratic form, we have $q_X(\alpha) = q_{X'}(\alpha')$. By 10.1 any birational correspondence Z respects the quadratic form as well. Hence, $q_{X'}(\alpha') = q_X([Z]_*(\alpha'))$. This yields

$$\begin{aligned} 0 &= q_X(\alpha) - q_X([Z]_*(\alpha')) \\ &= q_X(\alpha + [Z]_*(\alpha'), \alpha - [Z]_*(\alpha')) \\ &= q_X(\alpha + [Z]_*(\alpha'), \textstyle\sum [Y_i]_*(\alpha')). \end{aligned}$$

By Lemma 12.5 only those components Y_i contribute for which $V_i := p(Y_i)$ is a divisor. Moreover, in this case $[Y_i]_*(\alpha') = (\int_{C_i} \alpha')[V_i]$, where C_i is the generic fibre of $p : Y_i \to V_i$. Since α' is Kähler, $\int_{C_i} \alpha' > 0$. By 5.15 a Kähler class is positive (with respect to q) on any effective divisor. Hence, any $q_X((\int_{C_i} \alpha')[V_i], \,.\,)$ is positive on $\mathcal{K}_X$. Using the assumption on α, this implies $q_X(\sum (\int_{C_i} \alpha')[V_i], \alpha) > 0$. Since α' is Kähler, one also has $q_X([Z]_*(\alpha'), [V_i]) = q_{X'}(\alpha', [Z]_*[V_i]) > 0$ (cf. 5.15). Altogether, this yields that if $\sum [Y_i]_*(\alpha') \neq 0$ then $0 = q_X(\alpha + [Z]_*(\alpha'), \sum (\int_{C_i} \alpha')[V_i]) > 0$. Hence, $\sum [Y_i]_*(\alpha') = 0$ and, therefore, $\alpha = [\Gamma]_*(\alpha') = [Z]_*(\alpha')$.

If Z does not define an isomorphism $X \cong X'$ then there exists a (rational) curve $C \subset X$ such that $\int_C \alpha = \int_C [Z]_*(\alpha') < 0$, e.g. take a curve in the fibre of $Z \to X'$. On the other hand, there exists a rational class $\beta \in H^{1,1}(X)_{\mathbb{Q}}$ such that $cL_{\sigma\bar{\sigma}}^{n-1}(\beta) = [C]$ (cf. 6.8). Since any

Kähler class is positive on C, the linear form $q_X(\beta, \,.\,)$ is positive on $\mathcal{K}_X$. By the assumption this yields $\int_C \alpha > 0$. Contradiction. Thus $X \cong Z \cong X'$ and, therefore, $\alpha \in \mathcal{K}_X$. $\square$

The following is another instance where the Kähler cone can completely be described in terms of the period.

Corollary 13.7 — *Let X be an irreducible symplectic manifold and assume that* $\mathrm{Pic}(X)$ *is spanned by a line bundle L such that $q_X(c_1(L)) \geq 0$. Then $\mathcal{K}_X = \mathcal{C}_X$, i.e. any class $\alpha \in \mathcal{C}_X$ is Kähler.*

Proof. Of course, it suffices to prove that $\overline{\mathcal{C}}_X = \overline{\mathcal{K}}_X$. Thus we can apply the theorem. Without loss of generality we can assume that $q_X(c_1(L), \,.\,)$ is non-negative on $\overline{\mathcal{C}}_X$ (Hodge Index). Therefore, $\overline{\mathcal{C}}_X \subset \overline{\mathcal{K}}_X$. $\square$

14 Surjectivity of the Period Map

Recall, that for a lattice Γ of index $(3, b-3)$ we defined the period domain Q as the set $\{x \in \mathbb{P}(\Gamma_{\mathbb{C}}) | q_\Gamma(x) = 0,\ q_\Gamma(x + \bar{x}) > 0\}$, which is an open set of a smooth quadric. Also recall, that the period map $\mathcal{P} : \mathfrak{M}_\Gamma \to \mathbb{P}(\Gamma_{\mathbb{C}})$ takes values in Q. (cf. 7.9). The proof of the following theorem is, once again, modelled on Beauville's presentation in [1].

Theorem 14.1 — *Let $\mathfrak{M}_\Gamma^o$ be a non-empty connected component of the moduli space $\mathfrak{M}_\Gamma$ of marked irreducible symplectic manifolds. Then the period map*

$$\mathcal{P} : \mathfrak{M}_\Gamma^o \to Q$$

is surjective.

Proof. The proof is divided into two steps. The first one is a word-by-word copy of the arguments for K3 surfaces and has been given in Section 7 already. The result important for our purposes here is the fact that any two points $x, y \in Q \subset \mathbb{P}(\Gamma_{\mathbb{C}})$ are equivalent. Recall that two points are called equivalent if there exists a sequence $x = x_1, \ldots, x_n = y \in Q$ such that all $\langle P(x_i), P(x_{i+1} \rangle$ are positive 3-spaces (cf. 7.4). The second step of the proof consists of the following claim, which, in fact, directly implies the theorem.

Claim — If $x, y \in Q$ such that $\langle P(x), P(y) \rangle$ is a positive 3-space, then $x \in \mathcal{P}(\mathfrak{M}_\Gamma^o)$ if and only if $y \in \mathcal{P}(\mathfrak{M}_\Gamma^o)$.

Proof of the Claim. Assume $x = \mathcal{P}((X, \varphi))$, i.e. $P(x) = \varphi(P(X))$, with $(X, \varphi) \in \mathfrak{M}_\Gamma^o$.
I claim that one can deform X slightly such that

i) $\langle P(x), P(y) \rangle$ is still a positive 3-space.

ii) $\rho(X) = 0$.

This is proved as follows: Identify $Def(X)$ with a small open neighbourhood of x in Q via the period map (Local Torelli Theorem). Then consider the countable union of hypersurfaces $T = \bigcup S_\alpha \subset Def(X)$ where $\alpha \in H^2(X,\mathbb{Z})$ (see 7.15). To achieve *i)* and *ii)* it suffices to find $t \in Def(X) \setminus T$ such that $\langle\varphi(P(\mathcal{X}_t)), P(y)\rangle$ is a positive 3-space. Of course, the positivity is harmless as long as t is close to 0 and $\dim\langle\varphi(P(\mathcal{X}_t)), P(y)\rangle = 3$.

First, consider those t for which $\langle\varphi(P(\mathcal{X}_t)), P(y)\rangle = \langle P(x), P(y)\rangle$. They are parametrized by (an open subset of) the non-degenerate quadric $Q \cap \langle P(x), P(y)\rangle_{\mathbb{C}}$ (for a similar argument see 7.17). Moving t slightly in $Q \cap \langle P(x), P(y)\rangle_{\mathbb{C}}$ we can assume that the orthogonal complement $k_0 \cdot \mathbb{R} \subset \langle P(x), P(y)\rangle$ of $P(x)$ is not contained in $P(y)$. Now fix a basis $P(y) = \langle a, b\rangle$. Then to any k in a neighbourhood of k_0 in $\Gamma_{\mathbb{R}}$ we associate $P_k := \langle q(k,k)a - q(a,k)k, q(k,k)b - q(b,k)k\rangle$ – the orthogonal complement of k in $\langle k, P(y)\rangle$. Then $\langle P_k, P(y)\rangle$ is a positive 3-space.

Let $T' \subset Def(X)$ be the subset of those $t \in Def(X)$ such that $\varphi(P(\mathcal{X}_t)) = P_k$ for some k in a neighbourhood of $k_0 \in \Gamma_{\mathbb{R}}$. If $T' \not\subset T$ we are done. If $T' \subset T$, then there exists an α such that $T' \subset S_\alpha$ (note that T' is locally irreducible). Since T' certainly contains those points for which $\langle\varphi(P(\mathcal{X}_t)), P(y)\rangle = \langle P(x), P(y)\rangle$, we in particular have (locally!) $Q \cap \langle P(x), P(y)\rangle_{\mathbb{C}} \subset T' \subset S_\alpha$. Since S_α is the hyperplane section defined by $q(\alpha, \,.\,)$ and $Q \cap \langle P(x), P(y)\rangle_{\mathbb{C}}$ is a non-degenerate quadric, this shows that $q(\alpha, \,.\,)$ vanishes on $\langle P(x), P(y)\rangle$. Moreover, since $T' \subset S_\alpha$, the spaces P_k are all orthogonal to α. Hence, $q(q(k,k)a - q(a,k)k, \alpha) = 0$ and $q(q(k,k)b - q(b,k)k, \alpha) = 0$ for all k in a neighbourhood of k_0. Also $q(a,\alpha) = q(b,\alpha) = 0$ and, therefore, $q(a,k)q(k,\alpha) = q(b,k)q(k,\alpha) = 0$ for all k in a neighbourhood of k_0. Contradiction.

Corollary 12.6 then shows that for X satisfying *i)* and *ii)* above one has $\mathcal{C}_X = \mathcal{K}_X$. Thus, if β spans the orthogonal complement of $P(x)$ in $\langle P(x), P(y)\rangle$, then, after replacing β by $-\beta$ if necessary, $\alpha := \varphi^{-1}(\beta) \in \mathcal{C}_X$ is a Kähler class. On the other hand, $y \in \mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}})) \subset \mathbb{P}(\Gamma_{\mathbb{C}})$. Since there exists the twistor space $\mathcal{X}(\alpha) \to T(\alpha) = \mathbb{P}(\varphi(F(\alpha)_{\mathbb{C}}))$ and $T(\alpha) \subset \mathfrak{M}_\Gamma^o$, this suffices to conclude that $y \in \mathcal{P}(\mathfrak{M}_\Gamma^o)$. $\square$

The second part of the proof of the surjectivity of the period map deviates from the standard proofs even for K3 surfaces. In the original proof for K3 surfaces the class β is chosen to be rational and not orthogonal to any (-2)-class. Therefore, modulo the action of the Weyl-group, it is positive on all (-2)-curves and, hence, ample by the Nakai-Moishezon criterion. In the approach above the generalization of the Nakai-Moishezon criterion as proved in Section 13 does not suffice to prove the surjectivity along this line. Also note, that the order of the arguments is different compared to the original proof for K3 surfaces. For K3 surfaces one first proves the surjectivity of the period map using the Nakai-Moishezon criterion and then applies it to derive a description of the Kähler cone. Whereas here, we first described the Kähler cone and then applied the result, which is, however, far less explicit than the known one for K3 surfaces, to prove the surjectivity of the period map.

15 Automorphisms

Not much is known about the automorphism group of an irreducible symplectic manifold of dimension greater than two. In this section we collect some results related to this question and add some remarks.

Definition 15.1 — *Let X be an irreducible symplectic manifold. Then*

- *$Aut(X)$ (resp. $Birat(X)$) is the group of holomorphic (resp. birational) automorphisms of X.*
- *$A(X) \subset Aut(H^2(X,\mathbb{Z}))$ is the group of automorphisms of $H^2(X,\mathbb{Z})$ which are compatible with the Hodge structure and the quadratic form q_X and map the Kähler cone to the Kähler cone. $B(X) \subset Aut(H^2(X,\mathbb{Z}))$ is the group of automorphisms satisfying only the first two conditions.*
- *$a : Aut(X) \to Aut(H^2(X,\mathbb{Z}))$ respectively $b : Birat(X) \to Aut(H^2(X,\mathbb{Z}))$ maps a map f to f^*.*

Proposition 15.2 — *i)* $\mathrm{im}(a) \subset A(X)$*, ii)* $\mathrm{im}(b) \subset B(X)$*, iii)* $b^{-1}(A(X)) = Aut(X) \subset Birat(X)$*, iv)* $\ker(a) = \ker(b)$*, and v)* $\ker(a)$ *is finite.*

Proof. *i)* is trivial, *ii)* follows from 10.1. In order to prove *iii)* one has to show that any birational map which maps a Kähler class to a Kähler class can be extended to an automorphism. This was proved in [21]. *iv)* is an easy consequence of *iii)*. To prove *v)* one evokes two standard facts: Firstly, the group of isometries of a compact Riemannian manifold is compact and, secondly, the Calabi-Yau metric with respect to a fixed Kähler class is unique. Hence, an automorphism acting trivially on the second cohomology leaves invariant the Kähler class and, hence, the Calabi-Yau metric, i.e. it is an isometry. Since the group $Aut(X)$ is discrete, this suffices to conclude that $\ker(a)$ is finite. □

The natural inclusion $Aut(X) \subset Birat(X)$ is in general proper. Indeed, Beauville constructed in [3] an example of a birational automorphism of $\mathrm{Hilb}^n(S)$, where S is a special K3 surface, which does not extend to an automorphism (cf. 10.4).

However, one has

Proposition 15.3 — *If X is general, i.e. together with a marking φ it is a general point in $\mathfrak{M}_\Gamma$, then $Aut(X) = Birat(X)$.*

Proof. This stems from the fact that the general irreducible symplectic manifold X does not contain any (rational) curves (cf. 7.17). But a birational automorphism of a variety without rational curves extends to a holomorphic automorphism. □

For K3 surfaces one certainly has $Aut(X) = Birat(X)$. Moreover, the Global Torelli Theorem for K3 surfaces in particular asserts that $a : Aut(X) \to A(X)$ is an isomorphism. Using this fact one also shows that a is injective for the Hilbert scheme of a K3 surface [3]. However, due to an example of Beauville [3], we know that a is not injective in general (This time the counterexample is provided by a generalized Kummer variety). Since the graph of an automorphism in the kernel of a deforms (at least infinitesimally) in all directions with X, we shall not even expect that a is injective for general X.

Questions — *i)* Is a surjective? *ii)* Is $A(X) = \{1\}$ for general X?

The fact that a is not injective has the unpleasant consequence that the moduli space $\mathfrak{M}_\Gamma$ of marked irreducible symplectic manifolds is not fine (contrary to the K3 surface case). This was explained in 7.8. For a discussion of other questions related to the moduli space see the next section.

16 Further Remarks

As mentioned in the introduction, the two problems I consider the most important in the theory are:

- Is there a Global Torelli Theorem for irreducible symplectic manifolds in higher dimensions?
- What are the possible deformation (or diffeomorphism) types of irreducible symplectic manifolds?

The second question alludes to the rather easy fact that any two K3 surfaces are deformation equivalent and, hence, diffeomorphic. From dimension four on we know exactly two different deformation (diffeomorphism) types of irreducible symplectic manifolds (cf. Section 10). But there is no obvious reason why there should be no other possibilities.

Due to Debarre's counterexample [16] (cf. 10.4) we know that the the Global Torelli Theorem as formulated for K3 surfaces fails in higher dimensions. However, there seems to be no counterexample known to the following speculation which was first formulated by Mukai [44].

Speculation 16.1 — *Two irreducible symplectic manifolds with isomorphic periods are birational.*

The known proofs of the Global Torelli Theorem for K3 surfaces break down in higher dimensions. This is mainly due to a missing analogue of Kummer surfaces. Kummer surface are dense in the moduli spaces of K3 surfaces and whether a K3 surface is a Kummer surface can easily be read off its period. (Quartic hypersurfaces would be another such distinguished class of K3 surfaces.) In higher dimensions we neither have a good class of manifolds which are dense in $\mathfrak{M}_\Gamma$ nor do we know a 'typical' class of manifolds which could be recognized by its period. In this light, it would be interesting to find an answer to the following

Question — Let X be an irreducible symplectic manifold with the period of a Hilbert scheme $\mathrm{Hilb}^n(S)$ of a K3 surface S. Is X birational to $\mathrm{Hilb}^n(S)$?

Let us conclude with a few remarks on the moduli space $\mathfrak{M}_\Gamma$ of marked manifolds.

By definition $\mathfrak{M}_\Gamma$ is the set $\{(X,\varphi)\}/\sim$. Here X is an irreducible symplectic manifold and $\varphi : H^2(X,\mathbb{Z}) \cong \Gamma$ is an isomorphism of lattices. Two marked manifolds (X,φ) and (X',φ') are equivalent if there exists an isomorphism $g : X \cong X'$ with $g^* = \pm(\varphi^{-1}\circ\varphi')$. That this set can be given the structure of a smooth, although non-separated, manifold is a consequence of the unobstructedness of symplectic manifolds and the Local Torelli Theorem (see 7.7). Due to the existence of automorphisms $g : X \cong X$ with $g^* = \mathrm{id}$ the local Kuranishi families can in general not be glued to a universal family over $\mathfrak{M}_\Gamma$ (cf. 7.8). The period map $\mathcal{P} : \mathfrak{M}_\Gamma \to Q \subset \mathbb{P}(\Gamma_\mathbb{C})$ exhibits $\mathfrak{M}_\Gamma$ as a space étale over of the period domain Q.

The above speculation (using 10.7 and 14.1) is implied by:

Speculation 16.1' — *If $\mathfrak{M}_\Gamma \neq \emptyset$ then the fibre over a general point $x \in Q$ is exactly one point.*

Note also that once the first speculation is answered positively it can be used to generalize Theorem 10.12 to the effect that any two birational irreducible symplectic manifolds (projective or not) correspond to non-separated points in the moduli space.

It is tempting to define another moduli space which does not distinguish between birational manifolds. Let $\mathfrak{N}_\Gamma$ be the set $\{(X,\varphi)\}/\approx$, where the (X,φ) are as above and $(X,\varphi)\approx(X',\varphi')$ if and only if there exists a birational map $g : X - - \to X'$ with $g^* = \pm(\varphi^{-1}\circ\varphi')$. Of course, there is a natural map $\mathfrak{M}_\Gamma \to \mathfrak{N}_\Gamma$. Theorem 10.12 proves that two points in the same fibre of this map, at least if they are projective, are contained in the same component of $\mathfrak{M}_\Gamma$. In fact, they are non-separated there. Thus, the general form of Theorem 10.12, i.e. without the projectivity assumption, would prove that the number of components of $\mathfrak{M}_\Gamma$ and $\mathfrak{N}_\Gamma$ is the same. Can $\mathfrak{N}_\Gamma$ be endowed with the structure of a manifold? If a birational map can always be extended to birational maps between all nearby fibres in the Kuranishi family, then local patching would supply $\mathfrak{N}_\Gamma$ with the structure of a manifold. Of course, if the Global Torelli Theorem in the formulation of the first speculation above holds true, then this would be trivial. Also note that the period map $\mathcal{P} : \mathfrak{M}_\Gamma \to Q$ factorizes through a map $\mathfrak{N}_\Gamma \to Q$.

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