A. Weil: ... *il s'agit des variétés kählériennes dites K3, nommées en l'honneur de Kummer, Kähler, Kodaira, et de la belle montagne K3 au Cachemire.* Photograph <http://student.britannica.com/eb/art-55317>.

Holomorphic symplectic manifolds and derived categories

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Abelian and derived category for $X =$ smooth projective $/\mathbb{C}$

$\rightsquigarrow \mathrm{Coh}(X) =$ abelian category of coherent sheaves.

$\rightsquigarrow \mathrm{D^b}(X) := \mathrm{D^b}(\mathrm{Coh}(X)) =$ bounded derived category of X.

Objects: Bounded complexes of coherent sheaves:

$$\cdots 0 \longrightarrow F^i \longrightarrow F^{i+1} \longrightarrow \cdots \longrightarrow F^{j-1} \longrightarrow F^j \longrightarrow 0 \cdots$$

Morphisms: Morphisms of complexes

$$\begin{array}{ccccccccccccc} \cdots 0 & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & \cdots & \longrightarrow & F^{j-1} & \longrightarrow & F^j & \longrightarrow & 0 \cdots \\ & & \downarrow & \circlearrowleft & \downarrow & & & & \downarrow & \circlearrowleft & \downarrow & & \\ \cdots 0 & \longrightarrow & G^i & \longrightarrow & G^{i+1} & \longrightarrow & \cdots & \longrightarrow & G^{j-1} & \longrightarrow & G^j & \longrightarrow & 0 \cdots \end{array}$$

Add formal inverses of quasi-isomorphisms, i.e. morphisms inducing isomorphisms $H^i(F^\bullet) \xrightarrow{\sim} H^i(G^\bullet)$ for all i are declared isomorphisms in $\mathrm{D^b}(X)$.

For $F, G \in \mathrm{Coh}(X)$ one considers $\mathrm{Hom}(F, G)$ as a $\mathbb{C}$-vector space.
$\rightsquigarrow$ $\mathrm{Coh}(X)$ is a $\mathbb{C}$-linear abelian category.
$\rightsquigarrow$ $\mathrm{D^b}(X)$ is a $\mathbb{C}$-linear additive category.

$\mathrm{D^b}(X)$ has two further structures:

- Shift of complexes: $F^\bullet \longmapsto F^\bullet[1]$, where $(F^\bullet[1])^i = F^{i+1}$.
- Distinguished (or exact) triangles

$$E^\bullet \longrightarrow F^\bullet \longrightarrow G^\bullet \longrightarrow E^\bullet[1]$$

replacing short exact sequences in $\mathrm{Coh}(X)$.

This makes $\mathrm{D^b}(X)$ a $\mathbb{C}$-*linear triangulated category*. Equivalences will always assumed $\mathbb{C}$-linear and exact, i.e. to respect all the additional structures.

Main questions:

i) $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Leftrightarrow$?? and ii) $\mathrm{Aut}(\mathrm{D^b}(X)) = \{\Phi\}/\sim =$??

Orlov: Any $\mathbb{C}$-linear exact equivalence $\Phi : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X')$ is of *Fourier–Mukai type*, i.e. there exists $\mathcal{E} \in \mathrm{D^b}(X \times X')$ such that

$$\Phi(F^\bullet) \simeq Rp_*(Lq^* F^\bullet \otimes^L \mathcal{E})$$

in a functorial way. Here, $X \xleftarrow{q} X \times X' \xrightarrow{p} X'$.

Roughly: $\mathcal{E}$ defines an equivalence if X' is a very special 'moduli space' of the objects $\mathcal{E}|_{X \times \{y\}}$ on X. (Make this precise!)

i) Which varieties X' are special moduli spaces of complexes on X?
ii) In how many ways can X be seen as a special moduli space of complexes on itself?

Classical case: abelian varieties

A = abelian variety $\rightsquigarrow \hat{\mathrm{A}}$ = dual abelian variety, i.e.

$$\hat{\mathrm{A}} = \mathrm{Pic}^0(\mathrm{A}) = H^1(\mathrm{A}, \mathcal{O}_\mathrm{A})/H^1(\mathrm{A}, \mathbb{Z}).$$

$\mathcal{P}$ = Poincaré line bundle on $\mathrm{A} \times \hat{\mathrm{A}}$ such that $\mathcal{P}|_{\mathrm{A}\times\{[L]\}} \simeq L$.

Mukai: $\mathcal{P} \in \mathrm{D^b}(\mathrm{A} \times \hat{\mathrm{A}})$ induces $\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(\hat{\mathrm{A}})$.

Note: If $\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(X)$, then X is an abelian variety.

Orlov, Polishchuk:

- $\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(\mathrm{B}) \Leftrightarrow \mathrm{A} \times \hat{\mathrm{A}} \simeq \mathrm{B} \times \hat{\mathrm{B}}$ isometry.
- $\mathrm{Aut}(\mathrm{D^b}(\mathrm{A}))/(\mathbb{Z} \times \mathrm{A} \times \hat{\mathrm{A}}) = \mathrm{Aut}(\mathrm{A} \times \hat{\mathrm{A}}, q_\mathrm{A})$

$n \in \mathbb{Z}$ acts by shift $F^\bullet \longmapsto F^\bullet[n]$.
$x \in \mathrm{A}$ acts by translations $F^\bullet \longmapsto t_x^* F^\bullet$.
$L \in \hat{\mathrm{A}}$ acts by tensor product $F^\bullet \longmapsto F^\bullet \otimes L$.

Extreme cases: Ample $K_X^{\pm}$

Bondal, Orlov: Suppose $K_X^{\pm}$ is ample. Then

- $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Leftrightarrow X \simeq X'$
- $\mathrm{Aut}(\mathrm{D^b}(X)) \simeq \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X))$

Orlov: $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Rightarrow \mathrm{kod}(X) = \mathrm{kod}(X')$.

Recall: $\mathrm{kod}(X, K_X) := \mathrm{trdeg} \bigoplus_{n \geq 0} H^0(X, K_X^n) - 1$.

Kawamata: Suppose $\mathrm{kod}(X, K_X^{\pm}) = \dim(X)$. Then $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X') \Rightarrow X, X'$ birational (K-equivalent).

Restrict to varieties with trivial K_X!

$K_X \equiv 0$

Fact: The class of varieties X with $K_X \equiv 0$ is invariant under derived equivalences. More precisely, if $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$ and $K_X^n \simeq \mathcal{O}_X$, then $K_{X'}^n \simeq \mathcal{O}_{X'}$.

Question: Can one classify varieties X with $K_X \equiv 0$?

Decomposition theorem: Suppose $0 = \mathrm{c}_1(X) \in H^2(X, \mathbb{R})$. Then $\exists\ \widetilde{X} \longrightarrow X$ finite, étale (minimal) such that

$$\widetilde{X} \simeq \mathrm{A} \times \prod X_i \times \prod Y_j$$

with A = abelian variety; X_i = hyperkähler manifold (HK); Y_j = Calabi–Yau manifold (CY).

Remark: Works as well for compact Kähler manifolds.

HK and CY: Definitions

- $X =$ **HK** if X is projective, $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$ with $\sigma : \mathcal{T}_X \xrightarrow{\sim} \Omega_X$ and $\pi_1(X) = \{1\}$.
- $Y =$ **CY** if $K_Y \simeq \mathcal{O}_Y$, $H^0(Y, \Omega_Y^i) = 0$ for $0 < i < \dim(Y) \geq 3$ and $\pi_1(Y) = \{1\}$.

Remarks: i) For HK X the holomorphic form σ is called a *holomorphic symplectic structure*.
ii) HK = *irreducible holomorphic symplectic manifold*. Here 'irreducible' = '$\pi_1 = \{1\}$' + '$h^{2,0}(X) = 1$'.
iii) If $X =$ HK, then any Kähler class $\omega \in H^{1,1}(X)$ is uniquely represented by a special Ricci-flat Kähler form (hyperkähler metric).

Examples of HKs

i) dim $= 2$: K3 surfaces.
(By definition: K3 $=$ compact complex surface with $K_X \simeq \mathcal{O}_X$ and $b_1(X) = 0$. 'Kähler' and '$\pi_1(X) = \{1\}$' are automatic.)

ii) Hilbert schemes: $\mathrm{Hilb}^n(S)$ for $S =$ K3 surface and deformations thereof.

iii) Generalized Kummer varieties and their deformations: $\mathrm{K}^n(\mathrm{A})$ for $\mathrm{A} =$ abelian surface. Fibre of $\sum : \mathrm{Hilb}^n(\mathrm{A}) \longrightarrow \mathrm{A}$.

iv) Compact moduli spaces of stable sheaves on K3 surfaces. (Deformations of ii).) Similar for abelian surfaces.

iv) O'Grady's sporadic examples in dimensions 6 and 10. (Resolutions of singular moduli spaces. 'Sporadic' due to Kaledin/Lehn/Sorger and Choy/Kiem.)

How to study HKs?

Slogan:

- Complex tori are ruled by H^1.
- HKs are ruled by H^2, and
- CY n-folds by H^n.

$X = \mathrm{HK} \rightsquigarrow H^2(X,\mathbb{Z}) \simeq \mathbb{Z}^{b_2(X)}$ comes with the Hodge structure of weight two:

$$H^2(X,\mathbb{C}) \simeq H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

with $H^{2,0}(X) = H^0(X,\Omega_X^2) = \mathbb{C}\sigma$, $H^{0,2}(X) = \overline{H^{2,0}(X)} = \mathbb{C}\bar{\sigma}$.

Note: $\mathrm{Pic}(X) = H^{1,1}(X) \cap H^2(X,\mathbb{Z}) \simeq \mathbb{Z}^{\rho}$ with $0 \leq \rho \leq b_2(X) - 2$ (everything possible).

Quadratic forms

K3 surfaces:

- Intersection pairing $(\ ,\ )$ yields even unimodular quadratic form on $H^2(X,\mathbb{Z})$ abstractly isomorphic to $(-E_8)^{\bigoplus 2} \oplus U^{\bigoplus 3}$.
- *Mukai pairing* $\langle\ ,\ \rangle$ is an even unimodular quadratic form on $H^*(X,\mathbb{Z})$ abstractly isomorphic to $(-E_8)^{\bigoplus 2} \oplus U^{\bigoplus 4}$ given by $(\ ,\ )|_H^2 \oplus -(\ ,\ )|_{H^0 \oplus H^4}$. Write $\widetilde{H}(X,\mathbb{Z})$.

HKs:

- *Beauville–Bogomolov–Fujiki form* q_X on $H^2(X,\mathbb{Z})$ is a quadratic form of signature $(3, b_2(X)-3)$ which is a root of $\alpha \longmapsto \int_X \alpha^{2n}$.
-

H^2 and cones

K3 surfaces: Let $\mathcal{C}_X := \{\alpha \in H^{1,1}(X, \mathbb{R}) \mid \alpha^2 > 0\}^o$ and $\mathcal{K}_X \subset \mathcal{C}_X$ Kähler cone. Then

$$\mathcal{K}_X = \{\alpha \in \mathcal{C}_X \mid \int_C \alpha > 0 \ \forall\ \mathbb{P}^1 \simeq C \subset X\}.$$

HK: Let $\mathcal{C}_X := \{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0\}$ and $\mathcal{K}_X \subset \mathcal{C}_X$ Kähler cone. Then

- $\mathcal{K}_X = \{\alpha \in \mathcal{C}_X \mid \int_C \alpha > 0 \ \forall\ \mathbb{P}^1 \twoheadrightarrow C \subset X\}$.
- $\overline{\mathrm{Bir}\mathcal{K}_\mathrm{X}} = \{\alpha \in \bar{\mathcal{C}}_X \mid q_X(\alpha, [D]) > 0 \ \forall\ D \subset X \text{ uniruled}\}$.

H^2 and Global Torelli

K3 surfaces: $X \simeq X' \Leftrightarrow H^2(X,\mathbb{Z}) \simeq H^2(X',\mathbb{Z})$ respecting the Hodge structure and $(\ ,\ )$ (*Hodge isometry*).

Question: Which Hodge isometries are induced by isomorphisms? Will give natural explanation using $\mathrm{D^b}(X)$.

HKs: One knows that for birational HKs $X \sim X'$ there exists a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(X',\mathbb{Z})$ (using q). In fact, X and X' define non-separated points in the moduli space.

Namikawa example: $H^2(K^3(\mathrm{A}),\mathbb{Z}) \simeq H^2(K^3(\hat{\mathrm{A}}),\mathbb{Z})$ Hodge isometry, but $K^3(\mathrm{A})$ and $K^3(\hat{\mathrm{A}})$ not birational.

Question: GT for HKs?

Derived decomposition theorem?

Question: Suppose $\mathrm{c}_1(X) = 0$ and $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$. Let

$$\mathrm{A} \times \prod X_i \times \prod Y_j \longrightarrow X \quad \text{and} \quad \mathrm{A}' \times \prod X_i' \times \prod Y_j' \longrightarrow X'$$

be the minimal covers. Is then $\mathrm{D^b}(\mathrm{A}) \simeq \mathrm{D^b}(\mathrm{A}')$,

$$\mathrm{D^b}(X_i) \simeq \mathrm{D^b}(X'_{\sigma(i)}), \quad \text{and} \quad \mathrm{D^b}(Y_j) \simeq \mathrm{D^b}(Y'_{\tau(j)}) \ ?$$

Restrict to irreducible factors!

Abelian factors understood by work of Mukai, Orlov, Polishchuk.

$X = \mathrm{HK}$

H., Nieper-Wißkirchen: If $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X')$, then $X' = \mathrm{HK}$.

Conjecture (Bondal, Orlov): $X \sim X'$ birational HKs $\Rightarrow$ $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X')$.

Ploog: OK for Hilbert schemes.

Kawamata, Namikawa: OK for Mukai flops.

Expect:

$$\begin{array}{ccc} X \sim X' & \overset{\checkmark}{\Longrightarrow} & H^2(X,\mathbb{Z}) \simeq H^2(X',\mathbb{Z}) \\ & \searrow^{?} & \Downarrow ?? \\ & & \mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}(X') \end{array}$$

Test: Is $\mathrm{D}^{\mathrm{b}}(\mathrm{K}^3(\mathrm{A})) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{K}^3(\hat{\mathrm{A}}))$??

$\mathrm{Aut}(\mathrm{D^b}(X))$ for $K_X = \mathcal{O}_X$

- $E \in \mathrm{D^b}(X)$ is *spherical* if $\mathrm{Ext}^*(E,E) \simeq H^*(S^n, \mathbb{C})$.

Seidel, Thomas: $\mathrm{C}(E^\vee \boxtimes E \longrightarrow \mathcal{O}_\Delta)$ defines the *spherical twist*

$$T_E : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X), \quad F \longmapsto \mathrm{C}(\mathrm{Hom}^*(E,F) \otimes E \longrightarrow F).$$

- $E \in \mathrm{D^b}(X)$ is a $\mathbb{P}^n$*-object* if $\mathrm{Ext}^*(E,E) \simeq H^*(\mathbb{P}^n, \mathbb{C})$.

H., Thomas: $\mathrm{C}(\mathrm{C}(E^\vee \boxtimes E[-2] \longrightarrow E^\vee \boxtimes E) \longrightarrow \mathcal{O}_\Delta)$ defines the $\mathbb{P}^n$-twist

$$P_E : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X).$$

Examples: Line bundles on CYs and HKs are spherical respectively $\mathbb{P}^n$-objects. There is no spherical object E with $\mathrm{rk}(E) \neq 0$ on a HK of dimension > 2.

$X, X' =$ K3 surfaces

Derived Global Torelli (Mukai, Orlov): $\mathrm{D^b}(X) \simeq \mathrm{D^b}(X')$
$\Leftrightarrow H^*(X,\mathbb{Z}) \simeq H^*(X',\mathbb{Z})$ Hodge isometry of Mukai lattices.
$\Leftrightarrow X' \simeq X$ or $\simeq$ moduli space of slope stable vector bundles.

Study $\mathrm{Aut}(\mathrm{D^b}(X))$ via the action

$$\rho : \mathrm{Aut}(\mathrm{D^b}(X)) \longrightarrow \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$$

Theorem: $\mathrm{Im}(\rho) = \mathrm{O}_+(\widetilde{H}(X,\mathbb{Z})) =$ group of all Hodge isometries preserving the orientation of the positive directions (Mukai; Orlov; Hosono/Lian/Oguiso/Yau; Ploog; H./Macrì/Stellari).

What about the kernel?

Bridgeland's conjecture

Conjecture: $\mathrm{Ker}(\rho) = \pi_1(\mathcal{P}_0^+(X))$. **$(*)$**

$\mathcal{P}_0^+(X) =$ period domain for the space $\mathrm{Stab}(X)$ of stability conditions on $\mathrm{D^b}(X)$.

General: $\pi : \mathrm{Stab}(X) \longrightarrow \pi(\mathrm{Stab}(X)) \subset H^*(X, \mathbb{C})$ with equivalences $\Phi \in \mathrm{Ker}(\rho)$ acting continuously and fibrewise.

K3 surfaces (Bridgeland): There is a distinguished connected component $\Sigma \subset \mathrm{Stab}(X)$ such that:

- $\mathcal{P}_0^+(X) := \pi(\Sigma)$ admits explicit description similar to description of the Kähler cone.
- If $\Sigma = \mathrm{Stab}(X)$ and simply connected, then $(*)$.

The idea of stability: abelian

Recall: For a coherent sheaf E on a curve C one has the unique *Harder–Narasimhan filtration*:

$$0 \subset E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E,$$

where $E_0 =$ is the torsion of E and $F_{i+1} := E_{i+1}/E_i$ are semi-stable vector bundles with slopes $\mu(F_1) > \ldots > \mu(F_n)$.

$\rightsquigarrow$ Decomposes $\mathrm{Coh}(C)$ in smaller abelian categories

$$\mathcal{P}(\phi) \subset \mathrm{Coh}(C), \qquad \phi \in (0,1]$$

st. $\mathcal{P}(1) =$ subcategory of torsion sheaves and $\mathcal{P}(\phi)$ for $\phi \in (0,1)$ is the subcategory of semi-stable vector bundles of slope $-\cot(\pi\phi)$.

Classical stability: $\mathrm{Hom}(\mathcal{P}(\phi), \mathcal{P}(\phi')) = 0$ for $\phi > \phi'$.

The idea of stability: derived

Same idea for $\mathrm{D^b}(X)$: Decompose into abelian subcategories

$$\mathcal{P}(\phi) \subset \mathrm{D^b}(X), \quad \phi \in \mathbb{R} \ (!).$$

Need in addition additive *stability function*

$$Z : K(X) \longrightarrow H^*(X) \longrightarrow \mathbb{C}$$

such that $Z(E) = r(E)\exp(i\pi\phi(E)), \quad r(E) \in \mathbb{R}_{>0}$ for $E \in \mathcal{P}(\phi)$.

Bridgeland:

- $\mathrm{Stab}(X) := \{(\mathcal{P}, Z)\}$ has a natural topology.
- The projection $\pi : \mathrm{Stab}(\mathcal{T}) \longrightarrow H^*(X,\mathbb{C})^*$, $(\mathcal{P}, Z) \longmapsto Z$, is a local homeomorphism from each connected component Σ to a linear subspace $V_\Sigma \subset H^*(X,\mathbb{C})^*$.