# ON THE HODGE AND BETTI NUMBERS OF HYPERKÄHLER MANIFOLDS

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Abstract. Let $X$ be a compact Kähler manifold of dimension $m$. One consequence of the Hirzebruch–Riemann–Roch theorem is that the coefficients of the $\chi_y$-genus polynomial

$$p_X(y) := \sum_{p,q=0}^{m} (-1)^q h^{p,q}(X) y^p \in \mathbf{Z}[y]$$

are (explicit) universal polynomials in the Chern numbers of $X$. In 1990, Libgober–Wood determined the first three terms of the Taylor expansion of this polynomial about $y = -1$ and deduced that the Chern number $\int_X c_1(X) c_{m-1}(X)$ can be expressed in terms of the coefficients of the polynomial $p_X(y)$ (Proposition 2.1).

When $X$ is a hyperkähler manifold of dimension $m = 2n$, this Chern number vanishes. The Hodge diamond of $X$ also has extra symmetries which allowed Salamon to translate the resulting identity into a linear relation between the Betti numbers of $X$ (Corollary 2.4).

## Contents



## 1. Symmetries of the Hodge diamond of a hyperkähler manifold

Let $X$ be a compact hyperkähler manifold of dimension $2n$ and let $\sigma$ be a symplectic form on $X$. Apart from the usual symmetries

$$h^{p,q}(X) = h^{q,p}(X) = h^{2n-p,2n-q}(X)$$

coming from Kähler theory and Serre duality, there is another symmetry

$$h^{p,q}(X) = h^{2n-p,q}(X) \tag{1}$$

coming from the fact that the wedge product $\wedge \sigma^{\wedge(n-p)}$ is an isomorphism $\Omega_X^p \xrightarrow{\sim} \Omega_X^{2n-p}$. So the Hodge diamond of $X$ has a $D_8$-symmetry.

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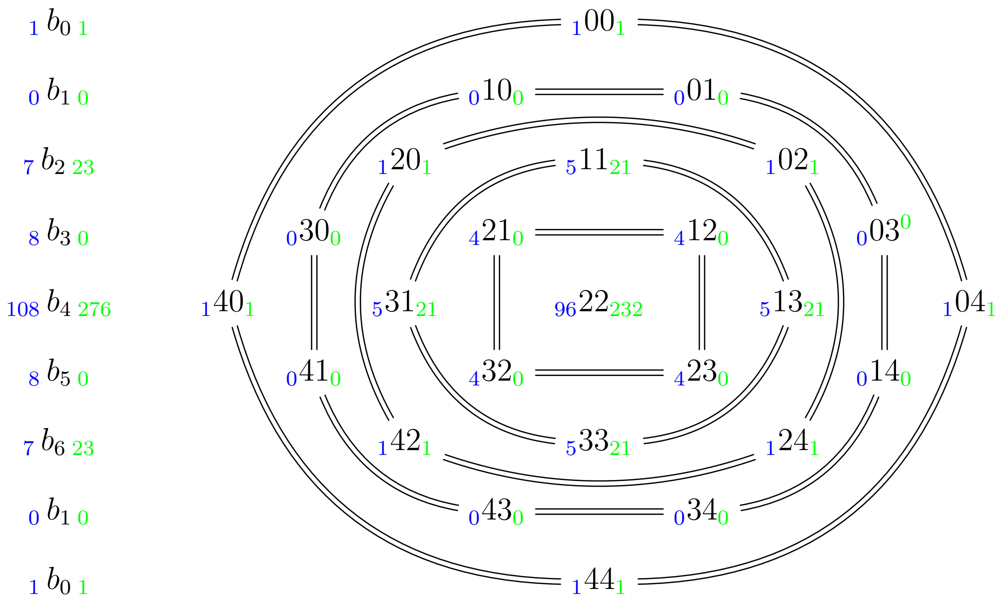
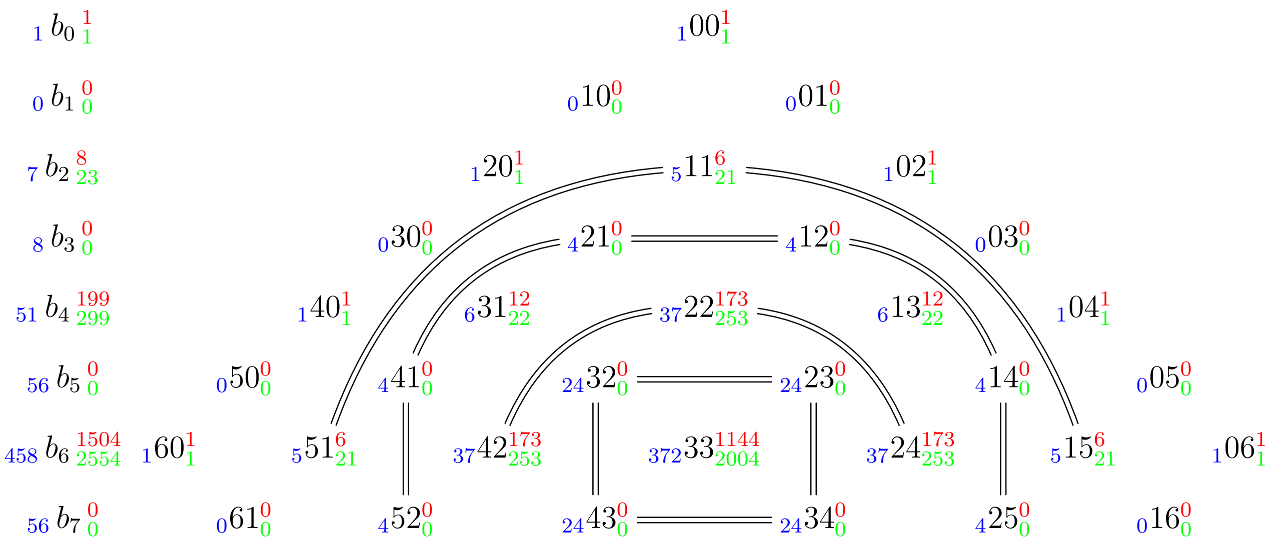
**Example 1.1** ($n = 2$)**.** We represents the various symmetries of the Hodge diamond for an irreducible hyperkähler fourfold (note that the extra "mirror" symmetry (1) is only visible here on the outer edges of the diamond). The numbers in blue are for the $\mathrm{Kum}_2$-type and the numbers in green are for the $\mathrm{K3}^{[2]}$-type.




So, a priori, there are only three "free" Hodge numbers: $h^{11}$, $h^{21}$, and $h^{22}$. We will see in Example 2.6 that there is a relation between them.

**Example 1.2** ($n = 3$)**.** We represents some of the symmetries of the Hodge diamond of an irreducible hyperkähler sixfold. The numbers in blue are for the $\mathrm{Kum}_3$-type, the numbers in green are for the $\mathrm{K3}^{[3]}$-type, and the numbers in red are for the OG6-type.




So, a priori, there are only six "free" Hodge numbers: $h^{11}$, $h^{21}$, $h^{31}$, $h^{22}$, $h^{32}$, and $h^{33}$. We will see in Example 2.7 that there is a relation between them.

## 2. Salamon's results on Betti numbers

2.1. **Hirzebruch–Riemann–Roch.** Let $X$ be a compact Kähler manifold of dimension $m$. Following [H], we set

$$\chi^p(X) := \sum_{q=0}^{m} (-1)^q h^{p,q}(X) = \chi(X, \Omega_X^p),$$

which satisfy

$$\chi^p(X) = (-1)^m \chi^{m-p}(X) \tag{2}$$

by Serre duality, and we define the $\chi_y$-genus

$$p_X(y) := \sum_{p=0}^{m} \chi^p(X) y^p = \sum_{p,q=0}^{m} (-1)^q h^{p,q}(X) y^p \in \mathbf{Z}[y]. \tag{3}$$

For instance,

- $p_X(0) = \chi^0(X) = \chi(X, \mathscr{O}_X)$,
- $p_X(-1) = \chi_{\mathrm{top}}(X) = e(X)$,
- $p_X(1)$ is the signature of the intersection form on $H^m(X, \mathbf{R})$ (which vanishes when $m$ is odd).

Serre duality translates into the reciprocity property $(-y)^m p_X\big(\frac{1}{y}\big) = p_X(y)$.

One consequence of the Hirzebruch–Riemann–Roch theorem is that $\chi^p(X)$ can be expressed as a universal polynomial $T_{m,p}(c_1, \dots, c_m)$ in the Chern classes of $X$ evaluated on $X$ ([H, Section IV.21.3, (10)]), that is,

$$p_X(y) = \sum_{p=0}^{m} y^p \int_X T_{m,p}(c_1(X), \dots, c_m(X)) = \int_X T_m(y)(c_1(X), \dots, c_m(X)), \tag{4}$$

where $T_m(y) := \sum_{p=0}^{m} T_{m,p} y^p$, a polynomial with coefficients in $\mathbf{Q}[c_1, \dots, c_m]$. One has

- $T_{m,p} = (-1)^m T_{m,m-p}$ and $(-y)^m T_m\big(\frac{1}{y}\big) = T_m(y)$;
- $T_{m,0} = \mathrm{td}_m(c_1, \dots, c_m)$.

Libgober–Wood found in [LW, Lemma 2.2] the first three terms[1] of the Taylor expansion of the polynomial $T_m(y)$ about $y = -1$:

$$T_m(y-1) = c_m - \tfrac{1}{2} m c_m y + \tfrac{1}{12}\big(\tfrac{1}{2} m(3m-5) c_m + c_1 c_{m-1}\big) y^2 + \cdots \tag{5}$$

The following is [LW, Proposition 2.3] (reproved later in [S, Theorem 4.1]).

**Proposition 2.1** (Libgober–Wood). *If $X$ is a compact Kähler manifold of dimension $m$, one has the relation*

$$\int_X c_1(X) c_{m-1}(X) = \sum_{p=0}^{m} (-1)^p \big(6p^2 - \tfrac{1}{2} m(3m+1)\big) \chi^p(X). \tag{6}$$

*Proof.* The Taylor expansion of the polynomial $p_X$ about the point $-1$ is

$$p_X(y-1) = \sum_{p=0}^{m} \chi^p(X)(y-1)^p$$

[1] It is not difficult to find the next term in this expansion:

$$T_m(y-1) = c_m - \tfrac{1}{2} m c_m y + \tfrac{1}{12}\big(\tfrac{1}{2} m(3m-5) c_m + c_1 c_{m-1}\big) y^2 - \tfrac{1}{24}(m-2)\big(\tfrac{1}{2} m(m-3) c_m + c_1 c_{m-1}\big) y^3 + \cdots$$

But this does not bring any new information since it is in fact a formal consequence of the reciprocity property $(-y)^m T_m\big(\frac{1}{y}\big) = T_m(y)$. The $y^4$-term involves 7 Chern numbers. On a hyperkähler manifold, where all odd Chern classes vanish, perhaps this term only involves $c_m$ and $c_2 c_{m-2}$ (this holds for $m \le 8$; to be checked in general).

$$= \sum_{p=0}^{m} (-1)^p \chi^p(X) + y \sum_{p=0}^{m} (-1)^{p-1} \binom{p}{1} \chi^p(X) + y^2 \sum_{p=0}^{m} (-1)^p \binom{p}{2} \chi^p(X) + \cdots$$

Using the Hirzebruch–Riemann–Roch theorem (4) and comparing with (5), we get, by identifying the coefficients, the relations[2]

$$\begin{aligned} p_X(-1) &= \int_X c_m(X) = \sum_{p=0}^{m} (-1)^p \chi^p(X), \\ p'_X(-1) &= -\tfrac{1}{2} m \int_X c_m(X) = \sum_{p=0}^{m} (-1)^{p-1} p \chi^p(X), \qquad (7) \\ p''_X(-1) &= \tfrac{1}{6} \int_X \big(\tfrac{1}{2} m(3m-5) c_m(X) + c_1(X) c_{m-1}(X)\big) = 2 \sum_{p=0}^{m} (-1)^p \binom{p}{2} \chi^p(X), \end{aligned}$$

from which it is not difficult to get (6). □

The following consequence of the proposition was obtained in [G, (1.14) and Proposition 2.4] using modular forms (but seems to have been known to Hirzebruch).[3]

**Corollary 2.2** (Gritsenko)**.** *If $X$ is a compact Kähler manifold of dimension $m$ that satisfies $c_1(X)_{\mathbf{R}} = 0$, one has*

$$\frac{1}{12} m e(X) = \sum_{p=0}^{m} (-1)^p \big(\tfrac{1}{2} m - p\big)^2 \chi^p(X) = 2 \sum_{0 \le p < m/2} (-1)^p \big(\tfrac{1}{2} m - p\big)^2 \chi^p(X). \qquad (8)$$

*In particular, when $m$ is even,[4] $me(X)$ is divisible by 24.*

*Proof.* The first equality in (8) is easily obtained from the relations (7), and the second equality from the symmetries (2). □

**Remark 2.3.** The polynomials $T_m$ can be computed. Setting for simplicity $c_1 = 0$ (the case of interest for us), we have, for even dimensions $m \in \{2, 4, 6\}$ (see [LW] or [D, Section 9]),

$$\begin{aligned} T_2(y-1) &= c_2 - c_2 y + \tfrac{1}{12} c_2 y^2, \\ T_4(y-1) &= c_4 - 2c_4 y + \tfrac{7}{6} c_4 y^2 - \tfrac{1}{6} c_4 y^3 + \tfrac{1}{720}(3c_2^2 - c_4) y^4, \\ T_6(y-1) &= c_6 - 3c_6 y + \tfrac{13}{4} c_6 y^2 - \tfrac{3}{2} c_6 y^3 + \tfrac{1}{240}(-c_3^2 + c_2 c_4 + 62 c_6) y^4 \\ &\quad + \tfrac{1}{720}(3c_3^2 - 3c_2 c_4 - 6c_6) y^5 + \tfrac{1}{60480}(10 c_2^3 - c_3^2 - 9 c_2 c_4 + 2 c_6) y^6. \end{aligned}$$

Setting $\chi := \mathrm{td}_m$ (this is the constant term and leading coefficient of $T_m$), we get

$$T_2(y) = \chi + (2\chi - c_2) y + \chi y^2,$$

[2]The first two relations are in fact formally equivalent upon using the symmetries (2), which give

$$p'_X(-1) = \sum_{p=0}^{m} (-1)^{p-1} (m-p) \chi^p(X) = -m p_X(-1) - p'_X(-1)$$

(see footnote 1).

[3]Gritsenko also gives in [G, (1.13)] relations between the $\chi^p(X)$ when $m \in \{4, 6, 8, 10\}$, but they are all rewritings of (8).

[4]Gritsenko does not make this assumption, but when $m$ is odd and we write $m = 2n + 1$, we have

$$\tfrac{m-3}{12} e(X) = 2 \sum_{0 \le p \le n} (-1)^p \Big(\big(\tfrac{1}{2} m - p\big)^2 - \tfrac{1}{4}\Big) \chi^p(X) = 2 \sum_{0 \le p \le n} (-1)^p \big(n(n+1) - p(2n+1) + p^2\big) \chi^p(X),$$

which is divisible by 4. So what we get is that $\frac{m-3}{2} e(X)$ is divisible by 24.

$$T_4(y) \quad = \quad \chi + (4\chi - \tfrac{1}{6}c_4)y + (6\chi + \tfrac{2}{3}c_4)y^2 + (4\chi - \tfrac{1}{6}c_4)y^3 + \chi y^4. \tag{9}$$

2.2. **Application to hyperkähler manifolds.** Assume now that $m$ is even and that we have the extra "mirror" symmetry $h^{p,q}(X) = h^{m-p,q}(X)$ like we do when $X$ is a hyperkähler manifold. We define polynomials

$$h_X(s,t) := \sum_{p,q=0}^{m} h^{p,q}(X) s^p t^q \in \mathbf{Z}[s,t],$$

$$b_X(t) := \sum_{j=0}^{2m} b_j(X) t^j = h_X(t,t).$$

The polynomial $h_X$ is symmetric and $p_X(y) = h_X(-1,y)$. Now we use the evenness of $m$ and the extra symmetry to get

$$\begin{aligned}
\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) &= \sum_{p,q=0}^{m} pq(-1)^{p+q} h^{p,q}(X) \\
&= \sum_{p,q=0}^{m} (m-p)q(-1)^{m-p+q} h^{p,q}(X) \\
&= -\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) + m \sum_{p,q=0}^{m} q(-1)^{p+q} h^{p,q}(X) \\
&= -\frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) - m\, \frac{\partial h_X}{\partial t}(-1,-1),
\end{aligned}$$

so that

$$2\, \frac{\partial^2 h_X}{\partial s \partial t}(-1,-1) = -m\, \frac{\partial h_X}{\partial t}(-1,-1) = -m p'_X(-1). \tag{10}$$

In terms of the polynomial $b_X$, we have, by symmetry of $h_X$,

$$\begin{aligned}
b'_X(t) &= 2\, \frac{\partial h_X}{\partial t}(t,t), \\
b''_X(t) &= 2\, \frac{\partial^2 h_X}{\partial s \partial t}(t,t) + 2\, \frac{\partial^2 h_X}{\partial t^2}(t,t),
\end{aligned}$$

so that we get, using (10),

$$b'_X(-1) = 2p'_X(-1) \qquad , \qquad b''_X(-1) = -m p'_X(-1) + 2p''_X(-1). \tag{11}$$

Proceeding as in the proof of Proposition 2.1, we write the Taylor expansion of the polynomial $b_X$ about the point $-1$:

$$\begin{aligned}
b_X(t-1) &= \sum_{j=0}^{2m} b_j(X)(t-1)^j \\
&= \sum_{j=0}^{2m} b_j(X)(-1)^j + t \sum_{j=0}^{2m} b_j(-1)^{j-1} \binom{j}{1} + t^2 \sum_{j=0}^{2m} b_j(X)(-1)^j \binom{j}{2} + \cdots
\end{aligned}$$

Using (11) and (7), we get

$$\sum_{j=0}^{2m} b_j(-1)^j j = -b'_X(-1) = -2p'_X(-1) = m \int_X c_m(X),$$

$$\sum_{j=0}^{2m} b_j(X)(-1)^j \binom{j}{2} = \tfrac{1}{2} b''_X(-1) = -\tfrac{1}{2} m p'_X(-1) + p''_X(-1)$$
$$= \tfrac{1}{4} m^2 \int_X c_m(X) + \tfrac{1}{6} \int_X \big(\tfrac{1}{2} m(3m-5) c_m(X) + c_1(X) c_{m-1}(X)\big).$$

Putting everything together, we obtain the analogue of (6) ([S, Theorem 4.1]):

$$2 \int_X c_1(X) c_{m-1}(X) = \sum_{j=0}^{2m} (-1)^j (6j^2 - m(6m+1)) b_j(X).$$

**Corollary 2.4** (Salamon). *If $X$ is a compact hyperkähler manifold of dimension $2n$, one has*[5]

$$\sum_{j=0}^{4n} (-1)^j (3j^2 - n(12n+1)) b_j(X) = 0.$$

Using the symmetry $b_j = b_{4n-j}$, one checks that one gets the equivalent relations (in the spirit of (8))

$$ne(X) = 6 \sum_{j=1}^{2n} (-1)^j j^2 b_{2n-j}(X) \quad , \quad n b_{2n}(X) = 2 \sum_{j=1}^{2n} (-1)^j (3j^2 - n) b_{2n-j}(X).$$

**Example 2.5** ($n = 1$)**.** We obtain $b_2(X) = 22$ and $e(X) = 24$.

**Example 2.6** ($n = 2$)**.** Salamon's relation reads

$$b_4(X) = 46 + 10 b_2(X) - b_3(X).$$

On an irreducible hyperkähler fourfold, because of the symmetries, there are only 3 unkown Hodge numbers: $h^{11}(X)$, $h^{21}(X)$, and $h^{22}(X)$. One has

$$b_2(X) = 2 + h^{11}(X) \quad , \quad b_3(X) = 2h^{21}(X) \quad , \quad b_4(X) = 2 + 2h^{11}(X) + h^{22}(X).$$

Salamon's relation translates into

$$h^{22}(X) = 64 + 8h^{11}(X) - 2h^{21}(X).$$

There are two Chern numbers, $c_4 := \int_X c_4(X) = e(X)$ and $c_2^2 := \int_X c_2(X)^2$. They satisfy

$$3 = \chi(X, \mathscr{O}_X) = T_4(0) = \mathrm{td}_4(X) = \tfrac{1}{720}(3c_2^2 - c_4).$$

But we also have, using (9),

$$\chi^1(X) = 12 - \tfrac{1}{6} c_4 \qquad , \qquad \chi^2(X) = 18 + \tfrac{2}{3} c_4.$$

A priori though, the value of $c_4$ is not enough to determine all the Hodge numbers but, once we know $c_4$, one Hodge number determines all the others.

The Chern numbers for the two known deformation types of irreducible hyperkähler fourfolds are in the following table.

| | $\chi_{\mathrm{top}} = e = c_4$ | $c_2^2$ |
|---|---|---|
| $\mathrm{Kum}_2$ | 108 | 756 |
| $\mathrm{K3}^{[2]}$ | 324 | 828 |

[5] There is a misprint in [Hu, 24.4.2].

**Example 2.7** ($n = 3$)**.** Salamon's relation reads

$$b_6(X) = 70 + 30b_2(X) - 16b_3(X) + 6b_4(X).$$

Because of the symmetries, there are only 6 Hodge numbers: $h^{11}(X)$, $h^{21}(X)$, $h^{31}(X)$, $h^{22}(X)$, $h^{32}(X)$, and $h^{33}(X)$. One has

$$\begin{aligned} b_2(X) &= 2 + h^{11}(X), \\ b_3(X) &= 2h^{21}(X), \\ b_4(X) &= 2 + 2h^{31}(X) + h^{22}(X), \\ b_5(X) &= 2h^{41}(X) + 2h^{32}(X), \\ b_6(X) &= 2 + 2h^{11}(X) + 2h^{22}(X) + h^{33}(X). \end{aligned}$$

Salamon's relation translates into

$$h^{33}(X) = 140 + 28h^{11}(X) - 32h^{21}(X) + 12h^{31}(X) + 4h^{22}(X).$$

There are three Chern numbers, $c_6 := \int_X c_6(X) = e(X)$, $c_2c_4 := \int_X c_2(X)c_4(X) = e(X)$, and $c_2^3 := \int_X c_2(X)^2$. They satisfy

$$4 = \chi(X, \mathscr{O}_X) = T_6(0) = \mathrm{td}_6(X) = \tfrac{1}{60480}(10c_2^3 - 9c_2c_4 + 2c_6).$$

The three known examples in dimension 6 are in the following table taken from [N2, Remark 4.13] (see also [N1, Appendix A]) and [MRS, Corollary 6.8].

| | $\chi_{\mathrm{top}} = e(X) = c_6$ | $c_2c_4$ | $c_2^3$ |
|---|---|---|---|
| $\mathrm{Kum}_3$ | 448 | 6784 | 30208 |
| $\mathrm{K3}^{[3]}$ | 3200 | 14720 | 36800 |
| OG6 | 1920 | 7680 | 30720 |

## References


[C] Chern, S.S., On a generalization of Kähler geometry, *Algebraic geometry and topology. A symposium in honor of S. Lefschetz,* 103–121, Princeton University Press, Princeton, N.J., 1957.

[D] Debarre, O., Cohomological characterizations of the complex projective space, eprint `arXiv:1512.04321`.

[F] Fujiki, A., On the de Rham cohomology group of a compact Kähler symplectic manifold, *Algebraic geometry, Sendai, 1985,* 105–165, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987.

[G] Gritsenko, V., Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms, *Algebra i Analiz* **11** (1999), 100–125; reprinted in *St. Petersburg Math. J.* **11** (2000), 781–804.

[H] Hirzebruch, F., *Topological methods in algebraic geometry,* reprint of the 1968 edition, Springer-Verlag, Berlin, 1995.

[Hu] Huybrechts, D., Compact Hyperkähler Manifolds, in *Calabi-Yau manifolds and related geometries*, Lectures from the Summer School held in Nordfjordeid, June 2001, Universitext, Springer-Verlag, Berlin, 2003.

[LW] Libgober, A.S., Wood, J.W., Uniqueness of the complex structure on Kähler manifolds of certain homotopy types, *J. Differential Geom.* **32** (1990), 139–154.

[MRS] Mongardi, G., Rapagnetta, A., Saccà, G., The Hodge diamond of O'Grady's six-dimensional example, *Compos. Math.* **154** (2018), 984–1013.

[N1] Nieper-Wißkirchen, M., On the Chern numbers of generalised Kummer varieties, *Math. Res. Lett.* **9** (2002), n 597–606.

[N2] Nieper-Wißkirchen, M., Calculation of Rozansky-Witten invariants on the Hilbert schemes of points on a K3 surface and the generalised Kummer varieties, *Doc. Math.* **8** (2003), 591–623.

[S] Salamon, S.M., On the cohomology of Kähler and hyper-Kähler manifolds, *Topology* **35** (1996), 137–155.

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