

DUAL LAGRANGIAN FIBRATIONS

T. BECKMANN & D. HUYBRECHTS

ABSTRACT. These are notes of a survey talk on fibrations naturally associated with a Lagrangian fibration of a hyperkähler manifold and recent results by Arinkin–Fedorov [AF16], Kim [Ki21], Nagai [Na05], and Saccà [Sa20, Sa21] addressing the possibility of (partially) compactifying them.

1. INTRODUCTION

We consider a Lagrangian fibration $f: X \dashrightarrow B$ of a projective hyperkähler manifold X and denote by $X_0 \subset X$ the open union of all smooth fibres which comes with a smooth, projective morphism $f_0: X_0 \rightarrow B_0$ over some open subset $B_0 \subset B$. With this situation, two abelian schemes are naturally associated:

$$P_0 \rightarrow B_0 \text{ and } P_0^\vee \rightarrow B_0.$$

The first one, $P_0 \simeq \text{Aut}_0(X_0/B_0) \simeq \text{Alb}(X_0/B_0)$, can be described as the relative Albanese scheme or, alternatively, as a relative scheme of automorphisms, while the second one $P_0^\vee \simeq \text{Pic}^0(X_0/B_0)$ is the (identity component of the) relative Picard scheme. The latter is the dual abelian scheme of the former. Note that the original $X_0 \rightarrow B_0$ is a torsor for $P_0 \rightarrow B_0$, which naturally compactifies to $X \dashrightarrow B$.

From here, two parallel stories develop, but the main questions are the same for both:

- (1) Can $P_0 \rightarrow B_0$ and $P_0^\vee \rightarrow B_0$ be (partially) compactified to a (smooth) symplectic variety with a Lagrangian fibration over B ?
- (2) Can torsors for $P_0 \rightarrow B_0$ or over $P_0^\vee \rightarrow B_0$ be (partially) compactified to (smooth) symplectic varieties with a Lagrangian fibration over B ?

We will discuss work of Arinkin–Fedorov [AF16], Kim [Ki21], Nagai [Na05], and Saccà [Sa20, Sa21] addressing these questions.

- In [AF16] a construction is described that partially compactifies $P_0 \rightarrow B_0$ to a smooth commutative group scheme $P_1 \dashrightarrow B_1$ over the open subscheme $B_1 \subset B$ of all points with integral fibres, see Section 6. Furthermore, P_1 acts faithfully on X over B .

This review was prepared in the context of the seminar organized by the ERC Synergy Grant HyperK, Grant agreement ID 854361. The talk was delivered on December 17, 2021.

- In [Na05] a smooth commutative group scheme (or, rather, a separated algebraic space) $P_1^\vee \twoheadrightarrow B_1$ is constructed that partially compactifies $P_0^\vee \twoheadrightarrow B_0$ and admits a symplectic structure, see Section 5.¹ The construction extends to the open subset of B of all points with fibres with at least one reduced component.

- In [Ki21], at least for the standard deformation types of compact hyperkähler manifolds, the natural torsor for $P_0^\vee \twoheadrightarrow B_0$ associated with the torsor $X_0 \twoheadrightarrow B_0$ for $P_0 \twoheadrightarrow B_0$ is compactified to a projective but singular hyperkähler variety $X^\vee \twoheadrightarrow B$, see Section 7.

- In the forthcoming work [Sa21], assuming $B_1 = B$, any torsor for the Arinkin–Fedorov partial compactification $P_1 \twoheadrightarrow B_1 = B$ is compactified to a smooth projective hyperkähler manifold with Lagrangian fibration over B , see Section 8.

Apart from [AF16] the existence of the compact hyperkähler manifold X as the starting point of the consideration is crucial.

Acknowledgement: We are grateful to D. Arinkin for answering our questions. Thanks also to P. Belmans for comments on the first version.

2. PREPARATIONS I: ABELIAN SCHEMES, ALBANESE, PICARD

2.1. Let us start with a smooth complex projective variety Y (or, more generally, a compact Kähler manifold). Classically, one associates two abelian varieties with Y (or, in the Kähler case, two complex tori): The Albanese variety $\text{Alb}(Y)$ and the Picard variety $\text{Pic}^0(Y)$. The Albanese variety is characterized by its universality property.²

Both, Picard and Albanese variety, admit an explicit description as complex tori naturally associate with Hodge structures of weight one:

$$\text{Pic}^0(Y) \simeq H^{0,1}(Y)/H^1(Y, \mathbb{Z}) \quad \text{and} \quad \text{Alb}(Y) \simeq H^{1,0}(Y)^*/H^1(Y, \mathbb{Z})^*.$$

Note that using this description, there exists a natural isomorphism of integral Hodge structures of weight one

$$H^1(\text{Alb}(Y), \mathbb{Z}) \simeq H^1(Y, \mathbb{Z})$$

which immediately leads to the following classical consequence.

Corollary 2.1. *The Albanese variety $\text{Alb}(Y)$ and the Picard variety $\text{Pic}^0(Y)$ are dual abelian varieties, i.e. there exist natural isomorphisms*

$$\text{Alb}(Y)^\vee := \text{Pic}^0(\text{Alb}(Y)) \simeq \text{Pic}^0(Y) \quad \text{and} \quad \text{Alb}(Y) \simeq \text{Pic}^0(\text{Pic}^0(Y)).$$

¹The notation suggests that P_1 and P_1^\vee are dual, but it is not clear whether (and in what sense) this is true.

²For any point $y \in Y$ there exists a morphism $\text{alb}_y: Y \rightarrow \text{Alb}(Y)$ with $\text{alb}(y) = 0_{\text{Alb}(Y)}$ such that every other morphism $Y \rightarrow T$, $y \mapsto 0_T$, to an abelian variety T factorizes through a unique morphism of abelian varieties $\text{Alb}(Y) \rightarrow T$.

Remark 2.2. (i) Although, both abelian varieties $\text{Pic}^0(Y)$ and $\text{Alb}(Y)$ can be constructed purely algebraically, i.e. they can be constructed for varieties over any field, they are very different in nature. The Picard variety is a moduli space of sheaves on Y , while the Albanese variety is described by its universal property. Once duality has been established, $\text{Alb}(Y)$ can also be viewed as a moduli space of sheaves on $\text{Pic}(Y)$.

(ii) Also the Albanese morphism $\text{alb}_y: Y \rightarrow \text{Alb}(Y)$ can be reconstructed using the description of $\text{Alb}(Y)$ as the dual of $\text{Pic}^0(Y)$. More precisely, consider the Poincaré bundle \mathcal{P} on $Y \times \text{Pic}^0(Y)$ normalized such that $\mathcal{P}|_{y \times \text{Pic}^0(Y)}$ is trivial. Then, \mathcal{P} as a family of line bundles on $\text{Pic}^0(Y)$ parametrized by Y determines the classifying morphism

$$Y \rightarrow \text{Pic}^0(\text{Pic}^0(Y)) \simeq \text{Alb}(Y), \quad x \mapsto \mathcal{P}|_{x \times \text{Pic}^0(Y)},$$

which is nothing but alb_y .

(iii) This point of view has the advantage to work more generally for normal projective (or complete) varieties Y . In this case, $\text{Pic}^0(Y)$ is still an abelian variety (in positive characteristic one needs to pass to the reduction) and the dual abelian variety of $\text{Pic}^0(Y)$ can be taken as the definition of $\text{Alb}(Y)$.

The automorphism group $\text{Aut}(Y)$ acts naturally on $\text{Alb}(Y)$ and $\text{Pic}^0(Y)$. More precisely, any $g \in \text{Aut}(Y)$ acts on both varieties as an isomorphism of abelian varieties, i.e. preserving the origin. Note however, that this action is trivial on the connected component $\text{id}_Y \in \text{Aut}_0(Y) \subset \text{Aut}(Y)$, as any continuous group acts trivially on the discrete cohomology $H^1(Y, \mathbb{Z})$.³ However, there is another action of $\text{Aut}_0(Y)$ on the Albanese variety defined via the universality property by the commutativity of the diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\text{alb}_y} & \text{Alb}(Y) \\ g \downarrow & & \downarrow \text{Alb}(g) \\ Y & \xrightarrow{\text{alb}_y} & \text{Alb}(Y). \end{array}$$

Note that with this definition $\text{Alb}(g)$ is only an isomorphism of varieties but does not necessarily fix the origin. The definition of $\text{Alb}(g)$ is independent of the choice of $y \in Y$. A point $x \in Y$ is being sent to the functional given by integrating 1-forms along \int_y^x and its image under $\text{Alb}(g)$ is given by integration along $\int_y^{g(x)} = \int_y^x + \int_x^{g(x)}$. The second summand is independent of y .

In particular, with this definition, the action of $\text{Aut}_0(Y) \subset \text{Aut}(Y)$ on $\text{Alb}(Y)$ is usually not trivial.⁴ Using that $\text{Alb}(Y)$ is an abelian variety and $\text{Aut}_0(Y)$ is connected, every $\text{Alb}(g)$ is in

³We consider $\text{Aut}_0(Y)$ as a group variety, which in general is neither compact nor abelian. Also note that $\text{Alb}(Y)$ is a birational invariant while $\text{Aut}_0(Y)$ is not.

⁴Note that for the same reason as before this action of $\text{Aut}_0(Y)$ on the variety(!) $\text{Alb}(Y)$ induces a trivial action on its Picard variety $\text{Pic}^0(\text{Alb}(Y)) \simeq \text{Pic}^0(Y)$.

fact given by translation. This yields a (natural) morphism of group schemes

$$(2.1) \quad \mathrm{Aut}_0(Y) \longrightarrow \mathrm{Alb}(Y).$$

2.2. We next consider the case that Y is isomorphic to an abelian variety A , but we do not fix any such isomorphism and, in particular, we do not fix a distinguished point of Y as the origin. In this case, the universality property of the Albanese variety shows that as varieties $Y \simeq \mathrm{Alb}(Y)$, but there is no natural choice of such an isomorphism, a point $y \in Y$ needs to be fixed. In fact, the more natural thing to do is to view Y as a torsor.

Lemma 2.3. *If Y is isomorphic to an abelian variety, then there exists a canonical isomorphism of group varieties*

$$\mathrm{Aut}_0(Y) \simeq \mathrm{Alb}(Y).$$

Via this isomorphism, Y can be naturally considered as a torsor for $\mathrm{Alb}(Y)$.

Proof. Since Y is isomorphic to an abelian variety, the scheme $\mathrm{Aut}_0(Y)$ is isomorphic to Y and acts on Y as well as $\mathrm{Alb}(Y)$ free and transitively. Hence, the morphism

$$\mathrm{Aut}_0(Y) \longrightarrow \mathrm{Alb}(Y)$$

from (2.1) is injective as well as surjective and sends the identity to $0_{\mathrm{Alb}Y}$. \square

2.3. Let us recall another bit of the classical theory of abelian varieties. Any line bundle L on a smooth projective variety Y induces a natural homomorphism of algebraic groups

$$(2.2) \quad \phi_L: \mathrm{Aut}_0(Y) \longrightarrow \mathrm{Pic}^0(Y), \quad g \longmapsto g^*L \otimes L^{-1}.$$

In the case that Y is isomorphic to an abelian variety, i.e. for any point $y \in Y$ the Albanese map $\mathrm{alb}_y: Y \xrightarrow{\sim} \mathrm{Alb}(Y)$ is an isomorphism, the morphism ϕ_L can be described alternatively as

$$(2.3) \quad \varphi_{L_y}: \mathrm{Aut}_0(Y) \simeq \mathrm{Alb}(Y) \longrightarrow \mathrm{Alb}(Y)^\vee \simeq \mathrm{Pic}^0(Y).$$

Here, L_y is the line bundle on $\mathrm{Alb}(Y)$ satisfying $\mathrm{alb}_y^*L_y \simeq L$ and $\varphi_M: A \longrightarrow A^\vee$ for a line bundle M on an abelian variety A is the standard morphism $x \longmapsto t_x^*M \otimes M^{-1}$ (which only depends on the class $M \in \mathrm{NS}(A)$). Note although L_y depends on the choice of the point $y \in Y$, the morphism φ_{L_y} does not, simply because it coincides with φ_L . Also, as explained earlier, the two isomorphisms involved in (2.3) are canonical.

2.4. We now turn to the relative situation. Assume $f: \mathcal{Y} \longrightarrow T$ is a smooth projective morphism with connected fibres. The relative versions of Albanese and Picard provide two abelian schemes

$$(2.4) \quad g: \mathrm{Alb}(\mathcal{Y}/T) \longrightarrow T \quad \text{and} \quad h: \mathrm{Pic}^0(\mathcal{Y}/T) \longrightarrow T.$$

By definition, an abelian scheme $\mathcal{A} \longrightarrow T$ is a smooth projective group scheme over T with connected fibres. In particular, it comes with a zero-section $\sigma: T \longrightarrow \mathcal{A}$ and each fibre \mathcal{A}_t , $t \in T$, is an abelian variety with its origin given by $\sigma(t) \in \mathcal{A}_t$. For the two abelian schemes (2.4) the fibres are $\mathrm{Alb}(\mathcal{Y}_t)$ and $\mathrm{Pic}^0(\mathcal{Y}_t)$. For the latter, σ is simply $t \longmapsto \mathcal{O}_{\mathcal{Y}_t}$.

The Hodge-theoretic description and the duality in the absolute case carry over to the relative description. For example, there exists a natural isomorphism of variation of Hodge structures

$$R^1 f_* \mathbb{Z} \simeq R^1 g_* \mathbb{Z}$$

and the dual abelian scheme of $\text{Alb}(\mathcal{Y}/T) \rightarrow T$ is $\text{Pic}^0(\mathcal{Y}/T) \rightarrow T$, i.e.

$$\text{Pic}^0(\text{Alb}(\mathcal{Y}/T)/T) \simeq \text{Pic}^0(\mathcal{Y}/T) \text{ and } \text{Alb}(\mathcal{Y}/T) \simeq \text{Pic}^0(\text{Pic}^0(\mathcal{Y}/T)/T).$$

Remark 2.4. As in Remark 2.2, the situation generalizes in a straightforward way to projective families $f: \mathcal{Y} \rightarrow T$ with connected normal fibres over a smooth (reduced is enough) base. The relative Picard variety $\text{Pic}^0(\mathcal{Y}/T) \rightarrow T$ is known to exist as a smooth abelian scheme over T , see [FAG]. The Albanese $\text{Alb}(\mathcal{Y}/T) \rightarrow T$ is defined as the dual abelian scheme $\text{Pic}^0(\text{Pic}^0(\mathcal{Y}/T)/T)$.

For any smooth projective morphism $f: \mathcal{Y} \rightarrow T$ one can also consider

$$\text{Aut}_0(\mathcal{Y}/T) \rightarrow T,$$

which in general is only a group scheme, neither abelian nor proper, cf. [Br18, Sec. 2] for standard facts and references. The natural morphisms (2.1) for all fibres glue to a morphism

$$\text{Aut}_0(\mathcal{Y}/T) \rightarrow \text{Alb}(\mathcal{Y}/T)$$

of group schemes over T . Similarly, if \mathcal{L} is a line bundle on \mathcal{Y} , then the morphisms $\phi_{\mathcal{L}_t}$ in (2.7) glue to a morphism of group schemes over T

$$(2.5) \quad \phi_{\mathcal{L}}: \text{Aut}_0(\mathcal{Y}/T) \rightarrow \text{Pic}^0(\mathcal{Y}/T).$$

In fact, we only need the data of line bundles \mathcal{L}_t on all fibres \mathcal{Y}_t varying algebraically with t and not the global line bundle \mathcal{L} on \mathcal{Y} . In other words, there exists a natural morphism $\phi: \text{Aut}_0(\mathcal{Y}/T) \rightarrow \text{Pic}^0(\mathcal{Y}/T)$ for any section of $\text{Pic}(\mathcal{Y}/T) \rightarrow T$.

2.5. Assume now that all fibres of the smooth projective morphism $f: \mathcal{Y} \rightarrow T$ are isomorphic to abelian varieties (but not assuming that any such isomorphism has been fixed). Again, the discussion in the previous sections carries over and shows that there exists a canonical isomorphism

$$(2.6) \quad P_0 := \text{Aut}_0(\mathcal{Y}/T) \simeq \text{Alb}(\mathcal{Y}/T)$$

and that $\mathcal{Y} \rightarrow T$ is naturally a torsor for $\text{Aut}_0(\mathcal{Y}/T) \simeq \text{Alb}(\mathcal{Y}/T)$. We denote the dual abelian scheme by

$$P_0^\vee := \text{Pic}^0(\mathcal{Y}/T) \simeq \text{Pic}^0(P_0/T).$$

Thus, for any relative ample line bundle \mathcal{L} on $\mathcal{Y} \rightarrow T$ we have have a finite étale morphism

$$(2.7) \quad \phi_{\mathcal{L}}: P_0 \twoheadrightarrow P_0^\vee$$

of abelian schemes over T . Note that (2.5) does not necessarily admit an alternative description that globalizes the morphisms φ_{L_y} in (2.3), since typically we cannot choose points $y_t \in \mathcal{Y}_t$

in a uniform manner. As noted before, in order to define $\phi_{\mathcal{L}}$ one only needs a section of $\text{Pic}(\mathcal{Y}/T) \rightarrow T$ and not necessarily a global line bundle. Moreover, since $\varphi_M: A \rightarrow A^\vee$ for an abelian variety A only depends on the class $M \in \text{NS}(A)$, (2.7) is well defined for any global section of $R^2 f_* \mathbb{Z}$ that fibrewise is contained in $H^{1,1}(\mathcal{Y}_t, \mathbb{Z})$ and is locally (in T) induced by a line bundle.

2.6. Still assuming $\mathcal{Y} \rightarrow T$ to be a smooth projective morphism with fibres isomorphic to abelian varieties, we consider the finite étale morphism $\phi_{\mathcal{L}}: P_0 \twoheadrightarrow P_0^\vee$. Its kernel

$$\mathcal{K} := \text{Ker}(\phi_{\mathcal{L}}) \subset P_0$$

is a finite subgroup scheme over T . Furthermore, its quotient is P_0^\vee :

$$P_0/\mathcal{K} \simeq P_0^\vee.$$

As $\mathcal{K} \subset P_0$, it acts on P_0 but since $P_0 \simeq \text{Aut}_0(\mathcal{Y}/T)$ it also acts (over T) on \mathcal{Y} . We denote the quotient of this action by $\mathcal{Y}^\vee := \mathcal{Y}/\mathcal{K}$. It is naturally a torsor for P_0^\vee :

$$\begin{array}{ccc} \mathcal{Y} & \twoheadrightarrow & \mathcal{Y}^\vee = \mathcal{Y}/\mathcal{K} \\ \curvearrowright & & \curvearrowright \\ P_0 & \twoheadrightarrow & P_0^\vee = P_0/\mathcal{K} \end{array}$$

In the applications, \mathcal{K} will be a constant group scheme associated with a finite group of automorphisms $K \subset \text{Aut}(\mathcal{Y})$. Conversely, any automorphism of \mathcal{Y} that commutes with f and is fibrewise isotopic to the identity defines a section of $P_0 \rightarrow T$. Thus, the subgroup $\text{Aut}(\mathcal{Y}, f) \subset \text{Aut}(\mathcal{Y})$ of all such gives a constant subgroup scheme $\text{Aut}(\mathcal{Y}, f)_T \hookrightarrow P_0$.

3. PREPARATIONS II: ELLIPTIC K3 SURFACES

This section can be read as a preparation for the subsequent parts or as an illustration of the abstract results to be presented there. The reader should feel free to skip this section at a first reading. The theory of Lagrangian fibrations of K3 surfaces, so elliptic K3 surfaces, is rather explicit and can help to understand the more complicated picture in higher dimensions. However, as elliptic curves are principally polarized, at certain points the difference between the Albanese fibration and the Picard fibration is blurred and certain general features are less clear.

Let $f: X \rightarrow B := \mathbb{P}^1$ be an elliptic K3 surface. As we are not assuming the existence of a section, this is often just called a genus one fibration. Denote by

$$X_0 \subset X' \subset X$$

the two open subsets consisting of all smooth fibres respectively all points at which f is regular. Hence, the image of the restriction $f_0 := f|_{X_0}: X_0 \rightarrow B_0$ is the proper open subset $B_0 \subset B = \mathbb{P}^1$ of all points with smooth fibre and $f': X' \rightarrow B$ is still surjective, as every fibre admits at least one reduced component, cf. [Hu16, Sec. 11.1.3].

3.1. As in the general situation (2.6), two out of the three abelian schemes naturally associated with $f: X_0 \rightarrow B_0$ are isomorphic:

$$P_0 := \text{Alb}(X_0/B_0) \simeq \text{Aut}_0(X_0/B_0)$$

and $X_0 \rightarrow B_0$ is naturally a torsor for $g_0: P_0 \rightarrow B_0$. Furthermore, the dual abelian scheme is

$$P_0^\vee := \text{Pic}^0(X_0/B_0) \simeq \text{Pic}^0(P_0/B_0).$$

To a line bundle \mathcal{L} on X (on X_0 suffices) one associates a morphism $\phi_{\mathcal{L}}: P_0 \rightarrow P_0^\vee$. If \mathcal{L} is fibrewise ample, $\phi_{\mathcal{L}}$ is étale of degree $\deg(\mathcal{L}|_{X_t})^2$. In general, X or, equivalently, X_0 may not admit a line bundle \mathcal{L} that is fibrewise of degree one. However, as explained above, $\phi_{\mathcal{L}}$ really only depends on the induced section of $R^2 f_{0*} \mathbb{Z}$ over B_0 . Moreover, one can in fact define $\phi_{\mathcal{L}}$ for any given section of $R^2 f_{0*} \mathbb{Z}$. Since the fibres are curves, the condition to be of type (1, 1) is automatic.

It turns out that the local system $R^2 f_{0*} \mathbb{Z}$ is trivial and that a global trivializing section restricts to a generator of $H^2(X_t, \mathbb{Z})$ for any smooth fibre X_t . Indeed, the class of a fibre $[X_t] \in H^2(X, \mathbb{Z})$ is primitive and the intersection form is unimodular. Thus, there is a distinguished isomorphism

$$\phi: P_0 \xrightarrow{\sim} P_0^\vee.$$

3.2. Clearly, $X_0 \rightarrow B_0$ can be compactified to an elliptic K3 surface, namely to $X \rightarrow B$ itself. What about P_0 and P_0^\vee ? It turns out that the latter can in fact be compactified to a K3 surface. As a first step, one considers the partial compactification provided by the relative Picard variety

$$\text{Pic}^0(X/B) \rightarrow B,$$

which is projective only over B_0 . Thinking of $\text{Pic}^0(X/B)$ as parametrizing sheaves on X , namely those of the form $i_* M$ for M a line bundle of degree zero on a fibre X_t , it is natural to compactify it further by the moduli space $M_H(v)$ of sheaves with Mukai vector $v = (0, [X_t], 0)$ that are stable with respect to a generic polarization. Then the general theory, see [Hu16, Sect. 11.4], shows that $\overline{P_0^\vee} := M_H(v)$ is a K3 surface and the support map extends $\text{Pic}^0(X/B) \rightarrow B$ to an elliptic fibration

$$\overline{P_0^\vee} \rightarrow B.$$

Moreover, it comes with a zero section given by $t \mapsto \mathcal{O}_{X_t}$. Note that $\overline{P_0^\vee}$ does not depend on the choice of the polarization H on X , simply because two birational K3 surfaces are isomorphic.

Remark 3.1. (i) Since $P_0 \simeq P_0^\vee$, the dual K3 surface $\overline{P_0^\vee}$ can also be viewed as a compactification $\overline{P_0}$ of $P_0 \rightarrow B_0$. Note however, that a priori we do not have a natural compactification of P_0 that would not make use of the isomorphism $P_0 \simeq P_0^\vee$, one that would only depend on the interpretation of P_0 as Alb or Aut_0 . This has been attempted only recently by Saccà, in the context of higher-dimensional hyperkähler manifolds and using MMP, see Section 8.

(ii) Since $\overline{P_0} \simeq \overline{P_0^\vee} \rightarrow B$ restricted to B_0 is isomorphic to $P_0 \simeq P_0^\vee$, the elliptic fibration $X \rightarrow B$, restricted to $B_0 \subset B$ can be viewed as a torsor for the restriction of $\overline{P_0} \simeq \overline{P_0^\vee}$. Hence, X defines a class $[X] \in \text{III}(\overline{P_0}/\mathbb{P}^1)$ in the Tate–Shafarevich group, cf. [Hu16, Sect. 11.5].

(iii) In fact, every class $\alpha \in \text{III}(\overline{P_0}/\mathbb{P}^1)$, so a torsor $X_0(\alpha) \rightarrow B_0$ for $P_0 \simeq P_0^\vee \rightarrow B_0$, can be compactified to an elliptic K3 surface $X(\alpha) \rightarrow B$, though most of these K3 surfaces $X(\alpha)$ are not projective.

3.3. There is another open set one can consider

$$X_0 \subset X_1 \subset X,$$

the union of all integral fibres, i.e. X_1 is obtained from X_0 by adding all fibres of type I₁ (rational curve with one node) and II (rational curve with one cusp). We can view $f_1 := f|_{X_1} : X_1 \rightarrow B_1 := f(X_1) \subset B$ as a partial compactification of $X_0 \rightarrow B_0$. In fact, the general elliptic K3 surface will have exactly 24 singular fibres all of type I₁, in which case $X_1 = X$.

Let us next consider the open subset $X'_1 = X_1 \cap X'$ of all f_1 -regular points. Hence, the fibre of $X'_1 \rightarrow B_1$ are either smooth curves of genus one, isomorphic to $\mathbb{P}^1 \setminus \{0, 1\}$ (in case I₁), or $\mathbb{P}^1 \setminus \{0\}$ (in case II).

Remark 3.2. There exists an abelian group scheme $g_1 : P_1 \rightarrow B_1$ acting on $X_1 \rightarrow B_1$ and such that restriction turns $X'_1 \rightarrow B_1$ into a torsor for $P_1 \rightarrow B_1$. The fibres of g_1 are either smooth elliptic curves, \mathbb{G}_m (case I₁), or \mathbb{G}_a (case II). More precisely, P_1 is constructed as a partial compactification of P_0 and there are two ways to do this:

(i) Since $P_0 \simeq P_0^\vee$, we can construct P_1 as a partial compactification of P_0^\vee . Namely, $P_1 := \text{Pic}^0(X_1/B_1)$. Note that Pic^0 of a rational curve with one node is indeed just \mathbb{G}_m which parametrizes the gluing of the fibres of $\mathcal{O}_{\mathbb{P}^1}$ at the two points $0, 1 \in \mathbb{P}^1$. Similarly, Pic^0 of a rational curve with one cusp is the additive group \mathbb{G}_a . With this construction it is easy to see that $X_1 \rightarrow B_1$ is a torsor for $P_1 \rightarrow B_1$.

(ii) In the second approach $X_1 \rightarrow B_1$ is a torsor for $P_1 \rightarrow B_1$ by construction, but the construction of P_1 itself is more involved. Consider $\text{Aut}_0(X_1/B_1) \rightarrow B_1$. Its restriction to B_0 gives back P_0 . The fibre over $t \in B_1$ with X_t a singular curve of type I₁ is the subgroup of $\text{Aut}(\mathbb{P}^1)$ fixing $0, 1 \in \mathbb{P}^1$, which is indeed \mathbb{G}_m . The automorphism group of a fibre of type II is isomorphic to $\mathbb{G}_m \rtimes \mathbb{G}_a$ and for our purposes of constructing P_1 we restrict to the action of \mathbb{G}_a , see Example 6.2 for details. The natural action $\text{Aut}_0(X_1/B_1) \times_{B_1} X_1 \rightarrow X_1$ restricted to P_1 has all the required properties.

(iii) This example also shows why, when one wants to construct the abelian scheme P_1 for certain singular fibres, one needs to use Aut_0 instead of Alb . Indeed, the Albanese variety is a birational invariant and the normalization of singular fibres of type I₁ or II is isomorphic to \mathbb{P}^1 , which has trivial first cohomology. Thus, we see that the dimension of the relative automorphism scheme can increase in singular fibres and we can restrict to a certain closed subgroup of

automorphisms that deform infinitesimally to nearby fibres, whereas for the Albanese variety no such construction is possible.

For K3 surfaces, this procedure can be extended to all of B . More precisely, there exists an abelian group scheme $P' \twoheadrightarrow B$ acting on $X \rightarrow B$ and such that with respect to this action $X' \rightarrow B$ is a torsor for P' .⁵

4. LAGRANGIAN FIBRATIONS OF ABELIAN SCHEMES AND HYPERKÄHLER MANIFOLDS

We explain how symplectic structures are transferred between abelian schemes and torsors for them. Eventually, this is applied to the smooth part of a Lagrangian fibration $X \rightarrow B$ of a projective hyperkähler manifold.

4.1. Let us begin with an abelian scheme $g: \mathcal{A} \rightarrow T$ and assume that \mathcal{A} is endowed with a symplectic structure, i.e. we are given a non-degenerate (closed) two-form $\sigma \in H^0(\mathcal{A}, \Omega_{\mathcal{A}}^2)$ for which g is a Lagrangian fibration. In particular, each fibre \mathcal{A}_t is a Lagrangian torus and $\dim(\mathcal{A}) = 2 \dim(T)$.

Assume now that we are given another section $\tau: T \rightarrow \mathcal{A}$ of $g: \mathcal{A} \rightarrow T$ (different from the given zero-section). Translation by τ defines an automorphism $\tau: \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ (over T) which induces a map τ^* on $H^0(\mathcal{A}, \Omega_{\mathcal{A}}^2)$.

Lemma 4.1. *Any symplectic structure on \mathcal{A} for which $g: \mathcal{A} \rightarrow T$ is a Lagrangian fibration is invariant under the action of translations by sections of g .*

Proof. Consider the short exact sequence $0 \rightarrow \mathcal{N}_{\mathcal{A}_t/\mathcal{A}}^* \rightarrow \Omega_{\mathcal{A}}|_{\mathcal{A}_t} \rightarrow \Omega_{\mathcal{A}_t} \rightarrow 0$ and the induced exact sequence

$$0 \rightarrow \mathrm{Sym}^2(\mathcal{N}_{\mathcal{A}_t/\mathcal{A}}^*) \rightarrow \mathcal{N}_{\mathcal{A}_t/\mathcal{A}}^* \otimes \Omega_{\mathcal{A}_t} \rightarrow \Omega_{\mathcal{A}}^2|_{\mathcal{A}_t} \rightarrow \Omega_{\mathcal{A}_t}^2 \rightarrow 0.$$

The fact that g is a Lagrangian fibration with respect to the given symplectic structure $\sigma \in H^0(\mathcal{A}, \Omega_{\mathcal{A}}^2)$ translates into the statement that the restriction $\sigma|_{\mathcal{A}_t} \in H^0(\mathcal{A}_t, \Omega_{\mathcal{A}}^2|_{\mathcal{A}_t})$ is the image of a class in $H^0(\mathcal{A}_t, \mathcal{N}_{\mathcal{A}_t/\mathcal{A}}^* \otimes \Omega_{\mathcal{A}_t})$ that induces an isomorphism $\mathcal{N}_{\mathcal{A}_t/\mathcal{A}} \simeq \Omega_{\mathcal{A}_t}$.

Since translation acts trivially on $H^0(\mathcal{A}_t, \mathcal{N}_{\mathcal{A}_t/\mathcal{A}}^* \otimes \Omega_{\mathcal{A}_t})$, the restriction $\sigma|_{\mathcal{A}_t} \in H^0(\mathcal{A}_t, \Omega_{\mathcal{A}}^2|_{\mathcal{A}_t})$ and hence σ itself remain unchanged under translation. \square

Let us now consider in addition a torsor $f: \mathcal{Y} \rightarrow T$ for the abelian scheme $g: \mathcal{A} \rightarrow T$ or, in other words, $\mathcal{A} \simeq \mathrm{Alb}(\mathcal{Y}/T)$. The next result establishes a link between symplectic structures on the two fibrations

Proposition 4.2. *There is a natural bijection between (closed) symplectic structures on \mathcal{A} and \mathcal{Y} for which the two fibrations g and f are Lagrangian.*

⁵Due to the existence of multiple fibres, this is not expected to generalize to higher dimensions. The analogue of P' in higher dimension could either be (i) Pic^0 as in Nagai, see Section 5, or (ii) the relative $P_1 \subset \mathrm{Aut}_0$ by Arininkin–Fedorov, see Section 6.

Proof. Étale locally, the two fibrations $f: \mathcal{Y} \rightarrow T$ and $g: \mathcal{A} \rightarrow T$ are isomorphic. The glueings for the two fibrations differ by translations, which according to the lemma leave invariant the symplectic structure. For example, a given symplectic structure σ on \mathcal{A} , restricts to symplectic structures $\sigma|_{U_i}$ on $\mathcal{A}|_{U_i} \simeq \mathcal{Y}|_{U_i}$ for some open covering $T = \bigcup U_i$ and those then glue back to a symplectic structure on \mathcal{Y} . The converse is similar. \square

4.2. We are interested in Lagrangian fibrations $f: X \rightarrow B$ of a projective hyperkähler manifold X of dimension $2n$. According to results of Matsushita, any morphism $f: X \dashrightarrow B$ to a normal variety of dimension $0 < \dim(B) < \dim(X)$ is a Lagrangian fibration. So, all fibres are of dimension n and the smooth part of the reduction of any fibre is Lagrangian. In particular, for a smooth fibre X_t the symplectic structure induces naturally an isomorphism $\Omega_{X_t} \simeq \mathcal{N}_{X_t/X}$ and since the normal bundle is $\mathcal{N}_{X_t/X} \simeq T_t B \otimes \mathcal{O}_{X_t}$ is trivial, smooth fibres are abelian varieties. The smoothness of B or, equivalently, the flatness of f is expected and has to be assumed for most of the things that will follow. We refer to [HM21] for further information on Lagrangian fibrations and references.

As for K3 surfaces, one denotes by $X_0 \subset X$ the union of all smooth fibres which comes with a proper smooth fibration $f_0: X_0 \rightarrow B_0$ over some open subset $B_0 \subset B$ with fibres isomorphic to abelian varieties.

4.3. As before, we associate two abelian schemes with the situation: The relative Albanese

$$g_0: P_0 = \text{Alb}(X_0/B_0) \rightarrow B_0$$

and its dual

$$P_0^\vee = \text{Pic}^0(X_0/B_0) \simeq \text{Pic}^0(P_0/B_0) \rightarrow B_0.$$

Remark 4.3. As in Remark 2.4, one could consider more generally the normal part, i.e. the union of all normal fibres. The relative Picard and Albanese would still exist as abelian schemes. However, it seems that normality of a fibre is often (always?) equivalent to its smoothness. So, we do not gain anything by this generalization.

As in Section 2.6, we view $X_0 \rightarrow B_0$ as a torsor for $P_0 \rightarrow B_0$. Following the discussion in Section 2, we consider

$$\phi: P_0 \dashrightarrow P_0^\vee$$

associated with the generator of the image of the restriction map $H^2(X, \mathbb{Z}) \rightarrow R^2 f_{0*} \mathbb{Z}$. Its kernel $\mathcal{K} = \ker(\phi) \subset P_0 = \text{Aut}(X_0/B_0)$ is a smooth finite group scheme over B_0 .

4.4. The restriction of the given symplectic structure on X endows the open subset X_0 with a closed symplectic structure for which $X_0 \rightarrow B_0$ is a Lagrangian fibration. According to the preceding discussion, this structure is passed on to the two abelian schemes $P_0 \rightarrow B_0$ and $P_0^\vee \rightarrow B_0$. More precisely, we draw two consequences of Proposition 4.2, see [DM96, Ma96, Na05].

Corollary 4.4 (Donagi–Markman, Markushevich). *The two abelian schemes $P_0 \rightarrow B_0$ and $P_0^\vee \rightarrow B_0$ are both naturally (up to scaling) endowed with a closed symplectic structure for which the projections are Lagrangian fibrations⁶.*

Proof. Since the polarization defines étale morphisms of abelian group schemes

$$P_0 \twoheadrightarrow P_0^\vee \twoheadrightarrow P_0,$$

the existence of a symplectic structure on one of the two abelian schemes induces a symplectic structure on the other one such that the Lagrangian properties are transferred.

By virtue of Proposition 4.2, $P_0 = \text{Alb}(X_0/B_0)$ is endowed with a symplectic structure for which g is Lagrangian. This suffices to conclude. \square

See also the more recent paper [BGO20], where one finds an explicit description of the symplectic structure on P_0 , similar to the one by Markushevich [Ma96], and on P_0^\vee .

Corollary 4.5. *All torsors $h: Y_0 \rightarrow B_0$ for $P_0 \rightarrow B_0$ are naturally (up to scaling) endowed with a closed symplectic structures for which h is a Lagrangian fibration. An analogous statement holds for torsors for $P_0^\vee \rightarrow B_0$.* \square

Note that in the above proof the composition $P_0 \twoheadrightarrow P_0^\vee \twoheadrightarrow P_0$ is multiplication by the square of the degree of the polarization. Under this pull-back, a symplectic structure on P_0 is multiplied by a scalar (the fourth power of the degree).

Remark 4.6. The abelian scheme $P_0^\vee \rightarrow B_0$ can be viewed as an open subset of a moduli space of sheaves on X , namely those of the form $i_*\mathcal{L}$, where $i: X_t \hookrightarrow X$ is the closed embedding of a smooth fibre and \mathcal{L} is a degree zero line bundle on X_t . The tangent space of X_t and X at the point $[\mathcal{L}]$ are canonically isomorphic to $\text{Ext}_{X_t}^1(\mathcal{L}, \mathcal{L})$ and $\text{Ext}_X^1(i_*\mathcal{L}, i_*\mathcal{L})$ respectively. It would be more natural to describe the symplectic form on P_0^\vee as given by the pairing

$$\text{Ext}_X^1(i_*\mathcal{L}, i_*\mathcal{L}) \times \text{Ext}_X^1(i_*\mathcal{L}, i_*\mathcal{L}) \longrightarrow \text{Ext}_X^2(i_*\mathcal{L}, i_*\mathcal{L}) \xrightarrow{\text{tr}} H^2(X, \mathcal{O}) \simeq \mathbb{C}.$$

The Lagrangian property should then be deduced from this pairing being trivial when restricted to $\text{Ext}_{X_t}^1(\mathcal{L}, \mathcal{L})$. Indeed, the pairing factorizes through

$$\begin{array}{ccc} H^2(X_t, \mathcal{H}om_{X_t}(\mathcal{L}, \mathcal{L})) & \xrightarrow{\sim} & \text{Ext}_{X_t}^2(\mathcal{L}, \mathcal{L}) \\ \downarrow & & \downarrow \\ H^2(X, \mathcal{H}om_X(i_*\mathcal{L}, i_*\mathcal{L})) & \longrightarrow & \text{Ext}_X^2(i_*\mathcal{L}, i_*\mathcal{L}) \xrightarrow{\text{tr}} H^2(X, \mathcal{O}_X), \end{array}$$

but $\text{tr}: \mathcal{H}om_X(i_*\mathcal{L}, i_*\mathcal{L}) \rightarrow \mathcal{O}_X$ is trivial.

⁶The associated zero sections are Lagrangian as well.

5. EXTENDED PICARD AFTER NAGAI

Let as before $f: X \rightarrow B$ be a Lagrangian fibration of a smooth projective hyperkähler manifold of dimension $2n$. We assume that B is smooth, which according to a result by Hwang [Hw08] is equivalent to $B \simeq \mathbb{P}^n$. Note that the smoothness of the base also implies that f is flat. It is convenient to also assume that f admits local analytic sections or, equivalently, that every fibre of f has at least one generically reduced component. If the latter assumption is not met, we tacitly shrink B .

5.1. The Picard functor is the étale sheafification of the functor that associates with each $T \rightarrow B$ the relative Picard group

$$\mathrm{Pic}(X_T/T) := \mathrm{Pic}(X_T)/\mathrm{Pic}(T).$$

The Picard variety $\mathrm{Pic}^0(X_0/B_0)$ represents a connected component of the Picard functor for the smooth morphism $f_0: X_0 \rightarrow B_0$. If f_0 admits a section, then there exists a universal line bundle on $X_0 \times_{B_0} \mathrm{Pic}^0(X_0/B_0)$. However, the Picard functor can be represented under weaker hypotheses, cf. [BLR90, Ch. 8] or [FAG, Part 5]:

- Grothendieck: Let $f_1: X_1 \rightarrow B_1$ be the union of all integral fibres. Then the identity component of the associated Picard functor can be represented by a quasi-projective morphism

$$P_1 := \mathrm{Pic}^0(X_1/B_1) \dashrightarrow B_1.$$

The fibres are the quasi-projective varieties $\mathrm{Pic}^0(X_t)$, which are projective for normal fibres.

- Artin: Without any assumption, the unit component of the Picard functor for the whole family $f: X \rightarrow B$ is represented by a group object in the category of algebraic spaces

$$\mathrm{Pic}^0(X/B) \dashrightarrow B$$

which is locally of finite type, but possibly not separated. The non-separatedness is caused by the non-integral fibres.

- Mumford: Assume all fibres of $f: X \rightarrow B$ are reduced (but possibly reducible). Then the unit component of its Picard functor is represented by a (possibly non-separated) scheme $\mathrm{Pic}^0(X/B) \rightarrow B$ locally of finite type.⁷

Note that the projections $\mathrm{Pic}^0(X_1/B_1) \dashrightarrow B_1$ resp. $\mathrm{Pic}^0(X/B) \dashrightarrow B$ come with a section induced by the structure sheaf.

The fact that $\mathrm{Pic}^0(X/B)$ is typically not separated makes it unsuitable for any geometric considerations. Nagai [Na05] suggested to remedy the situation by applying a procedure originally due to Raynaud [Ra70]: Let $G \subset \mathrm{Pic}^0(X/B)$ be the closure of the zero section, which is

⁷In [AF16, Cor. 3.8] Arinkin and Fedorov complement Mumford's result by showing that the open subspace $\mathrm{Pic}^\tau(X/B) \subset \mathrm{Pic}(X/B)$ of line bundles that are numerically trivial on all fibres is separated. The assumption here is that all fibres are reduced and connected.

a group scheme over B , and define

$$P^\vee := \text{Pic}^0(X/B)/G \longrightarrow B.$$

Then P^\vee is a separated algebraic space.⁸

Remark 5.1. Nagai [Na05, Thm. 1.6] shows that $P^\vee \longrightarrow B$ has the Néron property asserting that every morphism $Z_0 \longrightarrow P_0^\vee$ from a smooth family $Z \longrightarrow B$ extends to a morphism $Z \longrightarrow P^\vee$.

5.2. In order for P^\vee to be considered a partial compactification in the hyperkähler sense, one needs to make sure it inherits the symplectic structure.

Theorem 5.2 (Nagai). *Assume that every fibre of $f: X \longrightarrow B$ has a reduced component or, equivalently, that f admits local sections (in the analytic or étale topology). Then the natural closed symplectic structure on P_0^\vee extends to a closed symplectic structure on P^\vee for which $P^\vee \dashrightarrow B$ is a smooth (possibly non-proper) Lagrangian fibration⁹.*

Proof. The smoothness of $P^\vee \longrightarrow B$ and, therefore, of P^\vee itself follows from $R^1 f_* \mathcal{O}_X \simeq \Omega_B$ being locally free. The construction of the symplectic structure and the verification of the Lagrangian property rely heavily on the classification of the singular fibres of $X \longrightarrow B$ by Matsushita. \square

Remark 5.3. As in Remark 4.6, one could hope for a more functorial description of the symplectic structure on P^\vee in terms of a pairing on $\text{Ext}_X^1(i_* \mathcal{L}, i_* \mathcal{L})$.

Remark 5.4. The motivation for Nagai's work comes from results of Cho, Miyaoka, and Shepherd-Barron who attempted to prove that the base of a Lagrangian fibration $X \longrightarrow B$ with a section is \mathbb{P}^n . Nagai's idea now was to apply their reasoning to the Lagrangian fibration $P^\vee \longrightarrow B$. The extension of $P_0^\vee \longrightarrow B_0$ to a fibration over the whole B (and not only over B_0 or some smaller open subset) is of course essential for the idea to work. In fact, in his approach Nagai only needs to assume that $X \longrightarrow B$ admits local section over an open subset with a complement of codimension at least two.

6. EXTENDED ALBANESE AFTER ARINKIN–FEDOROV AND MARKUSHEVICH

As before, $f: X \longrightarrow B$ denotes a Lagrangian fibration of a projective hyperkähler manifold of dimension $2n$. In addition we assume that B is smooth, which implies that f is flat and, in

⁸It seems that Raynaud only treats the case of the base being the spectrum of a DVR, but this particular result should hold in our setting. Nagai does not address this issue. Also, it seems likely that $P^\vee \longrightarrow B$ is actually quasi-projective. It is possible that Raynaud's techniques actually also cover families without local sections. However, the next step in the program, the construction of a symplectic structure would fail in this broader generality.

⁹The zero section $\sigma: B \longrightarrow P^\vee$ is Lagrangian as well.

fact, $B \simeq \mathbb{P}^n$. By $B_1 \subset B$ we denote the open subset of all points with integral fibres and let $f_1: X_1 \rightarrow B_1$ be restriction of f . Clearly, $B_0 \subset B_1$ so that we are in the following situation

$$\begin{array}{ccccc} X_0 & \hookrightarrow & X_1 & \hookrightarrow & X \\ f_0 \downarrow & & \times & \downarrow f_1 & \times & \downarrow f \\ B_0 & \hookrightarrow & B_1 & \hookrightarrow & B. \end{array}$$

Recall from Section 2 that $f_0: X_0 \rightarrow B_0$ is naturally a torsor for the relative Albanese scheme $g_0: P_0 = \text{Alb}(X_0/B_0) \simeq \text{Aut}_0(X_0/B_0) \rightarrow B_0$. We will use the notation $X'_1 := X_1 \cap X'$, where $X' \subset X$ is the open subset of all f -regular points.

6.1. The following is one of the main results of [AF16].

Theorem 6.1 (Arinkin–Fedorov). *There exists a smooth group scheme $g_1: P_1 \rightarrow B_1$ with an action $P_1 \times_{B_1} X_1 \rightarrow X_1$ such that with this action $X'_1 \rightarrow B_1$ is a torsor for $P_1 \rightarrow B_1$.*

Note that the projection $P_1 \rightarrow B_1$ is necessarily surjective, but it is typically not proper over $B_1 \setminus B_0$.

We cannot give details of their proof, but here are some ideas.¹⁰ The group scheme P_1 is constructed as a closed subgroup-scheme of $\text{Aut}(X_1/B_1)$. A posteriori, it can be viewed as the closure of $\text{Aut}_0(X_0/B_0)$ inside $\text{Aut}(X_1/B_1)$

$$\text{Aut}_0(X_0/B_0) \subset P_1 = \overline{\text{Aut}_0(X_0/B_0)} \subset \text{Aut}(X_1/B_1),$$

but to ensure that this results in a smooth scheme, one needs to single out those automorphisms of the singular fibre X_t that deform in all infinitesimal directions of $t \in B$.

Recall that in the absolute case, that is $Y \rightarrow k$, the Lie algebra of $\text{Aut}_0(Y)$, i.e. the tangent space at id_Y is $H^0(Y, \mathcal{T}_Y)$. In the relative setting, one considers the sheaf of Lie algebras of $\text{Aut}_0(X_1/B_1)$. The key is to observe that on the level of vector fields, the automorphisms that deform in all infinitesimal directions correspond on the Lie algebra level to the Hamiltonian vector fields. In particular, the tangent space of the fibre $P_1 \rightarrow X_1$ over t is by construction naturally isomorphic to T_t^*B . The latter observation ensures that P_1 and $P_1 \rightarrow B_1$ are both smooth (since the flatness of $X \rightarrow B$ implies that B is smooth).

Example 6.2. Here is the simplest example showing that $\text{Aut}_0(X_1/B_1) \rightarrow B_1$ is in general not smooth. Consider the degeneration of a smooth elliptic curve to a curve C with a cusp. Let $\rho: \mathbb{P}^1 \rightarrow C$ be its normalization with coordinates $[x : y]$ such that the cusp has preimage $[1 : 0]$. Automorphisms of C can be described on the normalization as

$$[x : y] \mapsto [ax + by, y]$$

with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Thus, over the point in the base corresponding to the curve C , the dimension increases by one and therefore the morphism $\text{Aut}_0(X_1/B_1) \rightarrow B_1$ cannot be flat.

¹⁰Thanks to D. Arinkin for an instructive email exchange.

Looking at the Lie algebra, we have e.g. for the chart $t = x/y$

$$T_{\text{id}_C} \text{Aut}_0(C) = \left\{ (at + b) \frac{\partial}{\partial t} \mid a, b \in \mathbb{C} \right\}$$

which is of course also two-dimensional. The Hamiltonian vector fields are now exactly the abelian Lie sub-algebra of translations which correspond to setting $a = 0$. That these deform to nearby fibers can be seen easily in this example.

Remark 6.3. By construction, the restriction of $g_1: P_1 \rightarrow B_1$ to the open subset $B_0 \subset B_1$ gives back $g_0: P_0 \rightarrow B_0$. In this sense, $P_1 \rightarrow B_1$ has to be viewed as a partial compactification of the relative Albanese scheme of $f_0: X_0 \rightarrow B_0$:

$$\begin{array}{ccc} P_0 & \hookrightarrow & P_1 \\ g_0 \downarrow & \times & \downarrow g_1 \\ B_0 & \hookrightarrow & B_1 \hookrightarrow B. \end{array}$$

6.2. The group scheme $P_1 \rightarrow B_1$ as a partial compactification of $P_0 \rightarrow B_0$ is of interest to us only if it can be endowed with a symplectic structure. For this, the next result is [AF16, Prop. 7.1] is crucial.

Proposition 6.4 (Arinkin–Fedorov). *The étale morphism $\phi: P_0 \rightarrow P_0^\vee$ extends naturally to an étale morphism*

$$\phi_1: P_1 \rightarrow \text{Pic}(X_1/B_1).$$

Proof. The morphism is extended by the same formula as in Section 2.3. The assertion follows, on the one hand, from the construction and the observation above that $T_{\text{id}} g_1^{-1}(t) \simeq T_t^* B$ and, on the other hand, from $T_{\mathcal{O}_{X_t}} \text{Pic}(X_t) \simeq H^1(X_t, \mathcal{O}_{X_t}) \simeq T_t^* B$, where the last isomorphism is the fibre of Matsushita’s isomorphism $R^1 f_* \mathcal{O}_X \simeq \Omega_B$ (which makes use of the polarization). The proof then consists of showing that with these identifications the differential of ϕ induces the identity of $T_t^* B$. \square

Since $\phi: P_0 \rightarrow \text{Pic}(X_0/B_0)$ takes image in P_0^\vee , the extension ϕ_1 composed with the projection to the maximal separated quotient defines an étale morphism

$$P_1 \rightarrow P^\vee|_{B_1},$$

where $P^\vee \rightarrow B$ is the restriction of the Picard scheme introduced in 5.1. As a consequence of Theorem 5.2, this immediately yields the following.

Corollary 6.5. *The smooth group scheme $g_1: P_1 \rightarrow B_1$ admits a closed symplectic structure for which g_1 is a Lagrangian fibration.* \square

7. THE DUAL HYPERKÄHLER VARIETY AFTER KIM

7.1. Assume now that there exists a subgroup $K \subset \text{Aut}(X)$ of automorphisms of X that commutes with f and such that the constant group scheme K_{B_0} is isomorphic to K compatible with the action on $X_0 \rightarrow B_0$.

Under these assumptions, Kim [Ki21] introduces the notion of the dual hyperkähler fibration.

Definition 7.1. The *dual* of $f: X \rightarrow B$ is the fibration

$$f^\vee: X^\vee := X/K \rightarrow B.$$

Note that X^\vee has at most finite quotient singularities and the open part

$$f^{\vee-1}(B_0) = X_0/K \subset X^\vee$$

is a torsor for P_0^\vee . There are good reason to call X^\vee the dual variety, but be aware that $X^\vee \rightarrow B$ typically has no section (which one might expect from a dual fibration), see Remark 7.3. Note that $P_0^\vee \rightarrow B_0$ comes with a natural section, but not much is known about hyperkähler compactifications of it.

$$\begin{array}{ccc} X & \longrightarrow & B \\ \cup & & \cup \\ X_0 & \longrightarrow & B_0 \\ \circlearrowleft & \parallel & \circlearrowleft \\ P_0 & \longrightarrow & B_0 \end{array} \qquad \begin{array}{ccc} X^\vee = X/K & \longrightarrow & B \\ \cup & & \cup \\ X_0^\vee = X_0/K_{B_0} & \longrightarrow & B_0 \\ \circlearrowleft & \parallel & \circlearrowleft \\ P_0^\vee = P_0/K_{B_0} & \longrightarrow & B_0 \end{array}$$

7.2. The main result of [Ki21] is the following.

Theorem 7.2 (Kim). *Let $X \rightarrow B \simeq \mathbb{P}^n$ be a Lagrangian fibration of a projective hyperkähler manifold of one of the known deformation types $\text{K3}^{[n]}$, Kum_n , OG6 , or OG10 .*

Then there exists a subgroup $K \subset \text{Aut}(X)$ such that

$$K_{B_0} \simeq \ker(P_0 \rightarrow P_0^\vee).$$

More precisely, K is the subgroup of all automorphisms of X that act trivially on $H^2(X, \mathbb{Z})$ and commute with f :

$$K = \{\phi \in \text{Aut}(X) \mid f \circ \phi = \phi \circ f \text{ and } f^* = \text{id on } H^2(X, \mathbb{Z})\}.$$

Remark 7.3. The group K can be described explicitly in all known cases:

(i) Assume first that X is deformation equivalent to a Hilbert scheme of a K3 surface. Then $\text{Aut}(X) \rightarrow \text{O}(H^2(X, \mathbb{Z}))$ is injective [Be83]. Hence,

$$K = \{1\}, \quad X \simeq X^\vee, \quad \text{and } P_0 \simeq P_0^\vee.$$

Indeed, it is known that the fibres of $X \rightarrow B$ are principally polarized and that the image of the restriction map $H^2(X, \mathbb{Z}) \rightarrow H^2(X_t, \mathbb{Z})$ to the generic fibre is spanned by the principal polarization.

(ii) For hyperkähler manifolds deformation equivalent to O'Grady's ten-dimensional example the situation is analogous. That is, the smooth fibres are again known to be principally polarized abelian varieties, the restriction map $H^2(X, \mathbb{Z}) \rightarrow H^2(X_t, \mathbb{Z})$ is surjective onto the subgroup spanned by the principal polarization. Mongardi–Wandel [MW17] computed that again $K = \{1\}$ and, therefore, we have

$$X \simeq X^\vee, \text{ and } P_0 \simeq P_0^\vee.$$

(iii) In the other two cases, the situation is more involved, since $K \neq \{1\}$ and the fibres do not carry a principal polarization. Let X be deformation equivalent to a generalized Kummer variety of dimension $2n$. Then Kim shows, building on previous results of Wieneck [Wi16, Wi18], by studying moduli spaces of pure one-dimensional sheaves on abelian surfaces that

$$K = (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}.$$

Here, $d_1 d_2 = n + 1$ and the general fibre of $f: X \rightarrow B$ is a $(1, \dots, 1, d_1, d_2)$ -polarized abelian variety. The automorphisms of K are all induced from translations along $n + 1$ -torsion points on the abelian surface. The dual fibration

$$X^\vee = X/K \rightarrow B$$

in this case is a singular primitive symplectic orbifold.

(iv) This case is similar to the previous one. The smooth fibres are $(1, 2, 2)$ -polarized abelian threefolds and the group K was already computed by Mongardi–Wandel [MW17] to be

$$K = (\mathbb{Z}/2)^{\oplus 4}.$$

The dual fibration $X/K \rightarrow B$ is again an irreducible symplectic variety.

(v) Kim constructs in the case of hyperkähler manifolds deformation equivalent to Kum_n or OG6 a singular symplectic compactification of the smooth P_0^\vee -torsor X_0/K_{B_0} . In particular, this shows that there does not exist a smooth hyperkähler compactification $\tilde{X} \rightarrow B$ extending $X/K_{B_0} \rightarrow B_0$.

Indeed, these varieties would be birational and after passing to another birational model \tilde{X} of \tilde{X} the birational map would extend to a morphism $\tilde{X} \rightarrow X/K$, cf. [LP16]. In particular, X/K would admit a crepant resolution, which is a contradiction.

8. COMPACTIFYING TORSORS À LA SACCÀ

As before, let $f: X \rightarrow B$ be a Lagrangian fibration of a projective hyperkähler manifold X . Assume that $B_1 = B$, i.e. all fibres of f are integral.¹¹ In this situation, the result of Arinkin–Fedorov, see Section 6, provide a commutative group scheme $P_1 \rightarrow B_1 = B$ acting on $X \rightarrow B = B_1$ such that the open set $X' \subset X$ of f -regular points is a torsor for it.

Theorem 8.1 (Saccà). *Assume that all fibres of $X \rightarrow B = B_1$ are integral. Then any P_1 -torsor $Y_1 \rightarrow B$ can be compactified to a smooth projective hyperkähler manifold $Y \rightarrow B$ with a Lagrangian fibration.*

Example 8.2. For K3 surfaces, the result holds without any assumptions on the fibre, see Section 3, but in general the assumption on the integrality of all fibres is needed as shown by the following example.

Consider a K3 surface S with an ample divisor H of square $H^2 \geq 4$ and consider the Mukai vectors $v_1 = (0, mH, m\chi)$ and $v_2 = (0, mH, \chi')$ for $m \geq 3$ and χ coprime. For a generic polarization we obtain two moduli spaces of semistable sheaves with morphisms

$$f_1: M(v_1) \rightarrow |mH| \quad \text{and} \quad f_2: M(v_2) \rightarrow |mH|.$$

Over the open subset $U \subset |mH|$ of smooth curves, we have that both morphisms f_i restrict to $f_i: \text{Pic}^{d_i}(\mathcal{C}/U) \rightarrow U$ where $\mathcal{C} \rightarrow U$ is the universal curve. They are both torsors for $\text{Pic}^0(\mathcal{C}/U) \rightarrow U$, which using the principal polarization is seen to be isomorphic to the restriction of P_1 to U .

Note that the variety $M(v_2)$ is a smooth hyperkähler manifold, while the results of Kaledin–Lehn–Sorger [KLS] show that the variety $M(v_1)$ does not admit a smooth resolution and therefore the restriction of f_1 to the preimage of the open set U cannot be smoothly compactified. From this argument it is not easy to see why the Lagrangian fibration $f_1: \text{Pic}^{d_1} \rightarrow U$ cannot be smoothly compactified when just viewing it as a smooth Lagrangian fibration over a quasi-projective base.¹²

Remark 8.3. (i) A weaker form of the above result has been used by Saccà in [Sa20] where she studied the intermediate Jacobian fibrations associated to the Fano variety of lines of a smooth cubic fourfold.

(ii) If one only assumes that the complement of $B_1 \subset B$ has codimension at least two, then Saccà constructs a \mathbb{Q} -factorial terminal symplectic compactification $\bar{Y} \rightarrow B$ of P_1 -torsors $Y_1 \rightarrow B_1$. Similarly, given a finite subgroup $G \subset P_1 \subset \text{Aut}_0(X_1/B_1)$, e.g. $G = \mathcal{K}$, Saccà also constructs a \mathbb{Q} -factorial terminal symplectic compactification of $P_1/G \rightarrow B_1$.

¹¹Since then the fibration has local sections, the base is smooth and, therefore, $B \simeq \mathbb{P}^n$. Note that integrality of the fibre is not needed for the argument, the existence of local sections is enough.

¹²Saccà uses this example to argue that the assumption on the integrality of the fibres is necessary, but for this one would also need to view f_1 over the bigger open subset of all integral fibres as a torsor for P_1 .

REFERENCES

- [AF16] D. Arinkin, D. Fedorov *Partial Fourier–Mukai transform for integrable systems with applications to Hitchin fibration*. Duke Math. J. 165 (2016), 2991–3042. [1](#), [2](#), [12](#), [14](#), [15](#)
- [Be83] A. Beauville *Some remarks on Kähler manifolds with $c_1 = 0$* . Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, 1–26. [16](#)
- [BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud *Néron models*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer 1990. [12](#)
- [Br18] M. Brion *Notes on automorphism groups of projective varieties*. https://www-fourier.univ-grenoble-alpes.fr/~mbrion/autos_final.pdf [5](#)
- [BGO20] I. Biswas, T. Gómez, A. Oliveira *Complex Lagrangians in a hyperKähler manifold and the relative Albanese*. Complex Manifolds 7 (2020), 230–240. [11](#)
- [DM96] R. Donagi, E. Markman *Spectral curves, algebraically completely integrable Hamiltonian systems, and moduli of bundles*. LNM 1620 (1996), 1–119. [10](#)
- [FAG] B. Fantechi, L. Göttsche, L. Illusie, S. L. Kleiman, N. Nitsure, A. Vistoli *Fundamental algebraic geometry* American Mathematical Society, Providence, RI (2005). [5](#), [12](#)
- [Hu16] D. Huybrechts *Lectures on K3 surfaces*. Cambridge University Press, 2016. [6](#), [7](#), [8](#)
- [HM21] D. Huybrechts, M. Mauri *Lagrangian fibrations*. arXiv:2108.10193. [10](#)
- [Hw08] J.-M. Hwang *Base manifolds for fibrations of projective irreducible symplectic manifolds*. Invent. Math. 174 (2008), no. 3, 625–644. [12](#)
- [KLS] D. Kaledin, M. Lehn, C. Sorger *Singular symplectic moduli spaces*. Invent. Math. 164 (2006), no. 3, 591–614. [18](#)
- [Ki21] Y.-J. Kim *The dual Lagrangian fibration of known hyperkähler manifolds*. arXiv:2109.03987. [1](#), [2](#), [16](#)
- [LP16] C. Lehn, G. Pacienza *Deformations of singular symplectic varieties and termination of the log minimal model program*. Alg. Geom. 3 (2016), no. 4, 392–406. [17](#)
- [Ma96] D. Markushevich *Lagrangian families of Jacobians of genus 2 curves*. Alg. Geom. 5. J. Math. Sci. 82 (1996), 3268–3284. [10](#), [11](#)
- [MW17] G. Mongardi, M. Wandel *Automorphisms of O’Grady’s manifolds acting trivially on cohomology*. Algebr. Geom. 4 (2017), no. 1, 104–119 [17](#)
- [Na05] Y. Nagai *Dual fibration of a projective Lagrangian fibration*. secret preprint (2005). [1](#), [2](#), [10](#), [12](#), [13](#)
- [Ra70] M. Raynaud *Spécialisation du foncteur de Picard*. Publ. Math. Institut Hautes Études Sci. 38 (1970), 27–76. [12](#)
- [Sa20] G. Saccà *Birational geometry of the intermediate Jacobian fibration of a cubic fourfold*. arXiv:2002.01420. [1](#), [18](#)
- [Sa21] G. Saccà *??*. In preparation [1](#), [2](#)
- [Wi16] B. Wieneck *On polarization types of Lagrangian fibrations*. Manuscripta Math. 151 (2016), no. 3-4, 305–327. [17](#)
- [Wi18] B. Wieneck *Monodromy invariants and polarization types of generalized Kummer fibrations*. Math. Z. 290 (2018), no. 1-2, 347–378. [17](#)

MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY
 Email address: beckmann@math.uni-bonn.de, huybrech@math.uni-bonn.de