Seminar 'Fano varieties and cubics: Hodge theory and derived categories'

Summer term 2013

The following is open for discussion. We may decide to shift the emphasize or dwell longer on certain aspects. Personally, I would be more interested in the geometric picture than in the technical details and more in the Hodge theory than in the derived categories.

Fano varieties of lines. (Nicolò Sibilla.) We should collect some general fact on Fano varieties $F(X)$ on lines on a hypersurface X that are used in the later talks. E.g. general existence results, connectedness, and local description. Special emphasis should be put on the two examples: i) $F(X)$ is a surface for a cubic $X \subset \mathbb{P}^4$; ii) $F(X)$ is a fourfold for a cubic $X \subset \mathbb{P}^5$. (There are other examples, e.g. the Fano variety of lines on a quartic double solid $X \to \mathbb{P}^3$.) References [2, 4, 8]. (If someone feels inspired, one could also look at the Fano variety of rational curves of higher degree in hypersurfaces, see [10].)

Intermediate Jacobian of the cubic threefold: Torelli theorem and rationality. (Stefan Schreieder)

i) Clemens and Griffiths proved in [12] that two smooth cubic threefolds $X, X' \subset \mathbb{P}^4$ are isomorphic if and only if their intermediate Jacobians (as polarized abelian varieties) are isomorphic. The intermediate Jacobian $J(X)$ of a cubic $X \subset \mathbb{P}^4$ is isomorphic to the Albanese of the Fano variety of lines $F(X)$ (a surface). This holds true for other Fano threefolds.

ii) The intermediate Jacobian was also used in [12] to prove that a smooth cubic $X \subset \mathbb{P}^4$ is not rational (but it is unirational). The key point is to show that the intermediate Jacobian is not isomorphic to a Jacobian of a curve or to a product of thereof.

More algebraic proofs can be found in [5] and [19, 20] (based on ideas of Mumford). See also Beauville's article [6] and slides of recent talks on his webside.

Non-rationality à la Artin Mumford. (Stefanie Anschlag) Classically the easiest example of a unirational but not rational variety is due to Artin and Mumford. It uses torsion in $H^3(X, \mathbb{Z})$. See [3]. This is probably a 45 min talk.

Derived category of the cubic threefold. (Speaker?)

This should serve as a warm up for the fourfold case. The derived category $\mathrm{D}^{\mathrm{b}}(X)$ of a cubic threefold $X \subset \mathbb{P}^4$ can be decomposed into the category spanned by $\mathcal{O}_X$, $\mathcal{O}_X(1)$, $\mathcal{O}_X(2)$ and its orthogonal complement $\mathcal{A}_X$. In [11] it is shown that the latter category determines X. In a sense, this is a derived version of the Torelli theorem of Clemens and Griffiths. However, the relation between $J(X)$ and $\mathcal{A}_X$ remains mysterious. It would be nice to have a survey without entering a lengthy discussion of stability conditions (if possible). We could also do this later or just state the result (after a discussion of the orthogonal decomposition).

Fano variety of lines on a cubic fourfold. (Andrey Soldatenkov)

Recall the Beauville–Donagi result in [7] about $F(X)$ being an irreducible symplectic variety deformation equivalent to $\mathrm{Hilb}^2(S)$. Discuss the direct construction of the symplectic form on $F(X)$ following [15]. Survey the construction of Lagrangian surfaces in $F(X)$ following [15, Sec. 2]. Is there more in the recent literature? Can one see what these Lagrangian surfaces look like when $F(X) = \mathrm{Hilb}^2(S)$?

There is the obvious question here: Do we want to say anything about the Global Torelli for cubic fourfolds? There is a recent short proof by Charles [9]. (But warning: It relies on the Global Torelli for hyperkähler manifolds.)

Hassett's conjecture. (Giovanni Mongardi and Margherita Lelli Chiesa)

There is no cubic fourfold $X \subset \mathbb{P}^5$ known to be not rational. However, it is strongly expected that the generic cubic fourfold is only unirational and Hassett has put forward a conjecture characterizing the locus of rational cubic fourfolds as the union of certain Noether–Lefschetz components $\mathcal{C}_d$. Note that $\mathcal{C}_d$ is not empty iff $d \equiv 0(6)$ and $d > 6$. The cases $d = 8, 12$, and 14 have special geometric interpretations (are there other cases?), e.g. $\mathcal{C}_8$ is the divisor of cubics containing a plane. Moreover, Hassett proves that for generic $Y \in \mathcal{C}_d$ with $d = 2(n^2 + n + 2)$, $n \geq 2$ the Fano variety of lines $F(X)$ is isomorphic to $\mathrm{Hilb}^2(S)$ for some K3 surface S. (See [18] for the case $n = 1$.) We should also take the opportunity to collect a list of all known rational cubic fourfolds (see Hassett's paper for references, e.g. to articles by Tregub). Note that the paper [1] contains a more conceptual interpretation of some of Hassett's condition on d.

We should study both papers [13, 14] carefully. This may take more than one talk and maybe a team could take care of this. See also [21, Sec. 3].

Kuznetsov's conjecture. (Christoph Sorger)

The rationality of a cubic fourfold $X \subset \mathbb{P}^5$ is approached by its derived category $\mathrm{D}^{\mathrm{b}}(X)$. More precisely, the right orthogonal complement $\mathcal{A}_X$ of the subcategory generated by $\mathcal{O}_X$ and $\mathcal{O}_X(1)$ is a K3 category. It is called geometric if indeed $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S)$ for some K3 surface S. In [16] Kuznetsov conjectured that X is rational if and only if $\mathcal{A}_X$ is geometric.

Hodge theory versus derived categories. (Daniel Huybrechts)

This talk should review the recent paper by Addington and Thomas [1] that compares the Hodge theoretic (Hassett) and the derived category (Kuznetsov) approach to rationality of cubic fourfolds.

Schedule: In principle, we meet every Tuesday at 2.15-4 pm in 0.011, but we have reserved the room until 6pm. So there is no problem with going overtime. Also, some of us will be away at various times. So in order not to fall behind schedule too much, we should schedule some of the meetings for 2.15-6 pm right from the start.

List of dates and speakers so far (nothing completely fixed yet):

April 9 – Introduction (Daniel Huybrechts)

– Fano varieties of lines: general results (Nicolò Sibilla)

April 16 – Artin–Mumford example (Stefanie Anschlag)

– Cubic threefold, after Clemens–Griffiths I (Stefan Schreieder)

April 23 – Cubic threefold, after Clemens–Griffiths II (Stefan Schreieder)

May 7 – Fano variety of lines on a cubic fourfold. (Andrey Soldatenkov)

– Hassett's conjecture: examples and lattice theory (Margherita Lelli Chiesa)

May 14 – Hassett's conjecture II (Giovanni Mongardi and Margherita Lelli Chiesa)

June 4 – Hassett's conjecture III (Giovanni Mongardi)

– Kuznetsov's conjecture Part I (Christoph Sorger)

July 2 – Kuznetsov's conjecture Part II (Christoph Sorger)

July 16 – Hassett = Kuznetsov (after Addington–Thomas) Part I and II (Daniel Huybrechts)

References

[1] N. Addington, R. Thomas *Hodge theory and derived categories of cubic fourfolds*, arXiv:1211.3758.

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[8] S. Bloch, J. Murre *On the Chow group of certain types of Fano threefolds.* Comp. Math. 39 (1979), 47–105.

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[10] J. Harris, M. Roth, J. Starr *Rational curves on hypersurfaces of low degree*, J. Reine Angew. Math. 571 (2004), 73–106.

[11] M. Bernardara, E. Macri, S. Mehrotra, P. Stellari *A categorical invariant for cubic threefolds*, Adv. Math. 229 (2012), 770–803.

[12] Clemens, P. Griffiths *The intermediate Jacobian of the cubic threefold*, Ann. Math. (2) 95 (1972), 281–356.

[13] B. Hassett *Special cubic fourfolds*, Compositio 120 (2000), 1–23. (The original Harvard University Thesis (1996) has more material.)

[14] B. Hassett *Some rational cubic fourfolds*, J. Alg. Geom. 8 (1999), 103–114.

[15] A. Iliev, L. Manivel *Cubic hypersurfaces and integrable systems*, arXiv:math/0605260.

[16] A. Kuznetsov *Derived categories of cubic fourfolds*, In Cohomological and geomet- ric approaches to rationality problems, volume 282 of Progr. Math. 219–243. Birkhäuser Boston Inc., Boston, MA, 2010. arXiv:0808.3351.

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[18] E. Macri, P. Stellari *Fano varieties of cubic fourfolds containing a plane*, Math. Ann. 354 (2012), 1147–1176.

[19] J. Murre *Algebraic equivalence modulo rational equivalence on a cubic threefold*, Compositio Math. 25 (1972), 161–206.

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[21] M. Rempel *Positivité des cycles dans les variétés algébriques*, Thèse 2012. (I have a copy.)

[22] M. Shen *Hyperkähler manifolds of Jacobian type*, arXiv:1206.2063