Degree zero DT invariants and Behrend function

Summer term 2009/10[1]

The topic of the seminar is this formula

$$\sum_{n=0} |\mathrm{Hilb}^n(Y)| t^n = \left(\prod_{m=1} \left(\frac{1}{1-(-t)^m} \right)^m \right)^{\chi(Y)} . \tag{1}$$

Here, Y is a Calabi–Yau threefold and $|\mathrm{Hilb}^n(Y)|$ is the degree of the virtual fundamental class of the Hilbert scheme of zero-dimensional subschemes of length n on Y or, in other words, the degree zero Donaldson–Thomas invariant of length n. The formula is known as the MNOP conjecture (see [12]) and has been proved by at least three different techniques: i) Behrend–Fantechi [2] using the Behrend function [1], ii) Jun Li [11] (for compact complex threefolds, using cobordism) and iii) Levine and Pandharipande [10] (for arbitrary projective threefolds, also using cobordism). More recently, a motivic version of this formula was proved in [3].

The MNOP formula should be seen as a three-dimensional analogue of a formula that is well-known to all of us, i.e. Göttsche proved in 1990:

$$\sum_{n=0} \chi(\mathrm{Hilb}^n(S)) = \left(\prod_{m=1} \frac{1}{1-t^m} \right)^{\chi(S)}$$

for an algebraic surface S. The difficulty in generalizing this to higher dimensions is of course that the Hilbert schemes of points on a threefolds is badly behaved, i.e. singular, non-reduced, reducible.

The idea of the seminar would be to learn some of the important techniques that have been used to prove the MNOP conjecture. Most importantly, it should be the opportunity to learn what the Behrend function is and how it replaces virtual cycles. The hope would be to also learn about algebraic cobordism and how it can be used for concrete problems like the MNOP conjecture (or the more recent proof of the Göttsche formula).

Organization: The seminar will take place roughly every other Monday with at least two talks each time. The provisional schedule is this: **April 19, May 3, May 10, June 7, June 21, July 5**, (July 19 if necessary). (I tried to avoid collision with the conferences in Paris and Lisbon. June 21 is a problem for people going to Oberwolfach.)

Volunteers for the talks should contact me as soon as possible. Note that every meeting we will have at least two talks and 6. probably will consists of two rather independent talks.

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Program for the first meetings

1. **Introduction** Brief survey of GW and DT invariants following Sections 1 and 2 in [12]. Define DT-invariants (assuming the existence of a virtual fundamental class). Introduce $Z_{\rm GW}$ and $Z_{\rm DT}$ and their degree zero parts. State Conjecture 1, which is formula (1) above. The general version is Conjecture 1' for which Y need not be Calabi–Yau. Compare GW and DT in degree zero and length 1 (Lemma 3 and (2)).

2. **Göttsche's formula in higher dimensions** Although $\mathrm{Hilb}^n(Y)$ for a smooth variety is smooth only if $\dim(Y) \leq 2$ or $n \leq 3$ (it could be instructive to see an explicit example where it is not), there is an analogue of Göttsche's formula for the generating function of the Euler numbers in general. This is the content of [4] and the relevant result for us is Prop. 5.2 which resembles (1). There are usually two steps in the program: i) Using stratification one reduces to the affine case (equivariant) ii) Do the computation in the affine case. Follow [2] for i). More precisely Section 4.1 without Prop. 4.2 and Section 4.2 treating the function $\tilde{\chi}$ appearing in Theorem 4.11 formally. This talk should also contain something on the MacMahon function. Can our expert decide whether we actually want to do the calculation in the affine case?

3. **Normal cone and various classes - Preparation** For the definition of the Behrend function it might be useful to recall some basic facts from Fulton's book [5], eg. normal cone, Segre class, total Chern class. Explain how the normal cone behaves under smooth pull-back and under products (see [1, Prop. 1.2] for the application). Recall the Gauss–Bonnet formula (see [1, Prop. 1.6]).

4. **MacPherson's local Euler obstruction** (Speaker: D. Murfet) This talk should give an account of [6]. For the definition see also [5, Ch. 4]. In particular, the Nash blow-up should be explained. We shall also need to know how the Euler obstruction behaves under product and smooth pull-back (see [1, Prop. 1.5]). Also, why does the Euler obstruction yield constructible functions?

5. **Behrend function** (Speaker: D. van Straten) Define ν following [1] and maybe [9, Sect. 4.1]. Explain the main properties, see [1, Prop. 1.5] or [9, Thm. 4.3]. Depending on the taste of the speaker we could restrict to schemes here (but [9, Prop. 4.4] presents an extension to Artin stacks). State $\nu_X(x) = \pm(1 - \chi(\mathrm{MF}_f(x)))$ for X the critical set of a function f and MF_f its Milnor fibre (see [1, (4)] and [9, Thm. 4.7]). This is a theorem of Parusinski and Pragacz, but I am not sure we want to discuss the proof of it in detail.

6. **Behrend=DT** This part probably consists of two talks. We want to understand the main result of [1] which asserts that DT-invariants can be computed via Behrend's function (Theorem 4.18), which can be seen as a virtual Gauss–Bonnet formula. The first part should cover Sections 2 and 3 of [1] on obstruction theories and virtual fundamental classes (see also [2, Sect. 1]). The material is known to many of us, in particular to the participants of the seminar last term. But since the audience will not be identical, something should be said. In any case, Section 3.4 should be dealt with in detail. Section 4 contains the microlocal geometry needed for the proof. One of the central results is Theorem 4.9 (or rather Theorem 4.14) saying that the obstruction cone is Lagrangian.

7. **Proof of MNOP by Behrend–Fantechi** Since by Cheah's result a variant of (1) is known for the topological Euler characteristic and since Behrend's function computes local DT, it remains to show that topological Euler characteristic of $\mathrm{Hilb}^n(Y)$ is up to the sign $(-1)^n$ equal to the weighted Euler characteristic $\tilde{\chi}(\mathrm{Hilb}^n(Y)) = \chi(\mathrm{Hilb}^n(Y), \nu)$. This is the main result of [2]. The first main result is the computation of ν in an isolated fixed point of a $\mathbb{G}$-action (Prop. 3.3) and its global form (Theorem 3.4). For the latter I would suggest to simply assume that X is given as the zero locus of an invariant one-form and to skip the equivariant obstruction theory in Section 2. State and proof Proposition 4.2. Then pass straight to the proof of MNOP (Theorem 4.11 and 4.12) by using the usual stratification machinery as explained earlier.

This is the program up to the proof of the MNOP conjecture (degree zero) using Behrend functions. Maybe we should at this point decide democratically if we want to pass to cobordism techniques, to the motivic version as in [3] or learn more about Behrend function in relation to nearby and vanishing cycles as eg. in [9, Sect. 4]

References

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