Fourier–Mukai transforms

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Garda 2, March 2008

Serre functor

$\mathcal{A} = \mathbb{C}$-linear category with $\dim \mathrm{Hom}(A, B) < \infty$.

Serre functor: $\mathbb{C}$-linear equivalence $S : \mathcal{A} \longrightarrow \mathcal{A}$ st.

$$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, S(A))^*$$

functorial in $A, B \in \mathcal{A}$.

Facts:

- If S exists, then S is unique.
- Any equivalence is compatible with Serre functors.
- A Serre functor on a triangulated category is exact.

Geometric

Serre duality: For X smooth projective of dimension n is

$$S(E) := E \otimes \omega_X[n]$$

is a Serre functor on $\mathrm{D}^{\mathrm{b}}(X)$. As a special case, one has the classical Serre duality

$$\mathrm{Ext}^i(F, \omega_X) \simeq H^{n-i}(X, F)^*.$$

Fact: If $F : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X')$ is a $\mathbb{C}$-linear equivalence, then

$$\dim(X) = \dim(X') \ \text{ and } \ \omega_X \simeq \mathcal{O}_X \ \Leftrightarrow \ \omega_{X'} \simeq \mathcal{O}_{X'}.$$

Corollary: $X =$ K3 surface $\Rightarrow X' =$ K3 (or abelian) surface.

$X, X' =$ smooth projective varieties $/\mathbb{C}$ and $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X')$.
The *Fourier–Mukai transform* $\Phi_{\mathcal{E}}$ with *Fourier–Mukai kernel* $\mathcal{E}$ is the composition $p_* \circ (\mathcal{E} \otimes (\quad)) \circ q^*$, i.e. the functor

$$\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X'), \quad F \longmapsto p_*(\mathcal{E} \otimes q^* F)$$

$$\begin{array}{ccc} & X \times X' & \\ {}^{q}\swarrow & & \searrow^{p} \\ X & & X' \end{array}$$

Remark: As one could view $\mathcal{E}$ also as an object on $X' \times X$ and hence define $\mathrm{D}^{\mathrm{b}}(X') \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$ one sometimes writes $\Phi_{\mathcal{E}}^{X \to X'} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X')$ to indicate the direction.

Clear: $\Phi_{\mathcal{E}}$ is $\mathbb{C}$-linear and exact.

Example: Let $X = X'$ and $\mathcal{E} := \mathcal{O}_{\Delta}[n]$. Then $\Phi_{\mathcal{E}}$ is the shift functor $F \longmapsto F[n]$.

Orlov's result

Orlov: Suppose $\Phi : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X')$ is a fully faithful exact $\mathbb{C}$-linear functor (e.g. Φ an equivalence). Then there exists $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X')$ unique up to isomorphism such that $\Phi \simeq \Phi_{\mathcal{E}}$.

Remarks: i) Originally, Φ was assumed to have left and right adjoint. Automatic! Due to Bondal, van den Bergh: $\mathrm{D}^{\mathrm{b}}(X)$ is saturated, i.e. every contravariant cohomological functor of finite type is representable.
ii) The same results holds in other situations: smooth quotient stacks (Kawamata), twisted varieties (Canonaco, Stellari). The assumption 'fully faithful' can be weakened.
iii) It is generally(?) believed that any exact functor is of Fourier–Mukai type.

Adjoints

Grothendieck duality: For $f : X \longrightarrow Y$ between smooth projective varietes one defines $\omega_f := \omega_X \otimes f^*\omega_Y^*$. Then

$$f_*\mathcal{H}om(F, f^*E \otimes \omega_f[\dim f]) \simeq \mathcal{H}om(f_*F, E)$$

functorial in $E \in \mathrm{D^b}(Y)$ and $F \in \mathrm{D^b}(X)$.

For $\mathcal{E} \in \mathrm{D^b}(X \times X') \rightsquigarrow \Phi_{\mathcal{E}} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X')$.

$$\mathcal{E}_{\mathrm{L}} := \mathcal{E}^\vee \otimes p^*\omega_{X'}[\dim X'] \qquad \mathcal{E}_{\mathrm{R}} := \mathcal{E}^\vee \otimes q^*\omega_X[\dim X].$$

Mukai: $\Phi_{\mathcal{E}_{\mathrm{L}}} : \mathrm{D^b}(X') \longrightarrow \mathrm{D^b}(X)$ and $\Phi_{\mathcal{E}_{\mathrm{R}}} : \mathrm{D^b}(X') \longrightarrow \mathrm{D^b}(X)$ are left resp. right adjoint to $\Phi_{\mathcal{E}}$: $\mathrm{Hom}(\Phi_{\mathcal{E}_{\mathrm{L}}}(E), F) \simeq \mathrm{Hom}(E, \Phi_{\mathcal{E}}(F))$.

Remark: Note $\Phi_{\mathcal{E}_{\mathrm{R}}} \simeq S_X \circ \Phi_{\mathcal{E}_{\mathrm{L}}} \circ S_{X'}^{-1}$.

Corollary: If $\Phi_{\mathcal{E}}$ is an equivalence, then $\mathcal{E}_{\mathrm{R}} \simeq \mathcal{E}_L$.

Composition

Consider $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X')$ and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X' \times X'')$ and the induced

$$\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X') \qquad \Phi_{\mathcal{F}} : \mathrm{D}^{\mathrm{b}}(X') \longrightarrow \mathrm{D}^{\mathrm{b}}(X'').$$

Convolution: $\mathcal{E} * \mathcal{F} \in \mathrm{D}^{\mathrm{b}}(X \times X'')$ is $\pi_{XX''*}(\pi_{XX'}^{*}\mathcal{E} \otimes \pi_{X'X''}^{*}\mathcal{F})$ where $\pi_{XX'} : X \times X' \times X'' \longrightarrow X \times X'$ etc.

Mukai: $\Phi_{\mathcal{F}} \circ \Phi_{\mathcal{E}} \simeq \Phi_{\mathcal{E}*\mathcal{F}}$.

Corollary: $\Phi_{\mathcal{E}}$ equivalence $\Leftrightarrow \mathcal{E} * \mathcal{E}_{\mathrm{R}} \simeq \mathcal{E} * \mathcal{E}_{\mathrm{L}} \simeq \mathcal{O}_{\Delta_{X'}}$ and $\mathcal{E}_{\mathrm{R}} * \mathcal{E} \simeq \mathcal{E}_{\mathrm{L}} * \mathcal{E} \simeq \mathcal{O}_{\Delta_{X}}$.

Exercises: i) $\Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X') \Leftrightarrow \Phi_{\mathcal{E}} : \mathrm{D}^{\mathrm{b}}(X') \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(X)$.
ii) $\mathcal{O}_{\Delta}^{\vee} \simeq \mathcal{O}_{\Delta}[-\dim X] \otimes p^{*}\omega_{X}^{*}$

Spanning class

$\mathcal{D} = \mathbb{C}$-linear triangulated category.

Spanning class: $\Omega \subset \mathrm{Ob}(\mathcal{D})$ such that

- $\mathrm{Hom}(A, B[i]) = 0$ for all $A \in \Omega$, $i \in \mathbb{Z} \Rightarrow B \simeq 0$.
- $\mathrm{Hom}(B, A[i]) = 0$ for all $A \in \Omega$, $i \in \mathbb{Z} \Rightarrow B \simeq 0$.

Remark: Weaker notion than '(split) generating'.

Examples:

- $\{k(x) \mid x \in X\}$.
- $\{L^i \mid i \in \mathbb{Z}\}$, where $L \in \mathrm{Pic}(X)$ ample ('split generating').
- $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \in \mathrm{Pic}(\mathbb{P}^n)$ ('full exceptional').
- $\Omega := \{E\} \cup \{E\}^\perp$ with $E \in \mathrm{D^b}(X)$ arbitrary.
  Here $\{E\}^\perp := \{F \mid \mathrm{Hom}(E, F[i]) = 0 \ \forall i\}$.

FF via spanning classes

Formal: Suppose $\Omega \subset \mathrm{D^b}(X)$ is a spanning class and $\Phi : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X')$ an exact functor with left and right adjoints (eg. FM transform). Then Φ is fully faithful $\Leftrightarrow \forall A, B \in \Omega,\ i \in \mathbb{Z}$:

$$\mathrm{Hom}(A, B[i]) \simeq \mathrm{Hom}(\Phi(A), \Phi(B)[i]).$$

For the spanning class $\{k(x)\}$ one has the stronger version:

Bondal, Orlov: A FM transform $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X')$ is fully faithful $\Leftrightarrow \forall x, y \in X$:

$$\mathrm{Hom}(\Phi_{\mathcal{E}}(k(x)), \Phi_{\mathcal{E}}(k(y))[i]) = \begin{cases} k & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ or } i \notin [0, \dim(X)]. \end{cases}$$

Formal: Suppose $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X')$ is fully faithful.
Then $\Phi_{\mathcal{E}}$ is an equivalence $\Leftrightarrow$

$$\dim X = \dim X' \ \text{ and } \ \mathcal{E} \otimes q^*\omega_X \simeq \mathcal{E} \otimes p^*\omega_{X'}.$$

Bridgeland: Suppose $\Phi_{\mathcal{E}} : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X')$ is fully faithful.
Then Φ is an equivalence $\Leftrightarrow \forall x \in X$:

$$\Phi_{\mathcal{E}}(k(x)) \otimes \omega_{X'} \simeq \Phi_{\mathcal{E}}(k(x)).$$

Spherical objects

Spherical object: $E \in \mathrm{D^b}(X)$ with $\mathrm{Ext}^*(E,E) \simeq H^*(S^n, \mathbb{C})$ and $E \otimes \omega_X \simeq E$.

Remarks: i) By Serre duality: $n = \dim X$.
ii) Second condition is void for $\omega_X \simeq \mathcal{O}_X$.

Examples: i) $L \in \mathrm{Pic}(X)$ where $X =$ K3 surface.

$$\mathrm{Ext}^*(L,L) \simeq H^*(X, \mathcal{O}_X) \simeq H^*(S^2, \mathbb{C}) \simeq H^*(\mathbb{P}^1, \mathbb{C}).$$

ii) $\mathcal{O}_C(i)$, $i \in \mathbb{Z}$, where $X =$ K3 surface and $\mathbb{P}^1 \simeq C \subset X$.
iii) $\mathcal{O}_C(i)$, $i \in \mathbb{Z}$, $X =$ CY threefold and $\mathbb{P}^1 \simeq C \subset X$ with $\mathcal{N}_{C/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

For any $E \in \mathrm{D}^{\mathrm{b}}(X) \rightsquigarrow$ Define

$$t : E^\vee \boxtimes E \longrightarrow (E^\vee \boxtimes E)|_\Delta = \iota_*(E^\vee \otimes E) \xrightarrow{\mathrm{tr}} \mathcal{O}_\Delta$$

and

$$\mathcal{P}_E := \mathrm{C}\left(t : \ E^\vee \boxtimes E \longrightarrow \mathcal{O}_\Delta \right).$$

Spherical twist: associated to spherical object $E \in \mathrm{D}^{\mathrm{b}}(X)$:

$$T_E := \Phi_{\mathcal{P}_E} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X).$$

Then $T_E(E) \simeq E[1 - \dim X]$ and $T_E(F) \simeq F$ for $F \in \{E\}^\perp$.

Seidel, Thomas: T_E is an equivalence.
i) Fully faithful: Consider spanning class $\Omega := \{E\} \cup \{E\}^\perp$.
ii) Equivalence: Use

$$\mathcal{P}_E \otimes q^*\omega_X \simeq \mathrm{C}(E^\vee \boxtimes E \longrightarrow \iota_*\omega_X) \simeq \mathcal{P}_E \otimes p^*\omega_X.$$

$\mathbb{P}$-twists

Spherical twists work well for CYs and in dimension two where HK=CY.

$\mathbb{P}$-twists are the HK analogues of spherical twists.

$\mathbb{P}$-**object:** $E \in \mathrm{D^b}(X)$ with $\mathrm{Ext}^*(E, E) \simeq H^*(\mathbb{P}^n, \mathbb{C})$ and $E \otimes \omega_X \simeq E$.

Remarks: i) By Serre duality: $2n = \dim X$.
ii) Second condition is void for $\omega_X \simeq \mathcal{O}_X$.

Examples: i) $L \in \mathrm{Pic}(X)$, where $X =$HK.
ii) $\mathcal{O}_P(i)$, where $X =$ HK and $\mathbb{P}^n \simeq P \subset X$, $2n = \dim X$.

Define for $\mathbb{P}$-object $E \in \mathrm{D}^{\mathrm{b}}(X)$ the FM kernel

$$\mathcal{Q}_E := \mathrm{C}\left(\mathrm{C}\left(E^\vee \boxtimes E[-2] \longrightarrow E^\vee \boxtimes E\right) \xrightarrow{\ \mathrm{t}\ } \mathcal{O}_\Delta\right),$$

where $E^\vee \boxtimes E[-2] \longrightarrow E^\vee \boxtimes E$ is $h^\vee \boxtimes 1 - 1 \boxtimes h$ with $\mathbb{C}h^\vee = \mathrm{Ext}^2(E^\vee, E^\vee) \simeq \mathrm{Ext}^2(E, E) = \mathbb{C}h$. Show: t exists!

$\mathbb{P}$-**twist:** associated to $\mathbb{P}$-object $E \in \mathrm{D}^{\mathrm{b}}(X)$:

$$P_E = \Phi_{\mathcal{Q}_E} : \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X).$$

Then $P_E(E) \simeq E[-2n]$ and $P_E(F) = F$ for $F \in \{E\}^\perp$.

H., Thomas: P_E is an equivalence.

Same proof.

Comparison

$\dim(X) = 2$: $E \in \mathrm{D}^{\mathrm{b}}(X)$:

- E is $\mathbb{P}$-object $\Leftrightarrow$ E is spherical.
- In this case: $T_E^2 = P_E$.

$\dim(X) > 2$: $E \in \mathrm{D}^{\mathrm{b}}(X)$, $\mathrm{rk}(E) := \sum (-1)^i \mathrm{rk}(E^i)$. Then

$$\left(\mathcal{O}_X \xrightarrow{\mathrm{id}} E^\vee \otimes E \xrightarrow{\mathrm{tr}} \mathcal{O}_X \right) = \mathrm{rk}(E) \cdot \mathrm{id}.$$

- If $\mathrm{rk}(E) \neq 0$, then $\mathrm{Ext}^i(E,E) = H^i(X, E^\vee \otimes E) = H^i(X, \mathcal{O}_X) \oplus H^i(X, (E^\vee \otimes E)_0)$.
- If X symplectic, then $H^2(X, \mathcal{O}_X) \neq 0$. Hence, there are no spherical objects with $\mathrm{rk}(E) \neq 0$.