THE K3 CATEGORY OF A CUBIC FOURFOLD

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Abstract. Smooth cubic hypersurfaces $X \subset \mathbb{P}^5$ (over $\mathbb{C}$) are linked to K3 surfaces via their Hodge structures, due to work of Hassett, and via Kuznetsov's K3 category $\mathcal{A}_X \subset \mathrm{D^b}(X)$. The relation between these two viewpoints has recently been explained by Addington and Thomas.

In this paper, both aspects are studied further and extended to twisted K3 surfaces. In particular, we determine the group of autoequivalences of $\mathcal{A}_X$ of the general cubic fourfold, prove finiteness results for cubics with equivalent K3 categories and study periods of cubics in terms of generalized K3 surfaces.

1. Introduction

As shown by Kuznetsov [33, 36], the bounded derived category $\mathrm{D^b}(X)$ of coherent sheaves on a smooth cubic hypersurface $X \subset \mathbb{P}^5$ contains, as the semi-orthogonal complement of three line bundles, a full triangulated subcategory

$$\mathcal{A}_X \coloneqq \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle^{\perp} \subset \mathrm{D^b}(X)$$

that behaves in many respects like the bounded derived category $\mathrm{D^b}(S)$ of coherent sheaves on a K3 surface S. In fact, for certain special cubics $\mathcal{A}_X$ is equivalent to $\mathrm{D^b}(S)$ or, more generally, to the derived category $\mathrm{D^b}(S, \alpha)$ of α-twisted sheaves on a K3 surface S. Kuznetsov also conjectured that $\mathcal{A}_X$ is of the form $\mathrm{D^b}(S)$ if and only if X is rational. Neither of the two implications has been verified until now, although Addington and Thomas recently have shown in [1] that the conjecture is (generically) equivalent to a conjecture attributed to Hassett [17] describing rational cubics in terms of their periods.

1.1. This paper is not concerned with the rationality of cubic fourfolds, but with basic results on $\mathcal{A}_X$. Ideally, one would like to have a theory for $\mathcal{A}_X$ that parallels the theory for $\mathrm{D^b}(S)$ and $\mathrm{D^b}(S, \alpha)$. In particular, one would like to have analogues of the following results and conjectures:

– *For a given twisted K3 surface (S, α) there exist only finitely many isomorphism classes of twisted K3 surfaces (S', α') with $\mathrm{D^b}(S, \alpha) \simeq \mathrm{D^b}(S', \alpha')$.*

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– Two twisted K3 surfaces (S,α), (S',α') are derived equivalent, i.e. there exists a $\mathbb{C}$-linear exact equivalence $\mathrm{D^b}(S,\alpha)\simeq \mathrm{D^b}(S',\alpha')$, if and only if there exists an orientation preserving Hodge isometry $\widetilde{H}(S,\alpha,\mathbb{Z})\simeq\widetilde{H}(S',\alpha',\mathbb{Z})$.

– The group of linear exact autoequivalences of $\mathrm{D^b}(S,\alpha)$ admits a natural representation $\rho\colon \mathrm{Aut}(\mathrm{D^b}(S,\alpha))\longrightarrow \mathrm{Aut}(\widetilde{H}(S,\alpha,\mathbb{Z}))$, which is surjective up to index two and such that its kernel is at least conjecturally described as a fundamental group of a certain Deligne–Mumford stack.

Most of the theory for untwisted K3 surfaces is due to Mukai [45] and Orlov [50], whereas the basic theory of twisted K3 surfaces was developed in [21, 22]. See also [25, 23] for surveys and further references. Originally, the generalization to twisted K3 surfaces was motivated by the existence of non-fine moduli spaces [11]. However, more recently it has become clear that allowing twists has quite unexpected applications, e.g. to the Tate conjecture [14, 41]. Crucial for the purpose of this article is the observation proved in [24] that $\mathrm{Ker}(\rho)=\mathbb{Z}[2]$ for many twisted K3 surfaces (S,α). Note that for untwisted projective K3 surfaces the kernel is always highly non-trivial and, moreover, the conjectural description of $\mathrm{Ker}(\rho)$ has in this case only been achieved in the case of Picard rank one [4].

1.2. As a direct attack on $\mathcal{A}_X$ is difficult, we follow Addington and Thomas [1] and reduce the study of $\mathcal{A}_X$ via deformation to the case of (twisted) K3 surfaces. Central to our discussion is the Hodge structure $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ associated with $\mathcal{A}_X$ introduced in [1] as the analogue of the Mukai–Hodge structure $\widetilde{H}(S,\mathbb{Z})$ of weight two on the full cohomology $H^*(S,\mathbb{Z})$ of a K3 surface S or of the twisted version $\widetilde{H}(S,\alpha,\mathbb{Z})$ introduced in [21]. For example, any FM-equivalence $\mathcal{A}_X\simeq\mathcal{A}_{X'}$ induces a Hodge isometry $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})\simeq\widetilde{H}(\mathcal{A}_{X'},\mathbb{Z})$, cf. Proposition 3.4. This suffices to prove:

Theorem 1.1. *For any given smooth cubic $X\subset\mathbb{P}^5$ there are only finitely many cubics $X'\subset\mathbb{P}^5$ up to isomorphism together with a FM-equivalence $\mathcal{A}_X\simeq\mathcal{A}_{X'}$. See Corollary 3.5.*

Recall that due to a result of Bondal and Orlov a smooth cubic $X\subset\mathbb{P}^5$ itself does not admit any non-isomorphic Fourier–Mukai partners. This is no longer true if $\mathrm{D^b}(X)$ is replaced by its K3 category $\mathcal{A}_X$. In particular, there exist FM-equivalences $\mathcal{A}_X\simeq\mathcal{A}_{X'}$ that do not extend to equivalences $\mathrm{D^b}(X)\simeq\mathrm{D^b}(X')$. However, we will also see that general cubics X and X', i.e. those for which $\mathrm{rk}\,H^{2,2}(X,\mathbb{Z})=1$, admit a FM-equivalence $\mathcal{A}_X\simeq\mathcal{A}_{X'}$ if and only if $X\simeq X'$, see Theorem 1.5 or Corollary 3.6.

The following can be seen as an easy analogue of the result of Bayer and Bridgeland [4] describing $\mathrm{Aut}(\mathrm{D^b}(S))$ for general K3 surface S (namely those with $\rho(S)=1$) or rather of [26] describing this group for general non-projective K3 surfaces or twisted projective K3 surfaces (S,α) without (-2)-classes (see Section 6.1).

Theorem 1.2. *For the general*[1] *smooth cubic $X \subset \mathbb{P}^5$ the group $\mathrm{Aut}_s(\mathcal{A}_X)$ of symplectic FM-autoequivalences is infinite cyclic with*

$$\mathrm{Aut}_s(\mathcal{A}_X)/\mathbb{Z}\cdot[2] \simeq \mathbb{Z}/3\mathbb{Z}.$$

Moreover, the induced action on $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ of any FM-autoequivalence $\Phi\colon \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$ of a non-special cubic preserves the natural orientation.

In fact, for every smooth cubic $\mathrm{Aut}_s(\mathcal{A}_X)$ contains $\mathbb{Z}\cdot[2] \subset \mathbb{Z}$ as above, see Corollary 3.12. The theory of twisted K3 surfaces is crucial for the theorem, as eventually the problem is reduced to [24] which deals with general twisted K3 surfaces. The group $\mathrm{Aut}_s(\mathcal{A}_X)$ of an arbitrary cubic is described by an analogue of Brigdeland's conjecture, see Conjecture 3.14.

1.3. In [17] Hassett showed that in the moduli space of cubics $\mathcal{C}$ the set of those cubics X for which there exists a primitive positive plane $K_d \subset H^{2,2}(X,\mathbb{Z})$ of discriminant d containing the class $\mathrm{c}_1(\mathcal{O}(1))^2$ is an irreducible divisor $\mathcal{C}_d \subset \mathcal{C}$. Moreover, $\mathcal{C}_d$ is not empty if and only if

($*$) $d \equiv 0, 2\,(6)$ and $d > 6$.

Hassett also introduced the numerical condition

($**$) $d \equiv 0, 2\,(6)$ and d not divisible by $4, 9$ or any prime $2 \neq p \equiv 2\,(3)$

and proved that ($**$) is equivalent to the orthogonal complement of the corresponding lattice K_d in $H^4(X,\mathbb{Z})$ being (up to sign) Hodge isometric to the primitive Hodge structure $H^2(S,\mathbb{Z})_{\mathrm{prim}}$ of a polarized K3 surface. In [1] the condition was shown to be equivalent to the existence of a Hodge isometry $\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \simeq \widetilde{H}(S,\mathbb{Z})$ for some K3 surface S. We prove the following twisted version of it (cf. Proposition 2.11):

Theorem 1.3. *For a smooth cubic $X \subset \mathbb{P}^5$ the Hodge structure $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ is Hodge isometric to the Hodge structure $\widetilde{H}(S,\alpha,\mathbb{Z})$ of a twisted K3 surface (S,α) if and only if $X \in \mathcal{C}_d$ with*

($**'$) *$d \equiv 0, 2\,(6)$ and $n_i \equiv 0\,(2)$ for all primes $p_i \equiv 2\,(3)$ in $2d = \prod p_i^{n_i}$.*

Obviously, if d satisfies ($**$), then $k^2 d$ satisfies ($**'$) for all integers k, and vice versa.

As the main result of [1], Addington and Thomas proved that at least generically ($**$) is equivalent to $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S)$ for some K3 surface S. The following twisted version of it will be proved in Section 6.2.

Theorem 1.4. *i) If $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S,\alpha)$ for some twisted K3 surface (S,α), then $X \in \mathcal{C}_d$ with d satisfying ($**'$).*

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[1] A property holds for the *general* cubic if it holds for cubics in the complement of countably many proper closed subsets of the space of cubics under consideration. It holds for the *generic* cubic if it holds for a Zariski open, dense subset.

ii) Conversely, if d satisfies $(**')$, then there exists a Zariski open set $\emptyset \neq U \subset \mathcal{C}_d$ such that for all $X \in \mathcal{C}_d$ there exists a twisted K3 surface (S,α) and an equivalence $\mathcal{A}_X \simeq \mathrm{D^b}(S,\alpha)$.

Non-special cubics are determined by their associated K3 category $\mathcal{A}_X$ and for general special cubics $\mathcal{A}_X$ is determined by its Hodge structure (see Corollary 3.6 and Section 6.3):

Theorem 1.5. *Let X and X' be two smooth cubics.*

i) Assume X is not special, i.e. not contained in any $\mathcal{C}_d \subset \mathcal{C}$. Then there exists a FM-equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ if and only if $X \simeq X'$.

ii) For a Zariski dense open set of cubics $X \in \mathcal{C}_d$ with d satisfying $(**')$, there exists a FM-equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ if and only if there exists a Hodge isometry $\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_{X'},\mathbb{Z})$.

iii) For arbitrary d and general $X \in \mathcal{C}_d$ there exists a FM-equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ if and only if there exists a Hodge isometry $\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_{X'},\mathbb{Z})$.

We will also see that arguments of Addington [2] can be adapted to show that $(**')$ is in fact equivalent to the Fano variety of lines on X being birational to a moduli space of twisted sheaves on some K3 surface, see Proposition 4.1.

1.4. There are a few fundamental issues concerning $\mathcal{A}_X$ that we do not know how to address and that prevent us from developing the theory in full. Firstly, this paper only deals with FM-equivalences $\mathcal{A}_X \simeq \mathcal{A}_{X'}$, i.e. those for which the composition $\mathrm{D^b}(X) \to \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'} \hookrightarrow \mathrm{D^b}(X')$ is a Fourier–Mukai transform. One would expect this to be the case for all equivalences, but the classical result of Orlov [49] and its generalization by Canonaco and Stellari [13] do not apply to this situation. Secondly, it is not known whether $\mathcal{A}_X$ always admits bounded t-structures or stability conditions. This is problematic when one wants to study FM-partners of $\mathcal{A}_X$ as moduli spaces of (stable) objects in $\mathcal{A}_X$. As in [1], the lack of stability is also the crucial stumbling block to use deformation theory to prove statements as in Theorem 1.4 for all cubics and not only for generic or general ones.

1.5. The plan of the paper is as follows. Section 2 deals with all issues related to the lattice theory and the abstract Hodge theory. In particular, natural (countable unions of) codimension one subsets $D_{\mathrm{K3}} \subset D_{\mathrm{K3}'}$ of the period domain $D \subset \mathbb{P}(A_2^{\perp} \otimes \mathbb{C})$ are studied in great length. They parametrize periods that induce Hodge structures that are Hodge isometric to $\widetilde{H}(S,\mathbb{Z})$ resp. $\widetilde{H}(S,\alpha,\mathbb{Z})$ and which are described in terms of the numerical conditions $(**)$ resp. $(**')$. In particular, Theorem 1.3 is proved. We also provide a geometric description of all periods $x \in D$ in terms of generalized K3 surfaces, see Proposition 2.17.

In Section 3 we extend results in [1] from equivalences $\mathcal{A}_X \simeq \mathrm{D^b}(S)$ to the twisted case and prove the finiteness of FM-partners for $\mathcal{A}_X$, see Theorem 1.1. Moreover, we produce an action of the universal cover of $\mathrm{SO}(A_2)$ on $\mathcal{A}_X$ for all cubics (Remark 3.13) and formulate an analogue of Bridgeland's conjecture (Conjecture 3.14).

The short Section 4 shows that $(**')$ is equivalent to $F(X)$ being birational to a moduli space of stable sheaves on a K3 surface. In Section 5 we adapt the deformation theory of [1] to the twisted case. Finally, in Section 6 we conclude the proofs of Theorems 1.2, 1.4, and 1.5.

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2. Lattice theory and period domains

We start by discussing the relevant lattice theory. To make the reading self-contained, we will on the way also recall results due to Hassett resp. Addington and Thomas.

There are two kinds of lattices, those related to K3 surfaces, Λ, $\widetilde{\Lambda}$, etc., and those attached to cubic fourfolds, $\mathrm{I}_{21,2}$, K_d, etc.. The two types are linked by a lattice $A_2^\perp$ of signature $(2,20)$ and two embeddings

$$\mathrm{I}_{2,21} \longleftarrow\!\rhook A_2^\perp \lhook\joinrel\longrightarrow \widetilde{\Lambda}.$$

The induced maps between the associated period domains allows one to relate periods of cubic fourfolds to periods of (generalized) K3 surfaces.

2.1. By U we shall denote the hyperbolic plane, i.e. $\mathbb{Z}^2$ with the intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Furthermore, we let

$$\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$$

be the K3 lattice, i.e. the unique even, unimodular lattice of signature $(3,19)$. The extended K3 lattice is then defined as

$$\widetilde{\Lambda} := \Lambda \oplus U,$$

which is the unique even, unimodular lattice of signature $(4,20)$.

Next, A_2 denotes the standard root lattice of rank two, i.e. there exists a basis λ_1, λ_2 with respect to which the intersection matrix is given by $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The lattice A_2 is even and of signature $(2,0)$. Moreover, its discriminant group is $A_{A_2} := A_2^*/A_2 \simeq \mathbb{Z}/3\mathbb{Z}$ and, in particular, A_2 is not unimodular.

Due to [47, Thm. 1.14.4] there exist embeddings

$$A_2 \hookrightarrow \Lambda \text{ and } A_2 \hookrightarrow \widetilde{\Lambda},$$

which are both unique up to the action of $\mathrm{O}(\Lambda)$ resp. $\mathrm{O}(\widetilde{\Lambda})$. Note that all such embeddings are automatically primitive. In the following we will fix once and for all one such embedding $A_2 \hookrightarrow \Lambda \hookrightarrow \widetilde{\Lambda}$ and consider the orthogonal complement of $A_2 \subset \widetilde{\Lambda}$ as a fixed primitive sublattice

$$A_2^\perp \subset \widetilde{\Lambda}$$

of signature $(2,20)$. Its isomorphism type does not depend on the chosen embedding of A_2 and can be described explicitly by observing that the orthogonal complement of the embedding $A_2 \hookrightarrow \widetilde{\Lambda}$ given by

$$A_2 \hookrightarrow U \oplus U \hookrightarrow \widetilde{\Lambda},\ \lambda_1 \longmapsto e' + f',\ \lambda_2 \longmapsto e + f - e', \tag{2.1}$$

where e, f and e', f' denote the standard bases of the two copies of the hyperbolic plane, is isomorphic to

$$A_2^\perp \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus A_2(-1).^2 \tag{2.2}$$

Remark 2.1. For later use we recall that the group of isometries $\mathrm{O}(A_2)$ of the lattice A_2 is isomorphic to $\mathfrak{S}_3 \times \mathbb{Z}/2\mathbb{Z}$. Here, the Weyl group $\mathfrak{S}_3$ permutes the three positive roots and $\mathbb{Z}/2\mathbb{Z}$ acts by $-\mathrm{id}$. In fact, $\mathfrak{S}_3 \subset \mathrm{O}(A_2)$ is the kernel of the natural $\mathrm{O}(A_2) \twoheadrightarrow \mathrm{O}(A_{A_2}) \simeq \mathbb{Z}/2\mathbb{Z}$ (use the aforementioned $A_{A_2} \simeq \mathbb{Z}/3\mathbb{Z}$). The sign $\mathfrak{S}_3 \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ can be identified with the determinant $\mathrm{O}(A_2) \twoheadrightarrow \{\pm 1\}$. Thus, the group of orientation preserving isometries of A_2 acting trivially on A_{A_2} is just $\mathfrak{A}_3 \simeq \mathbb{Z}/3\mathbb{Z}$, where the generator can be chosen to act by $\lambda_1 \longmapsto -\lambda_1 - \lambda_2$, $\lambda_2 \longmapsto \lambda_1$.

2.2. Next, consider the unique odd, unimodular lattice

$$\mathrm{I}_{2,21} := \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 21} \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-1)^{\oplus 3}$$

of signature $(2,21)$ and an element $h \in \mathrm{I}_{2,21}$ with $(h)^2 = -3$, e.g. $h = (1,1,1) \in \mathbb{Z}(-1)^{\oplus 3}$. Then the primitive sublattice $h^\perp \subset \mathrm{I}_{2,21}$ is of signature $(2,20)$ and using (2.2) one finds

$$h^\perp \simeq A_2^\perp.$$

In the following, we will always consider $A_2^\perp$ with two fixed embeddings as above:

$$\mathrm{I}_{2,21} \hookleftarrow A_2^\perp \hookrightarrow \widetilde{\Lambda}.$$

Following Hassett [17], we now consider all primitive, negative definite sublattices

$$K_d \subset \mathrm{I}_{2,21}$$

of rank two containing h. Here, the index $d = \operatorname{disc} K_d$ denotes the discriminant of K_d, which is necessarily positive. Using [47, Sec. 1.5] one finds that up to the action of the subgroup of $\mathrm{O}(\mathrm{I}_{2,21})$ fixing h the lattice $K_d \subset \mathrm{I}_{2,21}$ is uniquely determined by d.

Moreover, Hassett shows that $d \equiv 0, 2\,(6)$ and that for a generator (unique up to sign) $v \in A_2^\perp$ of $K_d \cap A_2^\perp$ the following holds

$$-(v)^2 = \begin{cases} d/3 & \text{if } d \equiv 0\,(6) \\ 3d & \text{if } d \equiv 2\,(6). \end{cases} \tag{2.3}$$

[2] In [17] the last summand is instead described as a lattice with intersection matrix $\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$, which is of course isomorphic to $A_2(-1)$.

Viewing $v \in A_2^\perp \subset \widetilde{\Lambda}$ as an element of $\widetilde{\Lambda}$ leads to a lattice

$$A_2 \oplus \mathbb{Z} \cdot v \subset \widetilde{\Lambda}$$

of rank three and signature $(2,1)$. As it turns out, this is a primitive sublattice for $d \equiv 0\,(6)$ and it is of index three in its saturation for $d \equiv 2\,(6)$. This follows from [17, Prop. 3.2.2] asserting that $(v.A_2^\perp) = \mathbb{Z}$ resp. $3 \cdot \mathbb{Z}$ in the two cases.

In our discussion, the lattices K_d will occupy a place in the back, as it will be more natural to work with the generator $v \in A_2^\perp \cap K_d$ directly.

2.3. We shall be interested in the period domains, defined by the two conditions $(x.x) = 0$ and $(x.\bar{x}) > 0$, associated with the two lattices $A_2^\perp$ and $\widetilde{\Lambda}$. They will be denoted by

$$D \subset \mathbb{P}(A_2^\perp \otimes \mathbb{C}) \text{ resp. } Q \subset \mathbb{P}(\widetilde{\Lambda} \otimes \mathbb{C}).$$

In particular, $\dim D = 20$ and $\dim Q = 22$. Also note that D has two connected components, whereas Q is connected.

Points $x \in D$ correspond to Hodge structures of weight two on the lattice $A_2^\perp$. However, often we will think of $x \in D$ at the same time as defining a Hodge structure on $\widetilde{\Lambda}$ with A_2 contained in its $(1,1)$-part. In fact, for general points $x \in D$ the integral $(1,1)$-part of the corresponding Hodge structure is the lattice A_2.

We shall refer to D as the period domain of cubic fourfolds, although only an open subset really corresponds to smooth cubics. More concretely, for a smooth cubic $X \subset \mathbb{P}^5$ and any marking, i.e. an isometry, $\varphi\colon h^\perp \xrightarrow{\sim} A_2^\perp$, one defines the associated period as the image $x := [\varphi_{\mathbb{C}}(H^{3,1}(X))] \in D$. A description of the image of the period map, allowing cubics with ADE-singularities, has been given by Laza [39] and Looijenga in [42]. Points in Q are thought of as periods of generalized K3 surfaces, cf. Section 2.7.

Fixing embeddings $A_2 \hookrightarrow \widetilde{\Lambda}$ and the induced $A_2^\perp \hookrightarrow \widetilde{\Lambda}$ as before leads to the diagram

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, hook] & Q \arrow[d, hook] \\ \mathbb{P}(A_2^\perp \otimes \mathbb{C}) \arrow[r, hook] & \mathbb{P}(\widetilde{\Lambda} \otimes \mathbb{C}), \end{tikzcd}$$

which is central for the discussion. In particular, $D = \mathbb{P}(A_2^\perp \otimes \mathbb{C}) \cap Q$.

For later use we state the following technical observation.

Lemma 2.2. *The Hodge structure on $\widetilde{\Lambda}$ defined by an arbitrary $x \in D$ admits a Hodge isometry that reverses the orientation of the four positive directions.*

Proof. Consider a transposition $g := (12) \in \mathfrak{S}_3 \subset \mathrm{O}(A_2)$. Then g acts trivially on the discriminant A_{A_2} (see Remark 2.1) and can, therefore, be extended to $\tilde{g} \in \mathrm{O}(\widetilde{\Lambda})$ acting trivially on $A_2^\perp$. In particular, the Hodge isometry $\tilde{g}$ preserves the orientation of the two positive directions given by the $(2,0)$ and $(0,2)$-parts. On the other hand, it reverses the orientation of the two positive directions in A_2. $\square$

Remark 2.3. This result is the analogue of the fact that any Hodge structure on $\widetilde{\Lambda}$ with a hyperbolic plane U contained in its $(1,1)$-part admits an orientation reversing Hodge isometry. This statement in particular applies to the Hodge structure $\widetilde{H}(S,\mathbb{Z})$. Note that it is unknown whether $\widetilde{H}(S,\alpha,\mathbb{Z})$ always admits an orientation reversing Hodge isometry which prevents us from determining the image of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(S,\alpha)) \longrightarrow \mathrm{Aut}(\widetilde{H}(S,\alpha,\mathbb{Z}))$ completely, analogously to [26]. See [51] for partial results in this direction.

2.4. Let us now turn to the geometric interpretation of certain periods in Q. Recall that for a K3 surface S the extended K3 (or Mukai) lattice $\widetilde{H}(S,\mathbb{Z})$ is abstractly isomorphic to $\widetilde{\Lambda}$. Moreover, $\widetilde{H}(S,\mathbb{Z})$ comes with a natural Hodge structure of weight two defined by

$$\widetilde{H}^{2,0}(S) := H^{2,0}(S) \text{ and } \widetilde{H}^{1,1}(S) := H^{1,1}(S) \oplus (H^0 \oplus H^4)(S,\mathbb{C}).$$

For a Brauer class $\alpha \in \mathrm{Br}(S) \simeq H^2(S,\mathbb{G}_m) \simeq H^2(S,\mathcal{O}_S^*)_{\mathrm{tors}}$ we have introduced in [20] the weight-two Hodge structure $\widetilde{H}(S,\alpha,\mathbb{Z})$. As a lattice this is still isomorphic to $\widetilde{\Lambda}$ and its Hodge structure is determined by

$$\widetilde{H}^{2,0}(S,\alpha) := \mathbb{C}\cdot(\sigma + B \wedge \sigma) \text{ and } \widetilde{H}^{1,1}(S,\alpha,\mathbb{Z}) := \exp(B)\cdot \widetilde{H}^{1,1}(S).$$

Here, $0 \neq \sigma \in H^{2,0}(S)$ and $B \in H^2(S,\mathbb{Q})$ maps to α under the exponential map

$$H^2(S,\mathbb{Q}) \longrightarrow H^2(S,\mathcal{O}_S) \xrightarrow{\exp} H^2(S,\mathcal{O}_S^*).$$

The isomorphism type of the Hodge structure is independent of the choice of B.

Definition 2.4. A period $x \in Q$ is of *K3 type* (resp. *twisted K3 type*) if there exists a K3 surface S (resp. a twisted K3 surface $(S,\alpha \in \mathrm{Br}(S))$) such that the Hodge structure on $\widetilde{\Lambda}$ defined by x is Hodge isometric to $\widetilde{H}(S,\mathbb{Z})$ (resp. $\widetilde{H}(S,\alpha,\mathbb{Z})$).

The sets of periods of K3 type and twisted K3 type will be denoted

$$Q_{\mathrm{K3}} \subset Q_{\mathrm{K3}'} \subset Q.$$

There is also a geometric interpretation for points outside $Q_{\mathrm{K3}'}$ in terms of symplectic structures [20], but those are a priori inaccessible by algebro-geometric techniques, see however Section 2.7.

For the following recall that the twisted hyperbolic plane $U(n)$ is the rank two lattice with intersection matrix $\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$. The standard isotropic generators will be denoted e_n, f_n or simply e, f. Part i) of the next lemma is well known.

Lemma 2.5. *Consider a period point $x \in Q$. Then:*
i) $x \in Q_{\mathrm{K3}}$ if and only if there exists an embedding $U \hookrightarrow \widetilde{\Lambda}$ such that its image is contained in the $(1,1)$-part of the Hodge structure defined by x.
ii) $x \in Q_{\mathrm{K3}'}$ if and only if there exists a (not necessarily primitive) embedding $U(n) \hookrightarrow \widetilde{\Lambda}$ for some $n \neq 0$ such that its image is contained in the $(1,1)$-part of the Hodge structure defined by x.

Proof. We prove the second assertion, the first one is even easier. Start with a twisted K3 surface (S, α) and pick a lift $B \in H^2(S, \mathbb{Q})$ of α. Then the algebraic part $\widetilde{H}^{1,1}(S, \alpha, \mathbb{Z}) = \exp(B) \cdot \widetilde{H}^{1,1}(S, \mathbb{Q}) \cap \widetilde{H}(S, \mathbb{Z})$ contains the lattice $(\mathbb{Z} \cdot (1, B, B^2/2) \cap \widetilde{H}(S, \mathbb{Z})) \oplus H^4(S, \mathbb{Z})$, which is isomorphic to $U(n)$ for n minimal with $n(1, B, B^2/2) \in \widetilde{H}(S, \mathbb{Z})$.

Conversely, assume $U(n) \subset \widetilde{\Lambda}$ is of type $(1,1)$ with respect to x. Choosing n minimal, we can assume that the standard isotropic generator $e_n = e$ is primitive in $\widetilde{\Lambda}$. But then $e \in U(n)$ can be completed to a sublattice of $\widetilde{\Lambda}$ isomorphic to the hyperbolic plane $U = \langle e, f \rangle$, which therefore induces an orthogonal decomposition

$$\widetilde{\Lambda} \simeq \Lambda \oplus U \tag{2.4}$$

(usually different from the one defining $\widetilde{\Lambda}$).

With respect to (2.4) the second basis vector $f_n \in U(n)$ can be written as $f_n = \gamma + nf + ke$ with $\gamma \in \Lambda$. Similarly, a generator of the $(2,0)$-part of the Hodge structure determined by x is orthogonal to e and hence of the form $\sigma + \lambda e$ for some $\sigma \in \Lambda \otimes \mathbb{C}$ and $\lambda \in \mathbb{C}$. However, it is also orthogonal to f_n and so $(\gamma.\sigma) + n\lambda = 0$. Now set $B := -(1/n)\gamma$. Then $\sigma + \lambda e = \sigma + B \wedge \sigma$, where $B \wedge \sigma$ stands for $(B.\sigma)e$.

Eventually, the surjectivity of the period map implies that $\sigma \in \Lambda \otimes \mathbb{C}$ can be realized as the period of some K3 surface S, i.e. there exists an isometry $H^2(S, \mathbb{Z}) \simeq \Lambda$ identifying $H^{2,0}(S)$ with $\mathbb{C} \cdot \sigma \subset \Lambda \otimes \mathbb{C}$. Here one uses $(\sigma.\sigma) = (\sigma + \lambda e.\sigma + \lambda e) = 0$ and $(\sigma.\bar{\sigma}) = (\sigma + \lambda e.\bar{\sigma} + \bar{\lambda} e) > 0$. Mapping $H^4(S, \mathbb{Z})$ to $\mathbb{Z} \cdot e \subset U \subset \Lambda \oplus U$ in (2.4) and defining $\alpha \in \mathrm{Br}(S)$ as the Brauer class induced by B under $\Lambda \otimes \mathbb{Q} \simeq H^2(S, \mathbb{Q}) \twoheadrightarrow H^2(S, \mathbb{G}_m)$ yields a Hodge isometry of $\widetilde{H}(S, \alpha, \mathbb{Z})$ with the one defined by x on $\widetilde{\Lambda}$. $\square$

Corollary 2.6. *The sets $Q_{\mathrm{K3}} \subset Q_{\mathrm{K3}'} \subset Q$ can be described as the intersections of Q with countably many linear subspaces of codimension two:*

$$Q_{\mathrm{K3}} = Q \cap \bigcup U^{\perp} \subset Q_{\mathrm{K3}'} = Q \cap \bigcup U(n)^{\perp} \subset Q.$$

Here, the unions are over all embeddings $U \hookrightarrow \widetilde{\Lambda}$ resp. $U(n) \hookrightarrow \widetilde{\Lambda}$ with arbitrary $n \neq 0$. $\square$

2.5. However, it will turn out that the further intersection with D yields countable unions of codimension one subsets. These intersections are denoted by

$$D_{\mathrm{K3}} := D \cap Q_{\mathrm{K3}} \subset D_{\mathrm{K3}'} := D \cap Q_{\mathrm{K3}'} \subset D$$

and will be viewed as the sets of cubic periods that define generalized K3 periods of K3 type resp. of twisted K3 type.

We will first explain that $D_{\rm K3'}$ is a countable union of hyperplane sections. A second proof for the same assertion that also works for $D_{\rm K3}$ is provided in Section 2.6.

Lemma 2.7. *Any sublattice of the form $A_2+U(n)\subset\widetilde\Lambda$ contains a sublattice $U(m)\subset A_2+U(n)$ such that $A_2+U(m)$ is of rank three.*

Proof. If $A_2+U(n)$ is already of rank three, then there is nothing to show. So, assume $\mathrm{rk}(A_2+U(n))=4$, which is the generic case. For greater clarity we consider two cases, although only the second will be realized later.

i) Assume A_2 and $U(n)$ are orthogonal. Then define $e':=n\lambda_1+(e-nf)$ and $f':=n\lambda_2-(e-nf)$. Then $(e')^2=(f')^2=0$, and $(e'.f')=n^2=:m$, i.e. e' and f' span a sublattice $U(m)\subset A_2+\mathbb{Z}\cdot(e-nf)\subset A_2+U(n)$.

ii) If A_2 and $U(n)$ are not orthogonal, then we may assume that there exists an $x\in A_2$ with $(x.e)\neq 0$. Then $f':=(x.e)x-((x)^2/2)e$ satisfies $(f')^2=0$ and $(f'.e)=(x.e)^2=:m$. Hence, e,f' span $U(m)\subset\mathbb{Z}\cdot x+\mathbb{Z}\cdot e\subset A_2+U(n)$. $\Box$

Now, for $x\in D_{\rm K3'}$ there exists a sublattice $U(n)\subset\widetilde\Lambda$ contained in the $(1,1)$-part of the Hodge structure corresponding to x, i.e. $x\in D\cap U(n)^\perp$. However, due to the lemma there is another sublattice $U(m)\subset\widetilde\Lambda$ with $x\in U(m)^\perp$ and $\mathrm{rk}(A_2+U(m))=3$. Hence, the intersection $D\cap U(n)^\perp\subset D_{\rm K3'}$ of codimension two is subsumed by the codimension one set $D\cap U(m)^\perp\subset D_{\rm K3'}$.[3] A better way to phrase this is as follows:

Corollary 2.8. *The set of twisted K3 periods in D can also be described as the countable union of hyperplane sections:*

$$D_{\rm K3'}=D\cap\bigcup e^\perp.$$

Here, the union runs over all $0\neq e\in\widetilde\Lambda$ with $(e)^2=0$.

Proof. One inclusion follows from the fact that any $U(n)$ contains an isotropic vector. For the converse, recall that any primitive isotropic $0\neq e\in\widetilde\Lambda$ can be realized as the standard basis vector of a hyperbolic plane $U\subset\widetilde\Lambda$. If $e\in A_2^\perp$, then $D\cap e^\perp=\varnothing$. So we may assume that e is not orthogonal to A_2 and apply the arguments in part ii) of the above proof to A_2+U. This yields a sublattice $A_2+U(m)\subset A_2+U$ of rank three that contains e. However, then $A_2^\perp\cap U(m)^\perp=A_2^\perp\cap e^\perp$, which proves $D\cap e^\perp\subset D\cap U(m)^\perp\subset D_{\rm K3'}$. $\Box$

[3] It would be interesting to come up with a similar argument proving that also $D_{\rm K3}$ is of codimension one. Instead one has to pass via the description of $D_{\rm K3}$ in terms of divisors D_d and then use the more involved [1, Thm. 3.1]. See Proposition 2.11.

Remark 2.9. Any $0 \neq e \in \widetilde{\Lambda}$ with $(e)^2 = 0$ can be decomposed as $e = e_1 + e_2 \in (A_2 \oplus A_2^\perp) \otimes \mathbb{Q}$. Since A_2 is positive definite, we must have $e_2 \neq 0$. However, it can happen that $e_1 = 0$, but then $D \cap e^\perp = \varnothing$, as $A_2^\perp$ has signature $(2, 20)$ and so $e_2^\perp \subset A_2^\perp$ does not contain any positive plane. So classes $e \in \widetilde{\Lambda}$ with $e_1 = 0$ are of no interest to us and we can safely ignore them.

There is yet another class of hyperplane sections of D that is of importance to us. We let

$$D_{\mathrm{sph}} := D \cap \bigcup \delta^\perp,$$

where the union is over all $\delta \in \widetilde{\Lambda}$ with $(\delta)^2 = -2$. We call it the set of *periods with spherical classes*. Indeed, $x \in D$ is contained in D_{sph} if and only if the Hodge structure on $\widetilde{\Lambda}$ defined by x admits an integral $(1,1)$-class δ with $(\delta)^2 = -2$ and those classes typically appear as Mukai vectors of spherical objects, see Example 3.10.

Note that there are natural inclusions

$$D_{\mathrm{K3}} \subset D_{\mathrm{sph}} \subset D,$$

for every hyperbolic plane U contains a (-2)-class. However, $D_{\mathrm{K3}'}$ is not contained in D_{sph} and, more precisely, the inclusions

$$D_{\mathrm{K3}} \subsetneq D_{\mathrm{K3}'} \cap D_{\mathrm{sph}} \subsetneq D_{\mathrm{K3}'}, D_{\mathrm{sph}}$$

are all proper, see Example 2.14 and Proposition 2.15.

2.6. It is instructive to study the sets $D_{\mathrm{K3}} \subset D_{\mathrm{K3}'}$ and $D_{\mathrm{K3}} \subset D_{\mathrm{sph}}$ from a more cubic perspective, i.e. in terms of the lattices K_d.

For any $h \in K_d \subset \mathrm{I}_{2,21}$ as in Section 2.2 one introduces the hyperplane section

$$D \cap K_d^\perp \subset \mathbb{P}(A_2^\perp \otimes \mathbb{C})$$

of all cubic periods orthogonal to $K_d \cap A_2^\perp$. In other words, $D \cap K_d^\perp$ is the set of cubic periods for which the generator v of $K_d \cap A_2^\perp$ is of type $(1,1)$, i.e. $D \cap K_d^\perp = D \cap v^\perp$. Then one defines

$$D_d := D \cap \bigcup K_d^\perp,$$

where the union runs over all $h \in K_d \subset \mathrm{I}_{2,21}$ as above. So, for each positive $d \equiv 0, 2\,(6)$ the set D_d is a countable union of hyperplane sections of D. Dividing D_d by the subgroup $\tilde{\mathrm{O}}(h^\perp) = \mathrm{O}(\mathrm{I}_{2,21}, h) \subset \mathrm{O}(\mathrm{I}_{2,21})$ of elements fixing h yields Hassett's irreducible divisor

$$\mathcal{C}_d := D_d / \tilde{\mathrm{O}}(h^\perp).$$

Consider the following conditions for an even integer $d > 6$:

$(*)$ $d \equiv 0, 2\,(6)$.

$(**)$ The integer d is not divisible by $4, 9$, or any prime $2 \neq p \equiv 2\,(3)$.

$(**')$ $n_i \equiv 0\,(2)$ for any prime $p_i \equiv 2\,(3)$ in the prime factor decomposition $2d = \prod p_i^{n_i}$.

Obviously, $(**)$ implies $(**')$. More precisely, if d satisfies $(**)$ then $(**')$ holds for all k^2d, and vice versa.

Remark 2.10. Conditions $(*)$ and $(**)$ have first been introduced and studied by Hassett [17]. He shows that D_d is not empty if and only if $(*)$ is satisfied. Moreover, d satisfies $(**)$ if and only if for all cubics X with period x contained in D_d there exists a polarized K3 surface (S,H) such that its primitive cohomology $H^2(S,\mathbb{Z})_{\rm pr}$ is Hodge isometric to the Hodge structure on $K_d^\perp$ defined by x. To get polarized K3 surfaces and not only quasi-polarized ones, one has to use a result of Voisin [57, Sec. 4, Prop. 1] saying that $H^{2,2}(X,\mathbb{Z})_{\rm pr}$ does not contain any class of square 2.

Proposition 2.11. *With the above notations one has*

$$D_{\rm K3}=\bigcup_{(**)}D_d \text{ and } D_{\rm K3'}=\bigcup_{(**')}D_d,$$

where d runs through all d satisfying $(**)$ resp. $(**')$.

Proof. The first equality is due to Addington–Thomas [1, Thm. 3.1]. Indeed, they show that $x\in D_d$ with d satisfying $(**)$ if and only if there exists hyperbolic plane $U\subset\widetilde\Lambda$ which is of type $(1,1)$ with respect to x. The latter is in turn equivalent to $x\in D_{\rm K3}$, see Lemma 2.5.[4]

Maybe surprisingly, the second assertion is easier to prove. We include the elementary argument. Due to Corollary 2.8 we know $D_{\rm K3'}=D\cap\bigcup e^\perp$ with $0\neq e\in\widetilde\Lambda$ isotropic. So for one inclusion one has to show that each $D\cap e^\perp$ is of the form D_d with d satisfying $(**')$. Decompose $e=e_1+e_2\in(A_2\oplus A_2^\perp)\otimes\mathbb{Q}$ as in Remark 2.9. Let then $v\in A_2^\perp$ such that $\mathbb{Q}\cdot e_2\cap A_2^\perp=\mathbb{Z}\cdot v$ and define $K_d\subset {\rm I}_{2,21}$ as the sublattice spanned by $v\in A_2^\perp\subset {\rm I}_{2,21}$ and h. We have to show that the discriminant d of K_d satisfies $(**')$.

Assume first that $A_2\oplus\mathbb{Z}\cdot v\subset\widetilde\Lambda$ is primitive. Then $d\equiv 0\,(6)$ and $d=-3(v)^2$, see Section 2.2. As $e\in A_2\oplus\mathbb{Z}\cdot v$ in this case, the quadratic equation $2(x_1^2+x_2^2-x_1x_2)+(v)^2x^2=0$ admits an integral solution. However, it is a classical result that

$$2n=(w)^2 \tag{2.5}$$

[4]The 'only if' direction is a consequence of Hassett's original result saying that $x\in D_d$ with d satisfying $(**)$ if and only if there exists a polarized K3 surface (S,H) such that $H^2(S,\mathbb{Z})_{\rm prim}$ is Hodge isometric to the Hodge structure on $K_d^\perp$ given by x. As the orthogonal complement of $H^2(S,\mathbb{Z})_{\rm prim}\subset\widetilde H(S,\mathbb{Z})\simeq\widetilde\Lambda$ contains a hyperbolic plane, by [47, Thm. 1.14.4] this Hodge isometry extends to a Hodge isometry of $\widetilde H(S,\mathbb{Z})$ with the one on $\widetilde\Lambda$ given by x. For the other direction one has to show that any Hodge isometry between $\widetilde H(S,\mathbb{Z})$ and the one on $\widetilde\Lambda$ given by x can be used to get a Hodge isometry between the Hodge structure on $K_d^\perp\cap A_2^\perp\subset\widetilde\Lambda$ and $H^2(S,\mathbb{Z})_{\rm prim}$ for some polarization on S.

for some $w \in A_2$ if and only if $n = \prod p_i^{n_i}$ with $n_i \equiv 0\,(2)$ for all $p_i \equiv 2\,(3)$, see [32].[5] But clearly this holds for $n = -(v)^2/2$ if and only if $d = 6n$ satisfies $(**')$.

Next assume that $A_2 \oplus \mathbb{Z} \cdot v \subset \widetilde{\Lambda}$ has index three in its saturation. Hence, $d \equiv 2\,(6)$ and $3d = -(v)^2$. Then argue as before, but now with the isotropic vector $3e \in A_2 \oplus \mathbb{Z} \cdot v$ and with $n = -(v)^2/2 = 3d/2$.

Going the argument backwards proves the inverse inclusion. □

So in particular, although $Q_{\mathrm{K3}} \subset Q_{\mathrm{K3}'} \subset Q$ are countable unions of codimension two subsets, their restrictions $D_{\mathrm{K3}} \subset D_{\mathrm{K3}'} \subset D$ to D are countable unions of codimension one subsets. For $D_{\mathrm{K3}'}$ we have observed this already in Section 2.5.

Remark 2.12. As mentioned in [1, 2] and explained to me by Addington, condition $(**)$ is in fact equivalent to the existence of a primitive $w \in A_2$ with $d = (w)^2$. And, as has become clear in the above proof, condition $(**')$ is equivalent to the existence of a (not necessarily primitive) $w \in A_2$ with $d = (w)^2$.

The first values of $d > 6$ that satisfy the various conditions are

$(*)$	8	12	14	18	20	24	26	30	32	36	38	42	44	48
$(**)$			14				26				38	42		
$(**')$	8		14	18		24	26		32		38	42		

Example 2.13. For certain d the condition that the period $x \in D$ of a cubic X is contained in D_d has a geometric interpretation, see [17, Sec. 4]. For example, $x \in D_8$ if and only if X contains a plane $\mathbb{P}^2 \subset X$ or if X is a Pfaffian cubic, then $x \in D_{14}$.

Example 2.14. i) Consider $d = 24$ which obviously satisfies $(**')$ but not $(**)$, i.e. $D_d \subset D_{\mathrm{K3}'}$ but $D_d \not\subset D_{\mathrm{K3}}$. Also, $D_d \subset D_{\mathrm{sph}}$. Indeed, if v generates $A_2^\perp \cap K_d$, then $(v)^2 = -8$ and hence there exists $\delta \in A_2 \oplus \mathbb{Z} \cdot v$ with $(\delta)^2 = -2$, e.g. $2\lambda_1 + \lambda_2 + v$. So, as mentioned before, one has a proper inclusion

$$D_{\mathrm{K3}} \subsetneq D_{\mathrm{K3}'} \cap D_{\mathrm{sph}}.$$

ii) It is also easy to see that $D_{\mathrm{sph}} \not\subset D_{\mathrm{K3}'}$. Indeed, consider the standard embedding $A_2 \subset U \oplus U \subset \widetilde{\Lambda}$ (see Section 2.1) and let e, f be the standard basis of another copy of U inside $\widetilde{\Lambda}$ orthogonal to A_2. Then $\delta \coloneqq \lambda_1 + e - 2f$ is a (-2)-class. However, $A_2 + \mathbb{Z} \cdot \delta$ does not contain any isotropic vector (use again (2.5)) and hence $(A_2 + \mathbb{Z} \cdot \delta)^\perp$ cannot be of the form $(A_2 + U(m))^\perp$ as $A_2 + U(m)$ always contains isotropic vectors.

[5] E.g. a prime p can be written as $x^2 + 3y^2$ if and only if $p = 3$ or $p \equiv 1\,(3)$, see [16]. Since $4(x^2 + xy + y^2) = (2x + y)^2 + 3y^2$ and $(x_1^2 + 3y_1^2) \cdot (x_2^2 + 3y_2^2) = (x_1x_2 - 3y_1y_2)^2 + 3(x_1y_2 + x_2y_1)^2$, this proves one direction. The other one uses a computation with Hilbert symbols to determine when $-nx_1^2 + x_2^2 + 3x_3^2 = 0$ has a rational solution.

It would be interesting to find a numerical condition (†) such that $D_{\rm sph} = \bigcup D_d$ with the union over all d satisfying (†). The best we have to offer at this time is the following

Proposition 2.15. *Assume $D_d \subset D_{\rm K3'}$ and $9|d$. Then $D_d \not\subset D_{\rm sph}$.*

Proof. Consider a fixed K_d and the corresponding generator v of $K_d \cap A_2^\perp$. As $9|d$, clearly $d \equiv 0\,(6)$ and so $A_2 \oplus \mathbb{Z} \cdot v \subset \widetilde{\Lambda}$ is primitive. If there were a (-2)-class $\delta \in \widetilde{\Lambda}$ with $D \cap K_d^\perp = D \cap v^\perp \subset D \cap \delta^\perp$, then $\delta \in A_2 + \mathbb{Z} \cdot v$ and so $\delta = w + kv$ for some $w \in A_2$ and $k \in \mathbb{Z}$. But then $-2 = (w)^2 - k^2 d/3$. However, if $9|d$, then $k^2 d/3 \equiv 0\,(3)$ and hence $(w)^2 = 2m$ with $m \equiv 2\,(3)$, which contradicts (2.5). $\square$

The following immediate consequence is crucial for the proof of Theorem 1.2, see Section 6.1.

Corollary 2.16. *The locus of twisted K3 periods $D_{\rm K3'}$ contains infinitely many hyperplane sections D_d with $D_d \not\subset D_{\rm sph}$.* $\square$

2.7. In [20] we have shown that points in Q can be understood as periods of generalized K3 surfaces. It is useful to distinguish three types:[6]

i) Periods of ordinary K3 surfaces are parametrized by $Q_{\rm K3}$. Up to the action of $\mathrm{O}(\widetilde{\Lambda})$, the set of these periods is the intersection of Q with the linear codimension two subspace $\mathbb{P}(\Lambda \otimes \mathbb{C}) \subset \mathbb{P}(\widetilde{\Lambda} \otimes \mathbb{C})$.

ii) More generally, one can consider periods of the form $\sigma + B \wedge \sigma$, where $\sigma \in \Lambda \otimes \mathbb{C}$ is an ordinary period and $B \in \Lambda \otimes \mathbb{C}$ (but not necessarily $B \in \Lambda \otimes \mathbb{Q}$). Up to the action of $\mathrm{O}(\widetilde{\Lambda})$, these periods are parametrized by the intersection of Q with the linear subspace of codimension one $\mathbb{P}((\Lambda \oplus \mathbb{Z} \cdot f) \otimes \mathbb{C}) \subset \mathbb{P}(\widetilde{\Lambda} \otimes \mathbb{C})$. Here, f is viewed as the generator of H^4. Note that by definition $Q_{\rm K3'}$ is the subset of periods for which B can be chosen in $\Lambda \otimes \mathbb{Q}$.

iii) Periods of the form $\exp(B + i\omega) = 1 + (B + i\omega) + ((B^2 - \omega^2)/2 + (B.\omega)i)$ are geometrically interpreted as periods associated with complexified symplectic forms. Here, the first and third summand are considered in $U \simeq H^0 \oplus H^4$. Periods of this type are parametrized by an open dense subset of Q.

In particular, all cubic periods parametrized by $D \subset Q$ should have an interpretation in terms of these three types. This has been discussed above for type i) and has led to consider the intersection $D_{\rm K3} = D \cap Q_{\rm K3}$. For type ii) with B rational the intersection with the cubic period domain gives $D_{\rm K3'}$. It is now natural to ask whether the remaining periods, so the periods in $D \backslash D_{\rm K3'}$, are of type ii) with B not rational or rather of type iii), i.e. related to complexified symplectic forms. It is the latter, as shown by the following

[6] The discussion has been prompted by a question of Ben Bakker.

Proposition 2.17. *The Hodge structure of a cubic period $x \in D$ is Hodge isometric to the Hodge structure of a twisted projective K3 surface (S, α), i.e. $x \in D_{\mathrm{K3}'}$, or to the Hodge structure associated with* $\exp(B + i\omega)$.

Furthermore, if the Hodge structure of x is Hodge isometric to a Hodge structure of the type $\sigma + B \wedge \sigma$, then B can be chosen rational.

Proof. One first observes that, analogously to Lemma 2.5, ii), a period $x \in Q$ is of the type ii) if and only if the integral $(1,1)$-part of the Hodge structure associated with x contains an isotropic direction. Indeed, if x is of type ii), i.e. of the form $\sigma + B \wedge \sigma$, then H^4 provides an isotropic direction of type $(1,1)$. For the converse use that any isotropic direction can be completed to a hyperbolic plane U as a direct summand of $\widetilde{\Lambda}$. Now regard U as $H^0 \oplus H^4$ with H^4 as the given isotropic direction, which is of type $(1,1)$. Hence x is indeed of type ii).

Now let $x \in D \cap Q$ be of type ii). It is enough to show that then $x \in D_{\mathrm{K3}'}$. The integral $(1,1)$-part of the Hodge structure associated with x contains A_2 and an isotropic direction, say $\mathbb{Z} \cdot f$. Moreover, A_2 and $\mathbb{Z} \cdot f$ cannot be orthogonal, as otherwise the orthogonal complement $(A_2 \oplus \mathbb{Z} \cdot f)^\perp$ in $\widetilde{\Lambda}$ has only one positive direction and could, therefore, not accommodate for x. But then the arguments of part ii) of the proof of Lemma 2.7 show that $A_2 \oplus \mathbb{Z} \cdot f$ contains some $U(n)$ and hence $x \in Q_{\mathrm{K3}'}$ by Lemma 2.5, ii). $\square$

Note that both,

$$D \subset Q \text{ and } Q_{\mathrm{K3}} \subset Q,$$

are of codimension two and that they both parametrize periods that can be interpreted in complex geometric terms (in contrast to the 'symplectic periods' of the form $\exp(B + i\omega)$). In fact, periods in D are even algebro-geometric in the sense that essentially all of them are associated with cubic fourfolds $X \subset \mathbb{P}^5$, whereas most K3 surfaces are of course not projective. It would be interesting to exhibit other subsets (of codimension two) of periods in Q with complex geometric interpretations.

In categorical language one would want to interpret the inclusion $D \subset Q$ for points in the complement of $D_{\mathrm{K3}'}$ as saying that the cubic K3 category $\mathcal{A}_X$ associated with the cubic fourfold $X \subset \mathbb{P}^5$ corresponding to $x \in D \backslash D_{\mathrm{K3}'}$ is equivalent to the derived Fukaya category $\mathrm{DFuk}(B + i\omega)$ associated with a complexified symplectic form $B + i\omega$. Deciding which symplectic structures occur here is in principle possible, but establishing an equivalence

$$\mathcal{A}_X \simeq \mathrm{DFuk}(B + i\omega)$$

will be difficult even in special cases.

The categorical interpretation of $D_{\mathrm{K3}} \subset Q$ is the content of [1], where it is proved that at least for a Zariski open dense set of periods $x \in D_{\mathrm{K3}}$ the cubic K3 category $\mathcal{A}_X$ really is equivalent to $\mathrm{D^b}(S)$ of the K3 surface S realizing the Hodge structure associated with x. This paper deals with the categorical interpretation of $D_{\mathrm{K3}'} \subset Q$.

Remark 2.18. The period domain $Q \subset \mathbb{P}(\widetilde{\Lambda}\otimes\mathbb{C})$ contains $D \subset Q$ as a codimension two subset, but it also contains natural codimension one subspaces. E.g. for a K3 surface S and the Mukai vector $v = (1, 0, 1-n) \in \widetilde{H}^{1,1}(S, \mathbb{Z})$ the hyperplane section $Q \cap v^\perp$ can be seen as the period domain for deformations of the Hilbert scheme $S^{[n]}$. However, from a categorical point of view the situation is less satisfactory as deformations of the category $\mathrm{D}^{\mathrm{b}}(S)$ along $Q \cap v^\perp$ are at best understood to finite order thickenings of the period $[\sigma] \in Q \cap v^\perp$.

3. The cubic K3 category

Let $X \subset \mathbb{P}^5$ be a smooth cubic hypersurface. The *cubic K3 category* associated with X is the category

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle^\perp := \{E \in \mathrm{D}^{\mathrm{b}}(X) \mid \mathrm{Hom}(\mathcal{O}_X(i), E[*]) = 0 \text{ for } i = 0, 1, 2\}.$$

The category has first been studied by Kuznetsov in [33, 37]. It behaves in many respects like the derived category $\mathrm{D}^{\mathrm{b}}(S)$ of a K3 surface S. In particular, the double shift $E \longmapsto E[2]$ defines a Serre functor of $\mathcal{A}_X$ (see [34, 35, Cor. 4.3] and [38, Rem. 4.2]) and the dimension of Hochschild homology of $\mathcal{A}_X$ and of $\mathrm{D}^{\mathrm{b}}(S)$ coincide due to [35].

Example 3.1. Due to the work of Kuznetsov [33, 37], certain cubic K3 categories $\mathcal{A}_X$ are known to be equivalent to bounded derived categories $\mathrm{D}^{\mathrm{b}}(S, \alpha)$ of twisted K3 surfaces (S, α). For example, if the period $x \in D$ of a cubic X is contained in D_8, then X contains a plane and for generic choices there exists a twisted K3 surface (S, α) with $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$. Similarly, if X is a Pfaffian cubic and hence $x \in D_{14}$, then $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S)$ for the K3 surface S naturally associated with the Pfaffian X.

Remark 3.2. Despite the almost perfect analogy between the cubic K3 category $\mathcal{A}_X$ and the derived category $\mathrm{D}^{\mathrm{b}}(S)$ of K3 surfaces, certain fundamental issues are more difficult for $\mathcal{A}_X$. For example, to the best of my knowledge no $\mathcal{A}_X$, which is not equivalent to the derived category $\mathrm{D}^{\mathrm{b}}(S, \alpha)$ of some twisted K3 surface (S, α), has yet been endowed with a bounded t-structure, let alone a stability condition. See [54, 55] for a discussion of special stability conditions on certain $\mathcal{A}_X$ of the form $\mathrm{D}^{\mathrm{b}}(S, \alpha)$.

The semi-orthogonal decomposition $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, {}^\perp\mathcal{A}_X\rangle$ with ${}^\perp\mathcal{A}_X = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle$ comes with the full embedding $i_*\colon \mathcal{A}_X \hookrightarrow \mathrm{D}^{\mathrm{b}}(X)$ (which is often suppressed in the notation) and the left resp. right adjoint functors $i^*, i^!\colon \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathcal{A}_X$, see [34, Sec. 3] for a survey. In particular, for all $E \in \mathrm{D}^{\mathrm{b}}(X)$ there are exact triangles

$$E' \longrightarrow E \longrightarrow i_*i^*E \text{ and } i_*i^!E \longrightarrow E \longrightarrow E''$$

with $E' \in {}^\perp\mathcal{A}_X$ and $E'' \in \mathcal{A}_X^\perp$. The adjoint functors are related by functorial isomorphisms

$$i^*E \simeq i^!(E \otimes \omega_X)[2].$$

Moreover, according to [36], the compositions

$$i_* \circ i^*, i_* \circ i^! \colon \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathcal{A}_X \hookrightarrow \mathrm{D}^{\mathrm{b}}(X)$$

are of Fourier–Mukai type. More precisely, $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, {}^{\perp}\mathcal{A}_X \rangle$ induces a certain semi-orthogonal decomposition $\mathrm{D}^{\mathrm{b}}(X \times X) = \langle \mathrm{D}^{\mathrm{b}}(X) \boxtimes \mathcal{A}_X, \mathrm{D}^{\mathrm{b}}(X) \boxtimes {}^{\perp}\mathcal{A}_X \rangle$ and the $\mathrm{D}^{\mathrm{b}}(X) \boxtimes \mathcal{A}_X$-part of $\mathcal{O}_\Delta$, say $\mathcal{P} \in \mathrm{D}^{\mathrm{b}}(X) \boxtimes \mathcal{A}_X$, is the Fourier–Mukai kernel for $i_* \circ i^*$, i.e. there exists an exact triangle $\mathcal{Q} \longrightarrow \mathcal{O}_\Delta \longrightarrow \mathcal{P}$ with $\mathcal{Q} \in \mathrm{D}^{\mathrm{b}}(X \times X)$ contained in the category generated by objects of the form $F \boxtimes \mathcal{O}_X(i)$, with $F \in \mathrm{D}^{\mathrm{b}}(X)$ and $i = 0, 1, 2$.

3.1. For a K3 surface S the Mukai lattice $\widetilde{H}(S, \mathbb{Z})$ is endowed with the Hodge structure determined by $\widetilde{H}^{2,0}(S) = H^{2,0}(S)$ and by requiring $\widetilde{H}^{2,0} \perp \widetilde{H}^{1,1}$. Using the natural isomorphism $K_{\mathrm{top}}(S) \simeq H^*(S, \mathbb{Z})$ this Hodge structure can also be regarded as a Hodge structure on $K_{\mathrm{top}}(S)$.

In [1] Addington and Thomas introduce a similar Hodge structure associated with the category $\mathcal{A}_X$, defined on $K_{\mathrm{top}}(\mathcal{A}_X)$ and denoted by $\widetilde{H}(\mathcal{A}_X, \mathbb{Z})$. Here, $K_{\mathrm{top}}(\mathcal{A}_X) \subset K_{\mathrm{top}}(X)$ is the orthogonal complement of $\{[\mathcal{O}], [\mathcal{O}(1)], [\mathcal{O}(2)]\}$ with respect to the pairing $\chi(\alpha, \beta) = \langle v(\alpha), v(\beta) \rangle$ defined in terms of the Mukai vector $v \colon K_{\mathrm{top}}(X) \longrightarrow H^*(X, \mathbb{Q})$ and the Mukai pairing on $H^*(X, \mathbb{Q})$. It is not difficult to see that one has in fact an orthogonal direct sum decomposition

$$K_{\mathrm{top}}(X) = K_{\mathrm{top}}(\mathcal{A}_X) \oplus \langle [\mathcal{O}_X], [\mathcal{O}_X(1)], [\mathcal{O}_X(2)] \rangle.$$

As $H^*(X, \mathbb{Z})$ is torsion free, $K_{\mathrm{top}}(X)$ and

$$\widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \coloneqq K_{\mathrm{top}}(\mathcal{A}_X)$$

are as well. The Hodge structure is then defined by $\widetilde{H}^{2,0}(\mathcal{A}_X) \coloneqq v^{-1}(H^{3,1}(X))$ and the condition $\widetilde{H}^{2,0} \perp \widetilde{H}^{1,1}$. Furthermore, $N(\mathcal{A}_X)$ and the transcendental lattice $T(\mathcal{A}_X)$ of $\mathcal{A}_X$ are introduced in terms of this Hodge structure as $\widetilde{H}^{1,1}(\mathcal{A}_X, \mathbb{Z})$ resp. its orthogonal complement $\widetilde{H}^{1,1}(\mathcal{A}_X, \mathbb{Z})^{\perp}$. As a lattice $\widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ is independent of X and by [1] any equivalence $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S)$ (see Example 3.1) induces a Hodge isometry $\widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \simeq \widetilde{H}(S, \mathbb{Z})$ (cf. Proposition 3.3). In particular, $\widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ for all smooth cubics is abstractly isomorphic to $\widetilde{\Lambda}$.

As explained in [1, Prop. 2.3], the classes $\lambda_j \coloneqq [i^* \mathcal{O}_\ell(j)] \in \widetilde{H}^{1,1}(\mathcal{A}_X, \mathbb{Z})$, for a line $\ell \subset X$ and $j = 1, 2$, can be viewed as the standard generators of a lattice $A_2 \subset \widetilde{H}^{1,1}(\mathcal{A}_X, \mathbb{Z})$. Moreover, the Mukai vector $v \colon \widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \longrightarrow H^*(X, \mathbb{Q})$ induces an isometry

$$\langle \lambda_1, \lambda_2 \rangle^{\perp} \xrightarrow{\sim} h^{\perp} = H^4(X, \mathbb{Z})_{\mathrm{prim}}.$$

In particular, any marking $\varphi \colon h^{\perp} \xrightarrow{\sim} A_2^{\perp}$ induces a marking $\langle \lambda_1, \lambda_2 \rangle \oplus \langle \lambda_1, \lambda_2 \rangle^{\perp} \xrightarrow{\sim} A_2 \oplus A_2^{\perp}$ and further a marking

$$\widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\Lambda}. \tag{3.1}$$

Conversely, any marking (3.1) inducing the standard identification $\langle\lambda_1,\lambda_2\rangle \xrightarrow{\sim} A_2$ yields a marking $H^4(X,\mathbb{Z})_{\rm prim} \xrightarrow{\sim} A_2^\perp$. In this sense, (an open set of) points $x \in D$ will be considered as periods of cubic K3 categories $\mathcal{A}_X$ via their Hodge structures $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$.

Note that the positive directions of $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ come with a natural orientation, given by real and imaginary part of $\widetilde{H}^{2,0}(\mathcal{A}_X)$ and the oriented base λ_1,λ_2 of $A_2 \subset \widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$.

3.2. As we are also interested in equivalences $\mathrm{D}^{\mathrm{b}}(S,\alpha) \xrightarrow{\sim} \mathcal{A}_X$, we collect a few relevant facts dealing with the topological K-theory of twisted K3 surfaces (S,α). As it turns out, the topological setting does not require any new arguments. In order to speak of twisted sheaves or bundles, let us fix a class $B \in H^2(S,\mathbb{Q})$ which under the exponential map $H^2(S,\mathbb{Q}) \longrightarrow H^2(S,\mathcal{O}_S^*)$ is mapped to α. Next choose a Čech representative $\{B_{ijk}\}$ of $B \in H^2(S,\mathbb{Q})$ and consider the associated Čech representative $\{\alpha_{ijk} := \exp(B_{ijk})\}$ of α. This allows one to speak of $\{\alpha_{ijk}\}$-twisted sheaves and bundles, in the holomorphic as well as in the topological setting.

As explained in [21, Prop. 1.2], any $\{\alpha_{ijk}\}$-twisted bundle E can be untwisted to a bundle E_B by changing the transition functions φ_{ij} of E to $\exp(a_{ij}) \cdot \varphi_{ij}$, where the continuous functions a_{ij} satisfy $-a_{ij} + a_{ik} - a_{jk} = B_{ijk}$. The process can be reversed and so the categories of $\{\alpha_{ijk}\}$-twisted topological bundles is equivalent to the category of untwisted topological bundles. In particular,

$$K_{\rm top}(S,\alpha) \simeq K_{\rm top}(S)$$

which composed with the Mukai vector yields an isomorphism $K_{\rm top}(S,\alpha) \simeq \widetilde{H}(S,\alpha,\mathbb{Z})$ that identifies the image of $K(S,\alpha) \longrightarrow K_{\rm top}(S,\alpha)$ with $H^{1,1}(S,\alpha,\mathbb{Z})$.

The next result is the twisted version of the observation by Addington and Thomas mentioned earlier.

Proposition 3.3. *Any linear, exact equivalence $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S,\alpha)$ induces a Hodge isometry*

$$\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \simeq \widetilde{H}(S,\alpha,\mathbb{Z}).$$

Proof. By results due to Orlov in the untwisted case and due to Canonaco and Stellari [13] in the twisted case, any fully faithful functor $\Phi\colon \mathrm{D}^{\mathrm{b}}(S,\alpha) \longrightarrow \mathrm{D}^{\mathrm{b}}(X)$ is of Fourier–Mukai type, i.e. $\Phi \simeq \Phi_{\mathcal{E}}$ for some $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(S \times X, \alpha^{-1} \boxtimes 1)$. Therefore, Φ induces a homomorphism $\Phi_{\mathcal{E}}^K\colon K_{\rm top}(S,\alpha) \longrightarrow K_{\rm top}(X)$, see [18, Rem. 3.4].

If Φ is induced by an equivalence $\mathrm{D}^{\mathrm{b}}(S,\alpha) \xrightarrow{\sim} \mathcal{A}_X$, then $\Phi_{\mathcal{E}}^K\colon K_{\rm top}(S,\alpha) \xrightarrow{\sim} K_{\rm top}(\mathcal{A}_X)$ is an isomorphism and in fact a Hodge isometry $\widetilde{H}(S,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$. The compatibility with the Hodge structure follows from the twisted Chern character $\mathrm{ch}^{-\alpha\boxtimes 1}(\mathcal{E})$ of the Mukai kernel being of Hodge type. See [21, Sec. 1] for the notion of twisted Chern characters. That the quadratic form is respected as well is proved by mimicking the argument for FM-equivalences, see e.g. [23, Sec. 5.2].

(We are suppressing a number of technical details here. As explained before, the actual realization of the Hodge structure $\widetilde{H}(S,\alpha,\mathbb{Z})$ depends on the choice of a $B \in H^2(S,\mathbb{Q})$ lifting α. Similarly, the Chern character $\mathrm{ch}^{-\alpha\boxtimes 1}(\mathcal{E})$ also actually depends on B.) □

3.3. The above result generalizes to *FM-equivalences* $\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$, i.e. to equivalences for which the composition $\mathrm{D^b}(X) \longrightarrow \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'} \longrightarrow \mathrm{D^b}(X')$ admits a Fourier–Mukai kernel. It has been conjectured that in fact any linear exact equivalence is a FM-equivalence, but the existing results do not cover our case.

Proposition 3.4. *Any FM-equivalence $\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$ induces a Hodge isometry*

$$\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{X'},\mathbb{Z}).$$

Proof. The argument is a modification of the above. □

The following improves upon a result in [6] where it is shown that for a cubic X containing a plane there exist at most finitely many (up to isomorphisms) cubics $X_1,\ldots,X_n$ containing a plane with $\mathcal{A}_X \simeq \mathcal{A}_{X_1} \simeq \ldots \simeq \mathcal{A}_{X_n}$.

Corollary 3.5. *For any given smooth cubic $X \subset \mathbb{P}^5$ there exist only finitely many smooth cubics $X' \subset \mathbb{P}^5$ up to isomorphism such that there exists a FM-equivalence $\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$.*

Proof. The proof follows the argument for the analogous statement for K3 surfaces [8] closely, but needs a modification at one point that shall be explained.

Due to the proposition, it suffices to prove that up to isomorphism there exist only finitely many cubics X' such that there exists a Hodge isometry $\varphi\colon \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{X'},\mathbb{Z})$. Any such Hodge isometry induces a Hodge isometry $\varphi_T\colon T(\mathcal{A}_X) \xrightarrow{\sim} T(\mathcal{A}_{X'})$ and an isometry of lattices $N(\mathcal{A}_X) \xrightarrow{\sim} N(\mathcal{A}_{X'})$. We may assume

$$T(\mathcal{A}_X) \subset A_2^\perp \subset \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \text{ and } A_2 \subset N(\mathcal{A}_X)$$

and similarly for X'. Note however that these inclusions need not be respected by φ. The orthogonal complement of $T(\mathcal{A}_X)^\perp \subset A_2^\perp$ is just $N(\mathcal{A}_X) \cap A_2^\perp$ and the two inclusions of $A_2^\perp$ induce two Hodge structures on $A_2^\perp$. Note that if the Hodge isometry φ_T can be extended to a Hodge isometry $A_2^\perp \xrightarrow{\sim} A_2^\perp$, which can be interpreted as a Hodge isometry $H^4(X,\mathbb{Z})_{\mathrm{prim}} \simeq H^4(X',\mathbb{Z})_{\mathrm{prim}}$, then the Global Torelli theorem [57] implies that $X \simeq X'$.

A purely lattice theoretic argument, which is slightly more involved than the original one in [8], shows that the set of isomorphism classes of lattices Γ appearing as $N(\mathcal{A}_{X'}) \cap A_2^\perp$ is finite. Once this has been achieved one can assume that Γ is fixed. However, for two Fourier–Mukai partners realizing a fixed Γ, any Hodge isometry $T(\mathcal{A}_{X_1}) \simeq T(\mathcal{A}_{X_2})$ can be extended to a Hodge isometry $T(\mathcal{A}_{X_1}) \oplus \Gamma \simeq T(\mathcal{A}_{X_2}) \oplus \Gamma$. As the finite index overlattices $T(\mathcal{A}_{X_i}) \oplus \Gamma \subset H^4(X_i,\mathbb{Z})_{\mathrm{prim}}$ are all contained in $(T(\mathcal{A}_{X_i}) \oplus \Gamma)^*$, there are only finitely many choices for them, which allows one to reduce to the case that the Hodge isometry extends to a Hodge isometry $H^4(X_1,\mathbb{Z})_{\rm prim}\simeq H^4(X_2,\mathbb{Z})_{\rm prim}$.

For the lattice theory, fix even lattices Λ_1 and Λ. In our situation, $\Lambda_1=T(\mathcal{A}_X)$ and $\Lambda=A_2^\perp$. One proves that up to isomorphisms there exist only finitely many lattices Λ_2 occurring as the orthogonal complement of some primitive embedding $\Lambda_1\hookrightarrow\Lambda$. For unimodular Λ this is standard, but the proof can be tweaked to cover the more general statement. Of course, it suffices to show that only finitely many discriminant forms $(A_{\Lambda_2},q_{\Lambda_2})$ can occur. Now $G:=\Lambda/(\Lambda_1\oplus\Lambda_2)$ is naturally a finite subgroup of $\Lambda^*/(\Lambda_1\oplus\Lambda_2)$ of index $d=|{\rm disc}(\Lambda)|$. The first projection from $G\subset\Lambda^*/(\Lambda_1\oplus\Lambda_2)\subset A_{\Lambda_1}\oplus A_{\Lambda_2}$ defines an isomorphism of G with a finite subgroup of A_{Λ_1}. This leaves only finitely many possibilities for the finite groups G and $\Lambda^*/(\Lambda_1\oplus\Lambda_2)$. Note that $\Lambda/(\Lambda_1\oplus\Lambda_2)\subset A_{\Lambda_1}\oplus A_{\Lambda_2}$ is isotropic but not necessarily the bigger $\Lambda^*/(\Lambda_1\oplus\Lambda_2)\subset A_{\Lambda_1}\oplus A_{\Lambda_2}$. However, the restriction of the quadratic form to $\Lambda^*/(\Lambda_1\oplus\Lambda_2)$ takes values only in $(2/d^2)\mathbb{Z}/2\mathbb{Z}$. For fixed $G\subset A_{\Lambda_1}$ the restriction of q_{Λ_1} to G can be extended in at most finitely many ways to a quadratic form on $\Lambda^*/(\Lambda_1\oplus\Lambda_2)$ with values in $(2/d^2)\mathbb{Z}/2\mathbb{Z}$. Now use the other projection $\Lambda^*/(\Lambda_1\oplus\Lambda_2)\twoheadrightarrow A_{\Lambda_2}$ to see that there are only finitely many possibility for the group A_{Λ_2} and also for the quadratic form q_{Λ_1}. □

Two general cubics have FM-equivalent K3 categories only if they are isomorphic:

Corollary 3.6. *Let X be smooth cubic with ${\rm rk}\,H^{2,2}(X,\mathbb{Z})=1$. For a smooth cubic X' there exists a FM-equivalence $\mathcal{A}_X\simeq\mathcal{A}_{X'}$ if and only if $X\simeq X'$.*

Proof. The assumption implies that $N(\mathcal{A}_X)\simeq A_2$. As any FM-equivalence $\mathcal{A}_X\simeq\mathcal{A}_{X'}$ induces a Hodge isometry $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})\simeq\widetilde{H}(\mathcal{A}_{X'},\mathbb{Z})$, also $N(\mathcal{A}_{X'})\simeq A_2$. Moreover, the natural inclusions of the transcendental lattices $T(\mathcal{A}_X)\subset A_2^\perp$ and $T(\mathcal{A}_{X'})\subset A_2^\perp$ are in fact equalities and the induced Hodge isometry $T(\mathcal{A}_X)\simeq T(\mathcal{A}_{X'})$ can therefore be read as a Hodge isometry $H^4(X,\mathbb{Z})_{\rm prim}\simeq H^4(X',\mathbb{Z})_{\rm prim}$, which by the Global Torelli theorem [57] implies that $X\simeq X'$. □

Note that in contrast general K3 surfaces S, i.e. such that $\rho(S)=1$, usually have non-isomorphic FM-partners, see [48, 52]. The result may also be compared to the main result of [6] showing that for all cubic threefold $Y\subset\mathbb{P}^4$ the full subcategory $\langle\mathcal{O},\mathcal{O}(1)\rangle^\perp\subset{\rm D^b}(Y)$ determines Y.

Corollary 3.7. *Assume $X,X'\subset\mathbb{P}^5$ are two smooth cubics such that there exists a FM-equivalence $\mathcal{A}_X\simeq\mathcal{A}_{X'}$. If ${\rm rk}\,H^{2,2}(X,\mathbb{Z})>13$, then $X\simeq X'$.*

Proof. This follows more or less the standard argument. Any FM-equivalence $\mathcal{A}_X\simeq\mathcal{A}_{X'}$ induces a Hodge isometry $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})\simeq\widetilde{H}(\mathcal{A}_{X'},\mathbb{Z})$ and hence a Hodge isometry $T(\mathcal{A}_X)\simeq T(\mathcal{A}_{X'})$. The latter extends to a Hodge isometry $T(\mathcal{A}_X)\oplus\mathbb{Z}(-3)\simeq T(\mathcal{A}_{X'})\oplus\mathbb{Z}(-3)$ by id on $\mathbb{Z}(-3)$ which is declared to be of type $(1,1)$. Now use that there exists a primitive embedding of Hodge structures $T(\mathcal{A}_X)\oplus\mathbb{Z}(-3)\hookrightarrow H^4(X,\mathbb{Z})(-1)$, that sends $\mathbb{Z}(-3)$ to the line spanned by

$\mathrm{c}_1(\mathcal{O}(1))^2$, and similarly for X'. By Nikulin's [47, Thm. 1.14.4] the underlying lattice embedding is unique under our assumption, which leads to a Hodge isometry $H^4(X,\mathbb{Z}) \simeq H^4(X',\mathbb{Z})$ respecting the squares of the hyperplane classes. Hence, by the Global Torelli theorem $X \simeq X'$. □

Remark 3.8. In principle it should be possible to count FM-partners of $\mathcal{A}_X$ for general special cubics $X \in \mathcal{C}_d$ (i.e. $\mathrm{rk}\, H^{2,2}(X,\mathbb{Z}) = 2$). On the level of Hodge theory, this amounts to count the number of Hodge structures on $\widetilde{\Lambda}$ parametrized by D which are Hodge isometric to $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ up to those that are Hodge isometric on $A_2^{\perp}$. The arguments should follow [19, Thm. 1.4], see also [52], with the additional problem that $A_2^{\perp}$ is not unimodular.

However, as the period map for cubics is not surjective, this would only provide an upper bound for the number of isomorphism classes of cubics X' for which there exists a Hodge isometry $\widetilde{H}(\mathcal{A}_{X'},\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$. Note that due to Theorem 1.5 this would yield control over the number of isomorphism classes of X' with $\mathcal{A}_{X'} \simeq \mathcal{A}_X$, at least for general $X \in \mathcal{C}_d$.

3.4. We are interested in the group $\mathrm{Aut}(\mathcal{A}_X)$ of isomorphism classes of FM-equivalences $\Phi\colon \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$. As any FM-equivalence Φ induces a Hodge isometry

$$\Phi^H\colon \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_X,\mathbb{Z}),$$

there is a natural homomorphism

$$\rho\colon \mathrm{Aut}(\mathcal{A}_X) \longrightarrow \mathrm{Aut}(\widetilde{H}(\mathcal{A}_X,\mathbb{Z})),\ \ \Phi \longmapsto \Phi^H. \tag{3.2}$$

Here, $\mathrm{Aut}(\widetilde{H}(\mathcal{A}_X,\mathbb{Z}))$ denotes the group of Hodge isometries. We say that Φ is *symplectic* if the induced action on $\widetilde{H}^{2,0}(\mathcal{A}_X)$, or equivalently on $T(\mathcal{A}_X)$, is the identity. The subgroup of symplectic autoequivalences shall be denoted by $\mathrm{Aut}_{\mathrm{s}}(\mathcal{A}_X)$ and (3.2) induces

$$\rho\colon \mathrm{Aut}_{\mathrm{s}}(\mathcal{A}_X) \longrightarrow \mathrm{Aut}(\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})).$$

Remark 3.9. By $\mathrm{Aut}^+(\widetilde{H}(\mathcal{A}_X,\mathbb{Z}))$ one denotes the subgroup of Hodge isometries preserving a given orientation of the four positive directions. We expect that $\mathrm{Im}(\rho) = \mathrm{Aut}^+(\widetilde{H}(\mathcal{A}_X,\mathbb{Z}))$. This is known if $\mathcal{A}_X \simeq \mathrm{D^b}(S)$, see [26], and one inclusion can be proved for non-special cubics, see Theorem 1.2.

Example 3.10. The most important autoequivalences of K3 categories, responsible for the complexity of the groups $\mathrm{Aut}(\mathrm{D^b}(S))$ and $\mathrm{Aut}(\mathcal{A}_X)$ in particular, are spherical twists. Associated with any spherical object $A \in \mathcal{A}_X$, i.e. $\mathrm{Ext}^*(A,A) \simeq H^*(S^2)$, there exists a FM-equivalence

$$T_A\colon \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$$

that sends $E \in \mathcal{A}_X$ to the cone $T_A(E)$ of the evaluation map $R\mathrm{Hom}(A,E) \otimes A \longrightarrow E$. This is indeed a FM-equivalence – its kernel can be described as the cone of the composition $A^\vee \boxtimes A \xrightarrow{tr} \mathcal{O}_\Delta \longrightarrow (\mathrm{id},i)^*(\mathcal{O}_\Delta)$, where $(\mathrm{id},i)^*$ is the natural projection $\mathrm{D^b}(X\times X) \longrightarrow \mathrm{D^b}(X) \boxtimes \mathcal{A}_X$.

The action of the spherical twist $T_A\colon \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$ on $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ is given by the reflection $s_\delta\colon v \longmapsto v + \langle v,\delta\rangle\cdot\delta$, where $\delta\in\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$ is the Mukai vector of A.

In [35] Kuznetsov considers the functor

$$\Psi\colon \mathcal{A}_X \longrightarrow \mathcal{A}_X,\ E\longmapsto i^*(i_*E\otimes\mathcal{O}_X(1)),$$

which turns out to be an equivalence satisfying $\Psi^3\simeq[-1]$. Clearly, by construction Ψ is a FM-equivalence. In fact, for the proof that $\mathcal{A}_X$ is a K3 category this functor is crucial. Define

$$\Phi_0 \coloneqq \Psi[1],$$

which satisfies $\Phi_0^3\simeq[2]$.

Proposition 3.11. *The autoequivalence $\Phi_0\colon \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$ is symplectic and the induced action $\Phi_0^H\colon \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ corresponds to the element in $\mathrm{O}(A_2)$ that is given by the cyclic permutation of the positive roots.*

Proof. As the action on cohomology is independent of the specific cubic $X\subset\mathbb{P}^5$, we can assume that the transcendental lattice $T(\mathcal{A}_X)\subset\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ is of odd rank. However, $\pm\mathrm{id}$ are the only Hodge isometries of an irreducible Hodge structure of weight two of K3 type of odd rank, cf. [29, Cor. 3.3.5], and, as $\Phi_0^3\simeq[2]$ acts trivially on $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$, we must have $\Phi_0^H=\mathrm{id}$ on $T(\mathcal{A}_X)$, i.e. Φ_0 is symplectic.

If X is a cubic with $A_2\simeq\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$, then Φ_0^H corresponds to an element in $\mathrm{O}(A_2)$. As Φ_0 is symplectic, $\Phi_0^H=\mathrm{id}$ on $A_2^\perp$ and hence $\Phi_0^H=\mathrm{id}$ on the discriminant group A_{A_2}. Therefore, $\Phi_0^H\in\mathfrak{S}_3$, see Remark 2.1. For a cubic X such that $\mathcal{A}_X\simeq\mathrm{D^b}(S)$, we know that Φ_0^H must be orientation preserving by [26] and thus $\Phi_0^H\in\mathfrak{A}_3\simeq\mathbb{Z}/3\mathbb{Z}$ in general.

It remains to show that $\Phi_0^H\neq\mathrm{id}$. One way to see this relies on a direct computation. Another possibility is to use the recent result of Bayer and Brigeland [4] confirming Bridgeland's conjecture in [7] in the case of a K3 surface S of Picard rank one. More precisely, due to [4, Thm. 1.4] for a K3 surface S with $\rho(S)=1$ the subgroup of $\mathrm{Aut}(\mathrm{D^b}(S))$ of autoequivalences acting trivially on $\widetilde{H}(S,\mathbb{Z})$ is the product of $\mathbb{Z}[2]$ and the free group generated by squares of spherical twists T_E^2 associated with spherical vector bundles E on S. (That this is a reformulation of Bridgeland's original conjecture for $\rho(S)=1$ had also been observed by Kawatani [30].) Hence, if $\Phi_0^H=\mathrm{id}$, then $\Phi_0=(\ast_i T_{E_i}^2)\circ[2k]$, but then clearly Φ_0^3 could not be isomorphic to the double shift [2]. □

Corollary 3.12. *For every smooth cubic $X\subset\mathbb{P}^5$ the group of symplectic FM-autoequivalences $\mathrm{Aut}_s(\mathcal{A}_X)$ contains an infinite cyclic group $\mathbb{Z}\subset\mathrm{Aut}_s(\mathcal{A}_X)$ generated by Φ_0 such that*

$$\mathbb{Z}\cdot[2]\subset\mathbb{Z}$$

is a subgroup of index three and such that the natural map $\rho\colon \mathrm{Aut}_s(\mathcal{A}_X) \longrightarrow \mathrm{Aut}(\widetilde{H}(\mathcal{A}_X,\mathbb{Z}))$ *defines an isomorphism of* $\mathbb{Z}/\mathbb{Z}[2]$ *with the subgroup* $\mathfrak{S}_3 \subset \mathrm{O}(A_2) \subset \mathrm{O}(\widetilde{H}(\mathcal{A}_X,\mathbb{Z}))$ *of cyclic permutations of the positive roots of* A_2*.* □

Remark 3.13. The subgroup $\mathrm{SO}(A_2) \subset \mathrm{O}(A_2)$ of orientation preserving isometries of A_2 is $\mathfrak{A}_3 \times \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, see Remark 2.1. Its action can be 'lifted' to an action on $\mathcal{A}_X$ via the natural extension

$$0 \longrightarrow \mathbb{Z}\cdot[2] \times (\mathbb{Z}\cdot\Phi_0 \times \mathbb{Z}\cdot[1])/(\Phi_0^3 - [2]) \longrightarrow \mathrm{SO}(A_2) \longrightarrow 0,$$

which an be seen as induced by the universal cover of $\mathrm{SO}(A_2 \otimes \mathbb{R})$.

Inspired by Bridgeland's conjecture for K3 surfaces in [7], we state the following (see [28] explaining this reformulation):

Conjecture 3.14. *There exists an isomorphism*

$$\mathrm{Aut_s}(\mathcal{A}_X) \simeq \pi_1^{\mathrm{st}}[P_0/\mathrm{O}].$$

Here, $P \subset \mathbb{P}(\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})\otimes\mathbb{C})$ is the period domain defined analogously to D and Q in Section 2.3 and $P_0 := P\setminus\bigcup\delta^\perp$, with the union over all (-2)-classes $\delta \in \widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$. Moreover, $\mathrm{O} \subset \mathrm{O}(\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z}))$ the subgroup of all isometries acting trivially on the discriminant. However, contrary to the case of untwisted K3 surfaces we do not even have a natural map between these two groups at the moment.

3.5. The cubic K3 category $\mathcal{A}_X$ can also be described as a category of graded matrix factorizations, see [49]. More precisely, there exists an exact linear equivalence

$$\mathcal{A}_X \simeq \mathrm{MF}(W,\mathbb{Z}).$$

Here, $W \in R := k[x_0,\ldots,x_5]$ is a cubic polynomial defining X. The objects of $\mathrm{MF}(W,\mathbb{Z})$ are pairs $(K \xrightarrow{\alpha} L, L \xrightarrow{\beta} K(3))$, where K and L are finitely generated, free, graded R-modules and α,β are graded R-module homomorphisms with $\beta\circ\alpha = W\cdot\mathrm{id} = \alpha\circ\beta$. Recall that $K(n)$ for a graded R-module $K = \bigoplus K_i$ is the graded module with $K(n)_i = K_{n+i}$. Homomorphisms in $\mathrm{MF}(W,\mathbb{Z})$ are the obvious ones modulo those that are homotopic to zero (everything $\mathbb{Z}/2\mathbb{Z}$-periodic).

The shift functor that makes $\mathrm{MF}(W,\mathbb{Z})$ a triangulated category is given by

$$(K \xrightarrow{\alpha} L, L \xrightarrow{\beta} K(3))[1] = (L \xrightarrow{-\beta} K(3), K(3) \xrightarrow{-\alpha} L(3)).$$

Thus, the double shift is

$$(K \xrightarrow{\alpha} L, L \xrightarrow{\beta} K(3))[2] \simeq (K(3) \xrightarrow{\alpha} L(3), L(3) \xrightarrow{\beta} K(6)).$$

Viewing $\mathcal{A}_X$ as the category of graded matrix factorizations allows one to describe Φ_0 in Proposition 3.11 alternatively as follows. Consider the grade shift functor, i.e. the equivalence described by

$$\begin{array}{rcl}\Phi_0\colon \mathrm{MF}(W,\mathbb{Z}) & \xrightarrow{\sim} & \mathrm{MF}(W,\mathbb{Z})\\ (K \xrightarrow{\alpha} L, L \xrightarrow{\beta} K(3)) & \longmapsto & (K(1) \xrightarrow{\alpha} L(1), L(1) \xrightarrow{\beta} K(4)).\end{array}$$

Then, obviously,

$$\Phi_0^3 \simeq [2].$$

To actually show that the Φ_0 constructed in this way coincides with the one of Proposition 3.11, one would need to carry out a computation as in [3, Sec. 5] or [31, Sec. 5]. However, for abstract reasons Φ_0 with this description is certainly also of Fourier–Mukai type, i.e. $\Phi_0 \in \mathrm{Aut}(\mathcal{A}_X)$. To see this, note first that Φ_0 obviously has an enhancement, i.e. it lifts to the dg-enhancement of $\mathrm{MF}(W,\mathbb{Z})$ provided by the category $\mathrm{MF}^{\mathrm{dg}}(W,\mathbb{Z})$ (which has the same objects and its morphisms are $\mathbb{Z}/2\mathbb{Z}$-periodic morphisms of complexes of arbitrary degree). As i_* and i^* are of Fourier–Mukai type and thus admit dg-enhancements, also the composition $i_* \circ \Phi_0 \circ i^*$ does. By [56] this implies that $i_* \circ \Phi \circ i^*$ is a Fourier–Mukai functor and hence Φ_0 is of Fourier–Mukai type.

4. The Fano variety

For the sake of completeness, let us also mention the recent results of Addington [2] building upon an observation of Hassett [17], see also [43]. For this consider the Fano variety of lines $F(X)$, which, due to work of Beauville and Donagi [5], is a four-dimensional irreducible holomorphic symplectic variety deformation equivalent to $\mathrm{Hilb}^2(\mathrm{K3})$.

• For a smooth cubic X and its period $x \in D$ the following two conditions are equivalent:
i) $x \in D_d$ such that d satisfies $(**)$;
ii) $F(X)$ is birational to a moduli space of stable sheaves $M(v)$ on some K3 surface S.

• For a smooth cubic X and its period $x \in D$ the following two conditions are equivalent:
iii) $x \in D_d$ such that d divides $2(n^2+n+1)$ for some n;
iv) $F(X)$ is birational to the Hilbert scheme $\mathrm{Hilb}^2(S)$ of some K3 surface S.

Obviously, iv) implies ii) or, equivalently and after a moments thought, iii) implies i).

Proposition 4.1. *For the period x of a smooth cubic X the following two conditions are equivalent:*
i) $x \in D_d$ with d satisfying $(**')$;
ii) $F(X)$ is birational to a moduli space of stable twisted sheaves.

Proof. The argument is an adaptation of Addington's proof [2]. Note however that in the twisted case the transcendental lattice cannot play the same role as in the untwisted case. This was observed in [21], where it was shown that twisted K3 surfaces (S,α), (S',α') with Hodge isometric transcendental lattices, $T(S,\alpha)\simeq T(S',\alpha')$, need not be derived equivalent.

Following Markman [46] for every hyperkähler manifold Y deformation equivalent to $\mathrm{Hilb}^2(S)$ of a K3 surface S there exists a distinguished primitive embedding $H^2(Y,\mathbb{Z})\subset\widetilde{\Lambda}$ orthogonal to a vector $v\in\widetilde{\Lambda}$ with $(v.v)=2$. The Hodge structure of $H^2(Y,\mathbb{Z})$ extends to a Hodge structure on $\widetilde{\Lambda}$ such that v is of type $(1,1)$. Moreover, Y and Y' are birational if and only if there exists a Hodge isometry $H^2(Y,\mathbb{Z})\simeq H^2(Y',\mathbb{Z})$ that extends to a Hodge isometry $\widetilde{\Lambda}\simeq\widetilde{\Lambda}$. For a moduli space $M(v)$ of α-twisted stable sheaves on a K3 surface S with primitive $v\in\widetilde{H}^{1,1}(S,\alpha,\mathbb{Z})$ such that $(v.v)=2$ the universal family induces the distinguished embedding (see [59, Thm. 3.19])

$$H^2(M(v),\mathbb{Z})\simeq v^{\perp}\hookrightarrow\widetilde{H}(S,\alpha,\mathbb{Z}).$$

Similarly, and this is the other crucial input, Addington shows in [2, Cor. 8] that for the Fano variety of lines the universal family of lines induces this distinguished embedding

$$H^2(F(X),\mathbb{Z})\hookrightarrow\widetilde{H}(\mathcal{A}_X,\mathbb{Z})\simeq\widetilde{\Lambda}.$$

Hence, $F(X)$ and $M(v)$ are birational if and only if there exists a Hodge isometry

$$\widetilde{H}(\mathcal{A}_X,\mathbb{Z})\simeq\widetilde{H}(S,\alpha,\mathbb{Z}) \tag{4.1}$$

for some twisted K3 surface (S,α) that restricts to $H^2(F(X),\mathbb{Z})\simeq H^2(M(v),\mathbb{Z})$. Due to Proposition 2.11, the existence of a Hodge isometry (4.1) is equivalent to $x\in D_d$ with d satisfying $(**')$. This proves that ii) implies i).

Conversely, for a Hodge isometry (4.1) consider a primitive vector $v\in\widetilde{H}^{1,1}(S,\alpha,\mathbb{Z})$ in the orthogonal complement of $H^2(F(X),\mathbb{Z})\hookrightarrow\widetilde{H}(\mathcal{A}_X,\mathbb{Z})\simeq\widetilde{H}^2(S,\alpha,\mathbb{Z})$ and the induced moduli space $M(v)$ of stable α-twisted sheaves. Write $v=(r,\ell,s)$. If $r\neq 0$, then for v or $-v$ the moduli space $M(v)$ is indeed non-empty. For $r=0$ observe that $(v)^2>0$ and hence $(\ell)^2>0$. Again by passing to $-v$ if necessary, one can assume that $(\ell.H)>0$ for the polarization H. That the moduli space is non-empty in this case was shown in [60, Cor. 3.5]. (Note that for $r=0$ twisted sheaves can also be considered as untwisted ones.) In [2] the case $r=0$ is dealt with by a reflection associated with $\mathcal{O}$, which does not work in the twisted situation.

To conclude, compose the Hodge isometry $H^2(F(X),\mathbb{Z})\simeq v^{\perp}$, given by the choice of v, with $H^2(M(v),\mathbb{Z})\simeq v^{\perp}$, induced by the universal family as above. By construction, it extends to a Hodge isometry $\widetilde{\Lambda}\simeq\widetilde{\Lambda}$ and, therefore, $F(X)$ and $M(v)$ are birational. □

5. Deformation theory

This section contains two results on the deformation theory of equivalences $\mathrm{D}^{\mathrm{b}}(S,\alpha)\simeq\mathcal{A}_X$ resp. $\mathcal{A}_{X'}\simeq\mathcal{A}_X$ that are crucial for the main results of the paper. The techniques have been developed by Toda [53], Huybrechts–Macrì–Stellari [26], Huybrechts–Thomas [27], and in the present setting by Addington–Thomas [1]. We follow [1] quite closely and often only indicate the additional difficulties and how to deal with them.

5.1. We first consider FM-equivalences $\mathcal{A}_{X'} \simeq \mathcal{A}_X$ between the K3 categories of two cubics X and X' and study under which condition they deform sideways with X and X'.

Theorem 5.1. *Consider two families of smooth cubics $\mathcal{X}, \mathcal{X}' \longrightarrow T$ with distinguished fibres $X := \mathcal{X}_0$ resp. $X' := \mathcal{X}'_0$. Assume $\Phi\colon \mathcal{A}_{X'} \xrightarrow{\sim} \mathcal{A}_X$ is a FM-equivalence inducing a Hodge isometry $\varphi\colon \widetilde{H}(\mathcal{A}_{X'}, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ that remains a Hodge isometry $\varphi_t\colon \widetilde{H}(\mathcal{A}_{\mathcal{X}'_t}, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{\mathcal{X}_t}, \mathbb{Z})$ under parallel transport for all $t \in T$.*

Then Φ deforms sideways to FM-equivalences $\Phi_t\colon \mathcal{A}_{\mathcal{X}'_t} \xrightarrow{\sim} \mathcal{A}_{\mathcal{X}_t}$ for all t in a Zariski open neighbourhood $0 \in U \subset T$.

Proof. The argument is a variant of the deformation theory in [1]. We only indicate the necessary modifications.

As by assumption Φ is a FM-equivalence, the composition

$$\Phi_P\colon \mathrm{D^b}(X') \longrightarrow \mathcal{A}_{X'} \xrightarrow[\Phi]{\sim} \mathcal{A}_X \hookrightarrow \mathrm{D^b}(X)$$

is a FM-functor with some kernel $P \in \mathrm{D^b}(X' \times X)$. It suffices to show that P deforms to $P_t \in \mathrm{D^b}(\mathcal{X}'_t \times \mathcal{X}_t)$ for t in some open neighbourhood $0 \in U \subset T$, because the conditions for Φ_{P_t} to factorize via a functor $\Phi_t\colon \mathcal{A}_{\mathcal{X}'_t} \longrightarrow \mathcal{A}_{\mathcal{X}_t}$ and for this functor Φ_t to define an equivalence are both Zariski open. Indeed, Φ_t takes values in $\mathcal{A}_{\mathcal{X}_t}$ if and only if its composition with the projection $\mathrm{D^b}(\mathcal{X}_t) \longrightarrow {}^\perp\mathcal{A}_{\mathcal{X}_t} = \langle \mathcal{O}_{\mathcal{X}_t}, \mathcal{O}_{\mathcal{X}_t}(1), \mathcal{O}_{\mathcal{X}_t}(2)\rangle$ is trivial. The composition, however, is again of FM-type and the vanishing of a FM-kernel is a Zariski open condition. Similarly, whether Φ_t is an equivalence can be detected by composing it with its adjoints and then checking whether the natural map to the kernel of the identity is an isomorphism, again a Zariski open condition.

The crucial part is to understand the first order deformations, the higher order obstructions are dealt with by the T^1-lifting property, see [1, Sec. 7.2] and [26, Sec. 3.2]. First note that by results of Kuznetsov [34] one has

$$HH^*(\mathcal{A}_{X'}) \simeq \mathrm{Ext}^*_{X' \times X}(P, P) \simeq HH^*(\mathcal{A}_X).$$

This allows one to compare the first order deformations

$$\kappa_{X'} \in H^1(T_{X'}) \subset HH^2(X') \text{ and } \kappa_X \in H^1(T_X) \subset HH^2(X)$$

corresponding to some tangent vector $v \in T_0$ of T at 0. Due to a result of Toda [53] (cf. [1, Thm. 7.1]) it suffices to show that under $HH^2(X') \longrightarrow \mathrm{Ext}^2_{X' \times X}(P, P)$ resp. $HH^2(X) \longrightarrow \mathrm{Ext}^2_{X' \times X}(P, P)$

the classes $\kappa_{X'}$ and κ_X are mapped to the same class. For this consider the following diagram (cf. [1, Prop. 6.2])

$$
\begin{tikzcd}
HH_2(X') \arrow[r, "\sim"] \arrow[d, "\kappa_{X'}"'] \arrow[dr, phantom, "(1)"] & HH_2(\mathcal{A}_{X'}) \arrow[r, "\sim", "\Phi^{HH_*}"'] \arrow[d, "\alpha"] \arrow[dr, phantom, "(2)"] & HH_2(\mathcal{A}_X) \arrow[r, "\sim"] \arrow[d, "\bar{\kappa}_X"] \arrow[dr, phantom, "(3)"] & HH_2(X) \arrow[d, "\kappa_X"] \\
HH_0(X') \arrow[r, two heads] \arrow[d] & HH_0(\mathcal{A}_{X'}) \arrow[r, "\sim", "\Phi^{HH_*}"'] \arrow[dr, phantom, "(4)"] & HH_0(\mathcal{A}_X) \arrow[r, hook] & HH_0(X) \arrow[d] \\
H^*(X') \arrow[rrr, "\Phi_P^H"'] \arrow[dr] & & & H^*(X) \\
 & \widetilde{H}^*(\mathcal{A}_{X'}) \arrow[r, "\sim", "\varphi"'] & \widetilde{H}^*(\mathcal{A}_X) \arrow[ur] &
\end{tikzcd}
$$

By $H^*(X) \simeq HH_*(X)$ we denote the HKR-isomorphism (see [12]) post-composed with $\sqrt{\mathrm{td}} \wedge (\ )$ and, so in particular, $HH_2(X) \simeq H^1(\Omega_X^3)$ with chosen generator σ_X. Similarly for X', where we choose the generator $\sigma_{X'} \in H^1(\Omega_{X'}^3) \simeq HH_2(X')$ such that its image yields σ_X. Furthermore, $\bar{\kappa}_X$ denotes the image of κ_X under the projection $HH^2(X) \longrightarrow HH^2(\mathcal{A}_X)$, see [34], and $\alpha := \Phi^{HH^*}(\bar{\kappa}_X)$.

We aim at showing that (1) is commutative. For this note first that (4) is induced by the FM-transform $\Phi_P \colon \mathrm{D^b}(X') \longrightarrow \mathrm{D^b}(X)$ and hence commutative due to [44]. The commutativity of (2) is obvious, as Hochschild (co)homology is respected by equivalences, and commutativity of (3) is the analogue of [1, Prop. 6.1]. (Recall that Φ_P does not necessarily induce a map $\Phi_P^{HH^*}$, as it is not fully faithful.)

The first order version of the assumption on the Hodge isometry φ is the statement that the diagram

$$
\begin{tikzcd}
H^1(T_{X'}) \arrow[d, hook, "\sigma_{X'}"] & T_0 \arrow[l] \arrow[r] & H^1(T_X) \arrow[d, hook, "\sigma_X"] \\
H^{2,2}(X') \arrow[d, hook] & & H^{2,2}(X) \arrow[d, hook] \\
\widetilde{H}^{1,1}(\mathcal{A}_{X'}) \arrow[rr, "\sim", "\varphi"'] & & \widetilde{H}^{1,1}(\mathcal{A}_X)
\end{tikzcd}
$$

is commutative. (Note that φ might not map $H^{2,2}(X')$ into $H^{2,2}(X)$, i.e. it is in general different from Φ_P^H.) Using the ring-module isomorphism $(HH^*, HH_*) \simeq (H^*(\bigwedge^* T), H^*(\Omega^*))$ for X', this implies that the image in $H^*(X')$ of $\sigma_{X'} \in HH_2(X')$ under contraction with $\kappa_{X'}$ is mapped to the image of σ_X under contraction with κ_X. As $HH_2(X')$ is one-dimensional, this shows that also (1) is commutative.

Therefore, in the diagram

$$\begin{tikzcd}
HH^2(X') \arrow[r] \arrow[d, "\sigma_{X'}"'] & HH^2(\mathcal{A}_{X'}) \arrow[r, "\sim"] \arrow[d, hook, "\sigma_{\mathcal{A}_{X'}}"] \arrow[dr, phantom, "\circlearrowleft"] & HH^2(\mathcal{A}_X) \arrow[d, hook, "\sigma_{\mathcal{A}_X}"] \\
HH_0(X') \arrow[r] & HH_0(\mathcal{A}_{X'}) \arrow[r, "\sim"] & HH_0(\mathcal{A}_X)
\end{tikzcd}$$

the image of $\kappa_{X'} \in HH^2(X')$ under the two compositions $HH^2(X') \longrightarrow HH_0(\mathcal{A}_{X'}) \simeq HH_0(\mathcal{A}_X)$ coincide. As the contraction $HH^2(\mathcal{A}) \hookrightarrow HH_0(\mathcal{A})$ is injective (as for K3 surfaces), this implies that the image of $\kappa_{X'}$ under $HH^2(X') \longrightarrow HH^2(\mathcal{A}_X)$ is indeed $\bar{\kappa}_X$ as claimed.

As in [1], the deformation of P to first and then, by T^1-lifting property, to higher order is unique, for $\mathrm{Ext}^1_{X' \times X}(P, P) \simeq HH^1(\mathcal{A}_X) = 0$ by [34]. □

5.2. We now come to the more involved situation of equivalences $\mathrm{D^b}(S, \alpha) \simeq \mathcal{A}_X$ and their deformations.

Theorem 5.2. *Consider two families $\mathcal{X}, \mathcal{S} \longrightarrow T$ of smooth cubics resp. K3 surfaces with distinguished fibres $X := \mathcal{X}_0$ resp. $S := \mathcal{S}_0$ and let $\alpha_t \in \mathrm{Br}(\mathcal{S}_t)$ be a deformation of a Brauer class $\alpha := \alpha_0$ on S. Assume $\Phi\colon \mathrm{D^b}(S, \alpha) \xrightarrow{\sim} \mathcal{A}_X$ is an equivalence inducing a Hodge isometry $\varphi\colon \widetilde{H}(S, \alpha, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ that remains a Hodge isometry $\varphi_t\colon \widetilde{H}(\mathcal{S}_t, \alpha_t, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{\mathcal{X}_t}, \mathbb{Z})$ under parallel transport for all $t \in T$.*

Then Φ deforms sideways to equivalences $\Phi_t\colon \mathrm{D^b}(\mathcal{S}_t, \alpha_t) \xrightarrow{\sim} \mathcal{A}_{\mathcal{X}_t}$ for all t in a Zariski open neighbourhood $0 \in U \subset T$.

Proof. Let us fix representatives $\alpha_t = \{\alpha_{t,ijk}\}$ for the Brauer classes on $\mathcal{S}_t$ and a family E_t of locally free $\{\alpha_{t,ijk}\}$-twisted sheaves on the fibres $\mathcal{S}_t$ in a Zariski open neighbourhood of $0 \in U \subset T$.

The proof now consists of copying [1, Sec. 6, 7]. However, the techniques have to be adapted to the twisted case, which sometimes causes additional problems as certain fundamental issues related to Hochschild (co)homology have not been addressed in the twisted setting. For certain parts we choose ad hoc arguments to reduce to the untwisted case, for others we rely on Reinecke [51].

Section 6 in [1] deals with Hochschild (co)homology. For a twisted variety (Z, α) one defines $HH^n(Z, \alpha) := \mathrm{Ext}^n_{(Z,\alpha^{-1})\times(Z,\alpha)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. Here, $(Z, \alpha^{-1}) \times (Z, \alpha)$ denotes the twisted variety $(Z \times Z, \alpha^{-1} \boxtimes \alpha)$. Note that $\mathcal{O}_\Delta$ is indeed an $(\alpha^{-1} \boxtimes \alpha)$-twisted sheaf. Similarly, one defines $HH_n(Z, \alpha) := \mathrm{Ext}^{d-n}_{(Z,\alpha^{-1})\times(Z,\alpha)}(\Delta_* \omega_Z^{-1}, \mathcal{O}_\Delta)$, where $d = \dim(Z)$. Composition makes $HH_*(Z, \alpha)$ a right $HH^*(Z, \alpha)$-module. Moreover, there are natural isomorphisms

$$\begin{aligned}
HH^n(Z, \alpha) = \mathrm{Ext}^n_{(Z,\alpha^{-1})\times(Z,\alpha)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) &\simeq \mathrm{Ext}^n_Z(\Delta^* \mathcal{O}_\Delta, \mathcal{O}_Z) \\
&\simeq \mathrm{Ext}^n_{Z \times Z}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = HH^n(Z)
\end{aligned}$$

and

$$\begin{aligned} HH_n(Z,\alpha) = \mathrm{Ext}^{d-n}_{(Z,\alpha^{-1})\times(Z,\alpha)}(\Delta_*\omega_Z^{-1},\mathcal{O}_\Delta) &\simeq \mathrm{Ext}^{d-n}_Z(\mathcal{O}_Z,\Delta^*\mathcal{O}_\Delta)\\ &\simeq \mathrm{Ext}^{d-n}_{Z\times Z}(\Delta_*\omega_Z^{-1},\mathcal{O}_\Delta) = HH_n(Z). \end{aligned}$$

In particular, the HKR-isomorphisms post-composed with $\mathrm{td}^{-1/2}\lrcorner(\ )$ resp. $\mathrm{td}^{1/2}\wedge(\ )$ yield isomorphisms

$$I\colon HH^n(Z,\alpha) \xrightarrow{\sim} \bigoplus_{i+j=n} H^i(\Lambda^j T_Z) \text{ and } I\colon HH_n(Z,\alpha) \xrightarrow{\sim} \bigoplus_{j-i=n} H^i(\Omega_Z^j).$$

Note that these isomorphisms are again compatible with the ring and module structures on both sides, which follows from the fact that the isomorphisms $HH^*(Z,\alpha)\simeq HH^*(Z)$ and $HH_*(Z,\alpha)\simeq HH_*(Z)$ are. The latter is a consequence of the functoriality properties of $\Delta_!$, Δ_* and Δ^*.

For a twisted K3 surface (S,α) one has $HH_2(S,\alpha)\simeq H^0(\omega_S)=\mathbb{C}\cdot\sigma_S$ and the following diagram commutes

$$\begin{tikzcd} HH^2(S,\alpha) \arrow[r, "\sim", "I"'] \arrow[d, "\sigma_S"] & H^0(\bigwedge^2 T_S)\oplus H^1(T_S)\oplus H^2(\mathcal{O}_S) \arrow[d, "\lrcorner\sigma_S"] \\ HH_0(S,\alpha) \arrow[r, "\sim", "I"'] & H^{0,0}\oplus H^{1,1}(S)\oplus H^{2,2}(S). \end{tikzcd}$$

Let us now consider the fully faithful functor $\Phi_P\colon \mathrm{D^b}(S,\alpha)\xrightarrow{\sim}\mathcal{A}_X\hookrightarrow\mathrm{D^b}(X)$ between the twisted K3 surface (S,α) and the smooth cubic X, where $P\in\mathrm{D^b}((S,\alpha^{-1})\times X)$. Then as in [1, Sec. 6.1] one obtains natural maps

$$\Phi_P^{HH^*}\colon HH^*(X)\longrightarrow HH^*(S,\alpha) \text{ and } \Phi_P^{HH_*}\colon HH_*(S,\alpha)\longrightarrow HH_*(X)$$

compatible with the module structures, i.e. $\Phi_P^{HH_*}(a)\circ c=\Phi_P^{HH_*}(a\circ\Phi_P^{HH^*}(c))$ for all $a\in HH_*(S,\alpha)$ and $c\in HH^*(X)$. This has been checked by Reinecke in [51, Sec. 4].

The remaining input in the proof of [1, Prop. 6.2] is the commutativity of the untwisted version of the following diagram:

$$\begin{tikzcd} HH_0(S,\alpha) \arrow[r, "\Phi_P^{HH_*}"] \arrow[d, "I^B"', "\wr"] & HH_0(X) \arrow[d, "\wr"', "I"] \\ \bigoplus H^{p,p}(S) \arrow[r, "\Phi_P^{H,B}"] & \bigoplus H^{p,p}(X) \end{tikzcd} \tag{5.1}$$

Note that defining the induced action on cohomology requires the lift of α to a class $B\in H^2(S,\mathbb{Q})$, see [20, 21]. Moreover, the usual HKR isomorphism I post-composed with $\mathrm{td}^{1/2}\wedge(\ )$ needs to be twisted further to $I^B:=\exp(B)\circ I$.

In principle, one could try to prove the commutativity of (5.1) by rewriting the existing untwisted theory, in particular [12, 44], for the twisted situation. Instead, we follow Yoshioka [59] and reduce everything to the untwisted case by pulling back to a Brauer–Severi variety. We briefly review his approach and explain how to apply it to our situation.

Following [59] we pick a locally free $\alpha = \{\alpha_{ijk}\}$-twisted sheaf $E = \{E_i, \varphi_{ij}\}$ on a twisted variety (Z, α) and associate to it the projective bundle $\pi\colon Y := \mathbb{P}(E) \longrightarrow Z$, which naturally comes with a $\pi^*\alpha^{-1}$-twisted line bundle $L := \mathcal{O}_\pi(1)$. The pull-back of any α-twisted sheaf $F = \{F_i, \psi_{ij}\}$ tensored with L then naturally leads to the untwisted sheaf $\tilde{F} := \pi^*F \otimes L$. Analogously, any α^{-1}-twisted sheaf F can be turned into the untwisted sheaf $\pi^*F \otimes L^*$. The construction yields a functor $\mathrm{D^b}(Z, \alpha) \longrightarrow \mathrm{D^b}(Y)$ which in fact defines an equivalence of $\mathrm{D^b}(Z, \alpha)$ with a full subcategory

$$\tilde{(\ )}\colon \mathrm{D^b}(Z, \alpha) \xrightarrow{\sim} \mathrm{D^b}(Y/Z) \subset \mathrm{D^b}(Y).$$

The construction applied to E itself yields the sheaf $G := \tilde{E}$ that corresponds to the unique non-trivial extension class in $\mathrm{Ext}^1_Y(\mathcal{T}_\pi, \mathcal{O}_Y)$ and $\mathrm{D^b}(Y/Z) \subset \mathrm{D^b}(Y)$ can alternatively be described as the full subcategory of all objects H for which the adjunction map $\pi^*\pi_*(G^* \otimes H) \longrightarrow G^* \otimes H$ is an isomorphism. Analogously, $\mathrm{D^b}(Z, \alpha^{-1})$ is equivalent to the full subcategory of objects H for which $\pi^*\pi_*(G \otimes H) \xrightarrow{\sim} G \otimes H$.

We apply this construction to the twisted K3 surface (S, α) and consider $Y = \mathbb{P}(E) \longrightarrow S$ as above. Assume α is of order r and choose a lift $B = (1/r)B_0$ with $B_0 \in H^2(S, \mathbb{Z})$ of it. The FM-kernel of our given equivalence $\Phi_P\colon \mathrm{D^b}(S, \alpha) \xrightarrow{\sim} \mathcal{A}_X \subset \mathrm{D^b}(X)$, which is an object in $\mathrm{D^b}((S, \alpha^{-1}) \times X)$, is turned into the untwisted sheaf $\tilde{P} := \pi^*P \otimes (L^* \boxtimes \mathcal{O})$ on $Y \times X$. This leads to the commutative diagram

$$\begin{tikzcd}
\mathrm{D^b}(S,\alpha) \arrow[r, "\pi_1^*"] \arrow[d, "\tilde{(\ )}"] & \mathrm{D^b}((S,\alpha)\times X) \arrow[r, "\otimes P"] \arrow[d, "\tilde{(\ )}"] & \mathrm{D^b}(S\times X) \arrow[r, "\pi_{2*}"] \arrow[d, "\pi^*"] & \mathrm{D^b}(X) \arrow[d, "="] \\
\mathrm{D^b}(Y/S) \arrow[r, "\pi_1^*"] & \mathrm{D^b}((Y\times X)/(S\times X)) \arrow[r, "\otimes\tilde{P}"] & \mathrm{D^b}((Y\times X)/(S\times X)) \arrow[r, "\pi_{2*}"] & \mathrm{D^b}(X).
\end{tikzcd}$$

Therefore, the FM-functor $\Phi_P\colon \mathrm{D^b}(S, \alpha) \xrightarrow{\sim} \mathcal{A}_X \subset \mathrm{D^b}(X)$ can be written as the composition $\Phi_P = \Phi_{\tilde{P}} \circ \Phi_Q$ of a twisted FM-functor $\Phi_Q := \tilde{(\ )}$, with $Q = (\mathcal{O}_S \boxtimes L)|_{\Gamma_\pi}$, and an untwisted FM-functor $\Phi_{\tilde{P}}$:

$$\Phi_P\colon \mathrm{D^b}(S, \alpha) \xrightarrow{\Phi_Q} \mathrm{D^b}(Y) \xrightarrow{\Phi_{\tilde{P}}} \mathrm{D^b}(X). \tag{5.2}$$

This allows one to decompose the diagram (5.1) as

$$\begin{tikzcd} HH_0(S,\alpha) \arrow[r, "\Phi_Q^{HH_*}"] \arrow[d, "I^B"', "\wr"] & HH_0(Y) \arrow[r, "\Phi_{\tilde{P}}^{HH_*}"] \arrow[d, "\wr"] & HH_0(X) \arrow[d, "\wr"] \\ \bigoplus H^{p,p}(S) \arrow[r, "\Phi_Q^{H,B}"] & \bigoplus H^{p,p}(Y) \arrow[r, "\Phi_{\tilde{P}}^{H}"] & \bigoplus H^{p,p}(X). \end{tikzcd} \tag{5.3}$$

The right hand square is induced by the usual untwisted FM-functor $\Phi_{\tilde{P}}$ and its commutativity therefore follows from the result of Macrì and Stellari [44, Thm. 1.2]. Hence, it suffices to prove the commutativity of the left hand square (which does not involve the cubic X anymore). For greater clarity we split this further by decomposing Φ_Q as

$$\Phi_Q \colon \mathrm{D}^{\mathrm{b}}(S,\alpha) \xrightarrow{\pi^*} \mathrm{D}^{\mathrm{b}}(Y,\pi^*\alpha) \xrightarrow{L\otimes} \mathrm{D}^{\mathrm{b}}(Y).$$

Let us first consider $\pi^* \colon \mathrm{D}^{\mathrm{b}}(S,\alpha) \longrightarrow \mathrm{D}^{\mathrm{b}}(Y,\pi^*\alpha)$ and the induced diagram

$$\begin{tikzcd} HH_0(S,\alpha) \arrow[r] \arrow[d, "\wr"] \arrow[ddd, bend right=60, "I^B"'] & HH_0(Y,\pi^*\alpha) \arrow[d, "\wr"] \arrow[ddd, bend left=60, "I^{\pi^*B}"] \\ HH_0(S) \arrow[r] \arrow[d, "I"'] & HH_0(Y) \arrow[d, "I"] \\ H^*(S) \arrow[r, "\pi^*"] \arrow[d, "\exp(B)"] & H^*(Y) \arrow[d, "\exp(\pi^*B)"'] \\ H^*(S) \arrow[r, "\pi^*"'] & H^*(Y). \end{tikzcd}$$

Note that the usual π^* on the bottom is indeed the map on cohomology induced by the functor $\pi^* \colon \mathrm{D}^{\mathrm{b}}(S,\alpha) \longrightarrow \mathrm{D}^{\mathrm{b}}(Y,\pi^*\alpha)$ which a priori depends on the choice of the lifts of α and $\pi^*\alpha$ to classes in $H^2(S,\mathbb{Q})$ resp. $H^2(Y,\mathbb{Q})$ for which we choose B resp. π^*B. The commutativity of the upper and the lower squares is trivial. The commutativity of the middle square is an easy case of [44, Thm. 1.2]. Next consider $\Psi := L \otimes (\ ) \colon \mathrm{D}^{\mathrm{b}}(Y,\alpha) \longrightarrow \mathrm{D}^{\mathrm{b}}(Y)$ and the induced diagram

$$\begin{tikzcd}
HH_0(Y,\pi^*\alpha) \arrow[r, "\Psi^{HH_*}"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & HH_0(Y) \arrow[d, "="] \\
HH_0(Y) \arrow[r, "\psi"] \arrow[d, "I"'] & HH_0(Y) \arrow[ddd, "I"] \\
H^*(Y) \arrow[d, "\exp(\pi^*B)"'] & \\
H^*(Y) \arrow[r, "\Psi^{H,\pi^*B}"] & H^*(Y).
\end{tikzcd}$$

By definition, Ψ^{H,π^*B} is given by multiplication with $\mathrm{ch}^{\pi^*(-B)}(L) = \exp(-\pi^*B)\cdot\exp(\mathrm{c}_1(L))$. Here, use that L^r is an untwisted line bundle and define $\mathrm{c}_1(L) := (1/r)\mathrm{c}_1(L^r) \in H^{1,1}(Y,\mathbb{Q})$. See [21, Sec. 1] for the conventions concerning twisted Chern classes. In particular, $\Psi^{H,\pi^*B} \circ \exp(\pi^*B) = \exp(\mathrm{c}_1(L))$ and, therefore, it suffices to prove the commutativity of the diagram

$$\begin{tikzcd}
HH_0(Y) \arrow[r, "\psi"] \arrow[d, "I"'] & HH_0(Y) \arrow[d, "I"] \\
H^*(Y) \arrow[r, "\exp(\mathrm{c}_1(L))"] & H^*(Y),
\end{tikzcd} \tag{5.4}$$

which no longer depends on B and is a special case of Lemma 5.3 below.

This concludes the proof of the commutativity of the diagram (5.1) and hence of [1, Prop. 6.2] in our twisted setting. More precisely, if a first order deformation of X in D_d given by a class $\kappa_X \in H^1(T_X)$ corresponds via the interpretation of D_d as period domain for X and S to a first order deformation $\kappa_S \in H^1(T_S)$, then $\Phi^{HH^2} \colon HH^2(X) \longrightarrow HH^2(S,\alpha)$ sends κ_X to κ_S.

To conclude the proof one has to prove that the kernel $P \in \mathrm{D}^{\mathrm{b}}((S,\alpha^{-1}) \times X)$ deforms sideways, for which we again apply Yoshioka's untwisting technique. Instead of attempting to deform the twisted P sideways with $(S,\alpha) \times X$ we deform the untwisted $\tilde{P}$. As the condition describing the full subcategory $\mathrm{D}^{\mathrm{b}}((S,\alpha^{-1}) \times X) \simeq \mathrm{D}^{\mathrm{b}}((Y \times X)/(S \times X)) \subset \mathrm{D}^{\mathrm{b}}(Y \times X)$ is open, any deformation of $\tilde{P}$ will automatically induce a deformation of P.[7] The decomposition

[7] This is confirmed by the observation that under the natural isomorphisms

$$\mathrm{Ext}^2_{(S,\alpha^{-1})\times X}(P,P) \simeq \mathrm{Ext}^2_{(Y,\pi^*\alpha^{-1})\times X}(\pi^*P,\pi^*P) \simeq \mathrm{Ext}^2_{Y\times X}(\tilde{P},\tilde{P})$$

the obstruction $o(P) \in \mathrm{Ext}^2_{(S,\alpha^{-1})\times X}(P,P)$ to deform P sideways to first order is first mapped to $o(\pi^*P)$ and then to $o(\tilde{P}) - \mathrm{id}_{\pi^*P} \otimes o(\mathcal{O}_\pi(-1))$. The latter, however, equals the obstruction $o(\tilde{P}) \in \mathrm{Ext}^2_{Y\times X}(\tilde{P},\tilde{P})$ for $\tilde{P}$, because $\mathcal{O}_\pi(-1)$ clearly deforms sideways.

(5.2) leads to a diagram

$$\begin{tikzcd} HH^2(X)=\mathrm{Ext}^2_{X\times X}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X}) \arrow[r] & \mathrm{Ext}^2_{Y\times X}(\tilde{P},\tilde{P}) \arrow[r, "\sim"] & \mathrm{Ext}^2_{(S,\alpha^{-1})\times X}(P,P) \\ & \mathrm{Ext}^2_{Y\times Y}(\mathcal{O}_{\Delta_Y},\mathcal{O}_{\Delta_Y}) \arrow[u] \arrow[r, dashed] \arrow[d, equal] & \mathrm{Ext}^2_{(S,\alpha^{-1})\times(S,\alpha)}(\mathcal{O}_{\Delta_S},\mathcal{O}_{\Delta_S}) \arrow[u, "\wr"'] \arrow[d, equal] \\ & HH^2(Y) \arrow[r, dashed] & HH^2(S,\alpha). \end{tikzcd}$$

Recall that $\Phi_R^{HH_*}$ is defined for any FM-functor Φ_R, whereas in order to define $\Phi_R^{HH^*}$ one needs Φ_R to be fully faithful, which is the case for Φ_P and $\Phi_Q = (\tilde{\ })$. So, both maps in

$$\begin{array}{ccccc} HH^2(X) \longrightarrow & HH^2(S,\alpha) & \longleftarrow HH^2(Y) \\ \kappa_X \longmapsto & \kappa_S & \longleftarrow\!\shortmid \quad \kappa_Y \end{array}$$

are well defined, where as above $\kappa_X \in H^1(T_X) \subset HH^2(X)$ corresponds to $\kappa_S \in H^1(T_S) \subset HH^2(S,\alpha)$ (via their periods or, equivalently, via Φ^{HH^2}) and κ_Y is determined by our pre-chosen deformation E_t of E.

Now by [27] the obstruction $o(\tilde{P})$ can be expressed as

$$o(\tilde{P}) = (\kappa_Y, \kappa_X) \circ \mathrm{At}(\tilde{P}).$$

(Unfortunately, an analogous formula in the twisted case is not available.) The crucial [1, Thm. 7.1], which goes back to Toda [53], proves that in the untwisted case $o(P) = 0$ if κ_X is mapped to κ_S under $HH^2(X) \longrightarrow HH^2(S)$. However, in the twisted situation one has to face the additional problem that there is no natural map $HH^2(X) \longrightarrow HH^2(Y)$. Nevertheless, the argument in [1] goes through essentially unchanged as follows. Using the same notation, one writes

$$o(\tilde{P}) = \pi_1^*\kappa_Y \circ \mathrm{At}_Y(\tilde{P}) + \pi_2^*\kappa_X \circ \mathrm{At}_X(\tilde{P}) \in \mathrm{Ext}^2(\tilde{P},\tilde{P}).$$

The first term is the image of $\pi_1^*\kappa_Y \circ \mathrm{At}_1(\mathcal{O}_{\Delta_Y}) \in \mathrm{Ext}^2(\mathcal{O}_{\Delta_Y},\mathcal{O}_{\Delta_Y}) = HH^2(Y)$ which is just κ_Y, whereas the second one is the image of $-\pi_1^*\kappa_X \circ \mathrm{At}_2(\mathcal{O}_{\Delta_X}) \in \mathrm{Ext}^2(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X}) = HH^2(X)$ which is just $-\kappa_X$. Hence, to compare κ_X and κ_Y we do not need a map $HH^2(X) \longrightarrow HH^2(Y)$ (which simply does not exist naturally), as we only need to compare their images in $\mathrm{Ext}^2(\tilde{P},\tilde{P}) \simeq HH^2(S,\alpha)$. Therefore it suffices to ensure that under $HH^2(X) \longrightarrow HH^2(S,\alpha)$ the class κ_X is mapped to κ_S, which was verified above.

This concludes the argument proving that the FM-kernel P deforms to first oder with $(S,\alpha)\times X$. The arguments in [1, Sec. 7.2] proving the existence of deformations of P to all orders apply verbatim. Note that at the very end of the argument one needs to apply a result of Lieblich saying that the space of objects with no negative self-Exts in the derived category is an Artin stack of locally finite presentation. Again, the result as such does not seem to be in the literature for the twisted situation, but once again it can be deduced from the untwisted case by Yoshioka's trick. □

It remains to check the commutativity of (5.4) which is a general fact. Consider a smooth variety Z and $\alpha_{ijk} := \beta_{ij} \cdot \beta_{jk} \cdot \beta_{ki}$ with $\beta_{ij} \in \mathcal{O}^*_{U_{ij}}$. The associated Brauer class $\alpha \in H^2(Z, \mathcal{O}^*_Z)$ is of course trivial and hence $\mathrm{D^b}(Z, \{\alpha_{ijk}\})$ and $\mathrm{D^b}(Z)$ are equivalent categories and an explicit equivalence can be given by 'untwisting by $\{\beta_{ij}\}$', i.e. by $E = \{E_i, \varphi_{ij}\} \longmapsto \{E_i, \varphi_{ij} \cdot \beta_{ij}^{-1}\}$. Note that changing β_{ij} by a cocycle $\{\delta_{ij}\}$, which would correspond to an untwisted line bundle say M, the equivalence would be modified by $M^* \otimes (\ )$.

Assume furthermore that $\alpha^r_{ijk} = 1$. Then $\{\beta^r_{ij}\}$ is a cocycle defining a line bundle H and we define $\mathrm{c}_1(\beta) := (1/r)\mathrm{c}_1(H) \in H^{1,1}(Z)$. Explicitly, $\mathrm{c}_1(\beta) = \{d \log \beta_{ij}\}$.

Lemma 5.3. *The 'untwisting by $\{\beta_{ij}\}$', i.e. the equivalence*

$$\Phi \colon \mathrm{D^b}(Z, \{\alpha_{ijk}\}) \xrightarrow{\sim} \mathrm{D^b}(Z),\ E = \{E_i, \varphi_{ij}\} \longmapsto \{E_i, \varphi_{ij} \cdot \beta_{ij}^{-1}\},$$

induces a commutative diagram

$$\begin{tikzcd} HH_*(Z, \{\alpha_{ijk}\}) \arrow[r, "\Phi^{HH_*}"] \arrow[d, "\mathrm{HKR}"'] & HH_*(Z) \arrow[d, "\mathrm{HKR}"] \\ H^*(Z) \arrow[r, "\exp(\mathrm{c}_1(\beta))"'] & H^*(Z) \end{tikzcd}$$

The commutativity of (5.4) then follows from the observation that $L \otimes (\ )$ can be written as the composition of the 'untwisting by $\{\beta_{ij}\}$' as above with the equivalence $\mathcal{L} \otimes (\ )$. Here, $\mathcal{L}$ is the untwisted line bundle given by $\{\psi_{ij} \cdot \beta_{ij}\}$, where L itself is the $\{\alpha_{ijk}^{-1}\}$-twisted line bundle given by $\{\psi_{ij}\}$.

Indeed, for $\Psi := \mathcal{L} \otimes (\ ) \colon \mathrm{D^b}(Z) \xrightarrow{\sim} \mathrm{D^b}(Z)$ the commutativity of

$$\begin{tikzcd} HH_*(Z) \arrow[r, "\Psi^{HH_*}"] \arrow[d, "\mathrm{HKR}"'] & HH_*(Z) \arrow[d, "\mathrm{HKR}"] \\ H^*(Z) \arrow[r, "\exp(\mathrm{c}_1(\mathcal{L}))"'] & H^*(Z) \end{tikzcd}$$

is an easy special case of [44, Thm. 1.2][8], which can be proved by a direct calculation. The proof of the lemma is a variant of this computation.

Proof. Consider the universal Atiyah class $\mathrm{At} \colon \mathcal{O}_\Delta \longrightarrow \Delta_*\Omega_Z[1]$. Twisted with a line bundle of the form $M \boxtimes M^*$ it yields a map $\mathrm{At}_M \colon \mathcal{O}_\Delta \longrightarrow \Delta_*\Omega_Z[1]$. The usual formula $\mathrm{c}_1(E \otimes M) = \mathrm{c}_1(E) + \mathrm{rk}\, E \cdot \mathrm{c}_1(M)$ corresponds to the universal formula $\mathrm{At}_M = \alpha + \Delta_*\mathrm{c}_1(M)$, which can

[8]Note that $\mathrm{td}^{1/2} \wedge$ can be dropped here and in the lemma, as it commutes with $\exp(\mathrm{c}_1(\mathcal{L}))$.

be checked by using arguments of [9, 10] or a direct cocycle computation. Here, $\mathrm{c}_1(M)$ is viewed as a map $\mathcal{O}_Z \longrightarrow \Omega_Z[1]$ which can be pushed forward via Δ. Similarly, the exponential $\exp(\mathrm{At})\colon \mathcal{O}_\Delta \longrightarrow \bigoplus \Delta_*\Omega_Z^i[i]$ (see [12]) twisted with $M \boxtimes M^*$ is given by $\exp(\mathrm{At})_M = \Delta_* \exp(\mathrm{c}_1(M)) \circ \exp(\mathrm{At})$.

Let now $f \in HH_j(Z) = \mathrm{Ext}^j_{Z\times Z}(\Delta_!\mathcal{O}_Z, \Delta_*\mathcal{O}_Z)$ and denote by $F \in \mathrm{Ext}^j_Z(\mathcal{O}_Z, \Delta^*\Delta_*\mathcal{O}_Z)$ its image under $\mathrm{Ext}^j_{Z\times Z}(\Delta_!\mathcal{O}_Z, \Delta_*\mathcal{O}_Z) \simeq \mathrm{Ext}^j_Z(\mathcal{O}_Z, \Delta^*\Delta_*\mathcal{O}_Z)$. So if $\eta\colon (\Delta_!\Delta^*)\Delta_*\mathcal{O}_Z \longrightarrow \Delta_*\mathcal{O}_Z$ denotes adjunction, then $f = \eta \circ \Delta_! F$. Due to [12, Prop. 4.4], the latter is under the HKR isomorphism given by $\exp(\mathrm{At})$, so

$$\eta\colon (\Delta_!\Delta^*)\Delta_*\mathcal{O}_Z \simeq \bigoplus \Delta_*(\Omega^i[i] \otimes \omega_Z^{-1}[-d])) \simeq \bigoplus \Delta_*(\Omega_Z^{d-i})^*[i-d] \xrightarrow{\exp(\mathrm{At})} \Delta_*\mathcal{O}_Z.$$

The image of f under $\mathcal{L} \otimes (\ )$ is given by tensoring with $\mathcal{L} \boxtimes \mathcal{L}^*$. The push-forward $\Delta_! F$ remains unchanged by tensoring with $\mathcal{L} \boxtimes \mathcal{L}^*$ and by the above η changes by composing with $\Delta_* \exp(\mathrm{c}_1(\mathcal{L}))$.

Literally the same argument applies to the untwisting by $\{\beta_{ij}\}$ for which one has to observe that the universal Atiyah class $\mathrm{At}\colon \mathcal{O}_\Delta \longrightarrow \Delta_*\Omega_Z[1]$ on $(Z, \{\alpha_{ijk}^{-1}\}) \times (Z, \{\alpha_{ijk}\})$ under untwisting by $\{\beta_{ij}\}$ becomes $\mathrm{At} + \Delta_*\mathrm{c}_1(\beta)\colon \mathcal{O}_\Delta \longrightarrow \Delta_*\Omega_Z[1]$ on $Z \times Z$. $\square$

6. Proofs

6.1. **Proof of Theorem 1.2.** According to Corollary 3.12, for every smooth cubic $X \subset \mathbb{P}^5$ there exists a distinguished FM-autoequivalence $\Phi_0\colon \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_X$ of infinite order which acts as the identity on $T(\mathcal{A}_X)$, so it is symplectic, and such that Φ_0^3 is the double shift $E \longmapsto E[2]$. We have to show that for the general cubic every symplectic FM-equivalence Φ is a power of Φ_0.

As $\widetilde{H}^{1,1}(\mathcal{A}_X, \mathbb{Z}) \simeq A_2$ for general X and $\Phi^H = \mathrm{id}$ on $T(\mathcal{A}_X) = A_2^\perp$, the induced action Φ^H is contained in $\mathrm{O}(A_2)$. Clearly, any Hodge isometry of $\widetilde{H}(\mathcal{A}_X, \mathbb{Z})$ that is the identity on $A_2^\perp$ stays a Hodge isometry for all deformations of X. Therefore, applying Theorem 5.1, Φ deforms to FM-autoequivalences $\Phi_t\colon \mathcal{A}_{\mathcal{X}_t} \simeq \mathcal{A}_{\mathcal{X}_t}$ for cubics $\mathcal{X}_t$ in a Zariski open neighbourhood $U \subset \mathcal{C}$ of X inside the moduli space of smooth cubics.

Then for all but finitely many d satisfying $(**)$ the intersection $U \cap \mathcal{C}_d$ is non-empty (and open) and, therefore, by [1, Thm. 1.1] there exists $t \in U$ such that $\mathcal{A}_{\mathcal{X}_t} \simeq \mathrm{D^b}(S)$ for some K3 surface S. Due to [26, Thm. 2], autoequivalences of $\mathrm{D^b}(S)$ are orientation preserving and hence $\Phi^H \in \mathfrak{A}_3 \simeq \mathbb{Z}/3\mathbb{Z}$, cf. Remark 2.1. Thus, by composing with some power of Φ_0, we may assume that $\Phi^H = \mathrm{id}$.

Now apply Corollary 2.16 and Theorem 1.4 to be proved below to conclude that there exists $t \in U$ such that $\mathcal{A}_{\mathcal{X}_t} \simeq \mathrm{D^b}(S, \alpha)$ for a twisted K3 surface (S, α) not admitting any (-2)-class. Indeed,

$$(\mathcal{C}_{\mathrm{K3}'} \cap U) \backslash \mathcal{C}_{\mathrm{sph}} \neq \varnothing,$$

where $\mathcal{C}_{\mathrm{K3}'} := \bigcup_{(**)} \mathcal{C}_d \subset \mathcal{C}$ and $\mathcal{C}_{\mathrm{sph}} \subset \mathcal{C}$ is the image of D_{sph}. By [24, Thm. 2], we know that then Φ_t is isomorphic to an even shift $E \longmapsto E[2k]$. It is easy to see that k is independent of t.

The locus of points $U_0 \subset U$ such that $\Phi_t \simeq [2k]$ for $t \in U_0$ is Zariski open and by the above non-empty. Therefore, for every $X \in \mathcal{C}$ in the intersection of all $U_0 \subset \mathcal{C}$ the assertion holds. But this intersection is certainly countable, as FM-kernels are parametrized by countably many products of Quot-schemes.

Now consider a non-special cubic X, i.e. $X \in \mathcal{C} \setminus \bigcup \mathcal{C}_d$, and an arbitrary $\Phi \in \mathrm{Aut}(\mathcal{A}_X)$. By composing with the shift functor [1] if necessary, we may assume that Φ^H acts trivially on the discriminant group $A_{A_2} \simeq A_{A_2^\perp}$. But then the induced Hodge isometry of $T(\mathcal{A}_X) \simeq A_2^\perp$ extends to a Hodge isometry of $H^4(X,\mathbb{Z})$ that respects h. By the Global Torelli theorem [57, 42, 15] it is therefore induced by an automorphism $f \in \mathrm{Aut}(X)$, which clearly acts trivially on the orthogonal complement of $h^\perp \subset H^*(X,\mathbb{Z})$ and hence as the identity on $A_2 \subset \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$. Moreover, since f respects $H^{3,1}(X)$, the action of f in $\widetilde{H}(\mathcal{A}_X,\mathbb{Z})$ preserves the orientation.

So, composing, if necessary, Φ with the shift functor and an automorphism, we reduce to the case $\Phi \in \mathrm{Aut}_s(\mathcal{A}_X)$. As X is non-special, i.e. $A_2 \simeq \widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$, we can deform Φ sideways as above until it can be interpreted as an autoequivalence of a category of the form $\mathrm{D^b}(S)$, which implies that it is orientation preserving. This eventually proves that for every non-special cubic the image of $\rho\colon \mathrm{Aut}(\mathcal{A}_X) \longrightarrow \mathrm{Aut}(\widetilde{H}(\mathcal{A}_X,\mathbb{Z}))$ is the subgroup of orientation preserving Hodge isometries. □

Remark 6.1. We expect the first assertion in Theorem 1.2 to hold for every non-special cubic, i.e. for all $X \in \mathcal{C} \setminus \bigcup \mathcal{C}_d$, but this would require to show that if $\Phi \in \mathrm{Aut}(\mathcal{A}_X)$ deforms to the identity functor and $\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z}) \simeq A_2$, then $\Phi \simeq \mathrm{id}$. The techniques of [24] should be useful here, but they require the existence of stability conditions.

Furthermore, one would also expect that any $\Phi \in \mathrm{Aut}(\mathcal{A}_X)$ of any cubic preserves the natural orientation.

6.2. **Proof of Theorem 1.4.** Assertion i) follows from Theorem 1.3 and Proposition 3.3. For the converse, fix d satisfying $(**')$. Then for any smooth cubic $X \in \mathcal{C}_d$ there exists a Hodge isometry

$$\varphi\colon \widetilde{H}(S,\alpha,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \tag{6.1}$$

for some twisted K3 surface (S,α). In fact, this Hodge isometry can be chosen globally over the period domain D_d (or some appropriately constructed covering $\tilde{\mathcal{C}}_d$ of $\mathcal{C}_d$, see [1]). The aim is to show that generically this Hodge isometry is induced by an equivalence $\mathcal{A}_X \simeq \mathrm{D^b}(S,\alpha)$ (up to changing the orientation).

The starting point for the argument is [1, Thm. 4.1], which is based on Kuznetsov's work [33] and on the description of the image of the period map for cubic fourfolds due to Laza [39] and Looijenga [42]. Combined, these results show that for every d satisfying $(**')$ (but in fact $(*)$ is enough) there exists a smooth cubic $X \in \mathcal{C}_8 \cap \mathcal{C}_d$, a K3 surface S_0 and an equivalence

$$\Phi_0 \colon \mathcal{A}_X \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S_0).$$

By [1] or Proposition 3.3, any such Φ_0 induces a Hodge isometry $\Phi_0^H \colon \widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S_0, \mathbb{Z})$ (usually completely unrelated to (6.1)). Consider now the composition

$$\psi := \Phi_0^H \circ \varphi \colon \widetilde{H}(S, \alpha, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(S_0, \mathbb{Z}).$$

By modifying φ (globally over D_d) if necessary (use Lemma 2.2.), we may assume that ψ preserves the orientation and then [22] applies and shows that there exists an equivalence $\Psi \colon \mathrm{D}^{\mathrm{b}}(S, \alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S_0)$ with $\Psi^H = \psi$. Then the equivalence

$$\Phi := \Phi_0^{-1} \circ \Psi \colon \mathrm{D}^{\mathrm{b}}(S, \alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S_0) \xrightarrow{\sim} \mathcal{A}_X$$

satisfies $\Phi^H = \varphi$.

We can now forget about S_0 and only keep X and S and the equivalence $\Phi = \Phi_P$ with $P \in \mathrm{D}^{\mathrm{b}}((S, \alpha^{-1}) \times X)$. Then consider the two families $\mathcal{X}_t$ and $(\mathcal{S}_t, \alpha_t)$ over D_d (or rather $\tilde{\mathcal{C}}_d$ in order to use the Zariski topology) of cubics resp. twisted K3 surfaces with $X = \mathcal{X}_0$, $S = \mathcal{S}_0$, for which φ defines Hodge isometries $\widetilde{H}(\mathcal{S}_t, \alpha_t, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{\mathcal{X}_t}, \mathbb{Z})$ for all t. As $\Phi \colon \mathrm{D}^{\mathrm{b}}(S, \alpha) \xrightarrow{\sim} \mathcal{A}_X$ induces φ, Theorem 5.2 applies and shows that Φ can be deformed to equivalences $\Phi_t \colon \mathrm{D}^{\mathrm{b}}(\mathcal{S}_t, \alpha_t) \xrightarrow{\sim} \mathcal{A}_{\mathcal{X}_t}$ for all t in a Zariski open neighbourhood of $0 \in \tilde{\mathcal{C}}_d$. □

6.3. **Proof of Theorem 1.5.** The first assertion of the theorem has been proved already as Corollary 3.6. For ii) and iii) recall that any FM-equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ induces a Hodge isometry $\widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_{X'}, \mathbb{Z})$, cf. Proposition 3.4. So it remains to prove the converse for generic $X \in \mathcal{C}_d$ with d satisfying $(**')$ resp. general $X \in \mathcal{C}_d$ for arbitrary d. The first case is easy, as then, by Theorem 1.4, $\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(S, \alpha)$ and $\mathcal{A}_{X'} \simeq \mathrm{D}^{\mathrm{b}}(S', \alpha')$ for twisted K3 surfaces (S, α) resp. (S', α'). The assertion then follows from [22] and Lemma 2.2.

For the second case consider the correspondence

$$Z := \{(X, X', \varphi) \mid X \in \mathcal{C}_d \text{ and } \varphi \colon \widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{X'}, \mathbb{Z})\}$$

of smooth cubics X, X' with $X \in \mathcal{C}_d$ and a Hodge isometry φ. (Note that with X also $X' \in \mathcal{C}_d$.) This correspondence consists of countably many components $Z_0 \subset Z$ and for the image of a component $Z_0 \subset Z$ under the first projection $\pi \colon Z \longrightarrow \mathcal{C}_d$ one either has $\pi(Z_0) \subset \mathcal{C}_d \cap \bigcup_{d' \neq d} \mathcal{C}_{d'}$ or $\pi(Z_0) \subset \mathcal{C}_d$ is dense (a priori it could happen that one of the two cubics becomes singular without the other one).

Suppose we are in the latter case. Then by [1, Thm. 1.1], cf. Section 6.1, one finds a $(X, X', \varphi) \in Z_0$ for which there exist K3 surfaces S and S' and FM-equivalences

$$\Psi \colon \mathcal{A}_X \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S) \text{ and } \Psi' \colon \mathcal{A}_{X'} \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S'). \tag{6.2}$$

By Proposition 3.3, Ψ and Ψ' induce Hodge isometries Ψ^H resp. Ψ'^H, which composed with φ yield a Hodge isometry

$$\varphi_0\colon \widetilde{H}(S,\mathbb{Z}) \xrightarrow[(\Psi^{-1})^H]{\sim} \widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \xrightarrow[\varphi]{\sim} \widetilde{H}(\mathcal{A}_{X'},\mathbb{Z}) \xrightarrow[\Psi'^H]{\sim} \widetilde{H}(S',\mathbb{Z}).$$

We may assume that φ_0 is orientation preserving and, thus, induced by a FM-equivalence $\Phi_0\colon \mathrm{D^b}(S) \xrightarrow{\sim} \mathrm{D^b}(S')$. Composing the latter with the equivalences (6.2) yields a FM-equivalence $\Phi\colon \mathcal{A}_X \simeq \mathcal{A}_{X'}$ inducing φ. Now use Theorem 5.1 to deform Φ sideways to FM-equivalences $\Phi_t\colon \mathcal{A}_{X_t} \xrightarrow{\sim} \mathcal{A}_{X'_t}$ for all points $(X_t, X'_t, \varphi_t \equiv \varphi)$ in a Zariski dense open subset $U_0 \subset Z_0$.

Hence, for all $X \in \bigcap \pi(U_i)$, with the intersection over all components $Z_0 \subset Z$ (dominating $\mathcal{C}_d$), the existence of a Hodge isometry $\widetilde{H}(\mathcal{A}_X,\mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_{X'},\mathbb{Z})$ implies the existence of a FM-equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$. □

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