THE LOOIJENGA-LUNTS-VERBITSKY ALGEBRA AND APPLICATIONS

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Abstract. In this notes, following Looijenga-Lunts and Verbitsky, we define a Lie algebra (called LLV algebra) that acts on the rational cohomology of a Kähler manifold. We describe this algebra in the case of an Hyperkähler manifold. Morever, we give a proof of Verbitsky's Theorem: the subalgebra generated by degree 2 classes inside the rational cohomology is an irreducible representation of the LLV algebra.

1. Introduction

1.1. Let $V = \oplus_{k \in Z} V_k$ be a finite dimensional graded vector space over a field k, and denote by h the operator:

$$h|_{V_k} = k\mathrm{id}.$$

Definition 1.1. Let $e : V \longrightarrow V$ be a degree 2 endomorphism. We say e has the *Lefschetz property* if

$$e^k : V_{-k} \longrightarrow V_k$$

is an isomorphism.

Theorem 1.2 (Jacobson-Morozov)**.** *The operator e has the Lefschetz property if and only if there exists a degree -2 endomorphism $f : V \longrightarrow V$ such that*

$$[e, f] = h.$$

We say that the triple (e, h, f) is a $\mathfrak{sl}_2$-triple. Indeed, we can define a representation of the lie algebra on the vector space V as follows

$$\mathfrak{sl}_2(k) \longrightarrow \mathrm{End}(V)$$
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e.$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h.$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto f.$$

This review was prepared in the context of the seminar organized by the ERC Synergy Grant HyperK, Grant agreement ID 854361. The talk was delivered on April 23, 2021.

Remark 1.3. The set of operators with the Lefschetz property is a Zariski open subset of $\mathrm{End}_2(V)$.

In particular we are interested in the graded vector space $V = H^*(X,\mathbb{Q})[n]$, where X is a compact kähler manifold of dimension n. To any class $\alpha \in H^2(X,\mathbb{Q})$ we can associate the operator in cohomology obtained by taking the cup product with alpha:

$$e_\alpha : H^*(X,\mathbb{Q}) \longrightarrow H^*(X,\mathbb{Q}),\ \omega \longmapsto \alpha.\omega.$$

The operator h becomes

$$h|_{H^{n-k}(X,\mathbb{R})} := (n-k)\mathrm{id}.$$

From Theorem 1.2 we see that if e_α has the Lefschetz property (for example if α is kähler), there is an operator f_α of degree -2 that makes (e_α, h, f_α) an $\mathfrak{sl}_2$-triple. Moreover, the map

$$f : H^2(X,\mathbb{Q}) \longrightarrow \mathrm{End}_{-2}(H^*(X,\mathbb{Q})),$$

that sends α to the operator f_α is defined on a Zariski open subset and rational.

Remark 1.4. If $\alpha \in H^{1,1}(X,\mathbb{Q})$ is kähler, from standard Hodge theory it follows that everything can be defined at the level of forms, where the dual operator is $f_\alpha = *^{-1}e_\alpha *$. The $\mathfrak{sl}_2$ action preserves the harmonic forms, and so induces the action in cohomology.

Definition 1.5. Let X be a compact kähler manifold. The *total lie algebra* $\mathfrak{g}_{tot}(X)$, or LLV algebra, of X is the Lie algebra generated by the $\mathfrak{sl}_2$-triples

$$(e_a, h, f_a),$$

where $a \in H^2(X,\mathbb{Q})$ is a class with the Lefschetz property.

We state without proof a general result about this Lie algebra for compact kähler manifolds. Denote by ϕ the pairing on $H^*(X,\mathbb{C})$ given by

$$\phi(\alpha,\beta) = (-1)^q \int \alpha.\beta,$$

if α has degree $n+2q$ or $n+2q+1$.

Proposition 1.6 ([LL97, Proposition 1.6])**.** *The lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ is semisimple and preserves ϕ infinitesimally. Moreover, the degree 0 part $\mathfrak{g}_{\mathrm{tot}}(X)_0$ is reductive.*

1.2. Now let X be a compact hyperkähler manifold of complex dimension $2n$. If we fix an hyperkähler metric g on X we get an action of the quaternion algebra $\mathbb{H}$ on the real tangent bundle TX. This means that we have three complex structures I, J, K such that

$$IJ = -JI = K.$$

To each of this complex structure we can associate a kähler form $\omega_I := g(I(-),-), \omega_J := g(J(-),-), \omega_K = g(K(-),-)$ and a holomorphic symplectic form $\sigma_I = \omega_J + i\omega_K, \sigma_J = \omega_K + i\omega_I, \sigma_K = \omega_I + i\omega_J$.

Definition 1.7. We call the 3-plane

$$< [\omega_I], [\omega_J], [\omega_K] >=< [\omega_I], [\Re\sigma_I], [\Im\sigma_I] >\subset H^2(X, \mathbb{R})$$

the *charateristic 3-plane* of the metric g, and denote it by $F(g)$.

Definition 1.8. Denote by $\mathfrak{g}_g \subset \mathrm{End}(H^*(X, \mathbb{R}))$ the Lie algebra generated by the $\mathfrak{sl}_2$-triples (e_α, h, f_α) where $\alpha \in F(g)$.

Remark 1.9. This algebra is can be generated just by the three $\mathfrak{sl}_2$-triples associated to $\alpha = [\omega_I], [\omega_J], [\omega_K]$.

2. The algebra $\mathfrak{g}_g$

In this section we study the smaller algebra $\mathfrak{g}_g$ and its action on cohomology.

2.1. Let V be a left $\mathbb{H}$-module, equipped with an $\mathbb{H}$-invariant inner product

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}.$$

As before we have three complex structures I, J, K on V with corresponding "kähler" forms

$$\omega_I, \omega_J, \omega_K \in \Lambda^2 V^*,$$

and holomorphic symplectic forms $\sigma_I, \sigma_J, \sigma_K$.

Definition 2.1. Let $\mathfrak{g}(V) \subset \mathrm{End}(\Lambda^* V^*)$ be the Lie algebra generated by the $\mathfrak{sl}_2$-triples

$$(e_\lambda, h, f_\lambda)_{\lambda = \omega_I, \omega_J, \omega_K}.$$

In particular this definition makes sense for the rank 1 module $\mathbb{H}$, and we get an algebra $\mathfrak{g}(\mathbb{H}) \subset \mathrm{End}(\Lambda^\bullet \mathbb{H}^*)$. We denote by $\mathbb{H}_0$ the pure quaternions, i.e. linear combinations of I, J, K. We also denote by $\mathfrak{g}(\mathbb{H})_0$ the degree 0 component of $\mathfrak{g}(\mathbb{H})$ (here the degree is meant as endomorphisms of a graded vector space). It is a Lie subalgebra, and we denote by $\mathfrak{g}(\mathbb{H})'_0 := [\mathfrak{g}(\mathbb{H})_0, \mathfrak{g}(\mathbb{H})_0]$ its derived Lie algebra.

Proposition 2.2. *In the notation above we have the following.*

1. *There is a natural isomorphism* $\mathfrak{g}(V) \simeq \mathfrak{g}(\mathbb{H})$.
2. *There is an isomorphism* $\mathfrak{g}(\mathbb{H}) \simeq \mathfrak{so}(4, 1)$.
3. *The algebra decomposes with respect to the degree as*

$$\mathfrak{g}(\mathbb{H}) = \mathfrak{g}(\mathbb{H})_{-2} \oplus \mathfrak{g}(\mathbb{H})_0 \oplus \mathfrak{g}(\mathbb{H})_2.$$

Furthermore, $\mathfrak{g}(\mathbb{H})_{\pm 2} \simeq \mathbb{H}_0$ *as Lie algebras, and* $\mathfrak{g}(\mathbb{H})_0 = \mathfrak{g}(\mathbb{H})'_0 \oplus \mathbb{R}h$ *with* $\mathfrak{g}(\mathbb{H})'_0 \simeq \mathbb{H}_0$*; this last isomorphism is compatible with the actions on* $\Lambda^\bullet V^*$.

Proof. Proof of (1). Since $\langle -,-\rangle$ is $\mathbb{H}$-invariant, we can find an orthogonal decomposition

$$V = \mathbb{H} \oplus \cdots \oplus \mathbb{H}.$$

Taking exterior powers we get $\Lambda^\bullet V^* = \Lambda^\bullet \mathbb{H}^* \otimes \cdots \otimes \Lambda^\bullet \mathbb{H}^*$. This gives an injective map $\mathfrak{g}(\mathbb{H}) \longrightarrow \mathrm{End}(\Lambda^\bullet V^*)$, given by the natural tensor product representation. It is a direct check that the image of this morphism is exactly the algebra $\mathfrak{g}(V)$.

Proof of (2). Consider the subrepresentation $W \subset \Lambda^\bullet \mathbb{H}^*$ given by

$$W = \Lambda^0 \mathbb{H}^* \oplus \langle \omega_I, \omega_J, \omega_K \rangle \oplus \Lambda^4 \mathbb{H}^*.$$

We equip it with the quadratic form given by setting $\Lambda^0\mathbb{H}^* \oplus \Lambda^4\mathbb{H}^*$ to be an hyperbolic plane, orthogonal to the 3-plane, and $\{\omega_I, \omega_J, \omega_K\}$ to be an orthonormal basis of the 3-plane. By a direct computation we can see that the action of $\mathfrak{g}(\mathbb{H})$ respects infinitesimally this quadratic form. This gives a map

$$\mathfrak{g}(\mathbb{H}) \longrightarrow \mathfrak{so}(W) \simeq \mathfrak{so}(4,1),$$

that we next show to be an isomorphism.

Since W has dimension 5 the Lie algebra $\mathfrak{so}(W)$ has dimension 10. Now consider the following 10 elements of $\mathfrak{g}(\mathbb{H})$:

$$h, e_I, e_J, e_K, f_I, f_J, f_K, K_{IJ}, K_{IK}, K_{JK},$$

where $K_{IJ} := [e_I, f_J], K_{IK} = [e_I, f_K]$ and $K_{JK} = [e_J, f_K]$. Verbitsky [Ver90] showed that K_{IJ} acts like the Weil operator associated with the Hodge structure given by K, and similarly K_{JK} and K_{IK}. This means that on a (p,q) form with respect to K it acts as multiplication by $i(p-q)$. It follows that the ten operators above are linearly independent over W, hence the map is surjective. Moreover they generate $\mathfrak{g}(\mathbb{H})$ as a vector space. Indeed, they generate $\mathfrak{g}(\mathbb{H})$ as a Lie algebra, and one has the following relations (see [Ver90]):

$$\begin{aligned} [K_{\lambda,\mu}, K_{\mu,\nu}] = K_{\lambda,\nu}, &\quad [K_{\lambda,\mu}, H] = 0, \\ [K_{\lambda,\mu}, e_\mu] = 2e_\lambda, &\quad [K_{\lambda,\mu}, f_\mu] = 2f_\lambda, \\ [K_{\lambda,\mu}, e_\nu] = 0, &\quad [K_{\lambda,\mu}, f_\nu] = 0, \end{aligned}$$

where $\lambda, \mu, \nu \in I, J, K$ and $\nu \neq \lambda, \nu \neq \mu$. This implies that they are a basis of $\mathfrak{g}(\mathbb{H})$, hence the map is an isomorphism.

Point (3) follows using this explicit basis. Indeed we have

$$\begin{aligned} \mathfrak{g}(\mathbb{H})_{-2} &= \langle f_I, f_J, f_K \rangle, \\ \mathfrak{g}(\mathbb{H})_2 &= \langle e_I, e_J, e_K \rangle, \\ \mathfrak{g}(\mathbb{H})_0 &= \langle K_{IJ}, K_{JK}, K_{IK} \rangle \oplus \mathbb{R}h. \end{aligned}$$

In particular we have

$$\begin{aligned}\mathfrak{g}(\mathbb{H})_0' &\xrightarrow{\sim} \mathbb{H}_0,\\ K_{IJ} &\longmapsto K,\\ K_{JK} &\longmapsto I,\\ K_{IK} &\longmapsto J.\end{aligned}$$

Since $I, J, K \in \mathbb{H}_0$ act like Weil operators in cohomology, the isomorphism is compatible with the actions. □

Now we can compute the algebra $\mathfrak{g}_g$. As above we denote by $(\mathfrak{g}_g)_0$ the degree 0 part, and by $(\mathfrak{g}_g)_0' := [(\mathfrak{g}_g)_0, (\mathfrak{g}_g)_0]$ its derived Lie algebra. One can show that $\mathfrak{g}_0$ is reductive in a similar way to Proposition 1.6, so the derived subalgebra is the semisimple part of $(\mathfrak{g}_g)_0$.

Proposition 2.3. *Let (X, g) be an hyperkähler manifold with a fixed hyperkähler metric.*

(1) *There is a natural isomorphism of graded Lie algebras $\mathfrak{g}_g \simeq \mathfrak{g}(\mathbb{H})$.*
(2) *The semisimple part $(\mathfrak{g}_g)_0'$ acts on $H^*(X, \mathbb{R})$ as derivations.*

Proof. Proof of (1). Consider the Lie subalgebra $\hat{\mathfrak{g}}_g \subset \mathrm{End}(\Omega_X^\bullet)$, generated by the $\mathfrak{sl}_2$-triples (e_a, h, f_a) at the level of forms, with $a \in F(g)$. From the previous proposition, we see that for every point $x \in X$ there is an inclusion $\mathfrak{g}(\mathbb{H}) \hookrightarrow \mathrm{End}(\Omega_{x,X}^\bullet)$. This gives an inclusion $\mathfrak{g}(\mathbb{H}) \hookrightarrow \prod_{x \in X} \mathrm{End}(\Omega_{x,X}^\bullet)$. It follows from the definitions that the two algebras of $\mathfrak{g}(\mathbb{H})$ and $\hat{\mathfrak{g}}_g$ are equal as subalgebras of $\prod_{x \in X} \mathrm{End}(\Omega_{x,X}^\bullet)$.

Since the metric g is fixed, the $\mathfrak{sl}_2$-triples (e_a, h, f_a) preserve the harmonic forms $\mathcal{H}^*(X)$, and so does $\mathfrak{g}(\mathbb{H})$. Since $\mathcal{H}^*(X) \simeq H^*(X, \mathbb{R})$ we get a morphism

$$\mathfrak{g}(\mathbb{H}) \longrightarrow \mathfrak{g}_g.$$

This map is surjective, because the image contains the $\mathfrak{sl}_2$-triples that generate $\mathfrak{g}_g$. Moreover, by explicit computations similar to the proof of the previous proposition, we can see that $\dim \mathfrak{g}_g \geqslant 10$. Hence the map is an isomorphism.

Now we prove (2). We have

$$(\mathfrak{g}_g)_0' \simeq \mathfrak{g}(\mathbb{H})_0' \simeq \mathbb{H}_0.$$

Hence it suffices to prove the statement for the action of I, J, K. Each of them gives a complex structure, and acts as the Weil operator on the associated Hodge decomposition. So, the action on (p, q) forms is given by multiplication by $i(p - q)$, which is a derivation. □

Remark 2.4. In particular, we see that $\mathfrak{g}_g \simeq \mathfrak{so}(4, 1)$.

3. The total Lie algebra

The goal of this section is to prove the following result.

Theorem 3.1. *In the above notation we have:*

(1) *The total lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ lives only in degrees $-2, 0, 2$, so it decomposes as:*
$$\mathfrak{g}_{\mathrm{tot}}(X) = \mathfrak{g}_{\mathrm{tot}}(X)_{-2} \oplus \mathfrak{g}_{\mathrm{tot}}(X)_0 \oplus \mathfrak{g}_{\mathrm{tot}}(X)_2.$$
(2) *There are canonical isomorphisms $\mathfrak{g}_{\mathrm{tot}}(X)_{\pm 2} \simeq H^2(X, \mathbb{R})$.*

(3) *There is a decomposition $\mathfrak{g}_{\mathrm{tot}}(X)_0 = \mathfrak{g}_{\mathrm{tot}}(X)_0' \oplus \mathbb{Q}h$ with $\mathfrak{g}_{\mathrm{tot}}(X)_0' \simeq \mathfrak{so}(H^2(X, \mathbb{Q}), q)$. Furthermore $\mathfrak{g}_{\mathrm{tot}}(X)_0'$ acts on $H^*(X, \mathbb{Q})$ by derivations.*

The main geometric input in the proof is the following result.

Lemma 3.2. *If X is a compact hyperkähler manifold, then $[f_a, f_b] = 0$ for every $a, b \in H^2(X, \mathbb{R})$ where f is defined.*

The proof relies on the following fact.

Proposition 3.3. *The set of charateristic 3-planes is open in the Grassmannian of 3-planes in $H^2(X, \mathbb{R})$.*

This fact is a consequence of a celebrated Theorem by Calabi and Yau.

Theorem 3.4. *Let X be an hyperkähler manifold, and let I be a complex structure on X. If ω is a kähler class, then there is a unique hyperkähler metric g such that $[\omega_I] = \omega$.*

Proof of the lemma. Fix an HK metric g on X, then for every $a, b \in F(g)$ we have $[f_a, f_b] = 0$. We can see this holds already at the level of forms, using the definition $f_a = *^{-1} e_a *$. Let $a \in H^2(X, \mathbb{R})$ be a class where f is defined. Since f is rational, the condition $[f_a, f_b] = 0$ is Zariski closed with respect to $b \in H^2(X, \mathbb{R})$. From Proposition 3.3 it follows that the set
$$\{b \in H^2(X, \mathbb{R}) \mid a, b \in F(g) \text{ for some metric } g\}$$
is open. Since $[f_a, f_b] = 0$ for every b in this open set, we get $[f_a, f_b] = 0$ for every b where f is defined. □

While the statement of Theorem 3.1 is over $\mathbb{Q}$, we will give the proof over $\mathbb{R}$ following [LL97].

Proof of the Proposition. Consider the subspace
$$V := V_{-2} \oplus V_0 \oplus V_2 \subset \mathfrak{g}_{\mathrm{tot}}(X),$$
where V_2 is the abelian subalgebra generated by e_α with $\alpha \in H^2(X, \mathbb{R})$, V_{-2} is the abelian subalgebra generated by the f_α with $\alpha \in H^2(X, \mathbb{R})$ where f is defined, and V_0 is the subalgebra generated by $[e_\alpha, e_\beta]$. To prove (1) and (2), it is enough to show that V is a subalgebra of $\mathfrak{g}_{\mathrm{tot}}(X)$. Indeed, since $\mathfrak{g}_{\mathrm{tot}}(X)$ is generated by elements contained in V this would imply

$V = \mathfrak{g}_{\mathrm{tot}}(X)$. Since V_2 and V_{-2} are abelian subalgebras, it suffices to show that $[V_0, V_2] \subset V_2$ and $[V_0, V_{-2}] \subset V_{-2}$.

For this, consider the subalgebra $V_0' := [V_0, V_0]$. We first show that $V_0 = V_0' \oplus \mathbb{R}h$ where V_0' acts on cohomology via derivations. Since the set $\{(a,b) \mid a, b \in F(g) \text{ for some metric } g\}$ is open, the argument in the lemma above shows that V_0 is generated by the elements $[e_a, f_b]$ with $a, b \in F(g)$. If we fix the HK metric g, the elements $[e_a, f_b]$ with $a, b \in F(g)$ generate the algebra $(\mathfrak{g}_g)_0$ and their brackets the subalgebra $(\mathfrak{g}_g)_0'$. Thus the subalgebra V_0' is generated by the various algebras $(\mathfrak{g}_g)_0'$ and their brackets. Since the algebras $(\mathfrak{g}_g)_0'$ act via derivations, the same is true for their the brackets, hence V_0' acts via derivations. This argument combined with point (3) of 2.2 also shows the decomposition $V_0 = V_0' + \mathbb{R}h$. The sum is direct, because $h \notin V_0' \subset \mathfrak{g}_{\mathrm{tot}}(X)_0'$, since $\mathfrak{g}_{\mathrm{tot}}(X)_0$ is reductive (Proposition 1.6) and h is in the center.

Now we show that $[V_0, V_2] \subset V_2$. Since the adjoint action of h gives the grading, it is enough to show that $[V_0', V_2] \subset V_2$. Let $u \in V_0'$ and $e_a \in V_2$. For every $x \in H^2(X, \mathbb{R})$ we have

$$[u, e_a](x) = u(a.x) - a.u(x) = u(a).x = e_a(x),$$

because u is a derivation.

The inclusion $[V_0, V_{-2}] \subset V_{-2}$ is more difficult. Consider $G_0' \subset GL(H^*(X, \mathbb{R}))$ the closed Lie subgroup with lie algebra V_0'. For every $t \in G_0'$ we have $te_at^{-1} = e_{t(a)}$ and $tht^{-1} = h$, by integrating the analogous relations at the level of Lie algebras. Since the third element of an $\mathfrak{sl}_2$-triple is unique, we get that $tf_at^{-1} = f_{t(a)}$. This implies that the adjoint action of G_0' leaves V_{-2} invariant, hence so does the lie algebra V_0'.

To summarize, at this point we showed (1) and (2), and also that $\mathfrak{g}_{\mathrm{tot}}(X)_0'$ acts via derivations. It remains to see that $\mathfrak{g}_{\mathrm{tot}}(X)_0' \simeq \mathfrak{so}(H^2(X, \mathbb{R}), q)$.

We begin by defining the map $\mathfrak{g}_{\mathrm{tot}}(X)_0' \longrightarrow \mathfrak{so}(H^2(X, \mathbb{R}), q)$.For this, we consider the restriction of the action of $\mathfrak{g}_{\mathrm{tot}}(X)_0'$ to $H^2(X, \mathbb{R})$, and show that it preserves infinitesimally the BBF form. We can fix an HK metric g and check this for $(\mathfrak{g}_g)_0'$, because these subalgebras generate $\mathfrak{g}_{\mathrm{tot}}(X)_0'$. From Proposition 2.2 it is enough to check it for the Weil operators associated to the three complex structures I, J, K induced from g. Fix one of them, say I, we have to verify that

$$q(I\alpha, \beta) + q(\alpha, I\beta) = 0,$$

for every $\alpha, \beta \in H^2(X, \mathbb{R})$. This follows with a direct verification using the q-orthogonal Hodge decomposition

$$H^2(X, \mathbb{R}) = (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}) \oplus H^{1,1}(X, \mathbb{R}),$$

with respect to the complex structure I. To conclude the proof it remains to show that this map is bijective, for this see [LL97, Proposition 4.5]. $\square$

Definition 3.5. We define the Mukai completion of the quadratic vector space $(H^2(X, \mathbb{Q}), q)$, as the quadratic vector space

$$(\tilde{H}(X, \mathbb{Q}), \tilde{q}) := (H^2(X, \mathbb{Q}), q) \oplus U$$

where U is an hyperbolic plane: a two dimensional vector space with quadratic form given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Corollary 3.6. *There is a natural isomorphism*

$$\mathfrak{g}_{\mathrm{tot}}(X) \simeq \mathfrak{so}(\tilde{H}(X, \mathbb{Q}), \tilde{q}).$$

Proof. Recall that for a rational quadratic space (V, q) there is an isomorphism

$$\Lambda^2 V \xrightarrow{\sim} \mathfrak{so}(V, q),$$
$$x \wedge y \longmapsto \frac{1}{2}(q(x, -)b - q(b, -)a)$$

The desired isomorphism follows from this, at least at the level of vector spaces. The computations to show that it is in fact an isomorphism of lie algebra are carried out in [GYJLR20, Proposition 2.7]. □

Example 3.7. If X is a $K3$ surface, then the Mukai completion $\tilde{H}(X, \mathbb{Q})$ is the rational cohomology $H^*(X, \mathbb{Q})$ with the usual Mukai pairing. This identification is compatible with the action of $\mathfrak{g}_{\mathrm{tot}}(X)$.

Corollary 3.8. *The Hodge structure on $H^*(X, \mathbb{R})$ is determined by the Hodge structure on $H^2(X, \mathbb{R})$ and by the action of $\mathfrak{g}_{\mathrm{tot}}(X)_{2,\mathbb{R}} \simeq H^2(X, \mathbb{R})$ on $H^*(X, \mathbb{R})$.*

Proof. Let I, J, K be the three complex structures associated to an HK metric g, and assume I is the given one. As recalled before, the commutator $K_{JK} = [e_J, f_K]$ acts like the Weil operator for I; hence it recovers the Hodge structure. By definition, it depends only on the classes $[\omega_I], [\omega_K]$ and their action on $H^*(X, \mathbb{R})$. Since the Hodge structure is given by the class of the symplectic form $[\sigma_I] = [\omega_J] + i[\omega_K]$, the thesis follows. □

Recall that if $\mathfrak{g}$ is a Lie algebra, the *universal enveloping algebra* of $\mathfrak{g}$ is the smallest associative algebra extending the bracket on $\mathfrak{g}$. It is defined as the quotient of the tensor algebra by the relations:

$$x \otimes y - y \otimes x - [x, y] \quad x, y \in \mathfrak{g}.$$

In particular if $\mathfrak{g}$ is abelian, then $U\mathfrak{g} = \mathrm{Sym}^* \mathfrak{g}$.

Corollary 3.9. *There is a natural decomposition:*

$$U\mathfrak{g}_{\mathrm{tot}}(X) = U\mathfrak{g}_{\mathrm{tot}}(X)_2 . U\mathfrak{g}_{\mathrm{tot}}(X)_0 . U\mathfrak{g}_{\mathrm{tot}}(X)_{-2}.$$

4. Primitive decomposition

Definition 4.1. If V is a $\mathfrak{g}_{\mathrm{tot}}(X)$-representation, we define the primitive subspace as:

$$\mathrm{Prim}(V) = \{x \in V \mid (\mathfrak{g}_{\mathrm{tot}}(X)_{-2}).x = 0\}.$$

If $V = H^*(X,\mathbb{Q})$ is the standard representation we denote the primitive subspace as $\mathrm{Prim}(X)$.

Remark 4.2. The primitive subspace $\mathrm{Prim}(V)$ is a $\mathfrak{g}_{\mathrm{tot},0}(X)$-subrepresentation. This follows from the fact that $[\mathfrak{g}_{\mathrm{tot}}(X)_0, \mathfrak{g}_{\mathrm{tot}}(X)_{-2}] \subset \mathfrak{g}_{\mathrm{tot}}(X)_{-2}$.

Definition 4.3. The Verbitsky component $SH^2(X,\mathbb{Q}) \subseteq H^*(X,\mathbb{Q})$ is the graded subalgebra generated by $H^2(X,\mathbb{Q})$.

Proposition 4.4. *The cohomology $H^*(X,\mathbb{Q})$ is generated by $\mathrm{Prim}(X)$ as a $SH^2(X,\mathbb{Q})$ module. Moreover, if $W \subset \mathrm{Prim}(X)$ is a $\mathfrak{g}_{\mathrm{tot}}(X)_0$ irreducible subrepresentation, then $SH^2(X,\mathbb{Q}).W \subset H^*(X,\mathbb{Q})$ is an irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$-module.*

Proof. Since $\mathfrak{g}_{\mathrm{tot}}(X)$ is semisimple, we can decompose the cohomology in irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$-representations:

$$H^*(X,\mathbb{Q}) = V_1 \oplus \cdots \oplus V_k.$$

The primitive part is compatible with this decomposition, so we get the decomposition

$$\mathrm{Prim}(X) = \mathrm{Prim}(V_1) \oplus \cdots \oplus \mathrm{Prim}(V_k),$$

of $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representations.

We first want to show that $SH^2(X,\mathbb{Q}).\mathrm{Prim}(V_i) = V_i$. We have

$$SH^2(X,\mathbb{Q}).\mathrm{Prim}(V_i) = U\mathfrak{g}_{\mathrm{tot}}(X)_2.\mathrm{Prim}(V_i) = U\mathfrak{g}_{\mathrm{tot}}(X).\mathrm{Prim}(V_i) \subset V_i, \tag{4.1}$$

where the first equality follows from the fact that $\mathfrak{g}_{\mathrm{tot}}(X)_2$ is abelian, and the second from Corollary 3.9. Thus $SH^2(X,\mathbb{Q}).\mathrm{Prim}(V_i)$ is a $\mathfrak{g}_{\mathrm{tot}}(X)$ subrepresentation of V_i, but V_i is irreducbile, so the equality holds. This proves the first part of the proposition.

To prove the second part it is enough to show that $\mathrm{Prim}(V_i)$ are irreducible as $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representations. Assume it is not and write $\mathrm{Prim}(V_i) = W_1 \oplus W_2$. The identities (4.1) show that acting with $SH^2(X,\mathbb{Q})$ gives a decomposition $V_i = SH^2(X,\mathbb{Q}).W_1 \oplus SH^2(X,\mathbb{Q}).W_2$. Again, this is a contradiction to the fact that V_i is irreducible. □

Corollary 4.5. *The Verbitsky component $SH^2(X,\mathbb{Q}) \subset H^*(X,\mathbb{Q})$ is an irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$ subrepresentation.*

Proof. By definition we have $SH^2(X,\mathbb{Q}) = SH^2(X,\mathbb{Q}).H^0(X,\mathbb{Q})$, and $H^0(X,\mathbb{Q}) \subset \mathrm{Prim}(X)$. So it is enough to show that $H^0(X,\mathbb{Q})$ is preserved by $\mathfrak{g}_{\mathrm{tot}}(X)_0$. Thanks to Theorem 3.1 we only need to show that it is preserved by the action of h and $\mathfrak{g}_{\mathrm{tot}}(X)_0'$. The first is obvious, and the second follows from the fact that the action of $\mathfrak{g}_{\mathrm{tot}}(X)_0'$ is via derivation. □

5. Verbitsky's Theorem

In this section we give a proof of a result by Verbitsky on the structure of the irreducible component $SH(X)$. The argument presented is due to due to Bogomolov [Bog96], and works over $\mathbb{C}$. For the statement over $\mathbb{Q}$ see [Ver95, Proposition 15.1], [GYJLR20, Proposition 2.15].

Theorem 5.1. *There is a natural isomorphism of algebras and $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representations:*

$$SH^2(X,\mathbb{C}) \simeq \mathrm{Sym}^*(H^2(X,\mathbb{C}))/\langle \alpha^{n+1} \mid q(\alpha)=0\rangle.$$

Lemma 5.2. *Denote by A the graded $\mathbb{C}$-algebra $\mathrm{Sym}^*(H^2(X,\mathbb{C}))/\langle \alpha^{n+1} \mid q(\alpha)=0\rangle$. Then we have:*

(1) $A_{2n} \simeq \mathbb{C}$.
(2) *The multiplication map $A_k \times A_{2n-k} \longrightarrow A_{2n}$ induces a perfect pairing.*

Proof of the Theorem. From the Local Torelli Theorem we have that $\alpha^{n+1}=0$ for an open subset of the quadric $\{\alpha \in H^2(X,\mathbb{C}) \mid q(\alpha)=0\}$. Since the condition $\alpha^{n+1}=0$ is Zariski closed, we get that it holds for the entire quadric. Consider the multiplication map

$$\mathrm{Sym}^*(H^2(X,\mathbb{C})) \longrightarrow SH^2(X,\mathbb{C}).$$

The kernel contains $\{\alpha^{n+1} \mid q(\alpha)=0\}$, hence it factors via the ring A. It is an algebra homorphism by construction, and a map of $\mathfrak{g}_{\mathrm{tot}}(X)_0$-representations because $\mathfrak{g}_{\mathrm{tot}}(X)_0'$ acts via derivations.

The induced map $A \longrightarrow SH^2(X,\mathbb{C})$ is surjective by construction. If it were not injective, by the above lemma, the kernel would contain A_{2n}. But this is impossible, because in top degree the map $A_{2n} \longrightarrow H^{4n}(X,\mathbb{C})$ is non-zero. Indeed if σ is a holomorphic symplectic form, the form $(\sigma+\overline{\sigma})$ is non-zero. $\square$

Example 5.3. If X is of K3$^{[2]}$-type, for dimensional reasons, the Verbitsky component $SH(X)$ is the only irreducible component in the cohomology, that is we have an equality $H^*(X,\mathbb{Q})) = SH(X)$.

6. Spin action

In this section we study how the action of $\mathfrak{so}(H^2(X,\mathbb{Q}),q)$ integrates to an action of the simply connected algebraic group $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$. Recall that there is an exact sequence of algebraic groups

$$1 \longrightarrow \pm 1 \longrightarrow \underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q) \longrightarrow \underline{\mathrm{SO}}(H^2(X,\mathbb{Q}),q) \longrightarrow 1.$$

Proposition 6.1. *The action of $\mathfrak{so}(H^2(X,\mathbb{Q}),q)$ on $H^*(X,\mathbb{Q})$ integrates to an action of the algebraic group $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ via ring homomorphism. On the even cohomology it induces an action of $\underline{\mathrm{SO}}(H^2(X,\mathbb{Q}),q)$.*

Proof. The first part of the statement is clear: we can always lift the action because the algebraic group $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ is simply connected, and it via ring homomorphism because the Lie algebra action is via derivation.

To show the second part of the statement we proceed as follows. We first notice that the inclusion $\mathfrak{so}(H^2(X,\mathbb{Q}),q) \subset \mathfrak{so}(\tilde{H}(X,\mathbb{Q}),\tilde{q})$ induces an inclusion

$$\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q) \subset \underline{\mathrm{Spin}}(\tilde{H}(X,\mathbb{Q}),\tilde{q}),$$

that sends -1 to -1. The last fact can be checked via the construction with the Clifford algebras. At this point it is not hard to show (see [Tae21, Lemma 5.1]) that $-1 \in \underline{\mathrm{Spin}}(\tilde{H}(X,\mathbb{Q}),\tilde{q})$ acts like $(-1)^k$ on $H^k(X,\mathbb{Q})$. Hence the action of $\underline{\mathrm{Spin}}(H^2(X,\mathbb{Q}),q)$ on even cohomology descends to an action of $\underline{\mathrm{SO}}(H^2(X,\mathbb{Q}),q)$. $\square$

References

[Bog96] F. A. Bogomolov. On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky). *Geom. Funct. Anal.*, 6(4):612–618, 1996.

[GYJLR20] Mark Green, Kim Yoon-Joo, Radu Laza, and Colleen Robles. The llv decomposition of hyper-kähler cohomology, 2020. arXiv:1906.03432.

[LL97] Eduard Looijenga and Valery A. Lunts. A Lie algebra attached to a projective variety. *Invent. Math.*, 129(2):361–412, 1997.

[Tae21] Lenny Taelman. Derived equivalences of hyperkähler varities, 2021. arXiv:1906.08081.

[Ver90] M. S. Verbitsky. Action of the Lie algebra of SO(5) on the cohomology of a hyper-Kähler manifold. *Funktsional. Anal. i Prilozhen. translation in Funct. Anal. Appl*, 24(3):70–71, 1990.

[Ver95] M. S. Verbitsky. Cohomology of compact hyperkaehler manifolds, 1995. PhD thesis at Harvard University.

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