Moduli spaces of twisted sheaves and applications

Daniel Huybrechts

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Moduli of sheaves

$X =$ projective variety (over field k), e.g. $X = \mathbb{P}^n \rightsquigarrow \mathrm{Coh}(X)$, $D^{\mathrm{b}}(X)$

Goal: $\exists\, \mathfrak{M}$ & $\tilde{\mathcal{E}}$ s.t. all T-flat $\mathcal{E}$ are pull-backs:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \tilde{\mathcal{E}} \\ \downarrow & & \downarrow \\ X \times T & \longrightarrow & X \times \mathfrak{M} \end{array}$$

Problems and Questions:

1. Boundedness; 2. Hausdorff; 3. Compactness; 4. Geometric description

- Mumford ' 62, Gieseker & Maruyama '77
- Simpson '94 (Le Potier)
- Langer '04
- Halpern-Leistner, Heinloth

- Fix ample $\mathcal{O}(1) \rightsquigarrow$ stability
- Fix c_i or Hilbert polynomial
- Translate into GIT stability
- θ-stability

Example: Picard

C = smooth projective curve

$\rightsquigarrow$ Picard group $\mathrm{Pic}(C) = \{ L \mid \text{line bundle} \} \cong H^1(C, \mathcal{O}_C^*)$

$$\mathrm{Pic}(C) \quad \subset \quad \mathbf{Pic}(C)$$

$$= \bigsqcup \mathrm{Pic}^d(C) \qquad = \bigsqcup \mathbf{Pic}^d(C) \quad \text{varieties}$$

- $k = \mathbb{C}$:
  - $\mathrm{Pic}^0(C) \cong H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z}) \cong \mathbb{C}^g/\Gamma = \mathbf{Pic}^0(C)_{\mathrm{cl}}$ torus
  - $\mathbf{Pic}^d(C) \cong \mathbf{Pic}^0(C)$ not canonical

  Better: torsor $\mathbf{Pic}^0(C) \times \mathbf{Pic}^d(C) \to \mathbf{Pic}^d(C),\ (L, M) \mapsto L \otimes M$

- k = general:
  - $\mathbf{Pic}^0(C)$ is an abelian variety
  - $\mathbf{Pic}^0(C) \ncong \mathbf{Pic}^d(C)$

$$0 \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathbf{Pic}(C)(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(C)$$

Moduli of twisted sheaves

$X =$ projective variety & $\mathcal{A} = \mathcal{O}_X$-algebra (associative but not commutative) & ...

Examples:
- $\mathcal{A} = \mathcal{D}_X$ differential operators; $\mathcal{A} = \mathrm{Sym}^\bullet(\Omega_X)$
- $\mathcal{A} = \mathcal{E}nd(E)$ with E locally free
- $\mathcal{A} =$ Azumaya algebra, i.e. étale locally

Concretely: $L, M \in \mathrm{Pic}(X)$, $L^2 \xrightarrow{\sim} \mathcal{O}_X \xleftarrow{\sim} M^2$ $\qquad L^{\otimes 2} \hookrightarrow \mathcal{O}_X \hookleftarrow M^{\otimes 2}$

$\rightsquigarrow \mathcal{A} = \mathcal{O}_X \oplus L \oplus M \oplus (L \otimes M)$ with $\ell \cdot m = -m \cdot \ell$ and $\ell^2 = 1 = m^2$

i.e. $L \quad \rightsquigarrow \quad \widetilde{X} \xrightarrow{(2:1)} X$ étale and then $\mathcal{A} = \mathcal{O}_{\widetilde{X}} \oplus (\mathcal{O}_{\widetilde{X}} \otimes M)$

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \to \{ \; Azumaya \; \}$$

$$H^1(X,\mu_n) = \{(L, L^n \cong \mathcal{O})\} \qquad (L, M) \mapsto \bigoplus L^i \otimes M^j, \text{ with } \ell\, m = \zeta \cdot m\, \ell,\ \ell^n = 1 = m^n$$

Moduli of twisted sheaves

$\rightsquigarrow$ coherent $\mathcal{A}$-modules: $\mathrm{Coh}(X, \mathcal{A})$

Simpson: $(\mathrm{char} = 0)$ $\exists\, \mathfrak{M} =$ locally projective moduli space of semi-stable coherent $\mathcal{A}$-modules

Special case $\mathcal{A} =$ Azumaya. When is $\mathcal{A} \cong \mathcal{E}nd(E)$ with E locally free?

$\rightsquigarrow$ obstruction $\alpha = \{\alpha_{ijk}\} \in H^2(X, \mathbb{G}_m)$: $\mathrm{Coh}(X, \mathcal{A}) \cong \mathrm{Coh}(X, \{\alpha_{ijk}\})$ 'twisted sheaves'

Lieblich '07, Yoshioka '06, Hoffmann–Stuhler '04, ...

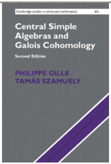
$C =$ smooth projective curve over a field k & Azumaya $\mathcal{A}$

$\rightsquigarrow \mathbf{Pic}_{\mathcal{A}}(C) =$ moduli of locally free $\mathcal{A}$-modules of minimal rank $\mathrm{rk}(\mathcal{A})^{1/2}$

- $k = \bar{k} \;\Rightarrow\; \mathbf{Pic}_{\mathcal{A}}(C) \cong \mathbf{Pic}(C)$
- k general $\;\Rightarrow\; \mathbf{Pic}_{\mathcal{A}}(C)$ is a $\mathbf{Pic}(C)$-torsor

For later: $\mathrm{Pic}_{\mathcal{A}}(C) \neq \emptyset \Rightarrow \mathbf{Pic}_{\mathcal{A}}(C) \cong \mathbf{Pic}(C)$ & $[\mathcal{A}] = 1$

Brauer group: algebraic



$k = K(X)$, $\mathbb{R}$, number field, ... Brauer–Hasse–Noether, Albert, Witt, ...

Cohomology: $k =$ field $\rightsquigarrow H^1(k, \mathbb{G}_m) = 0$ (Hilbert 90)

$$H^2(k, \mathbb{G}_m) = ?$$

Algebraic: $\mathrm{Br}(k) = \{\, A \mid \text{csa}/k \,\}/_\sim = \{\, D \mid \text{division algebra}/k \,\}/_\cong$ with $A \cong M_n(D) \sim D$

$$\cong H^2(k, \mathbb{G}_m) \qquad \text{(csa = Azumaya over } \mathrm{Spec}(k)\text{)}$$

- $K/k \rightsquigarrow \mathrm{Br}(k) \to \mathrm{Br}(K)$, $\alpha = [A] \mapsto \alpha_K = [A \otimes K]$
- $[K : k] < \infty \Rightarrow \mathrm{Br}(K/k) := \mathrm{Ker}(\alpha \mapsto \alpha_K)$ is finite

$$\cong H^2(\mathrm{Gal}(K/k), K^*) \cong k^*/N(K^*) \text{ for cyclic Galois } K/k$$

[ABHN '32]: If k is a number field with $\mu_n \subset k$, then everyone is cyclic!

$$H^1(k, \mu_n) \times H^1(k, \mu_n) \twoheadrightarrow \mathrm{Br}(k)[n], \qquad (\ell, m) \mapsto \bigoplus k \cdot (x^i y^j)$$

$$H^1(k, \mu_n) = k^*/(k^*)^n \qquad \text{with } x\,y = \xi \cdot y\,x,\ x^n = \ell,\ y^n = m.$$

Splitting, period, and index

Splitting field: $\alpha \in \mathrm{Br}(k) \quad \Rightarrow \quad \exists\, K/k$ with $1 = \alpha_K \in \mathrm{Br}(K)$

Classical theory: maximal fields $K \subset D$ split $[D] \in \mathrm{Br}(k)$

Period = exponent = order: $\mathrm{per}(\alpha) := |\alpha|$

Index:

$$\mathrm{ind}(\alpha) := \min\{\, [K : k] \mid \alpha_K = 1 \,\} = \min\{\, \dim(A)^{1/2} \mid \alpha = [A] \,\}$$

Index vs period:

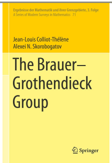
$$\mathrm{per}(\alpha) \mid \mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^N$$

For number fields: $N = 1$

Witt, Amitsur, Roquette, Clark, Saltman, ... :

Use K/k with $\mathrm{trdeg}_k(K) > 0$ to split $\alpha \in \mathrm{Br}(k)$

Brauer group: geometric



Grothendieck '64 - 66 (Bourbaki & 'Dix exposés... '): Brauer I–III

$X =$ projective variety over $k \rightsquigarrow \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \cong H^1(X, \mathbb{G}_m)$

$$H^2(X, \mathbb{G}_m) = ?$$

Grothendieck, Gabber, de Jong: $\mathrm{Br}(X) \cong H^2(X, \mathbb{G}_m)_{\mathrm{tors}}$

$\mathrm{Br}(X) = \{$ Azumaya $\}/_\sim$ Morita equivalence: $\mathcal{A} \sim \mathcal{A} \otimes \mathcal{E}nd(E)$

$\big\downarrow$ X factorial

$\mathrm{Br}(K(X))$

Recall: $L^{\otimes 2} \hookrightarrow \mathcal{O}_X \hookleftarrow M^2 \quad \rightsquigarrow \quad \mathcal{A} = \mathcal{O}_X \oplus L \oplus M \oplus (L \otimes M)$

- $\xrightarrow{\sim} \quad \xleftarrow{\sim} \quad \Rightarrow$ Azumaya $\rightsquigarrow [\mathcal{A}] \in \mathrm{Br}(X)$
- else $\quad \Rightarrow [\mathcal{A}|_U] \in \mathrm{Br}(U) \hookrightarrow \mathrm{Br}(K(X))$

$$\uparrow$$

$$\mathrm{Br}(X)$$

Brauer group: Hodge theory

$X =$ smooth projective over $\mathbb{C}$

$\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$

$\cong \mathbb{C}^q/\Gamma$ abelian variety

$\bigcup$

torsion $\cong (\mathbb{Q}/\mathbb{Z})^{\oplus 2q}$

$\mathrm{Br}^0_{\mathrm{an}}(X) \cong H^2(X, \mathcal{O}_X)/H^2(X, \mathbb{Z})$

$\cong \mathbb{C}^p/\Gamma$ wild

$\bigcup$

$\mathrm{Br}(X) \approx$ torsion $\cong (\mathbb{Q}/\mathbb{Z})^{\oplus b_2 - \rho}$

$(\Rightarrow \ \mathrm{Br}(C) = \{1\})$

weight $= 1$ *Hodge structures* weight $= 2$

$\mathrm{Pic}(X)_{\mathrm{tors}} = \{ L \mid \exists\, \widetilde{X} \xrightarrow{f} X$ finite étale $\}$ with $f^*L \cong \mathcal{O}_X$

$\mathrm{Br}(X) = \{ \alpha \mid \exists\, \widetilde{X} \xrightarrow{f} X$ finite $\}$ with $f^*\alpha = 1$

Geometric [ABHN]: $H^1(X, \mu_n) \times H^1(X, \mu_n) \to \mathrm{Br}(X)[n]$ (not surjective!)

Splitting Brauer classes

Change of perspective:

$$\mathrm{Br}(X) = \{\ \pi\colon P \to X \mid \text{Brauer–Severi variety}\ \}/_\sim$$

$$\alpha = [P] \ \Rightarrow\ \langle\alpha\rangle = \ker(\pi^*\colon \mathrm{Br}(X) \to \mathrm{Br}(P)) = \ker(\mathrm{Br}(K(X)) \hookrightarrow \mathrm{Br}(K(P)))$$

Corollary:

- $\exists$ $\begin{array}{ccc} \widetilde{X} & & \\ \downarrow & \searrow^{\text{gen. finite}} & \\ P & \longrightarrow & X \end{array}$ $\quad \alpha_{\widetilde{X}} = 1 \quad\Big|\quad \Rightarrow \quad \mathrm{per}(\alpha) \mid \deg(\widetilde{X} \to X)$

- $\exists$ $\begin{array}{ccc} C & & \\ \downarrow & \searrow^{\text{rel. curve}} & \\ P & \longrightarrow & X \end{array}$ $\quad \alpha_C = 1 \quad\Big|\quad \Rightarrow \quad \mathrm{per}(\alpha) \mid 2g(C) - 2$

Question: [Clark, Saltman, ..., Roquette, ...] Fix $\alpha \in \mathrm{Br}(X)$.

(i) $\exists\ C_\alpha \to X$, $g(C_\alpha) = 1$ with $\alpha_{C_\alpha} = 1$? [de Jong–Ho '12, Antieau–Auel '15]: $\mathrm{ind}(\alpha) \le 6$

(ii) $\exists\ A_\alpha \to X$, abelian torsor with $\alpha_{A_\alpha} = 1$? [Ho–Lieblich '21]: $\dim(A_\alpha) \approx \mathrm{ind}(\alpha)^{\mathrm{ind}(\alpha)}$

Theorem [H.– Mattei '23]: $\exists\, A \to X$ generic abelian scheme such that

$$\forall\, \alpha \in \mathrm{Br}(X)\ \exists\, A\text{-torsor } A_\alpha \to X \text{ with } \alpha_{A_\alpha} = 1$$

Note: $\dim(A_\alpha) = \dim(A) \approx g(C_t)$ where $\bigcup C_t = X$

In progress: $\alpha \in \mathrm{Br}(K(X))$?

Idea of proof: Pick $C \to X$ birational & moduli space $\mathfrak{M} = \mathbf{Pic}_{\mathcal{A}}(C/B)$.

$$\alpha = [\mathcal{A}] \qquad \begin{array}{c} C \to X \\ \downarrow \\ B = \mathbb{P}^{n-1} \end{array} \quad \rightsquigarrow \quad A_\alpha := \mathfrak{M} \times_B C \to C \to X \text{ torsor over}$$

$$A := \mathbf{Pic}(C/B) \times_B C \to C \to X$$

& numerical condition: $\exists$ universal Poincaré $\mathcal{P} \to \mathfrak{M} \times_B C$

$$\Rightarrow\ \mathcal{P} \text{ is a } \mathcal{A}_{A_\alpha}\text{-module of rank } \mathrm{rk}(\mathcal{A})^{1/2} \quad \Rightarrow \quad \alpha_{A_\alpha} = 1$$

Period–index conjecture

Conjecture [Colliot-Thélène]: For X over $k = \bar{k}$ and $\alpha \in \mathrm{Br}(K(X))$:

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\dim(X)-1}$$

OK if $\dim(X) = 1$ (Tsen), $\dim(X) = 2$ (de Jong), $X =$ abelian 3fold (Hotchkiss–Perry)

Conjecture [de Jong–Perry]: Weaker version for $\alpha \in \mathrm{Br}(X)$: $\exists\, e(X)$ with

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{e(X)}.$$

OK if Lefschetz standard conjecture holds for $H^2(X) \cong H^{2n-2}(X)$

Theorem [H.– Mattei '23]: OK always with $e(X) \approx 2g(C)$.

Idea of proof: Same construction & $\varphi\colon \mathbf{Pic}_\alpha(C_\eta) \to \mathbf{Pic}(C_\eta)$, $L \mapsto L^{\otimes|\alpha|}$

$\rightsquigarrow$ $\mathcal{A}_\alpha$-module $H^0(\mathbf{Pic}_\alpha(C_\eta), \mathcal{P}_\eta \otimes \varphi^*\Theta)$

$$\Rightarrow \qquad \mathrm{ind}(\alpha) \mid \dim(\quad) = \frac{(\varphi^*\Theta)^g}{g!} \approx \mathrm{per}(\alpha)^{2g}$$

Alternative (Antieau–Auel, Lieblich): Restrict to $\varphi^{-1}(0)$.

Example: $S \to \mathbb{P}^1$ elliptic K3, $\alpha \in \mathrm{Br}(S)$, $\rightsquigarrow$ $S_\alpha = \mathbf{Pic}^0_\alpha(S/\mathbb{P}^1) \to \mathbb{P}^1$

Then $\mathrm{ind}(\alpha) =$ minimal degree of multisection of $S_\alpha \to \mathbb{P}^1$

Using $S_\alpha \to S$, $L \mapsto L^{\otimes|\alpha|}$ $\rightsquigarrow$ expect: $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^2$

But [Ogg, Shafarevich, Lichtenbaum, de Jong]: $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$

Use $\mathbf{Pic}^1(S_\alpha/\mathbb{P}^1) = S_\alpha$ and $\mathbf{Pic}^d(S_\alpha/\mathbb{P}^1) = S_{\alpha^d}$ ($= S$ if $d = |\alpha|$) & section of $S \to \mathbb{P}^1$

General: $\exists\, C \twoheadrightarrow S$ genus one family $\Rightarrow$ $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha) \cdot \deg(C/S)$

Conjecture: $X =$ hyperkähler, $\dim(X) = 2n$, $\alpha \in \mathrm{Br}(X)$:

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^n$$

OK if:
- $X = \mathrm{K3}^{[n]}$
- $X \to \mathbb{P}^n$ of $\mathrm{K3}^{[n]}$-type
- $X \to \mathbb{P}^n$ general but with exponent $2n$