Finite dimensionality of Chow motives and cohomology of complete intersections

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Introduction

The present bachelor thesis concerns the notion of finite dimensionality for Chow motives and how it enables us to lift properties from the Betti cohomology of a complex smooth projective variety to properties of its Chow group, i.e. its group of algebraic cycles modulo rational equivalence.

The algebraic cycles together with the intersection product form a theory that contains a lot of information about the topology of varieties. This theory can be encapsulated into the category of Chow motives, which has a richer structure. To a smooth projective variety X, one can associate its Chow motive $\mathfrak{h}(X)$ and cohomology factors through this category.

However, the intersection theory from which Chow motives are constructed only sees those cohomological cycles that come from algebraic cycles, hence it does not seem to see the whole cohomology of varieties. Grothendieck's standard conjectures [Gro69; Kle68] are an attempt to fill this gap and would imply that an essential part of the cohomological information is already contained in Chow motives.

More recently, Kimura [Kim05] and O'Sullivan independently introduced a notion of finite dimensionality for Chow motives. It is related to the former conjectures but has a more algebraic nature from which important nilpotence consequences can be deduced. In a note, Peters [Pet17] used these nilpotence consequences of finite dimensionality in order to lift a splitting from the cohomology of complete intersections to their Chow motives. The goal of this bachelor thesis is to understand this latter result.

The first chapter briefly introduces the notions of intersection theory that are needed for the construction of the category of motives, which is the subject of the second chapter. We give several examples along the way to illustrate the theory.

The third chapter introduces the notion of finite dimensionality in a $\mathbb{Q}$-linear pseudo-abelian tensor category. Due to its algebraic nature, this part has a different flavour from the rest of the text. Then we specialize this notion to the category of Chow motives to understand its geometric meaning.

In the last chapter, we present a proof of the Lefschetz theorem on hyperplane sections relying on Morse theory [AF59]. This theorem induces a splitting in the cohomology of a smooth complete intersection X into a fixed part, which is entirely determined by the surrounding variety, and a variable part, which really depends on X. Finally, following Peters, we apply the results of the previous chapter in order to deduce a splitting in the Chow motive of X inducing the corresponding splitting in cohomology, assuming that the surrounding variety has a finite dimensional Chow motive and satisfies the Lefschetz type standard conjecture.

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Einleitung

Die vorliegende Bachelorarbeit beschäftigt sich mit dem Begriff der Endlichdimensionalität für Chow-Motive und der Frage, welche Eigenschaften der Betti-Kohomologie einer komplexen, glatten, projektiven Varietät zu deren Chow-Gruppe übertragen werden können. Die Chow-Gruppe ist dabei die Gruppe der algebraischen Zykeln modulo rationaler Äquivalenz.

Algebraische Zykel, zusammen mit dem Schnittprodukt, bilden eine Theorie, die viele Informationen über die Topologie einer Varietät enthält. Diese Theorie kann in der Kategorie der Chow-Motive kodiert werden, die eine reichhaltigere Struktur besitzt. Zu einer glatten, projektiven Varietät X kann ihr Chow-Motiv $\mathfrak{h}(X)$ assoziiert werden und Kohomologie faktorisiert über diese Kategorie. Die Schnitttheorie, von der aus Chow-Motive konstruiert werden, ist jedoch nur in der Lage kohomologische Zykel zu sehen die von algebraischen Zykeln stammen, und damit nicht die ganze Kohomologie der Varietät. Die Grothendieck-Standard-Vermutungen [Gro69; Kle68] sind ein Versuch diese Lücke zu füllen, und würden implizieren dass ein essentieller Teil der kohomologischen Information bereits in den Chow-Motiven enthalten ist.

Kimura [Kim05] und O'Sullivan führten beide unabhängig voneinander einen Begriff der Endlichdimensionalität für Chow-Motive ein. Er steht im Zusammenhang zu früheren Vermutungen, ist aber algebraischer, sodass er Folgerungen über Nilpotenz ermöglicht. In einem Preprint hat Peters [Pet17] diese Implikationen über Nilpotenz aus der Endlichdimensionalität genutzt, um eine Hebung einer Spaltung von der Kohomologie vollständiger Schnitte zu deren Chow-Motiven zu erhalten. Das Ziel dieser Bachelorarbeit ist es, dieses Ergebnis besser zu verstehen.

Das erste Kapitel gibt eine kurze Einführung in Begriffe der Schnitttheorie, die in der Konstruktion der Kategorie der Motive benötigt werden, was der Inhalt des zweiten Kapitels ist. Wir werden mehrere Beispiele geben, um die Theorie verständlicher zu machen.

Das dritte Kapitel führt den Begriff der Endlichdimensionalität in einer $\mathbb{Q}$-linearen pseudo-abelschen Tensorkategorie ein. Dieser Teil unterscheidet sich vom Rest der Arbeit durch einen stärkeren algebraischen Hintergrund.

Danach werden wir diese Begriffe auf die Kategorie der Chow-Motive anwenden, um deren geometrischen Hintergrund besser zu verstehen.

In dem letzten Kapitel behandeln wir einen Beweis des Lefschetz-Theorems über Hyperebenenschnitte, basierend auf Methoden der Morse-Theorie [AF59]. Dieser Satz induziert eine Spaltung in der Kohomologie eines glatten, vollständigen Schnittes X in einen fixen Teil, der quasi durch die umgebende Varietät bestimmt ist, und einen variablen Teil, der tatsächlich von X abhängt. Peters folgend werden wir schlussendlich diese Ergebnisse anwenden um eine Spaltung in dem Chow-Motiv von X abzuleiten, der die zugehörige Spaltung in der Kohomologie induziert, unter der Annahme, dass die umgebende Varietät ein endlichdimensionales Chow-Motiv hat und die Lefschetz-Standardvermutung erfüllt.

Contents

Notations. Throughout this text, we assume k to be an algebraically closed field. A variety is a reduced scheme over k, not necessarily irreducible. The category of smooth projective varieties over k is denoted SmProj(k). We write X_d to denote an irreducible variety of dimension d.

1 Algebraic cycles and equivalence relations

In this chapter, we introduce the different notions that will be needed for constructing the category of pure motives.

1.1 Adequate equivalence relation

The presentation below follows closely [MNP13]. Let X be a variety. For $i \geq 0$, we denote by

$$Z^i(X)$$

the free abelian group generated by codimension i irreducible subvarieties of X. Its elements are called *algebraic cycles*. Note that $Z^1(X)$ is the group of Weil divisors of X. Similarly, we denote by $Z_i(X)$ the free abelian group generated by dimension i irreducible subvarieties of X. If X_d has pure dimension d, we have $Z^i(X) = Z_{d-i}(X)$.

Let us introduce a few basic operations on cycles:

1.1.1 Cartesian product.

Given varieties X, X' and subvarieties $Z \subset X, Z' \subset X'$ of dimension i, i', respectively, their product $Z \times Z'$ is a subvariety of dimension $i + i'$ of $X \times X'$. This extends linearly to a morphism

$$Z_i(X) \times Z_{i'}(X') \to Z_{i+i'}(X \times X').$$

1.1.2 Pushforward.

Let $f\colon X \to Y$ be a proper morphism between k-varieties and let Z be an irreducible subvariety of X. We define

$$\deg(Z/f(Z)) = \begin{cases} [k(Z) : k(f(Z))] & \text{if } \dim f(Z) = \dim Z \\ 0 & \text{if } \dim f(Z) < \dim Z. \end{cases}$$

We get a homomorphism

$$f_* : Z_i(X) \to Z_i(Y)$$

given by $f_*(Z) = \deg(Z/f(Z))f(Z)$ and extended linearly.

When $Y = \operatorname{Spec}(k)$ and $f\colon X \to \operatorname{Spec}(k)$ is the structural morphism, the induced map

$$\deg \coloneqq f_*\colon Z_0(X) \to Z_0(\operatorname{Spec}(k)) = \mathbb{Z}$$

is called the *degree map*.

1.1.3 Intersection.

Two subvarieties V, W of a smooth variety X of codimension i and j intersect each other in a union of subvarieties of codimension $\leq i + j$. If all of them have codimension $i + j$, we say that V and W intersect *properly*, in which case we define the *intersection number* of an irreducible component $Z \subset V \cap W$ using *Serre's Tor formula*

$$i(V \cdot W; Z) \coloneqq \sum_r (-1)^r \ell_A(\operatorname{Tor}_r^A(A/I(V), A/I(W)))$$

where $A = \mathcal{O}_{X,Z}$ and $I(V)$ is the ideal of V in A [Har77, App. A]. Then we define the intersection product of V and W as

$$V \cdot W \coloneqq \sum_Z i(V \cdot W; Z)Z$$

where the sum runs over the irreducible components Z of $V \cap W$.

1.1.4 Pullback. Let $f\colon X \to Y$ be a morphism in SmProj(k) and let $T \subset Y$ be a subvariety. The graph of f is a subvariety $\Gamma_f \subset X \times Y$. If it meets $X \times T$ properly, we define

$$f^*T \coloneqq (\mathrm{pr}_X)_*(\Gamma_f \cdot (X \times T)).$$

1.1.5 Correspondences. A *correspondence* from X to Y is a cycle in $X \times Y$. For example, the graph of a morphism $X \to Y$ is a correspondence. Thus, correspondences generalize morphisms, and act on cycles in a similar way: given a correspondence $Z \in Z^t(X_d \times Y)$ and a cycle $T \in Z^i(X)$ we let

$$Z(T) = (\mathrm{pr}_Y)_*(Z \cdot (T \times Y)) \in Z^{i+t-d}(Y)$$

whenever the intersection product is defined. We call $t - d$ the *degree* of the correspondence. Note that correspondences of degree 0 preserve the codimension of the cycle.

1.1.6 Equivalence relations. We call an *equivalence relation* the data of an equivalence relation on $Z(X) \coloneqq \bigoplus_i Z^i(X)$ for each variety X. An equivalence relation $\sim$ is said to be *adequate* if restricted to the category SmProj(k) it satisfies the following axioms:

(R1) compatibility with grading and addition;

(R2) compatibility with products: if $Z \sim 0$ in $Z(X)$, then for all Y one has $Z \times Y \sim 0$ in $Z(X \times Y)$;

(R3) compatibility with intersections: if $Z_1 \sim 0$ and $Z_1 \cdot Z_2$ is defined, then $Z_1 \cdot Z_2 \sim 0$;

(R4) compatibility with projections: if $Z \sim 0$ on $X \times Y$, then $(\mathrm{pr}_X)_*(Z) \sim 0$ on X;

(R5) *moving lemma*: given $Z, W_1, \ldots, W_l \in Z(X)$ there exists $Z' \sim Z$ such that $Z' \cdot W_j$ is defined for $j = 1, \ldots, l$.

Note that the axioms **(R2,3,4)** are equivalent to the following single axiom [Sam60]:

(R') compatibility with correspondences: if $T \sim 0$ in X and $Z \in Z(X \times Y)$, then $Z(T) \sim 0$ if it is defined.

Given such an equivalence relation, denote by $Z^i_\sim(X)$ the subgroup of codimension i cycles that are equivalent to 0, and

$$C^i_\sim(X) = Z^i(X)/Z^i_\sim(X)$$
$$C_\sim(X) = \bigoplus_i C^i_\sim(X).$$

Lemma 1.1.6.1. *For any adequate equivalence relation $\sim$, we have:*

(i) *$C_\sim(X)$ is a graded ring with product induced from intersection of cycles;*

(ii) *For any proper*[1] *morphism* $f\colon X \to Y$ *in* SmProj(k) *the maps* f_* *and* f^* *induce group morphisms* $f_* : C_\sim(X) \to C_\sim(Y)$ *and a graded ring morphism* $f^* : C_\sim(Y) \to C_\sim(X)$*; Moreover, the* projection formula *holds:*

$$f_*(f^*(\alpha) \cdot \beta) = \alpha \cdot f_*(\beta)$$

for all cycles $\alpha \in C_\sim(Y)$*,* $\beta \in C_\sim(X)$*;*

(iii) *A correspondence* Z *from* $X \times Y$ *of degree* r *induces a group morphism* $Z_* : C^i_\sim(X) \to C^{i+r}_\sim(Y)$ *and equivalent correspondences induce the same morphism.*

Remark 1.1.6.2. For any irreducible variety X, the group $C^0_\sim(X)$ is freely generated by the class of X itself, which is the unit in the ring $C_\sim(X)$. △

Remark 1.1.6.3. It can be more convenient to consider rational coefficients. For example, we let

$$Z^i(X)_\mathbb{Q} \coloneqq Z^i(X) \otimes_\mathbb{Z} \mathbb{Q},$$

and similarly for $Z^i_\sim(X)_\mathbb{Q}$, $C^i_\sim(X)_\mathbb{Q}$, $C_\sim(X)_\mathbb{Q}$. The above results clearly extend to these groups when they are taken with rational coefficients. △

Remark 1.1.6.4. Usually, adequate equivalence relations are written $\sim_{\text{sub}}$ for some subscript sub, in which case we write

$$Z^i_{\text{sub}}(X) \coloneqq Z^i_{\sim_{\text{sub}}}(X)$$

and similarly for $C^i_{\text{sub}}(X)$, $C_{\text{sub}}(X)$ to ease notation. △

1.2 Rational equivalence and Chow rings

1.2.1 Rational equivalence.

Lemma 1.2.1.1 ([And04, Lem. 3.2.2.1])**.** *Any two points of* $\mathbb{P}^1$ *are equivalent with respect to any adequate equivalence relation.*

Proof. Let $\sim$ be an adequate equivalence relation. We first prove that $0 \sim \infty$. We write $\mathbb{P}^1 = k \cup \{\infty\}$ and $[x]$ the class of the point x in $Z_0(\mathbb{P}^1)$. By (R5) there exists a 0-cycle $\sum_i n_i[x_i] \sim [1]$ that intersects properly [1], i.e. , $x_i \neq 1$. The graph $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ of the polynomial $1 - \prod_i \left(\frac{x-x_i}{1-x_i}\right)^{m_i}$, for arbitrary $m_i > 0$, sends $\sum_i n_i[x_i] \sim [1]$ to $mn[1] \sim m[0]$ by (R'), where $m = \sum_i m_i$ and $n = \sum_i n_i$. Since the m_i were arbitrary, we get $n[1] \sim [0]$. Then using (R') again, the graph of $x \mapsto \frac{1}{x}$ sends the latter equivalence to $n[1] \sim [\infty]$. Therefore $[0] \sim [\infty]$. Since $\mathrm{Aut}(\mathbb{P}^1)$ acts 2-transitively on $\mathbb{P}^1$, the result follows by applying (R') to send any two distinct points to [0] and $[\infty]$. □

From this lemma, it follows that $C^1_\sim(\mathbb{P}^1)$ is freely generated by the class of any rational point. The intersection product of two codimension 1 cycles is zero for degree reasons. Together with Remark 1.1.6.2, this gives us the whole ring of cycles on the projective line with respect to any equivalence relation:

$$C_\sim(\mathbb{P}^1) \cong \mathbb{Z}[\alpha]/(\alpha^2) \quad \text{where } \alpha \text{ is the class of a point.}$$

[1]The morphism is automatically proper because it is a morphism between projective varieties.

Let $\sim$ be any adequate equivalence relation and $X \in \mathrm{SmProj}(\mathrm{k})$. For any cycle $H \subset X \times \mathbb{P}^1$ and any two points a, b of $\mathbb{P}^1$ such that H intersects $X \times a$ and $X \times b$ properly, the equivalence $[a] \sim [b]$ from the lemma above yields, by (R'),

$$H(a) \sim H(b).$$

In fact, this describes an adequate equivalence relation, called *rational equivalence* and denoted $\sim_{\mathrm{rat}}$, which is the finest one. Note the choice of the letter H suggesting an homotopy. This is the *homotopy description* of rational equivalence.

Rational equivalence has another description generalizing linear equivalence of divisors [Ful84, p. 15]. To prove that it is indeed an adequate equivalence relation, $(R1, 2, 3)$ are easy, $(R4)$ is more technical (see [Ful84, p. 12] or [Tra07, p. 29] for more details) and $(R5)$ takes more work (see [Stacks, Tag 0B0D] for a proof).

1.2.2 Chow rings. In the case of rational equivalence, the groups $C^i_{\mathrm{rat}}(X)$ deserve a special name:

Definition 1.2.2.1. Let $X \in \mathrm{SmProj}(\mathrm{k})$ be a smooth projective variety. The group

$$\mathrm{CH}^i(X) \coloneqq C^i_{\mathrm{rat}}(X)$$

is called the *i-th Chow group of* X for each $0 \leq i \leq \dim(X)$ and the ring

$$\mathrm{CH}(X) \coloneqq C_{\mathrm{rat}}(X)$$

is called the *Chow ring of* X. We also write $\mathrm{CH}_i(X_d) \coloneqq \mathrm{CH}^{d-i}(X_d)$.

Example 1.2.2.2. The Chow groups of affine spaces $\mathbb{A}^n$ are easy to compute:

$$\mathrm{CH}^0(\mathbb{A}^n) \cong \mathbb{Z} \quad \text{and} \quad \mathrm{CH}^i(\mathbb{A}^n) = 0 \text{ for } i \geq 1.$$

Indeed, any proper subvariety can be "moved to infinity":

Consider a proper subvariety $Y = V(f_1, \ldots, f_r) = V(f_i) \subset \mathbb{A}^n$. Up to translation, we can assume that Y does not contain the origin $(0, \ldots, 0) \in \mathbb{A}^n$. Then the following "homotopy"

$$H = V\left(\lambda_1^{\deg(f_i)} f_i\left(\frac{\lambda_0}{\lambda_1}x_1, \ldots, \frac{\lambda_0}{\lambda_1}x_n\right)\right) \subset \mathbb{A}^n_x \times \mathbb{P}^1_\lambda$$

"moves Y to infinity". Indeed,

$$H([1:1]) = V(f_i(x_1, \ldots, x_n)) = Y$$

and

$$H([0:1]) = V(f_i(0, \ldots, 0)) = \emptyset$$

since Y does not contain the origin. Therefore, $Y \sim_{\mathrm{rat}} 0$ as desired. △

Chow groups satisfy an *exactness* condition:

Proposition 1.2.2.3. *If $j \colon Y \hookrightarrow X$ is a closed embedding of arbitrary k-varieties and if we write $f \colon X - Y \hookrightarrow X$ the inclusion of its complement, then there is an exact sequence*

$$\mathrm{CH}_i(Y) \xrightarrow{j_*} \mathrm{CH}_i(X) \xrightarrow{f^*} \mathrm{CH}_i(X - Y) \to 0.$$

In particular, if a variety X has an affine stratification (see [EH16, Sec. 1.3.5]), then its Chow groups can be computed. For example, this is the case for the projective space $\mathbb{P}^n$ or for products of these:

$$\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathrm{CH}(\mathbb{P}^n)) \otimes \mathrm{CH}(\mathbb{P}^m) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{m+1}) \tag{1}$$

where α (resp. β) is the class of the pulback of an hyperplane from the first (resp. second) factor of $\mathbb{P}^n \times \mathbb{P}^m$ [EH16, Thm. 2.10].

1.2.3 The class of the diagonal in $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^n)$. To illustrate the above, let us compute the class of the diagonal $\Delta_{\mathbb{P}^n}$ in $\mathrm{CH}^n(\mathbb{P}^n \times \mathbb{P}^n)$. In the case $n = 1$, the diagonal $\Delta = \Delta_{\mathbb{P}^1}$ is rationally equivalent to $\mathbb{P}^1 \times e + e \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ where $e \in \mathbb{P}^1(k)$ is any closed point. Indeed, there is a linear homotopy

$$H = V(\lambda_0 x_0 y_0 + \lambda_1(x_0 y_1 - x_1 y_0)) \subset (\mathbb{P}^1_x \times \mathbb{P}^1_y) \times \mathbb{P}^1_\lambda$$

from $H([0:1]) = \Delta$ to $H([1:0]) = \mathbb{P}^1 \times 0 \cup 0 \times \mathbb{P}^1$. By the homotopy description of rational equivalence, we are done.

In the general case, translating the above argument is not obvious. However, under the isomorphism (1) above, we can write

$$[\Delta] = c_0 \alpha^n \beta^0 + c_1 \alpha^{n-1} \beta^1 + \cdots + c_n \alpha^0 \beta^n$$

for some integers c_i. To determine the c_i, take the product of each side with $\alpha^i \beta^{n-i}$ to get

$$c_i = \deg([\Delta] \cdot \alpha^i \beta^{n-i}) = \#(\Delta \cap (\Lambda \times \Gamma)) = \#(\Lambda \cap \Gamma) = 1$$

where Λ and Γ are linear subspaces of codimensions i and $n - i$ that intersect properly, i.e. in one point. Therefore, Δ is rationally equivalent to $\sum_i \mathbb{P}^{n-i} \times \mathbb{P}^i$.

1.3 Algebraic, homological and numerical equivalence

There are other interesting adequate equivalence relations.

1.3.1 Algebraic equivalence. The homotopy description of rational equivalence states that two cycles are rationally equivalent if we can go from one to the other through a rational family of cycles, i.e. a family of cycles parametrized by $\mathbb{P}^1$. This definition can be weakened by allowing families of cycles parametrized by any connected curve C.

Let $x \in \mathrm{SmProj(k)}$. For any smooth connected curve $C \in \mathrm{SmProj(k)}$ and any cycle $H \subset X \times C$, we let

$$H(a) \sim_{\mathrm{alg}} H(b)$$

for any points $a, b \in C$. This defines an adequate equivalence relation, called *algebraic equivalence* and denoted $\sim_{\mathrm{alg}}$.

By definition, $Z_{\mathrm{rat}}(X) \subset Z_{\mathrm{alg}}(X)$. Note that Bertini's theorem implies that any two points on a connected smooth variety $X_d \in \mathrm{SmProj(k)}$ can be joined by a smooth curve. This shows that $C^d_{\mathrm{alg}}(X) \cong \mathbb{Z}$, as one would expect from the similarity between algebraic and topological cycles. This is not at all the case for $\mathrm{CH}^d(X)$. For example, if X is an elliptic curve, then no two distinct points of X are rationally equivalent, but they are clearly algebraically equivalent: take $H = \Delta_X \subset X \times X$ the diagonal.

1.3.2 Homological equivalence. In this paragraph, $k = \mathbb{C}$. Given a complex variety $X \in \mathrm{SmProj}(\mathbb{C})$, consider

$$H^i(X, \mathbb{Q}) \coloneqq H^i_{\mathrm{sing}}(X_{\mathrm{an}}, \mathbb{Q}),$$

the singular cohomology with rational coefficients of the analytification X_{an} of X, called the *rational Betti cohomology* of X, or just *(rational) cohomology* of X when there is no ambiguity. These groups together with the cup-product constitute the *cohomology ring* of X

$$H(X, \mathbb{Q}) \coloneqq \bigoplus_i H^i(X, \mathbb{Q}),$$

whose elements are called *cohomological* or *topological cycles* on X.

A morphism $f\colon Y_e \to X_d$ in $\mathrm{SmProj}(\mathbb{C})$ naturally induces a pullback map $f^*\colon H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$, but there are several ways to define a map "in the other direction", a *pushforward* map f_*. One way to proceed is the following: Y, X are connected smooth complex projective varieties of respective dimension e, d, so their analytification Y_{an}, X_{an} are smooth oriented connected compact manifolds of respective real dimension $2e, 2d$. Poincaré duality yields isomorphisms $H^i(X_{\mathrm{an}}, \mathbb{Q}) \cong H_{2d-i}(X_{\mathrm{an}}, \mathbb{Q})$ for all i, similarly for Y. Then the pushforward f_* in cohomology is defined as the top map in the following diagram:

$$\begin{tikzcd}
H^i(Y, \mathbb{Q}) \arrow[r, "f_*"] \arrow[d, "\cong"'] & H^{2(d-e)+i}(X, \mathbb{Q}) \arrow[d, "\cong"] \\
H_{2e-i}(Y_{an}, \mathbb{Q}) \arrow[r, "f_*"] & H_{2e-i}(X_{an}, \mathbb{Q})
\end{tikzcd}$$

In particular, if $f\colon Y_{d-i} \hookrightarrow X_d$ is the inclusion of a subvariety Y of codimension i into X, then f_* maps the fundamental class $[Y] \in H^0(Y, \mathbb{Q})$ to a cycle $f_*[Y] \in H^{2i}(X, \mathbb{Q})$. Extended linearly, this yields a map

$$\gamma_X = \gamma_X^i\colon Z^i(X) \to H^{2i}(X, \mathbb{Q}), \tag{2}$$

called the *cycle class map.* It sends algebraic cycles to topological cycles.

An algebraic cycle $Y \in Z(X)$ is said to be *homologically equivalent to 0* if its cycle class is homologically trivial, i.e. if $\gamma_X(Y) = 0$ in $H(X, \mathbb{Q})$, we write $Y \sim_{\mathrm{hom}} 0$. This defines an adequate equivalence relation called *homological equivalence.* By definition, $Z^i_{\mathrm{hom}}(X) = \ker(\gamma_X^i)$ and $C^i_{\mathrm{hom}}(X)$ is isomorphic to a subgroup of $H^{2i}(X, \mathbb{Q})$, hence it is much smaller than $C^i_{\mathrm{rat}}(X)$.

Since $\sim_{\mathrm{rat}}$ is finer than $\sim_{\mathrm{hom}}$, γ_X factors through the Chow groups and

$$\gamma_X\colon\ \mathrm{CH}(X) \to H(X, \mathbb{Q})$$

is a ring homomorphism which doubles the grading, i.e. $\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$ for all algebraic cycles α, β on X.

Remark 1.3.2.1. Understanding the image of $\gamma_X^i : Z^i(X)_{\mathbb{Q}} \to H^{2i}(X, \mathbb{Q})$, i.e. which topological cycles are linear combinations of classes of algebraic cycles, is a very hard question. The *Hodge conjecture* is an attempt to answer it. See [EH16, App. C] for a nice introduction. The *Grothendieck's standard conjectures* are also related to it, see Section 2.4. △

Remark 1.3.2.2. Homological equivalence can be defined for any *classical Weil cohomology theory* in place of rational Betti cohomology, see [Stacks, Tag 0FGS]. Conjecturally, $\sim_{\mathrm{hom}}$ coincides with $\sim_{\mathrm{num}}$ introduced below. Thus, it is expected that $\sim_{\mathrm{hom}}$ does not depend on the chosen cohomology theory, which is not clear a priori. △

1.3.3 Numerical equivalence. Let $X_d \in \mathrm{SmProj}(\mathrm{k})$. For $0 \leq i \leq d$, the degree of the intersection product

$$\mathrm{CH}^i(X) \times \mathrm{CH}^{d-i}(X) \xrightarrow{\cdot} \mathrm{CH}^d(X) \xrightarrow{\deg} \mathbb{Z}$$

yields a pairing. Cycles $\alpha \in Z^i(X)$ whose rational class $[\alpha] \in \mathrm{CH}^i(X)$ is such that

$$\deg([\alpha] \cdot [\beta]) = 0$$

for all $\beta \in \mathbb{Z}^{d-i}(X)$ are said to be *numerically equivalent to 0*, denoted $\alpha \sim_{\mathrm{num}} 0$. This defines an adequate equivalence relation, called *numerical equivalence*.

Remark 1.3.3.1. Given two adequate equivalence relations $\sim$ and $\sim'$, we say that $\sim$ is finer than $\sim'$, written $\sim \leq \sim'$, if $\alpha \sim 0$ implies $\alpha \sim' 0$. The equivalence relations introduced above are linearly ordered as follows [MNP13]:

$$\sim_{\mathrm{rat}} < \sim_{\mathrm{alg}} < \sim_{\mathrm{hom}} \leq \sim_{\mathrm{num}} .$$

This induces a chain of surjections

$$\begin{tikzcd}
\mathrm{CH}(X) \arrow[r, two heads] & C_{\mathrm{alg}}(X) \arrow[r, two heads] & C_{\mathrm{hom}}(X) \arrow[r, two heads] \arrow[dr, hook] & C_{\mathrm{num}}(X) \arrow[d, dashed, "?"] \\
 & & & H(X, \mathbb{Q})
\end{tikzcd}$$

where the conjecture "$\sim_{\mathrm{hom}} = \sim_{\mathrm{num}}$" is equivalent to the existence of the dashed arrow. $\triangle$

2 Pure motives

The intersection theory of algebraic cycles introduced in the previous chapter can be encapsulated into the *category of correspondences*, which has smooth projective varieties as objects and where correspondences replace usual morphisms, modulo a fixed equivalence relation. Even though this category contains all the information we need, it does not have enough structure. We formally turn it into a pseudo-abelian rigid tensor category called the *category of pure motives*, with respect to the chosen equivalence relation.

The adjective "pure" refers to the fact that it only deals with smooth projective varieties. It is hoped that there exists a category of *mixed motives* dealing with all quasi-projective varieties. Although such a category has not been discovered yet, several constructions for what would be its derived category already exist. See [And04] for a nice introduction to mixed motives following Voevodsky's construction. This chapter only deals with pure motives, which will simply be called *motives*.

First, we describe the construction of the category of pure motives and we give several examples. Then we briefly talk about the Grothendieck's standard conjectures on algebraic cycles, which were the main motivation for the development of this framework.

2.1 Construction of the category of pure motives

The construction of the category of pure motives, with respect to a given adequate equivalence relation $\sim$, starts with the category SmProj(k) of smooth projective varieties over k and proceeds in three steps:

$$\mathrm{SmProj(k)}^{\mathrm{op}} \longrightarrow C_\sim \mathrm{SmProj(k)}_{\mathbb{Q}} \longrightarrow \mathrm{Mot}^{\mathrm{eff}}_\sim(k) \longrightarrow \mathrm{Mot}_\sim(k).$$

This construction is an idea of Alexander Grothendieck, although he never published anything about it. It first appeared in the litterature in [Man68].

Let us describe the steps of the construction, while giving examples along the way:

2.1.1 Category of correspondences. First, we enlarge morphisms by constructing the category of correspondences $C_\sim$ SmProj(k). Its objects are those of SmProj(k) and morphisms are the degree zero correspondences:

$$\mathrm{Hom}_{C_\sim \mathrm{SmProj(k)}}(X, Y) \coloneqq \mathrm{Corr}^0_\sim(X, Y) \coloneqq \bigoplus_i C^{d_i}_\sim(X_{d_i} \times Y)_{\mathbb{Q}}.$$

where $X = \bigcup_i X_{d_i}$ is the decomposition of X into connected components of respective dimension d_i. A correspondence $f \in \mathrm{Corr}^0_\sim(X, Y)$ is denoted $f : X \vdash Y$.

More generally, we let $\mathrm{Corr}_\sim(X, Y) \coloneqq C_\sim(X \times Y)_{\mathbb{Q}}$ and

$$\mathrm{Corr}^r_\sim(X, Y) \coloneqq \bigoplus_i C^{d_i+r}_\sim(X_{d_i} \times Y)_{\mathbb{Q}}.$$

Composition of correspondences $f \in \mathrm{Corr}_\sim(X, Y)$ and $g \in \mathrm{Corr}_\sim(Y, Z)$ is defined as follows:

$$g \circ f \coloneqq {\mathrm{pr}^{XYZ}_{XZ}}_* \left({\mathrm{pr}^{XYZ}_{XY}}^*(f) \cdot {\mathrm{pr}^{XYZ}_{YZ}}^*(g) \right).$$

If f, g have degree r, s, respectively, then $g \circ f$ has degree $r + s$. This yields a well defined composition in $C_\sim \operatorname{SmProj}(\mathrm{k})$:

$$\operatorname{Corr}^0_\sim(X, Y) \times \operatorname{Corr}^0_\sim(Y, Z) \longrightarrow \operatorname{Corr}^0_\sim(X, Z),$$

and the identity $\mathrm{id}_X \in \operatorname{Corr}^0_\sim(X, X)$ is the class of the diagonal $\Delta_X \subset X \times X$.

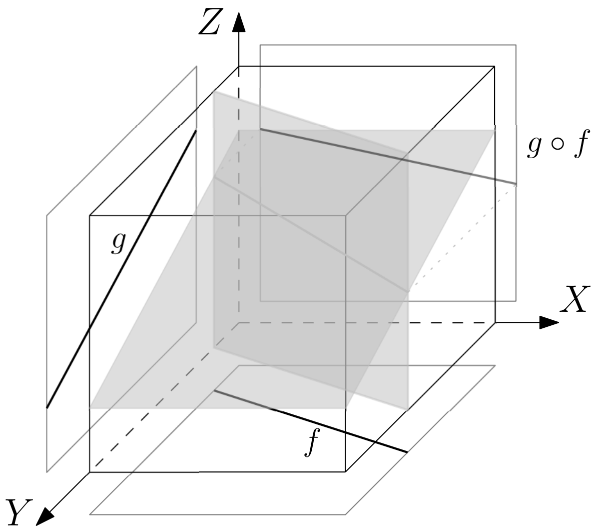


Figure 1: Composition of correspondences

Figure 1 illustrates the composition of correspondences $f \in \operatorname{Corr}_\sim(X, Y) = C_\sim(X \times Y)$ and $g \in \operatorname{Corr}_\sim(Y, Z) = C_\sim(Y \times Z)$. Consider the "cube" $X \times Y \times Z$. On the face $X \times Y$ lies f and on the face $Y \times Z$ lies g. We consider the intersection of their pullback $p^{XYZ*}_{XY}(f) = f \times Z$ and $p^{XYZ*}_{YZ}(g) = X \times g$ (in gray on the figure). Then projecting this intersection of the face $X \times Z$ yields the composition $g \circ f$.

There is a contravariant functor

$$\begin{aligned} \operatorname{SmProj}(\mathrm{k}) &\to C_\sim \operatorname{SmProj}(\mathrm{k}) \\ X &\mapsto X \\ (Y \xrightarrow{f} X_d) &\mapsto (\Gamma_f)^\intercal \in \operatorname{Corr}^0_\sim(X, Y) \end{aligned}$$

where $\Gamma_f \in C^d_\sim(Y \times X_d)$ is the graph of f and $(\Gamma_f)^\intercal \in C^d_\sim(X_d \times Y)$ its transpose, i.e. its image under the "swapping" automorphism $Y \times X \to X \times Y$. Note that the graph Γ_f also induces a correspondence $\Gamma_f \in \operatorname{Corr}^{d-e}_\sim(Y_e, X_d)$. In fact, $\Gamma_f = f_*$ and $(\Gamma_f)^\intercal = f^*$ as maps on cycles as defined in Section 1.

Remark 2.1.1.1. The category $C_\sim \operatorname{SmProj}(\mathrm{k})$ is a $\mathbb{Q}$-linear category, i.e. its Hom-sets are $\mathbb{Q}$-modules and composition is bilinear. Direct sums are given by disjoint unions of varieties. Product of varieties endows $C_\sim \operatorname{SmProj}(\mathrm{k})$ with the structure of a symmetric monoidal category, with the point $\operatorname{Spec}(k)$ as unit. △

Remark 2.1.1.2. If $f\colon Y_d \to X_d$ is a generically finite morphism of degree n, then f is surjective and we have $f_* \circ f^* = n\,\mathrm{id}$. Indeed, for any cycle $\alpha \in C_\sim(X)$, we have

$$f_*(f^*(\alpha)) = f_*([Y] \cdot f^*(\alpha)) = f_*([Y]) \cdot \alpha$$

since $[Y]$ is the unit in $C_\sim(Y)$ and using the projection formula (Lemma 1.1.6.1). By definition of the pushforward and generic finiteness, we have $f_*([Y]) = n[X]$, which concludes. △

To conclude this section on correspondences, let us mention a Lemma that will be useful at several points:

Lemma 2.1.1.3 (Lieberman's Lemma [Ful84, Prop. 16.1.1][MNP13, Lem. 2.1.3])**.** *Given correspondences $f \in \mathrm{Corr}_\sim(X,Y)$, $\alpha \in \mathrm{Corr}_\sim(X,X')$, $\beta \in \mathrm{Corr}_\sim(Y,Y')$, we have*

$$(\alpha \times \beta)_*(f) = \beta \circ f \circ \alpha^\intercal.$$

2.1.2 Category of effective motives.

The second step is purely formal: the category of *effective motives* $\mathrm{Mot}_\sim^{\mathrm{eff}}(k)$ is obtained by taking the *pseudo-abelian envelope*[2] of $C_\sim\,\mathrm{SmProj(k)}$, i.e. the smallest category containing it in which projectors split:

Definition 2.1.2.1. In an additive category $\mathcal{C}$, a *projector* p on X is an endomorphism $p : X \to X$ such that $p \circ p = p$. A projector p on X is said to *split* if there exists an object X' and a factorization of p as $X \xrightarrow{g} X' \xrightarrow{i} X$ such that $i \circ g = p$ and $g \circ i = \mathrm{id}_{X'}$.

Objects of $\mathrm{Mot}_\sim^{\mathrm{eff}}(k)$ are pairs (X,p) where $X \in \mathrm{SmProj(k)}$ and $p \in \mathrm{Corr}^0_\sim(X,X)$ is a projector. They are called *effective motives.* For two effective motives (X,p) and (Y,q), we let

$$\mathrm{Hom}_{\mathrm{Mot}_\sim^{\mathrm{eff}}(k)}((X,p),(Y,q)) \coloneqq q \circ \mathrm{Corr}^0_\sim(X,Y) \circ p, \tag{3}$$

and composition is given by composition of correspondences. The identity of (X,p) is p. We often write (X,id) to mean (X,Δ_X). Again, direct sum is given by disjoint union:

$$(X,p) \oplus (Y,q) \coloneqq (X \sqcup Y, p \sqcup q).$$

Remark 2.1.2.2. The effective motive (X,p) should be thought of as a piece of X corresponding to the image of p. For a morphism $f\colon (X,p) \to (Y,q)$, we have $f = q \circ f = f \circ p$. This illustrates the fact that f should be defined on the image of p and should arrive into the image of q. △

There is a fully faithful functor

$$\begin{aligned} C_\sim\,\mathrm{SmProj(k)} &\to \mathrm{Mot}_\sim^{\mathrm{eff}}(k) \\ X &\mapsto (X,\Delta_X) \\ f &\mapsto f \end{aligned}$$

which preserves the monoidal structure, where $\otimes$ is given on $\mathrm{Mot}_\sim^{\mathrm{eff}}(k)$ by

$$(X,p) \otimes (Y,q) \coloneqq (X \times Y, p \times q).$$

Remark 2.1.2.3. The above functor is universal among functors from $C_\sim\,\mathrm{SmProj(k)}$ to pseudo-abelian categories. This is what is meant by the pseudo-abelian envelope. △

Definition 2.1.2.4. The motive of a point is denoted $\mathbf{1} \coloneqq (\mathrm{Spec}(k),\mathrm{id})$ and called the *unit motive.* The motive of a variety $X \in \mathrm{SmProj(k)}$ is denoted $\mathfrak{h}(X) \coloneqq (X,\Delta_X) = (X,\mathrm{id})$.

[2]This is also called the *Karoubi envelope* after Max Karoubi, a student of Grothendieck.

Remark 2.1.2.5. Given two orthogonal projectors $p, q \in \mathrm{Corr}^0_\sim(X, X)$ on $X \in \mathrm{SmProj(k)}$, i.e. $p \circ q = q \circ p = 0$, we get an isomorphism of effective motives

$$(X, p + q) \cong (X, p) \oplus (X, q).$$

Indeed, we have $(X, p) \oplus (X, q) = (X \sqcup X, p \sqcup q)$ by definition. Candidates for isomorphisms between $(X \sqcup X, p \sqcup q)$ and $(X, p + q)$ are not abundant. The only natural morphisms are:

$$\begin{aligned}(\mathrm{id}_X \times \iota_1)_*(p) + (\mathrm{id}_X \times \iota_2)_*(q) &\in \mathrm{Corr}^0_\sim(X, X \sqcup X)\\ (\mathrm{id}_{X \sqcup X} \times \nabla)_*(p \sqcup q) &\in \mathrm{Corr}^0_\sim(X \sqcup X, X)\end{aligned}$$

where $\iota_1, \iota_2 : X \hookrightarrow X \sqcup X$ are the inclusions and $\nabla : X \sqcup X \to X$ is the "fold" map. We need to show that these are indeed morphisms of effective motives, and that their compositions yield the identities. A direct computation is not hard, but figures are more informative. The different correspondences in play are depicted on Figure 2 and their compositions on Figure 3. The meaning of these figures is explained on Figure 1. For example, In 3a, one "cube" computes $p \circ p = p$ and the other computes $q \circ q = q$. Projecting on the back face yield $p + q$. In 3b, orthogonality of p and q is crucial in order to obtain the identity on $(X \sqcup X, p \sqcup q)$.

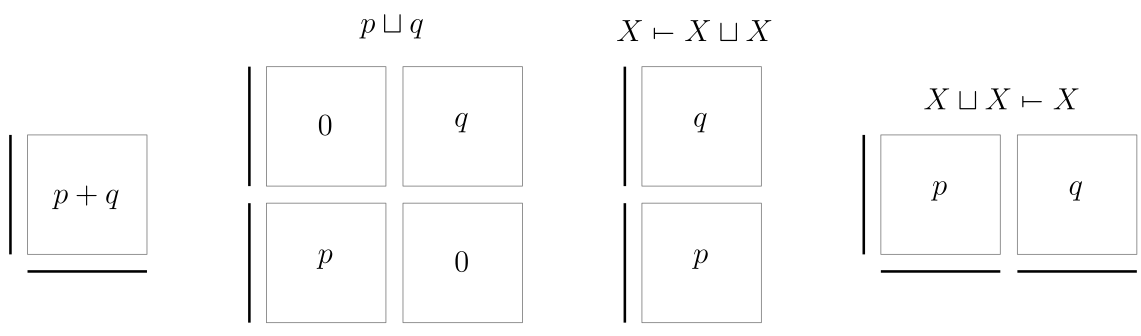


Figure 2: The different correspondences

In particular, if q is a projector on (X, p), then $p = q + (p - q)$ is a decomposition of $p = \mathrm{id}_{(X,p)}$ into orthogonal projectors: $q \circ (p - q) = q \circ p - q \circ q = 0$ and $(p - q) \circ q = 0$. We obtain the splitting

$$(X, p) \cong (X, q) \oplus (X, p - q)$$

where (X, q) is the image of q and $(X, p - q)$ is its kernel. Therefore, $\mathrm{Mot}^{\mathrm{eff}}_\sim(k)$ is indeed a pseudo-abelian category, as expected. $\triangle$

Example 2.1.2.6. Let $X_d \in \mathrm{SmProj(k)}$ and $e \in X(k)$ a rational point. The cycles

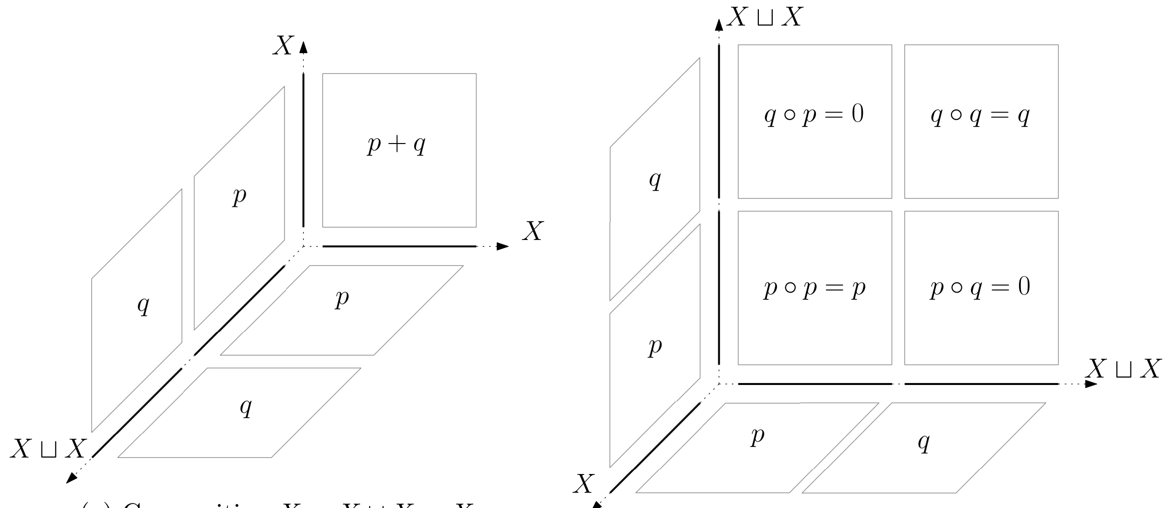
$$p_0(X) := e \times X, \;\; p_{2d}(X) := X \times e$$

define orthogonal projectors. By Remark 2.1.2.5, we obtain the decomposition

$$\mathfrak{h}(X) \cong \mathfrak{h}^0(X) \oplus \mathfrak{h}^+(X) \oplus \mathfrak{h}^{2d}(X)$$

where $\mathfrak{h}^?(X) := (X, p_?(X))$ for $? = 0, 2d, +$, with $p_+(X) = \mathrm{id}_X - p_0(X) - p_{2d}(X)$. In fact, $\mathfrak{h}^0(X)$ is isomorphic to the unit motive $\mathbf{1}$ and the "top dimensional part" $\mathfrak{h}^{2d}(X)$ only depends on the dimension d of X.

It follows that the nontrivial information about $\mathfrak{h}(X)$ is concentrated in the middle part $\mathfrak{h}^+(X)$. If X is a curve, this middle part $\mathfrak{h}^+(X)$ is related to its Jacobian, see Section 2.3. $\triangle$



(a) Composition $X \vdash X \sqcup X \vdash X$

(b) Composition $X \sqcup X \vdash X \vdash X \sqcup X$

Figure 3: Their compositions

Definition 2.1.2.7. The *Lefschetz motive* is $\mathbf{L} := \mathfrak{h}^2(\mathbb{P}^1) = (\mathbb{P}^1, \mathbb{P}^1 \times e)$.

Remark 2.1.2.8. As is mentioned in Example 2.1.2.6, the top dimensional part of the motive of a variety is the same for all d-dimensional varieties. In particular, they all coincide with $\mathfrak{h}^{2d}((\mathbb{P}^1)^d) = ((\mathbb{P}^1)^d, (\mathbb{P}^1)^d \times e^d) = \mathbf{L}^{\otimes d}$. As usual in this kind of result, finding an isomorphism $\mathfrak{h}^{2d}(X) \cong \mathfrak{h}^{2d}(Y)$ for two d-dimensional varieties $X, Y \in \mathrm{SmProj}(k)$ reduces to writing down the only natural morphisms there are. In fact, the proof is contained inside Example 2.1.2.9 below. △

Example 2.1.2.9. By the computation of the diagonal $\Delta_{\mathbb{P}^1}$ in Section 1.2.3, we have $p_+(\mathbb{P}^1) = 0$ and thus

$$\mathfrak{h}(\mathbb{P}^1) \cong \mathbf{1} \oplus \mathbf{L},$$

which is a sort of cellular decomposition: "$[\mathbb{P}^1] = [\text{point}] + [\text{line}]$". Thus, $\mathbf{L}$ can also be thought of as standing for *line*.

In the case of $\mathbb{P}^n$,

$$\Delta_{\mathbb{P}^n} = \sum_i \mathbb{P}^{n-i} \times \mathbb{P}^i$$

is a decomposition of the diagonal into orthogonal projectors, yielding

$$(\mathbb{P}^n, \mathrm{id}) \cong \bigoplus_i (\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}).$$

In fact, $(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}) \cong (Y, Y \times e)$ where we write $Y := \mathbb{P}^i$ in order not to confuse it with

$\mathbb{P}^i \subset \mathbb{P}^n$. This isomorphism is given by the following correspondences

$$\begin{aligned}
\mathbb{P}^i \times e = (Y \times e) \circ (\mathbb{P}^i \times e) \circ (\mathbb{P}^i \times \mathbb{P}^{n-i}) &\in \mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}), (Y, Y \times e)) \\
&= (Y \times e) \circ \mathrm{Corr}^0_\sim(\mathbb{P}^n, Y) \circ (\mathbb{P}^i \times \mathbb{P}^{n-i}) \\
&= (Y \times e) \circ C^n_\sim(\mathbb{P}^n \times Y) \circ (\mathbb{P}^i \times \mathbb{P}^{n-i}) \\
Y \times \mathbb{P}^{n-i} = (\mathbb{P}^i \times \mathbb{P}^{n-i}) \circ (Y \times \mathbb{P}^{n-i}) \circ (Y \times e) &\in \mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((Y, Y \times e), (\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i})) \\
&= (\mathbb{P}^i \times \mathbb{P}^{n-i}) \circ \mathrm{Corr}^0_\sim(Y, \mathbb{P}^n) \circ (Y \times e) \\
&= (\mathbb{P}^i \times \mathbb{P}^{n-i}) \circ C^i_\sim(Y \times \mathbb{P}^n) \circ (Y \times e).
\end{aligned}$$

Indeed, we have

$$\begin{aligned}
(\mathbb{P}^i \times e) \circ (Y \times \mathbb{P}^{n-i}) &= (p_{13})_*((Y \times \mathbb{P}^{n-i} \times Y) \cdot (Y \times \mathbb{P}^i \times e)) \\
&= (p_{13})_*(Y \times (\mathbb{P}^{n-i} \cdot \mathbb{P}^i) \times e) = Y \times e = \mathrm{id}_{(Y, Y \times e)}
\end{aligned}$$

since $\mathbb{P}^{n-i} \cdot \mathbb{P}^i$ is the class of a point. The other composition is

$$\begin{aligned}
(Y \times \mathbb{P}^{n-i}) \circ (\mathbb{P}^i \times e) &= (p_{13})_*((\mathbb{P}^i \times e \times \mathbb{P}^n) \cdot (\mathbb{P}^n \times Y \times \mathbb{P}^{n-i})) \\
&= (p_{13})_*(\mathbb{P}^i \times e \times \mathbb{P}^{n-i})) = \mathbb{P}^i \times \mathbb{P}^{n-i} = \mathrm{id}_{(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i})}\,.
\end{aligned}$$

This concludes the proof that $(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}) \cong (Y, Y \times e) \cong \mathbf{L}^{\otimes i}$. Note that the only fact about Y that we used is that it is a i-dimensional variety, hence this also proves what was claimed in Remark 2.1.2.8.

Therefore, the motive of the projective space of dimension n splits as follows

$$\mathfrak{h}(\mathbb{P}^n) \cong \mathbf{1} \oplus \mathbf{L} \oplus \mathbf{L}^{\otimes 2} \oplus \cdots \oplus \mathbf{L}^{\otimes n}\,.$$

Any *cellular variety* (e.g. Grassmanians) has a decomposition of that sort, with one $\mathbf{L}^d$ for each cell of dimension d. △

Example 2.1.2.10. Let $f\colon Y_d \to X_d$ be a surjective generically finite morphism of degree n as in Remark 2.1.1.2. We know that $f_* \circ f^* = n\,\mathrm{id}_{\mathfrak{h}(X)}$. Hence, $p \coloneqq \frac{1}{n} f^* \circ f_*$ is a projector on Y. In fact, $(Y, p) \cong \mathfrak{h}(X)$ and thus $\mathfrak{h}(Y) \cong \mathfrak{h}(X) \oplus (Y, \mathrm{id} - p)$. △

2.1.3 Category of motives. This last step is again purely formal. It consists in adding an inverse to the Lefschetz motive $\mathbf{L}$. Objects of the category $\mathrm{Mot}_\sim(k)$ of *pure motives* are triples (X, p, m) where $X \in \mathrm{SmProj(k)}$, p is a projector on X and $m \in \mathbb{Z}$. Morphisms are defined as follows:

$$\mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((X, p, m), (Y, q, n)) \coloneqq q \circ \mathrm{Corr}^{n-m}_\sim(X, Y) \circ p \tag{4}$$

and composition is given by composition of correspondences. The category of effective motives $\mathrm{Mot}^{\mathrm{eff}}_\sim(k)$ naturally embeds into the category of pure motives $\mathrm{Mot}_\sim(k)$ and all the computations from the previous section hold.

There is a contravariant functor

$$\begin{aligned}
\mathfrak{h}\colon \mathrm{SmProj(k)}^{\mathrm{op}} &\to \mathrm{Mot}_\sim(k) \\
X &\mapsto \mathfrak{h}(X) \coloneqq (X, \mathrm{id}, 0) \\
f &\mapsto f^* = (\Gamma_f)^\intercal,
\end{aligned}$$

sending a variety X to its *motive* $\mathfrak{h}(X)$.

The tensor product

$$(X,p,m)\otimes(Y,q,n):=(X\times Y,p\times q,m+n)$$

turns $\mathrm{Mot}_\sim(k)$ into a symmetric monoidal category, which is now *rigid*, i.e. every object has a dual:

$$(X_d,p,m)^\vee:=(X,p^\intercal,d-m)$$

such that $(-\otimes M^\vee,-\otimes M)$ and $(M^\vee\otimes-,M\otimes-)$ are adjoint pairs. Dual morphisms are defined by taking the transpose of (4). Moreover, the functor $\mathfrak{h}$ preserves the monoidal structure.

Remark 2.1.3.1. The functor $\mathfrak{h}\colon \mathrm{SmProj(k)}^{\mathrm{op}}\to\mathrm{Mot}_\sim(k)$ is not *conservative*, i.e. there are non-isomorphic varieties that have isomorphic motives. For example, two projective bundles of the same dimension over the same variety have isomorphic motives, see [Man68, §7]. △

Remark 2.1.3.2. There is a canonical isomorphism $\mathbf{L}=(\mathrm{Spec}(k),\mathrm{id},-1)$. By definition, $\mathbf{L}=(\mathbb{P}^1,\mathbb{P}^1\times e,0)$ where $e\in\mathbb{P}^1$ is any rational point. As before, candidates for isomorphisms are not abundant. Writing $*:=\mathrm{Spec}(k)$, the only natural morphisms are:

$$\begin{aligned}
f=(\mathbb{P}^1\times e)\circ(*\times e)&\in\mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((*,\mathrm{id},-1),(\mathbb{P}^1,\mathbb{P}^1\times e,0))\\
&=(\mathbb{P}^1\times e)\circ\mathrm{Corr}^1_\sim(*,X)\\
&=(\mathbb{P}^1\times e)\circ C^1_\sim(*\times\mathbb{P}^1)\\
g=(\mathbb{P}^1\times *)\circ(\mathbb{P}^1\times e)&\in\mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((\mathbb{P}^1,\mathbb{P}^1\times e,0),(*,\mathrm{id},-1))\\
&=\mathrm{Corr}^{-1}_\sim(\mathbb{P}^1,*)\circ(\mathbb{P}^1\times e)\\
&=C^0_\sim(\mathbb{P}^1\times *)\circ(\mathbb{P}^1\times e)
\end{aligned}$$

We have $f=*\times e$ and $g=\mathbb{P}^1\times *$, which is canonical. Let us compute their compositions:

$$\begin{aligned}
&g\circ f=p_{13*}((*\times e\times *)\cdot(*\times\mathbb{P}^1\times *))=p_{13*}(*\times e\times *)=*\times *=\Delta_*=\mathrm{id}_{(*,\mathrm{id},-1)}\\
&f\circ g=p_{13*}((\mathbb{P}^1\times *\times\mathbb{P}^1)\cdot(\mathbb{P}^1\times *\times e))=\mathbb{P}^1\times e=\mathrm{id}_{(\mathbb{P}^1,\mathbb{P}^1\times e,0)}\,.
\end{aligned}$$

Therefore, these morphisms yield a canonical isomorphism $\mathbf{L}\cong(\mathrm{Spec}(k),\mathrm{id},-1)$. △

It follows from the above remark that $\mathbf{L}^{\otimes r}=(\mathrm{Spec}(k),\mathrm{id},-r)$. For brevity, we sometimes write $\mathbf{L}^r$ for $\mathbf{L}^{\otimes r}$. We denote the dual of $\mathbf{L}$ by

$$\mathbf{L}^{-1}:=(\mathrm{Spec}(k),\mathrm{id},1),$$

and $\mathbf{L}^{-r}=(\mathbf{L}^{-1})^{\otimes r}$ for all $r>0$. In particular, we have $(X,p,m)=(X,p,0)\otimes\mathbf{L}^{-m}$ for any motive (X,p,m).

Remark 2.1.3.3. Sometimes, we use the notation

$$p\,\mathfrak{h}(X)(m):=(X,p,m)$$

to emphasize that (X,p,m) is the m-th twist of the "image" $p\,\mathfrak{h}(X)$ of p. Then, $\mathbf{L}=\mathbf{1}(-1)$ and $\mathbf{L}^r=\mathbf{1}(-r)$. △

Remark 2.1.3.4. Morphisms from $\mathbf{L}^{\otimes r}$ to the motive $\mathfrak{h}(X)$ of a variety $X \in \mathrm{SmProj(k)}$ correspond to cycles in $C^r_\sim(X)$. Indeed, $\mathbf{L}^r = (\mathrm{Spec}(k), \mathrm{id}, -r)$ and by definition

$$\mathrm{Hom}_{\mathrm{Mot}_\sim(k)}(\mathbf{L}^{\otimes r}, \mathfrak{h}(X)) = \mathrm{Corr}^r_\sim(\mathrm{Spec}(k), X) = C^r_\sim(X).$$

△

Remark 2.1.3.5. The construction of the category of motives can be done without tensoring with $\mathbb{Q}$. In this case, we end up with a category with integral coefficients, denoted $\mathrm{Mot}_\sim(k)_\mathbb{Z}$ and called the category of *integral motives* modulo $\sim$. Most of the above results hold for integral motives in the same way, but not all of them. For instance, rational coefficients are needed for Example 2.1.2.10 and they will be essential for defining symmetric and alternating powers of motives in Section 3. △

2.1.4 Chow motives. If $\sim = \sim_{\mathrm{rat}}$ is rational equivalence, then we call $\mathrm{Mot}_{\mathrm{rat}}(k)$ the category of *Chow motives*. It is sometimes denoted $\mathrm{CHM}(k)$. The object $\mathfrak{h}(X)$ is called the *Chow motive* of $X \in \mathrm{SmProj(k)}$.

Lemma 1.1.6.1 tells us that Chow groups are functorial over the category of Chow correspondences $\mathrm{CH}\,\mathrm{SmProj(k)} \coloneqq C_{\mathrm{rat}}\,\mathrm{SmProj(k)}$, i.e. the functor $\mathrm{CH}^i\colon \mathrm{SmProj(k)} \to \mathrm{Ab}$ factors through $\mathrm{CH}\,\mathrm{SmProj(k)}$.

Hence, a projector $p\colon X \vdash X$, $p \circ p = p$ induces a projector $p_*\colon \mathrm{CH}^i(X)_\mathbb{Q} \to \mathrm{CH}^i(X)_\mathbb{Q}$. For the motive $M = (X, p, m)$, we define the *i-th Chow group of M* as

$$\mathrm{CH}^i(M) \coloneqq \mathrm{Im}(p_*\colon \mathrm{CH}^{i+m}(X)_\mathbb{Q} \to \mathrm{CH}^{i+m}(X)_\mathbb{Q}). \tag{5}$$

In particular, $\mathrm{CH}(\mathfrak{h}(X)) = \mathrm{CH}(X)_\mathbb{Q}$ and $\mathrm{CH}(p\,\mathfrak{h}(X)) = p_*(\mathrm{CH}(X)_\mathbb{Q})$, i.e. the Chow group of the image of a projector is its image in the Chow group.

Remark 2.1.4.1. For a motive $M = (X, p, m)$, we have

$$\mathrm{CH}^i(M) \cong \mathrm{Hom}_{\mathrm{Mot}_{rat}(k)}(\mathbf{L}^i, M). \tag{6}$$

Indeed,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mot}_{rat}(k)}(\mathbf{L}^i, M) &= p \circ \mathrm{Corr}^{i+m}_{\mathrm{rat}}(\mathrm{Spec}(k), X)\\ &= p \circ \mathrm{CH}^{i+m}(\mathrm{Spec}(k) \times X)\\ &= \{p \circ \alpha : \alpha \in \mathrm{CH}^{i+m}(X)\}\end{aligned}$$

and a quick computation shows that $p \circ \alpha = p_*(\alpha)$ under the identification $\mathrm{Spec}(k) \times X \cong X$.

In particular, the functors CH^i become representable in the category $\mathrm{Mot}_{\mathrm{rat}}(k)$. △

The above remark enables us to introduce the following notation [Sch94]: Given a cycle $\alpha \in \mathrm{CH}^i(X)$, we denote $\alpha_* \in \mathrm{Hom}_{\mathrm{Mot}_{rat}(k)}(\mathbf{L}^i, \mathfrak{h}(X))$ the corresponding morphism, represented by $\mathrm{Spec}(k) \times \alpha$. Tensoring its dual $\alpha_*^\vee \in \mathrm{Hom}_{\mathrm{Mot}_{rat}(k)}(X \otimes \mathbf{L}^{-d}, \mathbf{L}^{-i})$ with $\mathbf{L}^d$ yields $\alpha^* \coloneqq \alpha_*^\vee \otimes \mathbf{L}^d \in \mathrm{Hom}_{\mathrm{Mot}_{rat}(k)}(X, \mathbf{L}^{d-i})$, which is simply represented by $\alpha \times \mathrm{Spec}(k)$.

Example 2.1.4.2. Taking the Chow group of the decomposition $\mathfrak{h}(\mathbb{P}^1) \cong \mathbf{1} \oplus \mathbf{L}$ yields

$$\mathrm{CH}(\mathbb{P}^1) = \mathrm{CH}(\mathbf{1}) \oplus \mathrm{CH}(\mathbf{L}).$$

In fact, under this decomposition we have $\mathrm{CH}(\mathbf{1}) = \mathrm{CH}^0(\mathbb{P}^1)$ and $\mathrm{CH}(\mathbf{L}) = \mathrm{CH}^1(\mathbb{P}^1)$. Thus, each summand is responsible for a specific part of the Chow group of $\mathbb{P}^1$. We will see below that this happens for cohomology as well. △

Remark 2.1.4.3. For any equivalence relation $\sim$, we can define the *cycles groups* $C^i_\sim(M)$ of a motive M in the same way and formula (6) holds. $\triangle$

2.2 Cohomology of motives

In this section $k = \mathbb{C}$ and H denotes *Betti cohomology.* Note that everything can be stated in terms of a *classical Weil cohomology.* If $\sim$ is an equivalence relation that is finer than $\sim_{\text{hom}}$, then the cohomology functor

$$H\colon \text{SmProj(k)} \to \text{GrVect}_{\mathbb{Q}}$$

to the category of graded $\mathbb{Q}$-vector spaces factors through $C_\sim \text{SmProj(k)}$.

Hence, a projector $p\colon X \vdash X$, $p \circ p = p$ induces a projector $p_*\colon H(X) \to H(X)$. For the motive $M = (X, p, m)$, we define the *cohomology groups* of M as

$$H^i(M) := \text{Im}(p_*\colon H^{i+2m}(X) \to H^{i+2m}(X)).$$

Remark 2.2.0.1. Let $X_d \in \text{SmProj(k)}$ be a variety of dimension d. Taking cohomology of the decomposition of example 2.1.2.6: $\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^+(X) \oplus \mathfrak{h}^{2d}(X)$ yields

$$H(X) = H(\mathfrak{h}(X)) = H(\mathfrak{h}^0(X)) \oplus H(\mathfrak{h}^+(X)) \oplus H(\mathfrak{h}^{2d}(X)).$$

In fact, $H(\mathfrak{h}^0(X)) = H^0(X)$ and $H(\mathfrak{h}^{2d}(X)) = H^{2d}(X)$. Hence all the nontrivial cohomological information about X is contained in the motive $\mathfrak{h}^+(X)$. If $\sim$ is strictly finer than $\sim_{\text{hom}}$, then $\mathfrak{h}^+(X)$ contains more information than just the cohomological one, as we shall see for curves in Section 2.3. $\triangle$

2.3 The Chow motive of a curve

In this section, $\sim = \sim_{\text{rat}}$ and we consider integral Chow motives, see Remark 2.1.3.5.

Let $X \in \text{SmProj(k)}$ be a curve, i.e. a one-dimensional irreducible variety, and pick a rational point $e \in X$. From Example 2.1.2.6, the motive of X splits as

$$\mathfrak{h}(X) = \mathfrak{h}^0(X) \oplus \mathfrak{h}^1(X) \oplus \mathfrak{h}^2(X)$$

where $\mathfrak{h}^0(X) = (X, p_0 = e \times X) \cong \mathbf{1}$, $\mathfrak{h}^2(X) = (X, p_2 = X \times e) \cong \mathbf{L}$ and the remaining part $\mathfrak{h}^1(X)$ is the projection of $\mathfrak{h}(X)$ along the projector $p_1 := \Delta_X - e \times X - X \times e$. These projectors are illustrated on Figure 4, where white lines depict cycles taken with a negative sign.

2.3.1 Chow groups of a curve. We have[3] $p_{0*}[X] = [X]$, $p_{2*}[X] = 0$ and $p_{1*}[X] = 0$. For a rational point $z \in X$, we have $p_{0*}[z] = 0$ and $p_{2*}[z] = [e]$, hence $p_{1*}[z] = [z] - [e]$. In particular,

$$\text{CH}^0(\mathfrak{h}^0(X)) = \mathbb{Z}.[X]$$
$$\text{CH}^1(\mathfrak{h}^2(X)) = \mathbb{Z}.[e]$$

[3] This follows from quick computations, but looking at the figure should be convincing.

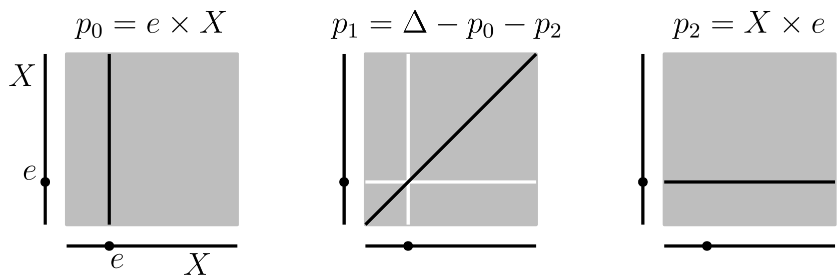


Figure 4: The projectors p_0, p_1, p_2 on the motive of a curve

are the only nontrivial Chow groups of $\mathfrak{h}^0(X)$ and $\mathfrak{h}^2(X)$, and the remaining part

$$\mathrm{CH}(\mathfrak{h}^1(X)) = \mathrm{CH}^1(\mathfrak{h}^1(X)) = \mathrm{Im}(p_{1*}\colon\ \mathrm{CH}^1(X) \to \mathrm{CH}^1(X))$$

consists of the degree zero 1-cycles, that is, $\mathrm{CH}^1(\mathfrak{h}^1(X)) = \mathrm{Pic}^0(X)$ is the subgroup of the Picard group consisting of the degree zero divisors.

Remark 2.3.1.1. Suppose $k = \mathbb{C}$. From Remark 2.2.0.1, we know that $H(\mathfrak{h}^1(X)) = H^1(X)$. It is free of rank $2g$ where g is the genus of the curve X. △

2.3.2 Morphisms between motives of curves. Let $X, X' \in \mathrm{SmProj(k)}$ be two curves. By definition of morphisms in $\mathrm{CHM}_{\mathbb{Z}}$, we have a surjective homomorphism

$$\begin{aligned} \psi\colon\ \mathrm{CH}^1(X \times X') &\longrightarrow \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Z}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(X')) \\ T &\longmapsto p_1(X') \circ T \circ p_1(X). \end{aligned}$$

We say that a 1-cycle T on $X \times X'$ is *degenerate* if it equals $D \times X' + X \times D'$ for some 1-cycles $D \in \mathrm{CH}^1(X)$ and $D' \in \mathrm{CH}^1(X')$. Degenerate 1-cycles on $X \times X'$ form a subgroup $\mathrm{CH}^1_{\equiv}(X \times X') \subset \mathrm{CH}^1(X \times X')$, which is precisely the kernel of the above homomorphism:

Lemma 2.3.2.1. *The homomorphism ψ induces an isomorphism*

$$\frac{\mathrm{CH}^1(X \times X')}{\mathrm{CH}^1_{\equiv}(X \times X')} \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Z}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(X')).$$

Proof. Write p_i, $i = 0, 1, 2$, the projectors associated to X and p'_i, $i = 0, 1, 2$, those associated to X', corresponding to fixed rational points $e \in X$ and $e' \in X'$, respectively.

Let $T \in \mathrm{CH}^1(X \times X')$, applying ψ yields

$$\psi(T) = (\Delta_{X'} - p'_0 - p'_2) \circ T \circ (\Delta_X - p_0 - p_2) = \begin{array}{cccccc} & T & - & T \circ p_0 & - & T \circ p_2 \\ - & p'_0 \circ T & + & p'_0 \circ T \circ p_0 & + & p'_0 \circ T \circ p_2 \\ - & p'_2 \circ T & + & p'_2 \circ T \circ p_0 & + & p'_2 \circ T \circ p_2 \end{array}$$

Using Lieberman's Lemma and the fact that p_0, p_2 are transposes of each other, we can compute these terms. For example, the terms

$$\begin{aligned} p'_2 \circ T \circ p_2 &= (p_0 \times p'_2)_*(T) = (\mathrm{pr}_2)_*((T \times X \times X') \cdot (e \times X' \times X \times e')) \\ &= \deg(T \cdot (e \times X')).X \times e' \\ p'_2 \circ T &= (\mathrm{pr}_{13})_*((T \times X') \cdot (X \times X' \times e')) \\ &= \deg(T \cdot (e \times X')).X \times e' \end{aligned}$$

cancel each other out. Similarly, the terms $T \circ p_0$ and $p_0' \circ T \circ p_0$ cancel each other out. The terms $p_0' \circ T \circ p_2$ and $p_2' \circ T \circ p_0$ are zero. There only remains

$$\begin{aligned}\psi(T) &= T - p_0' \circ T - T \circ p_2 \\ &= T - (\mathrm{pr}_{13})_*((T \times X') \cdot (X \times e' \times X')) - (\mathrm{pr}_{13})_*((X \times e \times X') \cdot (X \times T)) \\ &= T - (p_1)_*(T \cdot X \times e') \times X' - X \times (p_2)_*((e \times X') \cdot T)\end{aligned}$$

which is zero if and only if T is degenerate. This shows that $\mathrm{CH}^1_{\equiv}(X \times X')$ is precisely the kernel of ψ, which concludes the proof. □

To a curve $X \in \mathrm{SmProj(k)}$, one can associate an abelian variety $J(X)$ called the *Jacobian variety* of X, whose set of rational points is $\mathrm{Pic}^0(X)$. It satisfies a certain universal property, which can be used to show the existence of natural isomorphisms

$$\frac{\mathrm{CH}^1(X \times X')}{\mathrm{CH}^1_{\equiv}(X \times X')} \cong \mathrm{Hom}_{\mathrm{AV}}(J(X), J(X'))$$

for curves X, X', where AV is the category of abelian varieties [MNP13, App. A]. Together with the isomorphism given by the above lemma, we get:

Theorem 2.3.2.2 ([MNP13, Thm. 2.7.2])**.** *For two smooth projective curves X, X' over k, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{CHM}_{\mathbb{Z}}}(\mathfrak{h}^1(X), \mathfrak{h}^1(X')) \cong \mathrm{Hom}_{\mathrm{AV}}(J(X), J(X')).$$

Consequently, the full subcategory of $\mathrm{Mot}_{\mathrm{rat}}(k)$ whose objects are direct factors and sums of $\mathfrak{h}^1(X)$, for curves X, is equivalent to the category of abelian varieties up to isogenies.

2.4 Grothendieck's standard conjectures

In this section, $k = \mathbb{C}$ and $H(X)$ is the rational Betti cohomology of $X \in \mathrm{SmProj(k)}$, see Section 1.3.2.

The standard conjectures were introduced by Grothendieck in 1968 as a plan to prove the last parts of Weil's conjectures. The latter were finally proved by Deligne via other means, while the standard conjectures remain open. They concern existence problems for algebraic cycles.

The first conjecture was already mentioned in Section 1.3.2. Let us recall it:

Conjecture (Conjecture D(X))**.** Homological equivalence and numerical equivalence coincide for algebraic cycles on X.

Another conjecture is called the standard conjecture of *Hodge* type, which states that a certain pairing is positive definite. We do not describe it here, see [Gro69].

2.4.1 Künneth conjecture. Let $X_d \in \mathrm{SmProj(k)}$ be a smooth projective variety of dimension d and let $\Delta_X \subset X \times X$ be its diagonal. Under the Künneth decomposition, the cohomology class of the diagonal admits a decomposition

$$H^{2d}(X \times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$$
$$\gamma_{X\times X}(\Delta_X) = \sum_{i=0}^{2d} \Delta_i^{\text{topo}}$$

where Δ_i^{topo} is the *i-th Künneth component* of the diagonal.

Remark 2.4.1.1. A topological cycle $\gamma \in H(X \times Y_d)$ is called a *topological correspondence* from X to Y. It acts on cohomological cycles in the same way as an algebraic correspondence:

$$\gamma\colon H(X) \to H(Y)\colon \alpha \mapsto \gamma(\alpha) \coloneqq (\mathrm{pr}_Y)_*(\gamma \cup (\alpha \times Y)).$$

In fact, we can show $H(X\times Y) \cong H(X)\otimes H(Y) \cong H(X)^\vee \otimes H(Y) \cong \mathrm{Hom}(H(X), H(Y))$ using the Künneth formula and Poincaré duality. Hence, linear maps $H(X) \to H(Y)$ correspond to topological correspondences. △

As a topological correspondence, the i-th Künneth component Δ_i^{topo} is precisely the projector of $H(X)$ onto the degree i part $H^i(X)$.

Conjecture (Künneth conjecture C(X))**.** The Künneth components Δ_i^{topo} are algebraic, i.e. they come from algebraic cycles $\Delta_i \in \mathrm{CH}^d(X \times X)$.

If the Künneth conjecture holds for X, then we may wonder whether $\sum_{i=0}^{d} \Delta_i$ already equals Δ_X in $\mathrm{CH}^d(X \times X)$. If this is the case, then the Chow motive of X splits as

$$\mathfrak{h}(X) = \bigoplus_{i=0}^{2d} \mathfrak{h}^i(X)$$

where $\mathfrak{h}^i(X) = (X, \Delta_i)$, such that each summand contributes only to the corresponding degree in cohomology: $H(\mathfrak{h}^i(X)) = H^i(X)$. This motivates the following stronger version of the Künneth conjecture:

Conjecture (Chow–Künneth conjecture CK(X))**.** There exist orthogonal Chow projectors Δ_i whose cohomology classes are the Künneth components, such that $\Delta_X = \sum_{i=0}^{2d} \Delta_i$.

Example 2.4.1.2. The extremal Künneth components Δ_0^{topo} and $\Delta_{2d}^{\text{topo}}$ are always algebraic by Example 2.1.2.6.

In particular, if X is a curve, the 1-st Künneth component is automatically algebraic and $CK(X)$ holds. It also holds for projective spaces by Example 2.1.2.9. It is also known to hold for surfaces and abelian varieties. △

2.4.2 Lefschetz operator and Lefschetz type conjecture.

Let $X_d \in \mathrm{SmProj(k)}$ and $j\colon Y \hookrightarrow X$ the inclusion of a smooth hyperplane section. On the cohomology of X, taking cup-product with $j_*[Y]$ yields a degree 2 homomorphism

$$L\colon H^i(X) \to H^{i+2}(X)\colon \alpha \mapsto \alpha \cup j_*[Y]$$

called the *Lefschetz operator*. We write $L^r = L \circ \cdots \circ L$ for its r-iterated composition. Then we have the following:

Theorem 2.4.2.1 (Hard Lefschetz theorem, [Voi02, Thm. 6.25])**.** *For each $0 \leq i \leq d$, the homomorphism*

$$L^i \colon H^{d-i}(X) \to H^{d+i}(X)$$

is an isomorphism.

Using this, we can define an "almost inverse" $\Lambda \colon H(X) \to H(X)$ to L:

$$\begin{tikzcd}
H^{d-2}(X) \arrow[rr, bend left, "L"] \arrow[rrrr, bend left=40, "L^2"] & H^{d-1}(X) \arrow[rr, bend left, "L"] & H^d(X) \arrow[rr, bend left, "L"] \arrow[ll, dashed, bend left, "\Lambda"] & H^{d+1}(X) \arrow[ll, dashed, bend left, "\Lambda"] & H^{d+2}(X) \arrow[ll, dashed, bend left, "\Lambda"]
\end{tikzcd}$$

Concretely, let

$$\Lambda := \begin{cases} (L^{i+2})^{-1} \circ L^{i+1} & \text{on } H^{d-i}(X) \text{ with } i \geq -1 \\ L^{i-1} \circ (L^i)^{-1} & \text{on } H^{d+i}(X) \text{ with } i \geq 1, \end{cases} \tag{7}$$

then $\Lambda \colon H(X) \to H(X)$ is a degree -2 homomorphism and one can check that $\Lambda^i \colon H^{d+i}(X) \to H^{d-i}(X)$ is the inverse isomorphism to L^i. Moreover, $L \circ \Lambda$ is a cohomological projector, i.e. $L \circ \Lambda \circ L \circ \Lambda = L \circ \Lambda$, hence it induces a splitting

$$H(X) = H_{\mathrm{pr}}(X) \oplus L \circ \Lambda(H(X)) \tag{8}$$

where $H_{\mathrm{pr}}(X) := \ker(L \circ \Lambda)$ is called the *primitive cohomology* of X. In fact, $\ker(L \circ \Lambda) = \ker(\Lambda)$ as L is injective on $H^{\leq d-1}$ and $L \circ \Lambda = \mathrm{id}$ on $H^{\geq d+1}$. In other words,

$$\Pi_{\mathrm{pr}} = \mathrm{id} - L \circ \Lambda \tag{9}$$

is a cohomological projector onto the primitive cohomology of X.

Note that $L = j_* \circ j^*$ where $j \colon Y \hookrightarrow X$ is the inclusion. Indeed, for $\alpha \in H(X)$,

$$j_* \circ j^*(\alpha) = j_*(j^*(\alpha) \cup [Y]) = \alpha \cup j_*([Y]) = L(\alpha)$$

by the projection formula. We can define a Lefschetz operator on the level of Chow rings

$$L \colon \mathrm{CH}(X) \to \mathrm{CH}(X) \colon \alpha \mapsto \alpha \cdot [Y]$$

inducing the one on cohomology, and the same formula holds: $L = j_* \circ j^*$. This means that L is given by the algebraic cycle $j_* \circ j^* = \Gamma_j \circ (\Gamma_j)^\intercal \in \mathrm{CH}^{d+1}(X \times X)$.

Remark 2.4.2.2. Let us fix an embedding $X \subset \mathbb{P}^N$ and consider a smooth hypersurface section $j' \colon Y' := X \cap H \hookrightarrow X$ where $H \subset \mathbb{P}^N$ is an hypersurface of degree e. Since H is rationally equivalent to $e\mathbb{P}^{N-1}$, one has $j'_*[Y'] = e.j_*[Y]$, hence $j'_* \circ j'^* = eL$. △

The map Λ can be viewed as a topological correspondence, i.e. an element of $H^{2d-2}(X \times X)$, where $d = \dim(X)$.

Conjecture (Lefschetz type conjecture B(X))**.** The topological correspondence Λ is algebraic, i.e. it comes from an algebraic cycle in $\mathrm{CH}^{d-1}(X \times X)_{\mathbb{Q}}$.

Example 2.4.2.3. Let $X = \mathbb{P}^d$ be the projective space of dimension d. Its (rational) Chow ring is $\mathrm{CH}(\mathbb{P}^d)_{\mathbb{Q}} \cong \mathbb{Q}[\alpha]/(\alpha^{d+1})$ where α is the class of an hyperplane. The Lefschetz operator

$$L\colon \mathrm{CH}(\mathbb{P}^d) \to \mathrm{CH}(\mathbb{P}^d)$$

is simply multiplication by α. A quick computation shows that it is represented by the degree 1 correspondence

$$\alpha^d\beta + \alpha^{d-1}\beta^2 + \cdots + \alpha\beta^d \in \mathrm{CH}^{d+1}(\mathbb{P}^d \times \mathbb{P}^d) \cong \mathbb{Q}[\alpha, \beta]/(\alpha^{d+1}, \beta^{d+1}).$$

Similarly, the operator Λ is represented by the degree -1 correspondence

$$\lambda = \alpha^{d-1} + \alpha^{d-2}\beta + \cdots + \alpha\beta^{d-2} + \beta^{d-1} \in \mathrm{CH}^{d-1}(\mathbb{P}^d \times \mathbb{P}^d).$$

Thus, the Lefschetz type standard conjecture holds for the projective space. △

2.5 Grothendieck's dream: motives as a universal Weil cohomology

The title of the present section is taken from the very nice introduction to motives by Milne [Mil13]. A *(classical) Weil cohomology* is a functor

$$H\colon \mathrm{SmProj(k)} \longrightarrow \mathrm{GrVect}_F$$

from the category of smooth projective varieties into the category of finite dimensional graded vector spaces over a field F of characteristic 0, the *coefficient field*, satisfying certain axioms, see [Stacks, Tag 0FGS]. From these axioms, it follows that any Weil cohomology H factors through the category of Chow motives [Stacks, Tag 0FH2]:

$$\begin{tikzcd} \mathrm{SmProj(k)} \arrow[r, "\mathfrak{h}"] \arrow[dr, "H"'] & \mathrm{Mot_{rat}}(k) \arrow[d] \\ & \mathrm{GrVect}_F \end{tikzcd}$$

Example 2.5.0.1. If $k = \mathbb{C}$ and $F = \mathbb{Q}$, rational Betti cohomology is a Weil cohomology theory and the vertical arrow is defined in Section 2.2. △

We may wonder to what extent properties of cohomology are apparent on the level of Chow motives. Such properties would then be invariant on the chosen Weil cohomology, resulting in a better understanding of how they are related one to each other.

The target category of a Weil cohomology is an abelian semi-simple[4] category consisting of graded and finite dimensional objects. The Künneth standard conjecture aims to make the grading apparent on Chow motives. Similarly, the next chapter concerns an attempt to make finite dimensionality already apparent on the level of Chow motives. In other words, the grading and the finite dimensionality of Weil cohomologies are *conjecturally* induced from the structure of Chow motives.

[4] An abelian category is *semi-simple* if every object is a finite direct sum of *simple objects*, i.e. objects without non-trivial subobjects.

2.5.1 Jannsen's theorem. Unfortunately, the category of Chow motives is in general not an abelian category, as is shown by the following example [Sch94, Cor. 3.5]:

Example 2.5.1.1. Suppose that k is algebraically closed of characteristic 0, then we show that the category of Chow motives $\mathrm{Mot}_{\mathrm{rat}}(k)$ is not an abelian category.

Let E be an elliptic curve over k, with identity $0 \in E$. By the assumption on the field k, there is a non-torsion point[5] $x \in E$. Consider the cycle $\alpha := [x] - [0] \in \mathrm{CH}^1(E)$. Since α has degree 0, it is an element of $\mathrm{CH}^1(\mathfrak{h}^1(E))$ by Section 2.3.1. Under the isomorphism

$$\mathrm{CH}^1(\mathfrak{h}^1(E)) \cong \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(\mathbf{L}, \mathfrak{h}^1(E))$$

from Remark 2.1.4.1, the nonzero cycle α corresponds to a nonzero morphism $\alpha_* \colon \mathbf{L} \to \mathfrak{h}^1(E)$ represented by $* \times ([x] - [0])$. If we tensor its dual with $\mathbf{L}^2$, we get a nonzero morphism $\alpha^* \colon \mathfrak{h}^1(E) \otimes \mathbf{L} \to \mathbf{L}$ represented by $([x] - [0]) \times *$.

By definition of composition of correspondences, their composition

$$\alpha_* \circ \alpha^* \colon \mathfrak{h}^1(E) \otimes \mathbf{L} \to \mathfrak{h}^1(E)$$

is represented by the cycle in $\mathrm{CH}^2(E \times E)$

$$\begin{aligned}\eta := \big(* \times ([x] - [0])\big) \circ \big(([x] - [0]) \times *\big) &= \mathrm{pr}_{13}\left(((([x] - [0]) \times * \times E) \cdot (E \times * \times ([x] - [0]))\right)\\ &= ([x] - [0]) \times ([x] - [0])\\ &= ([x] \times [x]) + ([0] \times [0]) - ([x] \times [0]) - ([0] \times [x]).\end{aligned}$$

We can assume that $x = y + y$ in the group $E(k)$, for a point $y \in E$. Then we have

$$\eta = \Big(([x] \times [x]) + ([0] \times [0]) - 2([y] \times [y])\Big) + \Big(2([y] \times [y]) - ([x] \times [0]) - ([0] \times [x])\Big). \quad (10)$$

- The part on the left is rationally trivial because it is the pushforward along the diagonal of $[x] + [0] - [y] - [y]$, which is rationally trivial by the group law on E.

- The part on the right is symmetric, hence it is the pullback of the cycle $\eta' := [y, y] - [0, x]$ in $\mathrm{CH}^2(E^{(2)})$ along the two-fold projection

  $$E \times E \longrightarrow E^{(2)} := E \times E/\mathfrak{S}_2$$

  from $E \times E$ to the 2-nd symmetric product of E, where $[a, b]$ denotes the class $\{a \times b, b \times a\}$. Observe that the addition $E \times E \to E$ factors through $f \colon E^{(2)} \to E$, which is a $\mathbb{P}^1$-bundle (see Section 3.4.2 for more details). By [EH16, Thm. 9.6], one has

  $$\mathrm{CH}(E^{(2)}) \cong \mathrm{CH}(E) \cdot \zeta^0 \oplus \mathrm{CH}(E) \cdot \zeta^1$$

  for some $\zeta \in \mathrm{CH}^1(E^{(2)})$. In particular, the summand $\mathrm{CH}(E) \cdot \zeta^0$ has no element of degree two and

  $$f_* \colon \mathrm{CH}^2(E^{(2)}) \to \mathrm{CH}^1(E)$$

  is an isomorphism. Applied to η', we have

  $$f_*(\eta') = [y] + [y] - [x] - [0],$$

[5] For example, if k is uncountable, it follows from the fact that there are only countably many torsion points.

which is rationally trivial on E for the same reason as the left part. Therefore, η' is trivial and the part on the right of (10) vanishes[6].

Therefore, η is rationally trivial and $\alpha_* \circ \alpha^* = 0$, hence α_* is not a monomorphism. If $\mathrm{Mot}_{\mathrm{rat}}(k)$ was abelian, then $\ker(\alpha_*)$ would be a proper nonzero subobject of $\mathbf{L}$. Then, tensoring with $\mathbf{L}^{-1}$ would yield a nonzero proper subobject of the unit motive $\mathbf{1}$. But $\mathbf{1}$ is simple by [DM82, Prop. 1.17] since $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$ is a field. Thus $\mathrm{Mot}_{\mathrm{rat}}(k)$ cannot be an abelian category. △

The conjectures of Lefschetz and Hodge type together imply that $\mathrm{Mot}_{\mathrm{num}}(k)$ is an abelian semi-simple category. It was believed that they would be needed for a proof of that fact. Jannsen's unconditional proof of the following theorem came as a surprise.

Theorem 2.5.1.2 ([Jan92])**.** *Let $\sim$ be any adequate equivalence relation. The following properties are equivalent:*

(i) *The category $\mathrm{Mot}_\sim(k)$ is an abelian semi-simple category;*

(ii) *The relation $\sim$ is numerical equivalence;*

(iii) *The $\mathbb{Q}$-algebra $\mathrm{Corr}^0_\sim(X,X)_\mathbb{Q}$ is a finite-dimensional, semi-simple $\mathbb{Q}$-algebra for any smooth projective variety $X \in \mathrm{SmProj(k)}$.*

Remark 2.5.1.3*.* The geometric ingredient of the proof is the existence of a Weil cohomology theory over the field k. What remains can be abstracted, as we shall mention is Section 3.2.3. △

If homological and numerical equivalences coincide, then the Weil cohomology functor H factors through the category of numerical motives, which is abelian semi-simple by Jannsen's theorem. We can then decompose the numerical motive $\mathfrak{h}(X)$ of a variety into simple submotives, each of which being responsible for a certain cohomological property of X.

There is a lot more to say about motives. We close this chapter with the conclusion of Grothendieck's article introducing the standard conjectures of Hodge and Lefschetz type [Gro69]:

> The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called "theory of motives" which is a systematic theory of "arithmetic properties" of algebraic varieties, as embodied in their groups of classes of cycles for numerical equivalence. We have at present only a very small part of this theory in dimension one, as contained in the theory of abelian varieties. Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry. (Grothendieck 1968)

[6] This argument works without introducing y since the cycle η is already symmetric. But then we would have to use rational coefficients, while this argument works in the category of integral motives.

3 Finite dimensionality of Chow motives

This chapter concerns a notion of finite dimensionality for Chow motives, which was introduced by Kimura [Kim05] and O'Sullivan independently.

First, we state basic facts about the representation theory of the symmetric group which will be useful for subsequent results. Then we introduce the notion of finite dimensionality in a general setting. In the last section, we specialize this notion to the category of Chow motives and we see some consequences.

3.1 Representation theory of the symmetric group

This section concerns the representation theory of the symmetric group, see [FH91, Ch. 4] for more details. Let $\mathfrak{S}_n$ denote the symmetric group on n objects, i.e. the group of permutations of $\{1, \dots, n\}$. There are bijections between the sets of:

(i) Irreducible representations of $\mathfrak{S}_n$;

(ii) Partitions of n, i.e. tuples $\lambda = (\lambda_1, \dots, \lambda_s)$ with $\sum \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_s$;

(iii) Young diagrams of weight n, i.e. diagrams consisting of n boxes arranged in rows of decreasing size;

where a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ corresponds to the Young diagram with s rows of lengths $\lambda_1, \dots, \lambda_s$. A *Young tableau* on a Young diagram is a numbering of the boxes by the integers $1, \dots, n$. Given a tableau T on the Young diagram corresponding to λ, we define two subgroups of $\mathfrak{S}_n$

$$R_\lambda \coloneqq \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ only permutes elements in each row}\}$$
$$C_\lambda \coloneqq \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ only permutes elements in each column}\}$$

and corresponding elements in the group ring $\mathbb{Q}[\mathfrak{S}_n]$

$$a_\lambda(T) \coloneqq \sum_{\sigma \in R_\lambda(T)} \sigma \quad ; \qquad b_\lambda(T) \coloneqq \sum_{\sigma \in R_\lambda(T)} \operatorname{sgn}(\sigma)\sigma \qquad \text{and} \qquad c_\lambda(T) \coloneqq a_\lambda(T) b_\lambda(T),$$

the last one is called the *Young symmetrizer* of T.

Remark 3.1.0.1. Let V be a vector space and let $\mathfrak{S}_n$ act on $V^{\otimes n}$ by permuting the factors. Then

$$\operatorname{Im}(a_\lambda) \cong \operatorname{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \operatorname{Sym}^{\lambda_s}(V) \text{ and } \operatorname{Im}(b_\lambda) \cong \operatorname{Alt}^{\lambda_1}(V) \otimes \cdots \otimes \operatorname{Alt}^{\lambda_s}(V)$$

where $\mu = \lambda^\intercal$ is the *conjugate partition* to λ, i.e. the partition obtained by interchanging the rows and the columns in the corresponding Young diagram. △

Let W_λ be the irreducible representation corresponding to a partition λ. The elements

$$e_\lambda \coloneqq \frac{\dim(W_\lambda)}{n!} \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_\lambda(\sigma)} \cdot \sigma$$

in $\mathbb{Q}[\mathfrak{S}_n]$ are orthogonal idempotents: $e_\lambda \cdot e_\mu = 0$ if $\lambda \neq \mu$ and $e_\lambda \cdot e_\lambda = e_\lambda$. Moreover, their sum $\sum_\lambda e_\lambda$ equals 1.

The following proposition describes the correspondence between irreducible representations and partitions.

Proposition 3.1.0.2. (i) *$c_\lambda(T) \cdot c_\lambda(T) = n_\lambda(T) c_\lambda(T)$ for some $0 \neq n_\lambda(T) \in \mathbb{Q}$;*

(ii) *$\mathbb{Q}[\mathfrak{S}_n] \cdot c_\lambda(T)$ is a minimal left ideal in $\mathbb{Q}[\mathfrak{S}_n]$, hence it is an irreducible representation of $\mathfrak{S}_n$. Moreover, $\mathbb{Q}[\mathfrak{S}_n] \cdot a_\lambda(T) b_\lambda(T) = \mathbb{Q}[\mathfrak{S}_n] \cdot b_\lambda(T) a_\lambda(T)$;*

(iii) *$\mathbb{Q}[\mathfrak{S}_n] \cdot c_\lambda(T) = \mathbb{Q}[\mathfrak{S}_n] \cdot c_\mu(T')$ if and only if $\lambda = \mu$;*

(iv) *$e_\lambda \in \mathbb{Q}[\mathfrak{S}_n] \cdot c_\lambda(T)$, i.e. the idempotent e_λ is a linear combination of monomials in $c_\lambda(T)$.*

3.2 Finite dimensionality in a tensor category

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a pseudo-abelian $\mathbb{Q}$-linear rigid tensor category. That is, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor satisfying compatible associativity and commutativity constraints[7] with $\mathbf{1}$ as unit. Every projector splits, hence the *image* of a projector makes sense. Furthermore, assume $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$.

We shall keep in mind that $\mathcal{C}$ will eventually stand for the category of motives. The results are stated in a greater generality in order to emphasize the fact that they are not of geometric nature.

3.2.1 Even and odd objects.

Given an object $M \in \mathcal{C}$, the symmetric group $\mathfrak{S}_n$ acts on the n-th tensor power $M^{\otimes n}$ by permuting the factors. Formally this is done via the commutativity constraint and the hexagon axiom ensures that it is well-defined.

This action induces a ring morphism $\Gamma(M)\colon \mathbb{Q}[\mathfrak{S}_n] \to \mathrm{End}(M^{\otimes n})$. Given a partition λ of n, the idempotent $e_\lambda \in \mathbb{Q}[\mathfrak{S}_n]$ acts as a projector on M, denoted

$$d_\lambda(M) \coloneqq \Gamma_{e_\lambda}(M) : M^{\otimes n} \to M^{\otimes n}.$$

Definition 3.2.1.1. Let $M \in \mathcal{C}$ and λ a partition of n, define $\mathbf{T}_\lambda M \coloneqq \mathrm{Im}(d_\lambda(M))$. In particular, we define the n-th *symmetric* and *alternating* products of M as

$$\mathrm{Sym}^n(M) \coloneqq \mathbf{T}_{(n)} M = \left(\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \right) (M^{\otimes n})$$
$$\mathrm{Alt}^n(M) \coloneqq \mathbf{T}_{(1,\dots,1)} M = \left(\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\sigma \right) (M^{\otimes n})$$

Definition 3.2.1.2. An object $M \in \mathcal{C}$ is said to be

(i) *evenly finite dimensional* or *even* if $\mathrm{Alt}^n(M) = 0$ for some $n > 0$. In that case, its dimension $\dim(M)$ is the largest n such that $\mathrm{Alt}^n(M) \neq 0$;

(ii) *oddly finite dimensional* or *odd* if $\mathrm{Sym}^n(M) = 0$ for some $n > 0$. In that case, its dimension $\dim(M)$ is the largest n such that $\mathrm{Sym}^n(M) \neq 0$;

(iii) *finite dimensional* if it splits as a sum $M = M_+ \oplus M_-$ where M_+ is even and M_- is odd. In that case, $\dim(M) \coloneqq \dim(M_+) + \dim(M_-)$.

[7] These constraints are expressed as natural isomorphisms $\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ and $\psi_{X,Y} : X \otimes Y \to Y \otimes X$, satisfying certain commutative diagrams, called the *pentagon* and *hexagon* axioms. See [Del02] for more details.

Remark 3.2.1.3. The definition of dimension is ambiguous if M is both even and odd, but it will turn out that this can only happen if $M = 0$. Similarly, we will show that the splitting $M = M_+ \oplus M_-$ is essentially unique. △

Example 3.2.1.4. In the category of finite dimensional $\mathbb{Q}$-vector spaces, any object is even and its dimension is the usual one.

In the category of $\mathbb{Z}$-graded finite dimensional $\mathbb{Q}$-vector spaces with the *Koszul rule* as commutativity constraint, the even (resp. odd) objects are those concentrated in even (resp. odd) degree. In particular, any object is finite dimensional. △

Example 3.2.1.5. In $\mathrm{Mot}_{\mathrm{rat}}(k)$, a Chow motive $M = (X, p, m)$ is evenly finite dimensional of dimension n if $\mathrm{Alt}^{n+1}(M) = (M^{\otimes n+1}, d_{(1,\ldots,1)} \circ p^{\otimes n+1}, m) = 0$, i.e. if $d_{(1,\ldots,1)} \circ p^{\otimes n+1} = 0$. In particular, (X, p, m) is finite dimensional if and only if $(X, p, 0)$ is finite dimensional.

The unit motive $\mathbf{1} = (\mathrm{Spec}(k), \mathrm{id}, 0)$ is even of dimension 1 because

$$\mathrm{Alt}^2(\mathbf{1}) = (\mathrm{Spec}(k), d_{(1,1)} \circ \mathrm{id}, 0) = (\mathrm{Spec}(k), \tfrac{1}{2}(\mathrm{id} - \mathrm{id}), 0) = 0.$$

Consequently, $\mathbf{L}^{\otimes r} = (Spec(k), \mathrm{id}, -r)$ is also even of dimension 1 for all $r \in \mathbb{Z}$. In particular, the submotives $\mathfrak{h}^0(X) \cong \mathbf{1}$ and $\mathfrak{h}^{2d}(X) \cong \mathbf{L}^d$ from Example 2.1.2.6 are even of dimension 1. △

Remark 3.2.1.6. Following the same argument as in the above example, the unit object $\mathbf{1}$ in $\mathcal{C}$ is always even of dimension 1. △

Remark 3.2.1.7. The definition of finite dimensionality involves the vanishing of a symmetric or alternating power. There exists a weaker notion called *Schur-finiteness*. An object $M \in \mathcal{C}$ is said to be *Schur-finite* if $\mathbf{T}_\lambda M = 0$ for some partition λ of some positive integer n. It is clear that any even or odd object is Schur-finite. However, the converse does not hold, see [Maz04, Ex. 1.12, Cor. 5.20] for counterexamples. △

3.2.2 Sums and tensors products. As one would expect, sums, summands and tensors products of finite dimensional objects are finite dimensional. First, we only deal with even and odd objects. The case of summands of finite dimensional objects, i.e. sums of even and odd objects, will be studied in Section 3.3.2.

Proposition 3.2.2.1. *Sums and summands (and quotients) of even (resp. odd) objects are even (resp. odd). Moreover, if M and N are finite dimensional, then* $\dim(M \oplus N) = \dim(M) + \dim(N)$.

Proof. In the even case, this follows from the isomorphism

$$\mathrm{Alt}^n(M \oplus N) \cong \bigoplus_{p+q=n} \mathrm{Alt}^p(M) \otimes \mathrm{Alt}^q(N).$$

If M and N are even, then in each summand of $\mathrm{Alt}^{\dim(M)+\dim(N)+1}(M \oplus N)$ one factor will vanish, implying that $M \oplus N$ is even with $\dim(M \oplus N) \leq \dim(M) + \dim(N)$. An argument involving the rank shows that equality holds, see [AK02, Thm. 9.1.7].

Conversely, if $M \oplus N$ is even, then the $p = n$ summand $\mathrm{Alt}^n(M)$ will vanish for $n > \dim(M \oplus N)$.

The argument is the same in the odd case. □

Example 3.2.2.2. The Chow motive of the projective space $\mathbb{P}^d$ is evenly finite dimensional of dimension d. Indeed, we have

$$\mathfrak{h}(\mathbb{P}^d) = \bigoplus_{i=0}^{d} \mathbf{L}^{\otimes i}$$

by Example 2.1.2.9 and each term $\mathbf{L}^{\otimes i}$ is even of dimension 1 by Example 3.2.1.5. $\triangle$

Lemma 3.2.2.3 (Vanishing Lemma [Kim05])**.** *Let $q \geq n$ and $\lambda = (\lambda_1, \dots, \lambda_s)$ be a partition of q. Then*

(i) *if $\mathrm{Sym}^{n+1}(M) = 0$ and $\lambda_1 > n$, then $\mathbf{T}_\lambda\, M = 0$;*

(ii) *if $\mathrm{Alt}^{n+1}(M) = 0$ and $\lambda_{n+1} \neq 0$, then $\mathbf{T}_\lambda\, M = 0$.*

Proof. The proof relies on Remark 3.1.0.1 in the general context. We show (i). Let T be a Young tableau for λ. Then

$$\mathrm{Im}(a_{\lambda,T}) \cong \mathrm{Sym}^{\lambda_1}(M) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_s}(M).$$

Since $\lambda_1 > n$, the first factor $\mathrm{Sym}^{\lambda_1}(M)$ vanishes. Thus, $a_{\lambda,T}(M) = 0$ and hence, $c_{\lambda,T}(M) = 0$ as well by Proposition 3.1.0.2(ii). By (iv) of the same proposition, $e_\lambda = rc_{\lambda,T}$ for some $r \in \mathbb{Q}[\mathfrak{S}_n]$, hence $d_\lambda(M) = 0$.

For (ii), pass to the dual partition $\lambda^\intercal$ of λ and apply a similar argument. $\square$

Proposition 3.2.2.4. *If M and N are finite dimensional of the same (resp. different) parity, then $M \otimes N$ is even (resp. odd). In particular, if M, N are finite dimensional, then $M \otimes N$ is finite dimensional.*

Moreover, $\dim(M \otimes N) \leq \dim(M) \cdot \dim(N)$.

Proof. For example, assume M is even and N is odd, i.e. $\mathrm{Alt}^{m+1}(M) = 0 = \mathrm{Sym}^{n+1}(N)$. Put $q = mn + 1$. The other cases are similar. We need to prove $d_{(q)}(M \otimes N)^{\otimes q} = 0$. Using $\sum e_\lambda = 1 \in \mathbb{Q}[\mathfrak{S}_q]$ twice, we get

$$d_{(q)}(M \otimes N)^{\otimes q} \cong d_{(q)} \left(M^{\otimes q} \otimes N^{\otimes q}\right) = d_{(q)} \bigoplus_{\lambda,\nu} d_\lambda M^{\otimes q} \otimes d_\nu N^{\otimes q}. \tag{11}$$

By [FH91, Ex. 4.51], one has

$$e_{(q)} \cdot (e_\lambda \otimes e_\nu) = \begin{cases} 0 & \text{if } \lambda \neq \nu \\ e_{(q)} & \text{if } \lambda = \nu \end{cases}$$

hence the only remaining terms in (11) are projections of $d_\lambda M^{\otimes q} \otimes d_\lambda N^{\otimes q}$. If $\lambda_1 > n$, then $d_\lambda N^{\otimes q} = 0$ by the vanishing Lemma 3.2.2.3(i). Else, a pigeonhole argument yields $\lambda_{m+1} \neq 0$, hence $d_\lambda M^{\otimes q} = 0$ again by the vanishing Lemma 3.2.2.3(ii). This shows that (11) vanishes, i.e. $\mathrm{Sym}^q(M \otimes N) = 0$ as desired.

Distributivity of tensor products over sums yields the general result. $\square$

3.2.3 Tensor ideals and equivalence relations.

A $\mathbb{Q}$-linear tensor category is to a $\mathbb{Q}$-algebra what a groupoid is to a group. The notion of *ideal* generalizes well:

Definition 3.2.3.1. An *ideal* $\mathcal{I}$ in $\mathcal{C}$ is the data of a $\mathbb{Q}$-sub-module $\mathcal{I}(M,N) \subset \mathcal{C}(M,N)$ for each pair of objects M, N, satisfying the following property: for any $f \in \mathcal{C}(M', M), g \in \mathcal{C}(N, N')$, we have

$$g \circ \mathcal{I}(M,N) \circ f \subset \mathcal{I}(M', N').$$

Given an ideal $\mathcal{I}$ in $\mathcal{C}$, we can form the *quotient* $\mathcal{C}/\mathcal{I}$ whose objects are those of $\mathcal{C}$, with morphisms $\mathcal{C}/\mathcal{I}(M,N) \coloneqq \mathcal{C}(M,N)/\mathcal{I}(M,N)$.

An ideal $\mathcal{I}$ in $\mathcal{C}$ is a *tensor-ideal* if it is stable under tensor product: $\mathcal{I}(M,N) \otimes \mathcal{I}(M',N') \subset \mathcal{I}(M \otimes M', N \otimes N')$. In that case, $\mathcal{C}/\mathcal{I}$ inherits the structure of a tensor category.

Remark 3.2.3.2. There is a one-to-one correspondence between tensor ideals of $\mathrm{Mot}_{\mathrm{rat}}(k)$ and adequate equivalence relations on SmProj(k). Given an equivalence relation $\sim$, let $\mathcal{I}_\sim(M,N) \subset \mathrm{Mot}_{\mathrm{rat}}(M,N)$ be the sub-module of correspondences f with $f \sim 0$. Then $\mathrm{Mot}_\sim(k)$ is the pseudo-abelian envelope of $\mathrm{Mot}_{\mathrm{rat}}(k)/\mathcal{I}_\sim$ [And04, Lem. 4.4.1.1]. This gives a convenient way to study equivalence relations. △

Rigidity of $\mathcal{C}$ gives *evaluation* and *coevaluation* morphisms, defined for an object M as the adjuncts of id_M under $\mathcal{C}(M,M) \cong \mathcal{C}(M^\vee \otimes M, \mathbf{1})$ and $\mathcal{C}(M,M) \cong \mathcal{C}(\mathbf{1}, M \otimes M^\vee)$, respectively. We can define the *trace* of an endomorphism $f \in \mathcal{C}(M,M)$ as the element $\mathrm{tr}(f) \in \mathrm{End}(\mathbf{1}) = \mathbb{Q}$ given by

$$\mathrm{tr}(f) : \mathbf{1} \xrightarrow{\mathrm{coev}_M} M^\vee \otimes M \xrightarrow{\mathrm{id}_{M^\vee} \otimes f} M^\vee \otimes M \xrightarrow{\psi_{M^\vee, M}} M \otimes M^\vee \xrightarrow{\mathrm{ev}_M} \mathbf{1}\,.$$

Example 3.2.3.3. In the category $\mathrm{Vect}_\mathbb{Q}$ of finite dimensional $\mathbb{Q}$-vector spaces, the trace $\mathrm{tr}(f)$ of a morphism is the usual trace: the sum of the diagonal entries in a matrix representation of f.

Note that $\mathrm{tr}(\mathrm{id}_V)$ is the dimension of V as a $\mathbb{Q}$-vector space. This can be used to define another notion of *dimension* of an object. In fact, it coincides, up to a sign, with our definition of dimension for even and odd objects [AK02, Thm. 9.17]. △

Definition 3.2.3.4. The following defines a tensor ideal in $\mathcal{C}$:

$$\mathcal{N}(M,N) \coloneqq \{f \in \mathcal{C}(M,N) : \mathrm{tr}(g \circ f) = 0 \text{ for all } g \in \mathcal{C}(N,M)\}.$$

It is the largest proper tensor ideal in $\mathcal{C}$.

Example 3.2.3.5. In the category of finite dimensional (graded) vector spaces, the tensor ideal $\mathcal{N}$ is the zero ideal. △

Remark 3.2.3.6. In $\mathrm{Mot}_{\mathrm{rat}}(k)$, the ideal $\mathcal{N}$ corresponds to numerical equivalence $\sim_{\mathrm{num}}$.

Let $X_d \in$ SmProj(k). Pick a cycle $\alpha \in \mathrm{CH}^i(X)$. With the notation of Remark 2.1.4.1, we get a morphism $\alpha_* \in \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(\mathbf{L}^i, \mathfrak{h}(X))$. A cycle $\beta \in \mathrm{CH}^{d-i}(X)$ of complementary codimension induces a morphism $\beta^* \in \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(\mathfrak{h}(X), \mathbf{L}^i)$, and every morphism $\mathfrak{h}(X) \to \mathbf{L}^i$ is of that form.

Using that α_* is represented by $* \times \alpha$ and β^* is represented by $\beta \times *$, we have

$$\beta^* \circ \alpha_* = (p_{13})_*((* \times \alpha \times *) \cdot (* \times \beta \times *)) = (p_{13})_*(* \times (\alpha \cdot \beta) \times *)) = \deg(\alpha \cdot \beta),$$

hence $\alpha_* \in \mathcal{N}(\mathbf{1}, \mathfrak{h}(X))$ if and only if $\alpha \sim_{\text{num}} 0$.

Remark 2.1.4.1 allows us to consider morphisms as cycles, and the above argument applies to the general setting. For a motive $M = (X, p, m)$, the trace of an endomorphism $f \in \mathrm{End}(M)$ equals $\mathrm{tr}(f) = \deg(f \cdot p^{\mathsf{T}})$. △

Remark 3.2.3.7. Using the notion of tensor ideal, André and Kahn give an abstract version of Jannsen's Theorem [AK02, Thm. 8.2.2]. △

3.3 Finite dimensionality and nilpotence

Again, let $(\mathcal{C}, \otimes, \mathbf{1})$ be a pseudo-abelian $\mathbb{Q}$-linear rigid tensor category, with $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$. In this section, we shall see that an object being finite dimensional has strong nilpotency consequences.

3.3.1 Tensor nilpotence.

Definition 3.3.1.1. A morphism $f\colon M \to N$ in $\mathcal{C}$ is *tensor nilpotent*, or *$\otimes$-nilpotent*, if $f^{\otimes n}\colon M^{\otimes n} \to N^{\otimes n}$ vanishes for some $n > 0$.

In fact, tensor nilpotence implies nilpotence:

Proposition 3.3.1.2. *Let $f\colon M \to N$ be tensor nilpotent, say $f^{\otimes n} = 0$, and $g_i\colon N \to M$, $i = 1, \dots, n-1$, any morphisms. Then $f \circ g_{n-1} \circ f \circ \dots \circ f \circ g_1 \circ f = 0$.*

In particular, a tensor nilpotent morphism is nilpotent.

Proof. The idea is the following: Rigidity yields isomorphisms $\mathcal{C}(\mathbf{1}, X^\vee \otimes Y) \to \mathcal{C}(X, Y)$ enabling us to "pull back" everything to the tensor product $M^{\otimes n} \otimes N^{\otimes n-1}$ where tensor nilpotence applies [AK02, Lem. 7.4.2(ii)]. □

3.3.2 Morphisms between finite dimensional objects.

Proposition 3.3.2.1. *Any morphism between finite dimensional objects of different parity is tensor nilpotent.*

Proof. Let $f\colon M \to N$ be a morphism in $\mathcal{C}$, with M odd and N even, i.e. $\mathrm{Sym}^{m+1}(M) = 0 = \mathrm{Alt}^{n+1}(N)$. Let $q = nm + 1$ and consider, for partitions λ, ν of q, the composition

$$M^{\otimes q} \xrightarrow{d_\lambda(M)} M^{\otimes q} \xrightarrow{f^{\otimes q}} N^{\otimes q} \xrightarrow{d_\nu(N)} N^{\otimes q}. \tag{12}$$

The action of $\mathfrak{S}_q$ on q-th tensor powers is natural, i.e. the d_μ commute with morphisms. Hence, the above composition equals $f^{\otimes q} \circ d_\nu \circ d_\lambda$, which is zero unless $\lambda = \nu$. In that case, we claim that (12) vanishes: By the vanishing Lemma 3.2.2.3(i), if $\lambda_1 > m$, then $d_\lambda(M) = 0$; else, $\lambda_{n+1} \neq 0$ and $d_\lambda(N) = 0$. Thus, (12) is the zero map and, using $\sum_\lambda d_\lambda = \mathrm{id}$, also $f^{\otimes q}$ is the zero map. □

Corollary 3.3.2.2. *An object that is both even and odd is the zero object.*

Proof. Let M be such an object. The identity $\mathrm{id}_M : M \to M$ is a morphism from an even object to an odd object. Thus it is tensor nilpotent by Proposition 3.3.2.1, hence nilpotent by Proposition 3.3.1.2, say of order n. Then $0 = \mathrm{id}_M^n = \mathrm{id}$ and M is the zero object. □

Proposition 3.3.2.3. *Summands and quotients of finite dimensional objects are finite dimensional.*

Proof. Let M be finite dimensional and $M = M_+ \oplus M_-$ a decomposition into its even and odd parts.

Suppose $M = N \oplus K$, with projection $f\colon M \twoheadrightarrow N$ and inclusion $s\colon N \hookrightarrow M$, i.e. $f \circ s = \mathrm{id}_N$. We want to show that N is finite dimensional. More precisely, we will show that $N = N_+ \oplus N_-$ where f projects $M_\pm$ onto $N_\pm$. Then one concludes by Proposition 3.2.2.1.

The splitting $M = M_+ \oplus M_-$ comes from an orthogonal decomposition $\mathrm{id}_M = p_+ + p_-$. Then

$$\mathrm{id}_N = f \circ s = f \circ (p_+ + p_-) \circ s = \underbrace{f \circ p_+ \circ s}_{:= q'_+} + \underbrace{f \circ p_- \circ s}_{:= q'_-}$$

is almost the splitting we need. Indeed, $q'_+ \circ q'_- = f \circ p_+ \circ s \circ f \circ p_- \circ s$ factors through a morphism $p_+ \circ s \circ f \circ p_- : M_- \to M_+$ between objects of different parity, which is tensor nilpotent by Proposition 3.3.2.1. Hence, $q'_+ \circ q'_-$ is nilpotent by Proposition 3.3.1.2, say of order k. Hence, $0 = (q'_+ \circ q'_-)^k = q'^k_+ \circ q'^k_-$ since q'_+ and $q'_- = \mathrm{id}_N - q'_+$ commute. We claim that

$$q_+ := (\mathrm{id}_N - q'^k_-)^k \quad \text{and} \quad q_- := \mathrm{id}_N - q_+$$

are orthogonal projectors. To prove this, we need the following elementary formula: for any morphism $t\colon N \to N$, one has

$$\mathrm{id}_N - (\mathrm{id}_N - t)^k = P(t) \circ t$$

for some polynomial P which only depends on k.

Applying the above formula with $t = q'_+$ and $t = q'_-$ yields

$$q_+ = P(q'_+)^k \circ q'^k_+ \quad \text{and} \quad q_- = P(q'^k_-) \circ q'^k_-. \tag{13}$$

From the first one and using $q'^k_- \circ q'^k_+ = 0$, we deduce $q_+ \circ q'^k_+ = (\mathrm{id}_N - P(q'^k_-) \circ q'^k_-) \circ q'^k_+ = q'^k_+$. Thus, $q_+ \circ q_+ = q_+ \circ q'^k_+ \circ P(q'_+)^k = q'^k_+ \circ P(q'_+)^k = q_+$, i.e. q_+ is a projector. Therefore, $q_- = \mathrm{id}_N - q_+$ is also a projector and q_+, q_- are orthogonal.

To conclude, it remains to show that f projects $M_\pm$ onto $N_\pm := \mathrm{Im}(q_\pm)$. We have $q'_\pm = f \circ p_\pm \circ s$, hence $q'^k_\pm = f \circ t_\pm$ for some morphisms $t_\pm : N \to M$. (13) yields $\mathrm{id}_{N_\pm} = q_\pm = f \circ s_\pm$ for some morphisms $s_\pm : N \to M$, which concludes the proof. □

Corollary 3.3.2.4. *The decomposition $M = M_+ \oplus M_-$ of a finite dimensional object M into its even and odd parts is essentially unique, i.e. if $M = M'_+ \oplus M'_-$ is another decomposition, then $M_\pm \cong M'_\pm$. In particular, the dimension of M is well defined.*

Proof. The identity of M is an isomorphism $\phi : M_+ \oplus M_- \to M'_+ \oplus M'_-$. Composed with the projection $M'_+ \oplus M'_- \twoheadrightarrow M'_+$, we get a surjection

$$f\colon M_+ \oplus M_- \twoheadrightarrow M'_+.$$

Proposition 3.3.2.3 tells us that f induces a decomposition $M'_+ = (M'_+)_+ \oplus (M'_+)_-$ into even and odd parts. But $(M'_+)_- = 0$ by corollary 3.3.2.2, hence f is a surjection $M_+ \twoheadrightarrow M'_+$. By symmetry, the inverse isomorphism ϕ^{-1} induces a surjection $M'_+ \twoheadrightarrow M_+$. Therefore, $M_+ \cong M'_+$ and similarly for the odd part. □

3.3.3 Nilpotence theorem.

Proposition 3.3.3.1 ([Kim05, Prop. 7.2], [AK02, Prop. 7.2.7])**.** *Let $M \in \mathcal{C}$ be an object which is even or odd, of dimension $n-1$. Then,*

(i) *There is a nonzero polynomial $G(T) \in \mathbb{Q}[T]$ of degree n with $G(f) = 0$;*

(ii) *If $f \in \mathcal{N}(M, M)$, then f is nilpotent: $f^n = 0$;*

(iii) *The ideal $\mathcal{N}(M, M)$ is a nilpotent ideal, with order $\leq 2^n + 1$;*

(iv) *If M is a nonzero object, then its image in $\mathcal{C}/\mathcal{N}$ is nonzero.*

Proof. The proof is a formal computation. We only give a skecth, see [AK02] 7.2.7 for more details. Assume $\mathrm{Alt}^n(M) = 0$, then $d_{(1,\dots,1)} \circ f^{\otimes M} \colon M^{\otimes n} \to M^{\otimes n}$ is zero since it factors through $\mathrm{Alt}^n(M)$. Under the adjunction $\mathcal{C}(M^{\otimes n}, M^{\otimes n}) \cong \mathcal{C}(\mathbf{1}, (M^\vee)^{\otimes n} \otimes M^{\otimes n})$, the morphism $d_{(1,\dots,1)} \circ f^{\otimes M}$ corresponds to a morphism $\mathbf{1} \to (M^\vee)^{\otimes n} \otimes M^{\otimes n}$. Evaluating the middle $2(n-1)$ factors, i.e. composing with $\mathrm{ev}_{M^{\otimes(n-1)}} \colon (M^\vee)^{\otimes(n-1)} \otimes M^{\otimes(n-1)} \to \mathbf{1}$, yields a morphism $\mathbf{1} \to M^\vee \otimes M$. Adjunction gives back a morphism $G(f) \coloneqq (d_{(1,\dots,1)} \circ f^{\otimes n})_1 \colon M \to M$. Then a combinatorial argument shows that $G(f)$ is a nonzero polynomial of degree n, whose lower degree coefficients are traces of nonzero powers of f. Since $d_{(1,\dots,1)} \circ f^{\otimes n} = 0$, we have $G(f) = 0$. This proves (i).

For (ii), note that $f \in \mathcal{N}(M, M)$ implies that $\mathrm{tr}(f^i) = 0$ for all $i \geq 1$. In particular, only f^n remains in $G(f)$. Thus, $f^n = 0$, concluding the proof. Part (iii) follows from (ii) and Nagata–Higman Theorem.

For (iv), assume $M \neq 0$, then the identity id_M is not nilpotent, hence id_M represents a nonzero class in $\mathcal{C}(M, M)/\mathcal{N}(M, M)$, that is, $M \neq 0$ in $\mathcal{C}/\mathcal{N}$. □

Remark 3.3.3.2. There is a stronger version of the above proposition, with n replaced by $n-1$ in (ii) and (iii). See [Jan07, Thm. 6.4.3]. △

Theorem 3.3.3.3 ([Kim05, Thm. 7.5], [AK02, Thm. 9.1.14])**.** *Let $M \in \mathcal{C}$ be a finite dimensional object. Then any endomorphism $f \colon M \to M$ which belongs to $\mathcal{N}$ is nilpotent. In fact, $\mathcal{N}(M, M)$ is nilpotent with order bounded in function of* $\dim(n)$.

Proof. Let $M = M_+ \oplus M_-$ be the decomposition of M into its even and odd parts, induced by $\mathrm{id}_M = p_+ + p_-$. Then an endomorphism $f \colon M_+ \oplus M_- \to M_+ \oplus M_-$ can be written as

$$f = (p_+ + p_-) \circ f \circ (p_+ + p_-) = \underbrace{p_+ \circ f \circ p_+}_{f_+} + \underbrace{p_- \circ f \circ p_-}_{f_-} + \underbrace{(p_+ \circ f \circ p_- + p_- \circ f \circ p_+)}_{f_{\mathrm{mix}}}.$$

The non-parity preserving part f_{mix} is nilpotent by Propositions 3.3.2.1 and 3.3.1.2, say $f_{\mathrm{mix}}^r = 0$. The parity preserving parts f_+ and f_- belong to $\mathcal{N}(M, M)$, hence they are nilpotent by Proposition 3.3.3.1, say $f_+^s = f_-^s = 0$.

Since $f_+ \circ f_- = f_- \circ f_+ = 0$, a typical term in the expansion of $f^n = (f_+ + f_- + f_{\mathrm{mix}})^n$ looks like

$$m = f_\pm^{k_1} \circ f_{\mathrm{mix}} \circ f_\pm^{k_2} \circ f_{\mathrm{mix}} \circ \cdots \circ f_\pm^{k_{r-1}} \circ f_{\mathrm{mix}} \circ f_\pm^{k_r}$$

where $k_i \geq 0$ and $r - 1 + \sum_i k_i = n$. Indeed, if f_{mix} appears more than r times, then $m = 0$ by proposition 3.3.1.2. Hence, if $n \geq (r-1) + r(s-1) + 1 = rs$, then one of the k_i must be $\geq s$ and $m = 0$ because $f_\pm^s = 0$. □

Corollary 3.3.3.4. *Let $M \in \mathbb{C}$ be a finite dimensional object and $\overline{M}$ its image in $\mathcal{C}/\mathcal{N}$.*

(i) *Any projector $\overline{p}\colon \overline{M} \to \overline{M}$ in $\mathcal{C}/\mathcal{N}$ can be lifted to a projector $p\colon M \to M$ in $\mathcal{C}$.*

(ii) *More generally, any decomposition of $\mathrm{id}_{\overline{M}}$ into orthogonal projectors in $\mathcal{C}/\mathcal{N}$ lifts to a decomposition of id_M into orthogonal projectors in $\mathcal{C}$.*

Proof. The projector $\overline{p}$ in $\mathcal{C}/\mathcal{N}$ is represented by a morphism $p\colon M \to M$ in $\mathcal{C}$. However, p is not necessarily a projector in $\mathcal{C}$, i.e. the difference $p^2 - p$ does not necessarily vanish.

We still know that the induced morphism $\overline{p}^2 - \overline{p}$ is zero since $\overline{p}$ is a projector. This means that $p^2 - p$ belongs to $\mathcal{N}$, hence it is nilpotent by Theorem 3.3.3.3, say $(p^2 - p)^k = 0$ for some $k > 0$.

Consider the endomorphism $p' := (1 - (1 - p)^k)^k$ where $1 := \mathrm{id}_M$. In $\mathcal{C}/\mathcal{N}$, it induces $\overline{p'} = \overline{p}$ since $\overline{p}$ and $\overline{1} - \overline{p}$ are projectors. As in the proof of Proposition 3.3.2.3, we show that p' is indeed a projector.

On one hand, we can write

$$p' = (1 - (1 - p)^k)^k = (P(p) \circ p)^k = P(p)^k \circ p^k$$

for some polynomial P. On the other hand, we can write

$$p' = (1 - (1 - p)^k)^k = 1 - \sum_{j=1}^{k} (-1)^j \binom{k}{j} (1 - p)^{jk}.$$

Then

$$\begin{aligned} p' \circ p' &= p' \circ \left(1 - \sum_{j=1}^{k} (-1)^j \binom{k}{j} (1 - p)^{jk}\right) = p' - P(p)^k \circ p^k \circ \sum_{j=1}^{k} (-1)^j \binom{k}{j} (1 - p)^{jk} \\ &= p' \end{aligned}$$

because the second term has $p^k \circ (1 - p)^k = (p - p^2)^k = 0$ as a factor. This shows that p' is a projector.

For the second part, we argue by induction. Suppose there is a decomposition $\mathrm{id}_{\overline{M}} = \overline{p_1} + \cdots + \overline{p_n}$ into orthogonal projectors in $\mathcal{C}/\mathcal{N}$. By the first part, we can find a projector $p_1'\colon M \to M$ in $\mathcal{C}$ that lifts $\overline{p_1}$. Then $q' := \mathrm{id}_M - p_1'$ is a projector orthogonal to p_1' which lifts $\overline{p_2} + \cdots + \overline{p_n}$. Then $M' := \mathrm{Im}(q')$ is finite dimensional by Proposition 3.3.2.3 and the decomposition $\mathrm{id}_{\overline{M'}} = \overline{p_2} + \cdots + \overline{p_n}$ in $\mathcal{C}/\mathcal{N}$ lifts to an orthogonal decomposition in $\mathcal{C}$ by induction hypothesis. This concludes the proof. □

Corollary 3.3.3.5. *If a finite dimensional object $M \in \mathcal{C}$ becomes $\overline{M} = 0$ in $\mathcal{C}/\mathcal{N}$, then it is the zero object.*

Proof. From $\overline{M} = 0$, it follows that the identity id_M belongs to $\mathcal{N}$, hence it is nilpotent by Theorem 3.3.3.1. Therefore, $0 = \mathrm{id}_M^k = \mathrm{id}_M$, hence M is the zero object. □

3.4 Finite dimensionality of Chow motives

The category of Chow motives $\mathrm{Mot}_{\mathrm{rat}}(k)$ is a pseudo-abelian $\mathbb{Q}$-linear rigid tensor category, with $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$. Thus, all the results from the last sections hold for $\mathrm{Mot}_{\mathrm{rat}}(k)$. This section focuses on their application in the case of Chow motives.

3.4.1 Surjective morphisms. From Example 2.1.2.10, we know that a generically finite surjective morphism $X \to Y$ allows us to express $\mathfrak{h}(Y)$ as a submotive of $\mathfrak{h}(X)$. In particular, finite dimensionality of $\mathfrak{h}(X)$ implies finite dimensionality of $\mathfrak{h}(Y)$ by Proposition 3.3.2.3. In fact, this last statement also holds if f is surjective but not necessarily generically finite.

Lemma 3.4.1.1. *Let $X, Y \in \mathrm{SmProj(k)}$ be projective varieties and $f\colon X \to Y$ a surjective morphism. Then the induced map $f_*\colon Z(X)_{\mathbb{Q}} \to Z(Y)_{\mathbb{Q}}$ on algebraic cycles is surjective.*

Proof. Let $V \subset Y$ be an irreducible subvariety of Y and $\eta \in V$ its generic point. By surjectivity of f, we can pick a closed point $\xi \in X \times_Y \eta$ over η. Since $X \times_Y \eta$ is a finite type $K(\eta)$-scheme, the degree $d = [K(\xi) : K(\eta)]$ is finite by Hilbert Nullstellenstaz. Let $\widetilde{V}$ be the closure of ξ in X, it is a subvariety of X and $f_*(\widetilde{V}) = dV$ by definition of the push-forward f_*. Since we work with rational coefficients, this shows that f_* is surjective. □

Using the above lemma, the next Proposition generalizes Remark 2.1.1.2.

Proposition 3.4.1.2. *If $f\colon X \to Y$ is surjective, then $f_*\colon \mathfrak{h}(X) \to \mathfrak{h}(Y)$ is split surjective. In particular, $\mathfrak{h}(Y)$ is a submotive of $\mathfrak{h}(X)$.*

Proof. Let $f\colon X \to Y$ be a surjective morphism. Then $f \times \mathrm{id}_Y\colon X \times Y \to Y \times Y$ is surjective. By Lemma 3.4.1.1, it induces a surjection

$$(f \times \mathrm{id}_Y)_*\colon \mathrm{CH}(X \times Y) \twoheadrightarrow \mathrm{CH}(Y \times Y).$$

Take a preimage $s \in \mathrm{CH}(X \times Y)$ of the diagonal $\Delta_Y \in \mathrm{CH}(Y \times Y)$. Then

$$\Delta_Y = (f \times \mathrm{id}_Y)_*(s) = s \circ (f_*)^\intercal$$

by Lieberman's Lemma. Taking the transpose yields $\Delta_Y = f_* \circ s^\intercal$ where $s^\intercal \in \mathrm{CH}(Y \times X)$. Therefore, as a correspondence, $s^\intercal$ is a right-inverse of f_*. □

Remark 3.4.1.3. A morphism $f\colon M \to N$ between Chow motives is said to be *surjective* if the induced map

$$\mathrm{CH}(M \otimes \mathfrak{h}(Z)) \to \mathrm{CH}(N \otimes \mathfrak{h}(Z))$$

is surjective for all $Z \in \mathrm{SmProj(k)}$. The above result holds for surjective morphisms between Chow motives in the same way, see [Kim05, Lem. 6.8]. △

The next result follows immediately from the above Proposition and Proposition 3.3.2.3.

Corollary 3.4.1.4. *Let $f\colon X \to Y$ be a surjective morphism. If $\mathfrak{h}(X)$ is finite dimensional, then $\mathfrak{h}(Y)$ is finite dimensional.*

3.4.2 Curves have finite dimensional Chow motives. From Section 2.3, the motive of a curve $C \in \mathrm{SmProj(k)}$, i.e. a 1-dimensional irreducible smooth projective variety, splits as

$$\mathfrak{h}(C) = \mathbf{1} \oplus \mathfrak{h}^1(C) \oplus \mathbf{L}$$

where $\mathbf{1}$ and $\mathbf{L}$ are even of dimension 1 by Example 3.2.1.5.

Remark 3.4.2.1. If $k = \mathbb{C}$ and H denotes rational Betti cohomology, then we have $H(\mathfrak{h}^1(C)) = H^1(C) \cong \mathbb{Q}^{2g}$ where g is the genus of C. Thus $H(\mathrm{Sym}^{2g}(\mathfrak{h}^1(C))) = \mathrm{Alt}^{2g} H^1(C) \neq 0$, hence the dimension of $\mathfrak{h}^1(C)$ is at least $2g$. △

Let $C \in \mathrm{SmProj}(\mathrm{k})$ be a smooth projective curve. In fact, $\mathfrak{h}^1(C)$ is odd of dimension $2g$. We follow [Kim05, Thm. 4.2][MNP13, Thm. 4.6.1] and divide the proof into smaller steps. Before diving into the proof, we need to introduce some notations.

Since C is smooth projective of dimension 1, its n-th symmetric product

$$C^{(n)} \coloneqq C^n/\mathfrak{S}_n$$

is a smooth projective variety of dimension n, for each $n \geq 1$. Let $\varphi_n \colon C^n \to C^{(n)}$ be the natural morphism.

Let J be the Jacobian variety of C, it is an abelian variety of dimension g, where g is the genus of the curve C. Recall from Section 2.3.1 that rational points on J are in bijection with degree zero divisors on C, i.e. $J(k) \cong \mathrm{CH}^1(\mathfrak{h}^1(C))$. Given a fixed point $x_0 \in C$, the morphism $C^n \to J$ sending the tuple $(x_1, \ldots, x_n)$ to the class of the cycle $x_1 + \cdots + x_n - nx_0$ factors through $C^{(n)}$ and induces the morphism (see [Mil08, Ch. III.5] for more details)

$$\pi \colon C^{(n)} \longrightarrow J$$

whose fibers are either empty or projective spaces. More precisely, after identifying $J(k) \cong \mathrm{CH}^1(\mathfrak{h}^1(C))$ with the group $\mathrm{Pic}_0(C)$ of degree 0 line bundles on C, the morphism π becomes

$$[x_1, \ldots, x_n] \longmapsto \mathcal{O}(x_1 + \cdots + x_n - nx_0)$$

and the fiber over $\mathcal{L} \in \mathrm{Pic}_0(C)$ is the projective space $\mathbb{P}(H^0(C, \mathcal{L} \otimes \mathcal{O}(nx_0)))$, which may be empty. Riemann-Roch formula yields

$$h^0(C, \mathcal{L} \otimes \mathcal{O}(nx_0)) - h^0(C, \mathcal{L}^\vee \otimes \mathcal{O}(-nx_0) \otimes \omega_C) = \deg(\mathcal{L} \otimes \mathcal{O}(nx_0)) + 1 - g = n + 1 - g,$$

where $\mathcal{L}^\vee \otimes \mathcal{O}(-nx_0) \otimes \omega_C$ has degree $-n+2g-2$, hence it has no global section if $n > 2g-2$. Thus, if $n > 2g-2$, then $h^0(C, \mathcal{L} \otimes \mathcal{O}(nx_0)) = n+1-g$, hence π is surjective with fiber $\mathbb{P}^{n-g}$. In fact, $C^{(n)} = \mathbb{P}(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ on J of rank $n+1-g$. This allows us to express the Chow group of $C^{(n)}$ in terms of the Chow group of J (see [EH16, Thm. 9.6]):

$$\mathrm{CH}(C^{(n)}) \cong \mathrm{CH}(J)[1, \zeta, \ldots, \zeta^r] = \bigoplus_{i=0}^{r} \mathrm{CH}(J) \cdot \zeta^i \tag{14}$$

where $r + 1 = n + 1 - g$ is the rank of $\mathcal{E}$, and ζ is the divisor associated with the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on $\mathbb{P}(\mathcal{E})$.

for $n \geq 1$, we have $\mathrm{Sym}^n(\mathfrak{h}^1(C)) = (C^n, \alpha_n)$ where

$$\alpha_n \coloneqq d_{(n)} \circ p_1^{\otimes n} = \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} \sigma \right) \circ p_1^{\otimes n} \in \mathrm{CH}^n(C^n \times C^n).$$

Hence, the claim that $h^1(C)$ is odd of dimension $2g$, i.e. $\mathrm{Sym}^{2g+1}(\mathfrak{h}^1(C)) = 0$, reduces to the vanishing of α_{2g+1}.

The natural morphism $\varphi_n \colon C^n \to C^{(n)}$ from above allows us to transport the cycle α_n to the cycle

$$\beta_n \coloneqq \frac{1}{n!} (\varphi_n)_* \circ \alpha_n \circ \varphi_n^* = \frac{1}{n!} (\varphi_n \times \varphi_n)_* \alpha_n \in \mathrm{CH}^n(C^{(n)} \times C^{(n)}),$$

where the second equality follows from Lieberman's Lemma. The cycle β_n vanishes if only if α_n vanishes, as is shown by the following lemma:

Lemma 3.4.2.2. *For all $n \geq 1$, we have:*

(i) *β_n is a projector on $C^{(n)}$;*

(ii) *$\alpha_n = \frac{1}{n!}\varphi_n^* \circ \beta_n \circ (\varphi_n)_*$;*

(iii) *The morphisms $\frac{1}{n!}(\varphi_n)_*$ and φ_n^* induce an isomorphism $\mathrm{Sym}^n(\mathfrak{h}^1(C)) \cong (C^{(n)}, \beta_n)$.*

Proof. Observe that the morphism φ_n is surjective and generically finite of degree $n!$, hence $\frac{1}{n!}(\varphi_n)_* \circ \varphi_n^*$ is the identity as a correspondence on $C^{(n)}$ by Remark 2.1.1.2. On the other hand, one has

$$\varphi_n^* \circ (\varphi_n)_* = n! \left(\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \right) = n! d_{(n)}.$$

Using the fact that α_n is a projector, (i) and (ii) follow by computation. For (iii), recall that $\mathrm{Sym}^n(\mathfrak{h}^1(C)) = (C^n, \alpha_n)$ and consider the following morphisms:

$$\begin{tikzcd}
(C^n, \alpha_n) \arrow[loop left, "\alpha_n"] \arrow[rr, bend left, "\frac{1}{n!}\beta_n \circ (\varphi_n)_* \circ \alpha_n"] & & (C^{(n)}, \beta_n) \arrow[loop right, "\beta_n"] \arrow[ll, bend left, "\alpha_n \circ \varphi_n^* \circ \beta_n"]
\end{tikzcd}$$

A quick computation shows that both compositions indeed yield α_n and β_n, which are the identities on the considered objects. $\square$

Now assume that $n > 2g - 2$. Using the description (14) of the Chow group of $C^{(n)}$, we can write β_n in terms of cycles on J. We need two additional lemmata. The first one consists in making explicit the isomorphism (14):

Lemma 3.4.2.3 ([Kim05, Lem. 4.2.1])**.** *Let $\mathcal{E}$ be a rank $r + 1$ locally free sheaf on J, let $\pi \colon \mathbb{P}(\mathcal{E}) \to J$ be the associated projective bundle and ζ the divisor associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. The morphism*

$$f \colon \mathrm{CH}(\mathbb{P}(\mathcal{E})) \longrightarrow \bigoplus_{i=0}^{r} \mathrm{CH}(J) \colon \beta \longmapsto (\pi_*(\zeta^i \cdot \alpha))_i$$

is bijective.

Proof. From [EH16, Thm. 9.6], we have the isomorphism

$$g \colon \bigoplus_{i=0}^{r} \mathrm{CH}(J) \longrightarrow \mathrm{CH}(\mathbb{P}(\mathcal{E})) \colon (a_i)_i \longmapsto \sum_{i=0}^{r} \pi^*(a_i) \cdot \zeta^i$$

and the multiplicative structure on the right is given by $\zeta^{r+1} = -c_1(\mathcal{E})\zeta^r - \cdots - c_{r+1}(\mathcal{E})$, where $c_j(\mathcal{E})$ are the Chern classes of $\mathcal{E}$ [EH16, Ch. 5]. For $i, j = 0, \ldots, r$, we have by the projection formula

$$\pi_*(\zeta^j \cdot \pi^*(a_i) \cdot \zeta^i) = a_i \cdot \pi_*(\zeta^{i+j}) = \begin{cases} 0 & \text{if } i+j<r \\ 1 & \text{if } i+j=r \\ -c_1(\mathcal{E}) & \text{if } i+j=r+1 \\ \vdots & \end{cases}$$

All in all, the matrix representing $f \circ g$ has 1's on the anti-diagonal $i+j=r$ and zeros above, thus it is bijective. $\square$

Lemma 3.4.2.4 ([Kim05, Prop. 4.1])**.** *Let $e \in C$ be a point. Define*

$$i\colon C^{(n)} \longrightarrow C^{(n+1)}\colon (x_1,\dots,x_n) \longmapsto (x_1,\dots,x_n,e).$$

For $n > 2g-2$, one can choose $\mathcal{E}_{n+1}$ in $C^{(n+1)} = \mathbb{P}(\mathcal{E}_{n+1})$ so that $i_*[C^{(n)}] = \zeta_{n+1}$ where ζ_{n+1} corresponds to $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{n+1})}(1)$.

Finally, we have all the tools at hand to prove the desired result:

Theorem 3.4.2.5. *Let $C \in \mathrm{SmProj}(\mathrm{k})$ be a curve of genus g. The motive $\mathfrak{h}^1(C)$ is odd of dimension $2g$. Hence, $\mathfrak{h}(C)$ is finite dimensional of dimension $2g+2$.*

Proof. We need to show that $\mathrm{Sym}^{2g+1}(\mathfrak{h}^1(C)) = (C^{2g+1}, \alpha_{2g+1}) \cong (C^{(2g+1)}, \beta_{2g+1})$ vanishes, i.e. the cycle $\beta_{2g+1} \in \mathrm{CH}^{2g+1}(C^{(2g+1)} \times C^{(2g+1)})$ is zero.

Let $\mathrm{pr}_1, \mathrm{pr}_2\colon C^{(2g+1)} \times C^{(2g+1)} \to C^{(2g+1)}$ be the projections. The morphism $\pi\colon C^{(2g+1)} \to J$ is a $\mathbb{P}^{g+1}$-bundle and the cycle ζ corresponds to the associated tautological line bundle, chosen according to Lemma 3.4.2.4. Applying Lemma 3.4.2.3 twice, it suffices to show that

$$(\pi \circ \mathrm{pr}_1 \times \pi \circ \mathrm{pr}_2)_*(\mathrm{pr}_1^* \zeta^i \cdot \mathrm{pr}_1^* \zeta^j \cdot \beta_{2g+1}) = 0 \tag{15}$$

for $0 \le i,j \le g+1$. For $i=j=0$, it is zero for dimenion reasons: the cycle β_{2g+1} has dimension $2g+1$, hence its projection on the dimension $2g$ product $J \times J$ must be zero.

Now suppose $i>0$ or $j>0$. In fact, we shall see that the intersection product $\mathrm{pr}_1^* \zeta \cdot \beta_{2g+1}$ already vanishes, thus (15) vanishes. By definition of β_{2g+1} and by the projection formula, we have (where $n = 2g+1$)

$$\begin{aligned} \mathrm{pr}_1^* \zeta \cdot \beta_{2g+1} &= \frac{1}{n!} \mathrm{pr}_1^* \zeta \cdot (\varphi \times \varphi)_* \alpha_{2g+1} \\ &= \frac{1}{n!} (\varphi \times \varphi)_* \Big((\varphi \times \varphi)^* \mathrm{pr}_1^* \zeta \cdot \alpha_{2g+1} \Big). \end{aligned}$$

Since ζ was chosen so that it equals $i_*[C^{(2g)}] = C^{(2g)} \times e$, we have $\mathrm{pr}_1^* \zeta = C^{(2g)} \times e \times C^{(2g+1)}$. Hence,

$$\begin{aligned} \mathrm{pr}_1^* \zeta \cdot \beta_{2g+1} &= \frac{1}{n!} (\varphi \times \varphi)_* \Big((\varphi \times \varphi)^* C^{(2g)} \times e \times C^{(2g+1)} \cdot \alpha_{2g+1} \Big) \\ &= \frac{1}{n!} (\varphi \times \varphi)_* \left(\left(\sum_{j=1}^{2g+1} C \times \cdots \times \underset{j\text{-th}}{e} \times \cdots \times C \times C^{2g+1} \right) \cdot \alpha_{2g+1} \right) \\ &= \frac{1}{(n!)^2} (\varphi \times \varphi)_* \sum_{j=1}^{2g+1} \sum_{\sigma \in \mathfrak{S}_{2g+1}} \left(C \times \cdots \times \underset{j\text{-th}}{e} \times \cdots \times C \times C^{2g+1} \cdot \left(\Gamma_\sigma \circ p_1^{\otimes 2g+1} \right) \right) \end{aligned}$$

where $p_1 = \Delta - e \times C - C \times e$. By symmetry of what is inside the last parentheses, it suffices to show that $e \times C^{2g} \times C^{2g+1} \cdot (\Gamma_\sigma \circ p_1^{\otimes 2g+1})$ vanishes for all $\sigma \in \mathfrak{S}_{2g+1}$, which is the case since

$$(e \times C) \cdot p_1 = (e \times C) \cdot (\Delta - e \times C - C \times e) = 0.$$

This concludes the proof that (15) is zero for all $0 \leq i, j \leq g+1$, hence that β_{2g+1} is zero. Therefore, $\mathrm{Sym}^{2g+1}(\mathfrak{h}^1(C)) = 0$ and $\mathfrak{h}^1(C)$ is oddly finite dimensional of dimension $2g$. □

Remark 3.4.2.6. Using a similar computation as in the proof above, we can show that $\mathrm{Sym}^{2g}(\mathfrak{h}^1(C)) \cong \mathbf{L}^g$, see [Kim05, Rem. 4.5]. △

Corollary 3.4.2.7. *The Chow motive of a variety dominated by a product of curves is finite dimensional. In particular, abelian varieties have finite dimensional chow motives.*

Proof. The first part follows immediately from the fact that finite dimensionality is preserved under products (3.2.2.4), the above theorem and Corollary 3.4.1.4.

The second part follows from the fact that any abelian variety is a quotient of a Jacobian variety [Mil08, Thm. III.10.1], and the Jacobian of a curve is dominated by a product of the curve [Mil08, Thm. III.5.1] as we have seen above. □

3.4.3 Kimura–O'Sullivan conjecture.

Apart from cellular varieties, curves and abelian varieties, very few varieties are known to have a finite dimensional Chow motive. Motivated by the strong consequences of finite dimensionality of Chow motives, Kimura [Kim05] and O'Sullivan independently stated the following conjecture:

Conjecture (Kimura–O'Sullivan conjecture)**.** Every Chow motive is finite dimensional.

3.4.4 Nilpotence and phantom motives.

From Remark 3.2.3.6, we know that the ideal $\mathcal{N}$ corresponds to numerical equivalence in $\mathrm{Mot}_{\mathrm{rat}}(k)$. In other words, a morphism between Chow motives belongs to the ideal $\mathcal{N}$ if and only if it is numerically trivial as an algebraic cycle. The nilpotence Theorem 3.3.3.3 translates into:

Theorem 3.4.4.1. *Let M be a Chow motive and $f\colon M \to M$ an endomorphism of M. If f is numerically trivial (in particular, if f is homologically trivial), then f is nilpotent.*

A nonzero Chow motive $M \in \mathrm{Mot}_{\mathrm{rat}}(k)$ whose homological motive $M_{\mathrm{hom}}(k)$ is zero is called a *phantom motive*. In other words, M is a phantom motive if $M \neq 0$ but $H(M) = 0$. It is expected that phantom motives do not exist. If Kimura–O'Sullivan conjecture holds, it is the case by Corollary 3.3.3.5. In fact, if M is a finite dimensional Chow motive, we even have $\dim(M) = \dim_{\mathbb{Q}}(H(M))$.

Remark 3.4.4.2. Assuming finite dimensionality of Chow motives, Corollary 3.3.3.4(ii) suggests that, if a splitting appears naturally in cohomology and is given by projectors coming from algebraic cycles, then the splitting already appears on the level of Chow motives. In the next paragraph, this idea is applied to the Künneth decomposition. In Chapter 4, it will be applied to a splitting appearing in the cohomology of a complete intersection. △

3.4.5 Finite dimensionality and Künneth decomposition. In Section 2.4.1 were introduced the Künneth conjecture $C(X)$ and the Chow–Künneth conjecture $CK(X)$. The latter is stronger, but it is not known whether they are equivalent. For varieties with a finite dimensional Chow motive, it turns out that it is the case.

Proposition 3.4.5.1. *Let $X \in \mathrm{SmProj(k)}$ be a smooth projective variety with a finite dimensional Chow motive. The Künneth conjecture $C(X)$ holds for X if and only if the Chow–Künneth conjecture $CK(X)$ holds for X.*

Proof. Assume that $C(X)$ holds. Let d be the dimension of X. Then there exist algebraic cycles $\Delta_0, \ldots, \Delta_{2d}$ lifting the topological Künneth components $\Delta_0^{\mathrm{topo}}, \ldots, \Delta_{2d}^{\mathrm{topo}}$. In particular, the induced homological cycles $\overline{\Delta_i}$ are projectors and the equality

$$\Delta_X = \overline{\Delta_0} + \cdots + \overline{\Delta_{2d}}$$

holds in $\mathrm{Mot}_{\mathrm{hom}}(k)$, i.e. as homological cycles. Since $\mathfrak{h}(X)$ is finite dimensional, Corollary 3.3.3.4(ii) yields orthogonal Chow projectors lifting the Künneth components, whose sum is the diagonal in $\mathrm{Mot}_{\mathrm{rat}}(k)$. In other words, the Chow–Künneth conjecture holds for X. $\square$

4 Motivic interpretation of variable and fixed cohomology

Lifting a decomposition appearing in cohomology to a decomposition on the deeper level of Chow motives results in a better understanding of its essence. However, there are very few cases where this task has been achieved. The part of the theory which concerns curves and abelian varieties is well-understood, but the standard conjectures separate us from a complete understanding of this question.

This chapter focuses on a specific case where the Lefschetz type and Kimura–O'Sullivan conjectures bring a partial answer to the above general question. More precisely, we will focus on a smooth complete intersection X inside a surrounding variety V. The Lefschetz theorem on hyperplane sections, also known as the *weak Lefschetz* theorem, tells us that the cohomology of V determines almost all the cohomology of X. This induces a decomposition of the cohomology of X into a *fixed part*, the part that is determined by V, and what remains, the *variable part*.

In the first section of this chapter, we present a proof of the Lefschetz theorem on hyperplane sections which relies on Morse theory.

In the second section, we study how the splitting of the cohomology of X into fixed and variable parts can be lifted to the level of Chow motives if the Lefschetz type and Kimura–O'Sullivan conjectures hold for the surrounding variety V, following a note of C. Peters [Pet17].

4.1 Weak Lefschetz for singular cohomology: The Lefschetz theorem on hyperplane sections

Morse theory is a powerful tool that enables us to study the topology of smooth manifolds through the study of the critical points of an appropriate smooth function on it, which we will call a *Morse function*.

After introducing the basics of Morse theory, we apply it to complex varieties to prove the Lefschetz theorem on hyperplane sections.

4.1.1 Reconstructing a torus. Let us first consider an example. Let M be a torus, embedded vertically in $\mathbb{R}^3$ as depicted on Figure 5. Let $f\colon M \to \mathbb{R}$ be the restriction to M of the vertical coordinate of $\mathbb{R}^3$. This function has four *critical points*, i.e. points where $df = 0$, namely A, B, C, D. Their images under f are called *critical values*. These critical points are *non-degenerate* in the sense that the *Hessian quadratic form* of f (whose definition we recall below) is non-degenerate at these points.

For every $a \in \mathbb{R}$, write $M_{\leq a}$ for the subspace of M where $f \leq a$. Using f, we can now reconstruct the torus step by step by attaching cells, see Figure 6. The idea is to "scan" M via f and to study how its homotopy type changes whenever we pass through a critical point.

- For $a < f(A)$, the subspace $M_{\leq a}$ is empty.
- At the first critical value, $M_{\leq f(A)}$ is a point. It corresponds to attaching a 0-cell. It is homotopy equivalent to a 2-cell, see (0) on Figure 6.
- Between $f(A)$ and $f(B)$, there are no critical values, so $M_{\leq a}$ is a 2-cell for all a between these values.

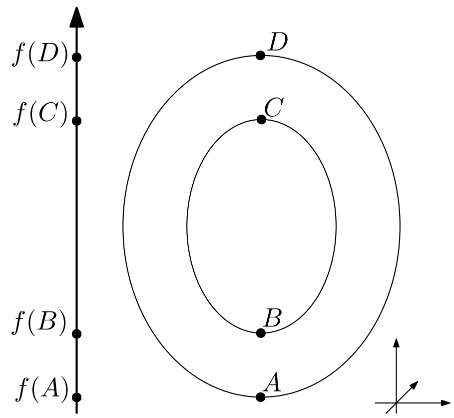


Figure 5: Torus

- At the next critical value, $M_{\leq f(B)}$ is a 2-cell with two points of its boundary identified. This corresponds to the attachment of a 1-cell (1), which results in a space homotopy equivalent to a cylinder (2). For all $a \in (f(B), f(C))$, $M_{\leq a}$ has the same homotopy type.

- Now, $M_{\leq f(C)}$ is a cylinder where we identify one point of each boundary. This corresponds to the attachment of a 1-cell (3), which results in a space homotopy equivalent to a torus minus a disk (4). For all $a \in (f(C), f(D))$, $M_{\leq a}$ has the same homotopy type.

- At the last critical value $f(D)$, $M_{\leq f(D)}$ is the whole torus M. This corresponds to the attachment of a 2-cell (5), ending the reconstruction.

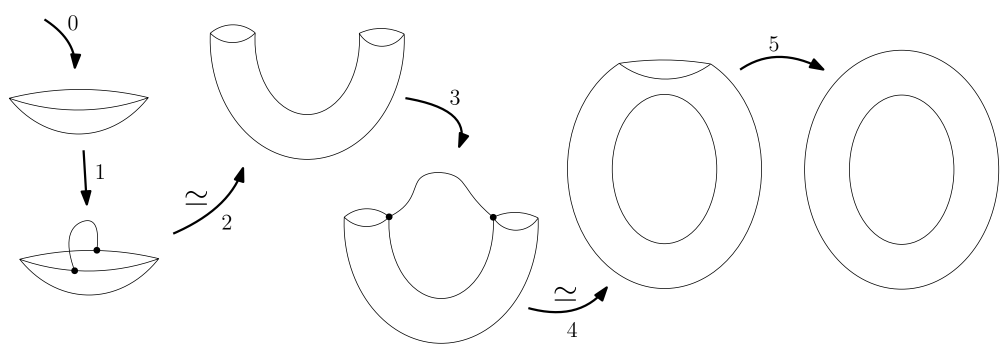


Figure 6: Reconstruction of the torus using the function f

From this we can conclude that the torus is homotopy equivalent to a CW-complex with one 0-cell, two 1-cells and one 2-cell.

4.1.2 Morse functions. Let us recall the definition of the Hessian bilinear form of a smooth function f on a smooth manifold M. Let $x_1, \ldots, x_n$ be local coordinates around a point $0 \in M$. Then the vectors $\partial/\partial x_i$, $i = 1, \ldots, n$, form a basis of the tangent space $T_{M,0}$ of

M at 0. The *Hessian quadratic form* of f at 0 is defined by

$$\mathrm{Hess}_0 f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(0).$$

Now assume $0 \in M$ is a critical point of f. If $\mathrm{Hess}_0 f$ is non-degenerate, it can be represented by a diagonal matrix with only 1's and -1's in an appropriate basis of $T_{M,0}$. In this case, we say that 0 is a *non-degenerate critical point* of f. The number of -1's is independent of the basis and is called the *Morse index* of f at 0, denoted $\mathrm{ind}_0 f$.

Around a non-degenerate critical point, the function f is characterized by its Morse index at that point. This local description is the basis of Morse theory:

Lemma 4.1.2.1 (Morse lemma)**.** *If* 0 *is a non-degenerate critical point of a function f on a smooth manifold M, then there exist local coordinates $x_1, \ldots, x_n$ around* 0 *such that, for $x = (x_1, \ldots, x_n)$ around* 0*, we have*

$$f(x) = f(0) - \sum_{i=1}^{r} x_i^2 + \sum_{i=r+1}^{n} x_i^2 \tag{16}$$

with $r = \mathrm{ind}_0 f$.

In particular, the critical points are isolated, and the critical values are isolated locally on M. A function $f \colon M \to \mathbb{R}$ is said to be an *exhaustion function* if $M_{\leq a}$ is compact for all $a \in \mathbb{R}$. If $f \colon M \to \mathbb{R}$ is a smooth function whose critical points are non-degenerate, it is called a *Morse function*. In that case, there are only finitely many critical values $\leq a$, for any $a \in \mathbb{R}$.

Note that, in the example of the torus above, the Morse indices of the critical points correspond to the dimension of the cells that we attached: $\mathrm{ind}_A f = 0$, $\mathrm{ind}_B f = \mathrm{ind}_C f = 1$ and $\mathrm{ind}_D f = 2$. This can be understood visually using the lemma above. Indeed, around A, f looks like (in appropriate coordinates) $f(x) = f(A) + x_1^2 + x_2^2$, so locally M looks like the graph of $x_1^2 + x_2^2$, i.e. a paraboloid. Around B and C it looks like a saddle. Around D it looks like a paraboloid again, but this time it is pointing downward.

This is in fact a general phenomenon, which can be proven by studying the topology of level sets of functions of the form of equation (16). This results in the following theorem, see [MSW69, Thm. 2.10], or [Voi02, Thm. 13.15]:

Theorem 4.1.2.2. *Let $f \colon M \to \mathbb{R}$ be a Morse function on a smooth manifold M, $\lambda \in \mathbb{R}$ a critical value and $\epsilon > 0$ such that it is the only critical value of f in $[\lambda - \epsilon, \lambda + \epsilon]$. Let 0_i, $i = 1, \ldots, r$ be the corresponding critical points of f and r_i their Morse index. Then $M_{\leq \lambda + \epsilon}$ deformation retracts onto the attachment of r_i-cells to $M_{\lambda - \epsilon}$. The images of the attaching maps are pairwise disjoint.*

The next natural question to ask concerns the existence of a Morse function on a given smooth manifold M. It turns out that, given a smooth embedding $M \hookrightarrow \mathbb{R}^N$, the function *square of distance* $f_0 : M \to \mathbb{R} : x \mapsto d(0, x)^2 = |x - 0|^2$ is a Morse function for almost any point 0 in $\mathbb{R}^N$, i.e. for all point in a dense subset of $\mathbb{R}^N$ [Voi02, Lem. 13.17].

4.1.3 Lefschetz theorem on hyperplane sections. Given a complex affine manifold M of (complex) dimension n, embedded in some affine complex space $\mathbb{C}^N$, the usual hermitian metric on $\mathbb{C}^N$ and a point $0 \in \mathbb{C}^N \cong \mathbb{R}^{2N}$ such that f_0 is a Morse function on M, it turns out that the Morse index of f_0 is $\leq n$ at any critical point of f_0. In other words, it implies that M has the homotopy type of a CW-complex of (real!) dimension $\leq n$.[8] From this result, we obtain a proof of the Lefschetz theorem on hyperplane sections [AF59]:

Theorem 4.1.3.1 (Lefschetz theorem on hyperplane sections)**.** *Let $X \subset \mathbb{P}^N$ be a smooth complex projective subvariety of dimension n and $Y = \mathbb{P}^{N-1} \cap X$ an hyperplane section of X. Then the morphism*

$$j^* \colon H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z}),$$

induced by the inclusion $j : Y \hookrightarrow X$, is an isomorphism for all $i < n-1$ and is injective for $i = n-1$. Similarly, the morphism

$$j_* \colon H_i(X, \mathbb{Z}) \to H_i(V, \mathbb{Z})$$

is an isomorphism for all $i < n-1$ and is injective for $i = n-1$.

Proof. We show the statement for cohomology, the argument for homology is similar. The inclusion j yields a long exact sequence in cohomology with compact support (which coincides with singular cohomology for X and Y since they are compact)

$$\cdots \to H^i_c(X - Y, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to H^{i+1}_c(X - Y, \mathbb{Z}) \to \ldots.$$

By Poincaré duality for orientable manifolds,

$$H^i_c(X - Y, \mathbb{Z}) \cong H_{2n-i}(X - Y, \mathbb{Z})$$

for all $0 \leq i \leq 2n$. Moreover, $X - Y$ is an n-dimensional affine complex manifold, hence it has the homotopy type of a $\leq n$-dimensional CW-complex by the discussion above. Therefore, its singular homology vanishes in all degrees $> n$. In particular, $H^i_c(X - Y, \mathbb{Z}) = 0$ for all $i < n$ and the result follows from the long exact sequence above. □

Remark 4.1.3.2. In the proof above, we only need that $X - Y$ is a smooth manifold, so it still works if X is not smooth and Y contains the singular locus of X. Note also that a Veronese embedding enables us to generalize this theorem to hypersurface sections. △

4.2 Splitting of the cohomology of a complete intersection

Let V be an irreducible smooth projective variety and $X = H_1 \cap \cdots \cap H_r \subset V$ be a smooth complete intersection of r smooth hypersurfaces in V. Let d be the dimension of X, thus V has dimension $d + r$. By applying Lefschetz theorem 4.1.3.1 r times, we find that the inclusion $j \colon X \hookrightarrow V$ induces an isomorphism

$$j^* \colon H^i(V) \to H^i(X)$$

[8] Note that it is clearly not the case for complex projective manifold, for example the complex projective line does not have the homotopy type of a ≤ 1 dimensional CW-complex because $H_2(\mathbb{P}^1_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$.

for all $i < d$, and it is injective for $i = d$. Hard Lefschetz tells us that $H^i(X) \cong H^{2d-i}(X)$ for all $i < d$. Thus, the knowledge of the cohomology of V implies the knowledge of the cohomology of X, except for its middle part $H^d(X)$. We still know that $j^*\colon H^d(V) \to H^d(X)$ is injective, its image is called the *fixed part* of the cohomology of X

$$H^d_{\text{fix}}(X) \coloneqq \operatorname{Im}(j^* : H^d(V) \to H^d(X)).$$

We define the *variable part* $H^d_{\text{var}}(X)$ as the orthogonal complement of $H^d_{\text{fix}}(X)$ in $H^n(X)$ with respect to the cup product, which yields a nondegenerate bilinear form on $H^d(X)$ by Poincaré duality. In fact,

$$H^d_{\text{var}}(X) = \ker(j_* : H^d(X) \to H^{d+2r}(V)).$$

This follows from the projection formula $j_*(j^*x \cup y) = x \cup j_*y$, for all $x \in H^d(V), y \in H^d(X)$ [Har75, Thm. 7.5], [Hat02, p. 241]. Therefore, we have

$$H^d(X) = H^d_{\text{fix}}(X) \oplus H^d_{\text{var}}(X) \tag{17}$$

where $H^d_{\text{fix}}(X)$ only depends on the cohomology of V and $H^d_{\text{var}}(X)$ is the part of the cohomology of X that really depends on X.

4.2.1 Projector on the fixed cohomology.

With the same notations as above, we define the *total fixed* and *total variable parts* of the cohomology of X as

$$\begin{aligned} H_{\text{fix}}(X) &\coloneqq \operatorname{Im}(j^*\colon H(V) \to H(X)) \\ H_{\text{var}}(X) &\coloneqq \ker(j_*\colon H(X) \to H(V)). \end{aligned}$$

In degree d, they coincide with the fixed and variable parts introduced above. In every other degree, as one would expect, the variable part is trivial and the fixed part is everything. Indeed, the morphism j^* is surjective in all degrees $\neq d$:

- In degrees $d - i$ with $0 < i \leq d$, j^* is even an isomorphism by Weak Lefschetz.
- In degrees $d + i$ with $0 < i \leq d$, consider the following commutative diagram:

$$\begin{tikzcd} H^{d+i}(V) \arrow[r, two heads, "j^*"] & H^{d+i}(X) \\ H^{d-i}(V) \arrow[u, hook, "L^i"] \arrow[r, "j^*"'] & H^{d-i}(X) \arrow[u, "L^i"'] \end{tikzcd}$$

  The right arrow is an isomorphism by Hard Lefschetz. The bottom arrow is an isomorphism by Weak Lefschetz. Thus, the top j^* must be surjective.

On the other hand, j_* is injective in all degrees $\neq d$:

- In degrees $d + i$ with $0 < i \leq d$, j_* is even an isomorphism by Weak Lefschetz for homology and Poincaré duality.

- In degrees $d-i$ with $0 < i \le d$, consider the following commutative diagram:

$$\begin{tikzcd} H^{d-i}(X) \arrow[r, hook, "j_*"] \arrow[d, "L^i"', "\cong"] & H^{d+2r-i}(V) \arrow[d, two heads, "L^i"] \\ H^{d+i}(X) \arrow[r, "\cong", "j_*"'] & H^{d+2r+i}(V) \end{tikzcd}$$

  The left arrow is an isomorphism by Hard Lefschetz. As above, the bottom arrow is an isomorphism by Weak Lefschetz in homology together with Poincaré duality. Thus, the top j_* must be injective.

The projection formula implies that $H_{\text{fix}}(X)$ and $H_{\text{var}}(X)$ are orthogonal. All in all, we have an orthogonal splitting

$$H(X) = H_{\text{fix}}(X) \overset{\perp}{\oplus} H_{\text{var}}(X) \tag{18}$$

of the total cohomology $H(X)$ of X into its fixed part, determined by V, and its variable part.

Since this splitting appears naturally in the cohomology of any complete intersection $X \hookrightarrow V$, we can wonder whether it already appears on the level of Chow motives. In the next section, we will see that it is the case if the surrounding variety V satisfies certain conditions. In order to make this precise, we first need to make explicit the projectors of $H(X)$ onto the fixed and variable parts.

Lemma 4.2.1.1. *The composition $L^r \circ \Lambda^r$ is the identity on $L^r H(V)$, i.e. $L^r \circ \Lambda^r \circ L^r = L^r$.*

Proof. We prove this by induction on r, using the definition (7) of Λ.

For $r = 1$, we have

$$L \circ \Lambda \circ L = \begin{Bmatrix} L \circ (L^{i+2})^{-1} \circ L^{i+1} \circ L \\ L \circ L^{i-1} \circ (L^i)^{-1} \circ L \end{Bmatrix} = L$$

where the cases subdivision and i depend on the degree we are considering in $H(V)$.

Let $r > 1$ and assume that the result holds for $r-1$. In the first case, we have

$$\begin{aligned} L^r \circ \Lambda^r \circ L^r &= L^r \circ \Lambda^{r-1} \circ (L^{i+2})^{-1} \circ L^{i+1} \circ L^r = L^r \circ \Lambda^{r-1} \circ \underbrace{(L^{i+2})^{-1} \circ L^{i+2}}_{=\mathrm{id}} \circ L^{r-1} \\ &= L^r \circ \Lambda^{r-1} \circ L^{r-1} = L \circ L^{r-1} \circ \Lambda^{r-1} \circ L^{r-1} = L \circ L^{r-1} = L^r \end{aligned}$$

and in the second case, we have

$$\begin{aligned} L^r \circ \Lambda^r \circ L^r &= L^r \circ \Lambda^{r-1} \circ L^{i-1} \circ (L^i)^{-1} \circ L^r \\ &\begin{cases} \overset{i \le r}{=} L^r \circ \Lambda^{r-1} \circ L^{i-1} \circ (L^i)^{-1} \circ L^i \circ L^{r-i} = L^r \circ \Lambda^{r-1} \circ L^{i-1} \circ L^{r-1} \\ \overset{i \ge r}{=} L^r \circ \Lambda^{r-1} \circ L^{r-1} \circ L^{i-r} \circ (L^i)^{-1} \circ L^r = L \circ L^{r-1} \circ L^{i-r} \circ (L^i)^{-1} \circ L^r \end{cases} \\ &= L^r. \end{aligned}$$

Thus, the result holds no matter which degree in $H(V)$ we are considering. This concludes the induction step, hence the proof. $\square$

Remark 4.2.1.2. The *degree* e of the complete intersection $X \subset V$ is the product of the degrees of the hypersurfaces $H_1, \dots, H_r$ defining it. It follows from 2.4.2.2 that the composition $j_* \circ j^*$ equals eL^i where L is the Lefschetz operator. △

We are finally ready to make explicit the projector of $H(X)$ onto $H_{\text{fix}}(X)$:

Proposition 4.2.1.3. *The cohomological correspondence*

$$\Pi_{\text{fix}} := \frac{1}{e} j^* \circ \Lambda^r \circ j_*$$

is a projector of $H(X)$ onto the fixed part $H_{\text{fix}}(X)$. Thus, the cohomological correspondence $\Pi_{\text{var}} := \mathrm{id}_{H(X)} - \Pi_{\text{fix}}$ is a projector of $H(X)$ onto the variable part $H_{\text{var}}(X)$.

Proof. First, it is clear that Π_{fix} acts as zero on $H_{\text{var}}(X) = \ker(j_*)$.

We show that Π_{fix} acts as the identity on $H_{\text{fix}}(X) = \mathrm{Im}(j^*)$, onto which j^*, resp. on which j_*, is surjective, resp. injective, by the above discussion. Hence, it suffices to show that

$$j_* \circ \Pi_{\text{fix}} \circ j^* = j_* \circ j^*$$

on $H(V)$. We have

$$j_* \circ \Pi_{\text{fix}} \circ j^* = \frac{1}{e} j_* j^* \Lambda^r j_* j^* = \frac{e^2}{e} L^r \Lambda^r L^r = eL^r = j_* \circ j^*$$

where the one before last equality follows from Lemma 4.2.1.1. This shows that Π_{fix} acts as the identity on $H^{\neq d}(X)$. This concludes the proof. □

4.2.2 Motivic nature of the splitting. In this section, $\sim = \sim_{\text{rat}}$ and we only deal with Chow motives. Following Peters [Pet17], we explain how the splitting (17) has a motivic nature for varieties V with finite-dimensional Chow motive and such that $B(V)$ holds. From now on, we make these assumptions: $\mathfrak{h}(V)$ is finite dimensional and $B(V)$ holds.

The standard conjecture of Lefschetz type $B(V)$ states that Λ is induced by an algebraic correspondence $\lambda \in \mathrm{Corr}^{-1}_{\text{rat}}(V, V)$. Thus, the projector Π_{fix} is algebraic: it comes from the algebraic correspondence $\frac{1}{e} j^* \circ \lambda^r \circ j_*$. However, this latter correspondence may not be a Chow projector, i.e. a projector on $\mathfrak{h}(X)$. A first idea would be to apply Corollary 3.3.3.4 in order to find a Chow projector lifting Π_{fix}. For this, we would need $\mathfrak{h}(X)$ to be finite dimensional but this was only assumed for $\mathfrak{h}(V)$. Thus this approach does not work. However, one can show that Π_{fix} factors through a projector on $H(V)$, for which we can apply Corollary 3.3.3.4.

Lemma 4.2.2.1. *If $\mathfrak{h}(V)$ is finite dimensional and $B(V)$ holds, then there exists a correspondence $\lambda_r \in \mathrm{Corr}^{-r}(V, V)$ such that $L^r \circ \lambda_r \colon \mathfrak{h}(V) \to \mathfrak{h}(V)$ is a projector inducing $L^r \circ \Lambda^r \colon H(V) \to H(V)$ in cohomology.*

Proof. The argument is almost the same as the one in the proof of Corollary 3.3.3.4. The Chow correspondence $p = L^r \circ \lambda^r$ induces a projector $\overline{p}$ on $H(V)$, hence $p^2 - p$ is nilpotent by Theorem 3.3.3.3. Thus $(p^2 - p)^k = 0$ for some $k > 0$. Then

$$p' := (1 - (1 - p)^k)^k = p^k \circ P(p)^k = L^r \circ \lambda^r \circ p^{k-1} \circ P(p)^k$$

is a projector on $\mathfrak{h}(V)$ and we put $\lambda_r := \lambda^r \circ p^{k-1} \circ P(p)^k$. □

Remark 4.2.2.2. For $r = 1$, the above lemma implies that the cohomological projector $\Pi_{\mathrm{pr}} = \mathrm{id}_{H(V)} - L \circ \Lambda$ (9) is induced by a Chow projector $\pi_{\mathrm{pr}} \coloneqq \mathrm{id}_{\mathfrak{h}(V)} - L \circ \lambda_1$. Therefore, the splitting (8) of $H(V)$ is induced by a splitting

$$\mathfrak{h}(V) = \mathfrak{h}_{\mathrm{pr}}(V) \oplus (L \circ \lambda_1)\, \mathfrak{h}(V)$$

where $\mathfrak{h}_{\mathrm{pr}}(V) \coloneqq (V, \pi_{\mathrm{pr}})$ could be defined as the primitive part of the Chow motive of V and its cohomology is precisely $H(\mathfrak{h}_{\mathrm{pr}}(V)) = H_{\mathrm{pr}}(V)$, that is, the primitive cohomology of V.

Note that we do not define $\mathfrak{h}_{\mathrm{pr}}(V)$ as *the* primitive Chow motive of V because the definition of π_{pr} is not canonical: defining λ_1 involves a choice.

On the level of homological motives, the correspondence $\Pi_{\mathrm{pr}} = \mathrm{id} - L \circ \lambda$ is already a projector and defines a canonical primitive *homological* motive of V. It is not clear whether there is a canonical *primitive Chow motive* of V. △

The idea from the above remark applies for the splitting into fixed and variable parts, as is made explicit in the following result, in which we recall the situation for clarity:

Proposition 4.2.2.3 ([Pet17])**.** *Let $V \in \mathrm{SmProj}(\mathbb{C})$, for which $B(V)$ holds and assume that $\mathfrak{h}(V)$ is finite dimensional. Let $X \hookrightarrow V$ be a smooth complete intersection of dimension d. Then there exist orthogonal Chow projectors π_{fix} and π_{var} on $\mathfrak{h}(X)$ inducing projection onto the fixed and variable parts in cohomology. In other words, the Chow motive of X admits a splitting*

$$\mathfrak{h}(X) = \mathfrak{h}_{\mathrm{fix}}(X) \oplus \mathfrak{h}_{\mathrm{var}}(X)$$

inducing the splitting (18) in cohomology.

Proof. Let r be the codimension of X in V, that is, the number of hypersurfaces defining it. Put

$$\pi_{\mathrm{fix}} \coloneqq \frac{1}{e} j^* \circ \lambda_r \circ L^r \circ \lambda_r \circ j_*.$$

It is a projector:

$$\begin{aligned}
(\pi_{\mathrm{fix}})^2 &= \frac{1}{e^2} j^* \circ \lambda_r \circ L^r \circ \lambda_r \circ \underbrace{j_* \circ j^*}_{=eL^r} \circ \lambda_r \circ L^r \circ \lambda_r \circ j_* \\
&= \frac{1}{e} j^* \circ \lambda_r \circ \underbrace{L^r \circ \lambda_r \circ L^r \circ \lambda_r \circ L^r \circ \lambda_r}_{=L^r \circ \lambda_r} \circ j_* \\
&= \pi_{\mathrm{fix}}
\end{aligned}$$

where we used that $L^r \circ \lambda_r$ is a projector by construction. Thus $\pi_{\mathrm{var}} \coloneqq \mathrm{id}_{\mathfrak{h}(X)} - \pi_{\mathrm{fix}}$ is also a projector and they are orthogonal.

It remains to show that π_{fix} induces projection onto the fixed part in cohomology. Write $\overline{\pi_{\mathrm{fix}}} = \frac{1}{e} j^* \circ \overline{\lambda_r} \circ L^r \circ \Lambda^r \circ j_*$ the projector in cohomology induced by π_{fix}. As in the proof of Proposition 4.2.1.3, it suffices to show that $j_* \circ \overline{\pi_{\mathrm{fix}}} \circ j^* = j_* \circ j^*$:

$$\begin{aligned}
j_* \circ \overline{\pi_{\mathrm{fix}}} \circ j^* &= \frac{1}{e} \underbrace{j_* \circ j^*}_{=eL^r} \circ \overline{\lambda_r} \circ L^r \circ \overline{\lambda_r} \circ \underbrace{j_* \circ j^*}_{=eL^r} \\
&= eL^r \circ \overline{\lambda_r} \circ L^r \circ \overline{\lambda_r} \circ L^r \\
&= eL^r = j_* \circ j^*
\end{aligned}$$

where we used that $L^r \circ \overline{\lambda_r} = L^r \circ \Lambda^r$ by construction and then Lemma 4.2.1.1. This concludes the proof. $\square$

Remark 4.2.2.4. Note that the fixed part $\mathfrak{h}_{\text{fix}}(X) = (X, \pi_{\text{fix}})$ as defined in the above result is a submotive of the motive $\mathfrak{h}(V)$ of the surrounding variety. This is made explicit by the following diagram:

$$\begin{tikzcd}
\mathfrak{h}(V) \arrow[loop left, "p_{\text{fix}} := \lambda_r \circ L^r \circ \lambda_r \circ L^r"] \arrow[rr, bend left, "\frac{1}{e}\pi_{\text{fix}} \circ j^* \circ \lambda_r \circ L^r"] & & \mathfrak{h}_{\text{fix}}(X) \arrow[loop right, "\mathrm{id}_{\mathfrak{h}_{\text{fix}}(X)} = \pi_{\text{fix}}"] \arrow[ll, bend left, "\lambda_r \circ j_* \circ \pi_{\text{fix}}"]
\end{tikzcd}$$

The correspondence p_{fix} is a projector by construction of λ_r and the two maps between $\mathfrak{h}(V)$ and $\mathfrak{h}(X)$ yield an isomorphism

$$p_{\text{fix}}\, \mathfrak{h}(V) \cong \mathfrak{h}_{\text{fix}}(X).$$

In particular, since we assumed $\mathfrak{h}(V)$ to be finite dimensional, $\mathfrak{h}(X)$ is also finite dimensional. $\triangle$

Example 4.2.2.5. Following Example 2.4.2.3, the r-th power of the Lefschetz and Λ operators on $CH(\mathbb{P}^{d+r})$ are represented by the correspondences

$$L^r = \sum_{j=r}^{d+r} \alpha_1^{d+2r-j} \alpha_2^j \quad \text{and} \quad \lambda^r = \sum_{j=0}^{d} \alpha_1^{d-j} \alpha_2^j$$

in $\mathrm{CH}(\mathbb{P}^{d+r} \times \mathbb{P}^{d+r}) = \mathbb{Q}[\alpha_1, \alpha_2]/(\alpha_1^{d+r+1}, \alpha_2^{d+r+1})$. Their composition $\lambda^r \circ L^r$ is given by the coefficient of α_2^{d+r} in the product

$$\left(\sum_{j=r}^{d+r} \alpha_1^{d+2r-j} \alpha_2^j \right) \cdot \left(\sum_{l=0}^{d} \alpha_2^{d-l} \alpha_3^l \right) = \sum_{j=r}^{d+r} \sum_{l=0}^{d} \alpha_1^{d+2r-j} \alpha_2^{d-l+j} \alpha_3^l.$$

We find

$$\lambda^r \circ L^r = \sum_{l=0}^{d} \alpha_1^{d+r-l} \alpha_2^l$$

which is already a projector. Hence the projector p_{fix} from Remark 4.2.2.4 is

$$p_{\text{fix}} = \lambda^r \circ L^r = \sum_{l=0}^{d} \alpha_1^{d+r-l} \alpha_2^l,$$

which is known from Example 2.1.2.9 as the projector onto the submotive $\mathbf{1} \oplus \cdots \oplus \mathbf{L}^d$ in $\mathfrak{h}(\mathbb{P}^{d+r})$.

As a result, the Chow motive of any smooth complete intersection X of dimension d in $\mathbb{P}^{d+r}$ splits as

$$\mathfrak{h}(X) = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{L}^d}_{\mathfrak{h}_{\text{fix}}(X)} \oplus \mathfrak{h}_{\text{var}}(X). \tag{19}$$

$\triangle$

In the example above, the splitting (19) is a Chow–Künneth decomposition. Indeed, it gives a decomposition

$$\Delta_X = \underbrace{q_0 + q_2 + \cdots + q_{2d}}_{\pi_{\mathrm{fix}}} + \pi_{\mathrm{var}}$$

of the diagonal of X into orthogonal projectors. For $i \neq d$, the projector q_i induces the i-th Künneth component. The middle degree Künneth component is induced by $q_d + \pi_{\mathrm{var}}$ if d is even and π_{var} if d is odd. In particular, this shows:

Corollary 4.2.2.6. *Complete intersections in projective spaces admit a Chow–Künneth decomposition.*

See [MNP13, App. C] for a different proof of that fact.

Example 4.2.2.7. If $d = r = 1$ in the above example, then X is a smooth curve in $\mathbb{P}^2$ and we get

$$\mathfrak{h}(X) = \mathbf{1} \oplus \mathbf{L} \oplus \mathfrak{h}_{\mathrm{var}}(X).$$

Thus, the variable part $\mathfrak{h}_{\mathrm{var}}(X)$ is precisely the middle part $\mathfrak{h}^1(X)$. $\triangle$

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