Generalized K3 surfaces

D. Huybrechts

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Rewriting K3s

Write a K3 surface X as (M, I) with $M =$ oriented differentiable manifold and $I \in \mathrm{End}(TM)$ a complex structure.
Then a holomorphic two-form $0 \neq \sigma \in H^0(X, \Omega_X^2)$ viewed as $\sigma \in \mathcal{A}^2(M)_{\mathbb{C}}$ satisfies:

i) $\sigma \wedge \sigma \equiv 0$.
ii) $\sigma \wedge \bar{\sigma} > 0$.
iii) $d\sigma = 0$.

Calabi–Yau structure

Converse (Andreotti): Suppose $\sigma \in \mathcal{A}^2(M)_{\mathbb{C}}$ satisfies i) - iii). Then there exists a unique complex structure I such that $X := (M, I)$ is a K3 surface with $\sigma \in H^0(X, \Omega_X^2)$.

i) $\sigma \wedge \sigma \equiv 0$, ii) $\sigma \wedge \bar{\sigma} > 0$, iii) $d\sigma = 0$

Idea: σ induces $\sigma, \bar{\sigma} : TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}^*$. Define

$$T^{0,1} := \mathrm{Ker}(\sigma) \ \text{ and } \ T^{1,0} := \mathrm{Ker}(\bar{\sigma}) = \overline{T^{0,1}}.$$

i) $\Rightarrow T^{1,0} \neq 0$, ii) $\Rightarrow T^{1,0} \cap T^{0,1} = 0$,
σ alternating $\Rightarrow \dim T^{0,1} \equiv 0(2)$.

Hence

$$TM_{\mathbb{C}} = T^{1,0} \oplus T^{0,1} \quad \rightsquigarrow \quad I(v) := iv^{1,0} - iv^{0,1}$$

defines an almost complex structure I.

iii) $\Rightarrow I$ is integrable.

Remark: σ and $\lambda\sigma$ define the same I ($\lambda \in \mathbb{C}^*$).

$\sigma \in \mathcal{A}^2(M)_{\mathbb{C}} \rightsquigarrow \varphi \in \mathcal{A}^{2*}(M)_{\mathbb{C}}$

$M =$ oriented differentiable manifold underlying a K3 surface. The *Mukai pairing for even forms* $\varphi = \varphi_0 + \varphi_2 + \varphi_4 \in \mathcal{A}^{2*}(M)_{\mathbb{C}}$ is:

$$\langle \varphi, \psi \rangle := \varphi_2 \wedge \psi_2 - \varphi_0 \wedge \psi_4 - \varphi_4 \wedge \psi_0 \in \mathcal{A}^4(M)_{\mathbb{C}}.$$

Definition (Hitchin): A *generalized Calabi–Yau structure* on M is a $\varphi \in \mathcal{A}^{2*}(M)_{\mathbb{C}}$ with

i) $\langle \varphi, \varphi \rangle \equiv 0$.

ii) $\langle \varphi, \bar{\varphi} \rangle > 0$.

iii) $d\varphi = 0$ (i.e. $\varphi_0 \in \mathbb{C}$ and $d\varphi_2 = 0$).

Examples:

- $\sigma =$ holomorphic two-form on K3 $X = (M, I) \rightsquigarrow \varphi := \sigma$.
- $\omega \in \mathcal{A}^2(M)$ symplectic $\rightsquigarrow \varphi := \exp(i\omega) := 1 + i\omega - \omega^2/2$.

B-field transforms

A closed real form $B \in \mathcal{A}^2(M)$ is called a *B-field*.

Easy: Multiplication with $\exp(B) := 1 + B + B^2/2$ is orthogonal, i.e.

$$\langle \varphi, \psi \rangle \equiv \langle \exp(B) \cdot \varphi, \exp(B) \cdot \psi \rangle.$$

Lemma: If $\varphi \in \mathcal{A}^2(M)_{\mathbb{C}}$ is a generalized CY structure, then the *B-field transform* $\exp(B) \cdot \varphi$ is one too.

Examples:

- $\exp(B) \cdot \sigma = \sigma + B \wedge \sigma = \sigma + B^{0,2} \wedge \sigma$.
- $\exp(B) \cdot \exp(i\omega) = \exp(B + i\omega) = 1 + B + i\omega + \frac{B^2 - \omega^2}{2} + iB \wedge \omega$.

Classification of generalized CY structures

Hitchin: Any generalized CY structure φ is of the form $\exp(B) \cdot \sigma$ with σ a holomorphic two-form or $\varphi_0 \cdot \exp(B + i\omega)$ with ω real symplectic and a B-field B.

Proof: If $\varphi_0 = 0$, then φ_2 satisfies i)-iii). Let $\sigma := \varphi_2$ and choose B such that $B \wedge \sigma = \varphi_4$.
If $\varphi_0 \neq 0$, let $\omega := \mathrm{Im}(\varphi_0^{-1} \cdot \varphi_2)$ and $B := \mathrm{Re}(\varphi_0^{-1} \cdot \varphi_2)$.
Check that:
i) $\Rightarrow \varphi_0^{-1} \cdot \varphi = \exp(B + i\omega)$ and
ii) $\Rightarrow \omega$ symplectic.

$\varphi \in \mathcal{A}^{2*}(M)_{\mathbb{C}} \rightsquigarrow [\varphi] \in H^*(M, \mathbb{C})$

Define the *Mukai pairing* by

$$\langle \varphi, \psi \rangle := \int (\varphi_2 \wedge \psi_2 - \varphi_0 \wedge \psi_4 - \varphi_4 \wedge \psi_0)$$

on $H^*(M, \mathbb{Z})$. Thus $\langle [\varphi], [\psi] \rangle = \int \langle \varphi, \psi \rangle$.

On $H^2(M, \mathbb{Z})$: standard intersection pairing.
On $H^0(M, \mathbb{Z}) \oplus H^4(M, \mathbb{Z})$: minus standard intersection pairing.

Write $\widetilde{H}(M, \mathbb{Z}) := (H^*(M, \mathbb{Z}), \langle\ ,\ \rangle)$. Thus abstractly

$$\widetilde{H}(M, \mathbb{Z}) \simeq 2(-E_8) \oplus 4U.$$

$[\varphi] \in H^*(M,\mathbb{C}) \rightsquigarrow$ Hodge structure

For $[\varphi] \in H^*(M,\mathbb{C})$ of a generalized CY structure $\varphi \in \mathcal{A}^{2*}(M)_{\mathbb{C}}$:

$$\text{i)}\ \ \langle[\varphi],[\varphi]\rangle = 0 \ \ \text{and} \ \ \text{ii)}\ \ \langle[\varphi],\overline{[\varphi]}\rangle > 0.$$

Define a weight two Hodge structure $\widetilde{H}(M,\varphi,\mathbb{Z})$ on $\widetilde{H}(M,\mathbb{Z})$ by

$$\widetilde{H}^{2,0}(M,\varphi) := \mathbb{C}[\varphi] \subset H^*(M,\mathbb{C}),$$
$$\widetilde{H}^{0,2}(M,\varphi) := \overline{H^{2,0}(M,\varphi)} = \mathbb{C}[\bar{\varphi}],$$

and

$$\widetilde{H}^{1,1}(M,\varphi) := (H^{2,0} \oplus H^{0,2})(M,\varphi)^{\perp}.$$

Example: $\varphi = \sigma$ holomorphic two-form on $X = (M, I)$, then

$$\widetilde{H}^{2,0}(M,\varphi) = H^{2,0}(X)$$

and

$$\widetilde{H}^{1,1}(M,\varphi) = H^{1,1}(X) \oplus H^0(X) \oplus H^4(X).$$

X = K3 surface

Brauer group:

$$\mathrm{Br}(X) := H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}.$$

Exponential sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$ yields

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow H^2(X,\mathcal{O}_X^*) \longrightarrow 0.$$

For $\alpha \in \mathrm{Br}(X)$ pick $B \in H^2(X,\mathbb{R})$ such that $\alpha = \alpha_B := e^{2\pi i B^{0,2}}$. Define the integral(!) weight two Hodge structure of the *twisted K3 surface* (X,α) as:

$$\widetilde{H}((X,\alpha),\mathbb{Z}) := \widetilde{H}(M, \exp(B)\cdot\sigma, \mathbb{Z}).$$

Lemma: The isomorphism type of $\widetilde{H}((X,\alpha),\mathbb{Z})$ is independent of the choice of B.

M as before

The *moduli space of generalized CY structures*:

$$\widetilde{\mathfrak{N}} := \{\mathbb{C}\varphi \mid \varphi \text{ generalized CY structure}\}/\simeq,$$

where $\varphi \simeq \varphi'$ if there exists $f \in \mathrm{Diff}_*(M)$ (i.e. $f^* = \mathrm{id}$ on $H^*(M, \mathbb{Z})$) and $B \in \mathcal{A}^2(M)$ exact such that $\varphi = \exp(B) \cdot f^*\varphi'$.

Classically

$$\mathfrak{N} := \{\mathbb{C}\sigma \mid \sigma \text{ CY structure}\}/\mathrm{Diff}_*(M).$$

Then: $\mathfrak{N} \subset \widetilde{\mathfrak{N}}$ and $\mathfrak{N} = \mathfrak{M} =$ moduli space of marked K3 surfaces (one connected component).

$\Gamma := H^2(M,\mathbb{Z})$, $\widetilde{\Gamma} := \widetilde{H}(M,\mathbb{Z})$

Classical period domain

$$Q := \{x \mid (x,x) = 0,\ (x,\bar{x}) > 0\} \subset \mathbb{P}(\Gamma_{\mathbb{C}})$$

and period map $\mathcal{P} : \mathfrak{N} \longrightarrow Q$, $\sigma \longmapsto [\mathbb{C}\sigma]$.
Generalized period domain

$$\widetilde{Q} := \{x \mid \langle x,x \rangle = 0,\ \langle x,\bar{x} \rangle > 0\} \subset \mathbb{P}(\widetilde{\Gamma}_{\mathbb{C}})$$

and period map $\widetilde{\mathcal{P}} : \widetilde{\mathfrak{N}} \longrightarrow \widetilde{Q}$, $\varphi \longmapsto [\mathbb{C}\varphi]$.
Then

$$\begin{array}{ccccc} \mathfrak{N} & \xrightarrow{\mathcal{P}} & Q & \hookrightarrow & \mathbb{P}(\Gamma_{\mathbb{C}}) \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathfrak{N}} & \xrightarrow[\widetilde{\mathcal{P}}]{} & \widetilde{Q} & \hookrightarrow & \mathbb{P}(\widetilde{\Gamma}_{\mathbb{C}}). \end{array}$$

Torelli etc.

Moser: $\widetilde{\mathcal{P}} : \widetilde{\mathfrak{N}} \setminus \mathfrak{N} \longrightarrow \widetilde{Q}$ is a local homeomorphism.

Andreotti: $\mathcal{P} : \mathfrak{N} \longrightarrow Q$ is a local homeomorphism.

Open: Is $\widetilde{\mathfrak{N}}$ connected?

Surjectivity: $\widetilde{\mathcal{P}}$ is surjective.

Restrict in $\widetilde{\mathfrak{N}}$ to symplectic forms which are hyperkähler with respect to at least one complex structure. Then

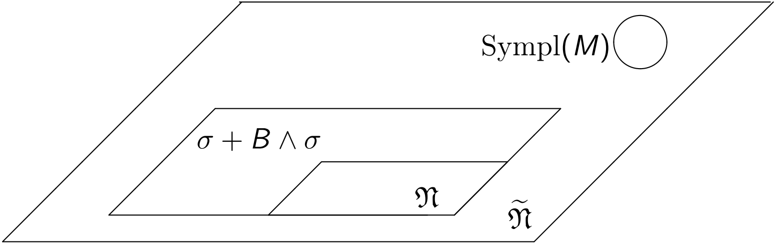
Global Torelli:

i) $\widetilde{\mathcal{P}} : \widetilde{\mathfrak{N}} \longrightarrow \widetilde{Q}$ is an isomorphism over $\widetilde{Q} \setminus \exp(H^2(M, \mathbb{R})) \cdot Q$.

ii) $\mathcal{P} : \mathfrak{N} \longrightarrow Q$ bijective up to non-Hausdorff phenomena.

(i) uses ii))

Identify non separated points:



Observe:

$$t \exp\left(\frac{\mathrm{Re}(\sigma) + i\mathrm{Im}(\sigma)}{t}\right) \longrightarrow \sigma \quad \text{for} \quad t \longrightarrow 0.$$

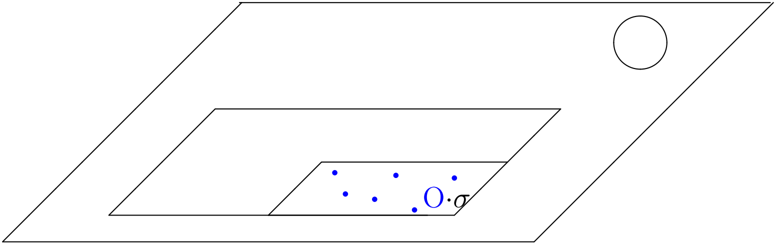
$\Gamma = H^2(M,\mathbb{Z})$, $\widetilde{\Gamma} = \widetilde{H}(M,\mathbb{Z})$

- $\mathrm{O} := \mathrm{O}(\Gamma)$-action on $\mathbb{P}(\Gamma_{\mathbb{C}})$ preserves Q.
- $\widetilde{\mathrm{O}} := \mathrm{O}(\widetilde{\Gamma})$-action on $\mathbb{P}(\widetilde{\Gamma}_{\mathbb{C}})$ preserves $\widetilde{Q}$.
- $\mathrm{O}(\widetilde{\Gamma})$-action on $\mathbb{P}(\widetilde{\Gamma}_{\mathbb{C}})$ does not preserve Q or $\exp(H^2(M,\mathbb{R}))\cdot Q$.

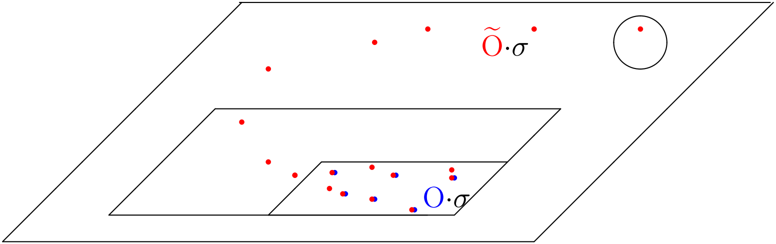
Explicitly: $\widetilde{\mathcal{P}}(\varphi) \in \widetilde{\mathrm{O}} \cdot \widetilde{\mathcal{P}}(\varphi')$
$\Leftrightarrow\ \exists g : \widetilde{H}(M,\mathbb{Z}) \simeq \widetilde{H}(M,\mathbb{Z})$ Hodge isometry.

Classically: $\mathcal{P}(\sigma) \in \mathrm{O} \cdot \mathcal{P}(\sigma')$
$\Leftrightarrow\ \exists g : H^2(X,\mathbb{Z}) \simeq H^2(X',\mathbb{Z})$ with $g(H^{2,0}(X)) = H^{2,0}(X')$.

O-orbits:



$\widetilde{\mathrm{O}}$-orbits



Global Torelli à la Weil:

- *Now it may happen that two classes S, S' may be distinct and still define isomorphic structures; this will be so when structures belonging to these classes can be transformed into one another by a differentiable homeomorphism... The latter will induce an automorphism of the homology group, and therefore a unit U of the quadratic form F, ie. a matrix with integral coefficients belonging to the orthogonal group of F.*
- *It seems plausible (but not easy to prove) that two such forms with the same periods must determine complex structures which can be transformed into one other by a differentiable homeomorphism,* **homotopic to the identity***;...*

i) OK, but ii) is open.

Open: Is $\mathrm{Diff}_0(M) \subset \mathrm{Diff}_*(M)$ equality?

The Global Torelli theorem for K3 surfaces is still not proven !