GUAN'S BOUNDS FOR BETTI NUMBERS OF HYPERKÄHLER MANIFOLD OF DIMENSION 4

PIETRO BERI

1. A bound on b_2

Theorem 1.1. [2, Theorem 1] *Let X be an irreducible compact hyperkähler manifold of dimension 4. Then $b_2 \leq 23$. Moreover, if $b_2 = 23$ the Hodge diamond of X is the same as the Hodge diamond of the Hilbert square of a K3 surface.*

Let X be an irreducible compact hyperkähler manifold of complex dimension $2n$; here we will always refer to the complex dimension. Let σ be a symplectic form on X and let g be the Riemannian metric on X. For I, J, K a basis of the quaternions $\mathbb{H}$ acting on the tangent bundle of X, we will consider the complex structures on X in the form $aI + bJ + cK$ for $a^2 + b^2 + c^2 = 1$. We have $b_1 = 0$, $b_2 \geq 3$ since $H^{2,0}(X) = \sigma \cdot \mathbb{C}$, $H^{0,2}(X) = \bar{\sigma} \cdot \mathbb{C}$ and g is Kähler with respect to any complex structure defined above. About the higher Betti numbers, we have the following result by Verbitsky.

Theorem 1.2. [6, Theorem 1.5] *For $k \leq n$, the canonical map $\operatorname{Sym}^k H^2(X, \mathbb{R}) \to H^{2k}(X, \mathbb{R})$ given by the cup product is injective.*

We denote by $H^{(2k)} \subset H^{2k}(X, \mathbb{C})$ the image of the map above.

Guan found a sharp bound for b_2 when X has dimension 4. To show it, we will need the Riemann-Roch formula for irreducible compact hyperkähler manifolds of dimension $2n$, proved by Salamon, see [5, Theorem 4.1],

$$nb_{2n} = 2\sum_{j=1}^{2n}(-1)^j(3j^2 - n)b_{2n-j}. \tag{1}$$

Proof of Theorem 1.1. For $n = 2$ then (1) reads $b_3 + b_4 = 46 + 10b_2$. We also have $b_4 \geq \frac{b_2(b_2+1)}{2}$, hence

$$b_3 + \frac{b_2(b_2+1)}{2} \leq 46 + 10b_2 \tag{2}$$

$$\frac{b_2(b_2+1)}{2} \leq 46 + 10b_2$$

$$b_2^2 - 19b_2 - 23 \cdot 4 \leq 0 \;\Leftrightarrow\; (b_2 + 4)(b_2 - 23) \leq 0.$$

So $b_2 \leq 23$. Now take $b_2 = 23$: substituting in the inequality above we get $b_3 + 276 \leq 46 + 230 = 276$, so $b_3 = 0$, which is equal to the third Betti number of a Hilbert square on a K3 surface. Now (1) gives b_4, which is equal to the fourth Betti number of a Hilbert square since the formula only depends on b_2 and b_3. Because of the symmetries in the Hodge diamond of a hyperkähler manifold, the only Hodge nubers for $n = 2$ are $h^{1,1}$, $h^{2,1}$ and $h^{2,2}$. It is then sufficient to observe that the $h^{p,q}$

are completely determined by the Betti numbers i.e. $h^{1,1} = b_2 - 2$, $h^{2,1} = b_3/2$, $h^{2,2} = b_4 - 2(1 + h^{1,1})$. $\square$

2. The generalized Chern number $N(c_2)$

For an irreducible compact hyperkähler manifold X of dimension $2n$, we have the Beauville-Bogomolov-Fujiki quadratic form q on $H^2(X, \mathbb{Q})$. We refer to [1] or [4, Section 1.11]. There exists a rational constant $c_X > 0$ such that

$$\int_X \beta^{2n} = c_X q(\beta)^n \text{ for any } \beta \in H^2(X, \mathbb{Q}). \tag{3}$$

More generally, let $\alpha \in H^{4j}(X, \mathbb{C})$ be of type $(2j, 2j)$ on all small deformations of X. Then there is a constant $c_\alpha \in \mathbb{C}$ such that

$$\int_X \alpha\beta^{2(n-j)} = c_\alpha q(\beta)^{n-j} \text{ for any } \beta \in H^2(X, \mathbb{C}). \tag{4}$$

For $\alpha = 1$, (4) gives exactly (3).

Remark 2.1. The Chern classes of X are classes of type $(2j, 2j)$ on all small deformations, since Chern classes are constant under small deformations. This happens because they are integral, hence they lie in a discrete subset of $H^2(X, \mathbb{C})$.

We can now define the generalized Chern numbers.

Definition 2.2. *Let $C \in H^{4j}(X, \mathbb{C})$ be a polynomial in the Chern classes. We call* generalized Chern number of degree $4j$ *the number*

$$N(C) = \frac{\int_X Cu^{2(n-j)}}{\left(\int_X u^{2n}\right)^{\frac{n-j}{n}}}$$

for any $u \in H^2(X, \mathbb{C})$ with $\int_X u^{2n} \neq 0$.

We prove that the definition does not depend on the choice of u. By Remark 2.1 we have $\int_X Cu^{2(n-j)} = aq(u)^{n-j}$, where a is the sum of the c_α as in (4) for all monomials α in C. Moreover, $\int_X u^{2n} = c_X q(u)^n$, so $N(C) = a/c_X^{\frac{n-j}{n}}$.

In our case, $n = 2$, we are interested in the generalized Chern number $N(c_2(X))$ of degree 4. Guan rewrote [3, (1)] as follows.

Lemma 2.3. [2, Lemma 2] *Let X be an irreducible compact hyperkähler manifold of dimension $2n$. Then*

$$\frac{((2n)!)^{n-1} N(c_2(X))^n}{(24n(2n-2)!)^n} = \int_X \mathrm{td}^{\frac{1}{2}}(X). \tag{5}$$

Moreover $N(c_2(X)) > 0$.

Proof. (Hitchin and Sawon, and then Guan, use the $\hat{A}^{\frac{1}{2}}$-genus instead of $\mathrm{td}^{\frac{1}{2}}$. In general $\hat{A} = e^{c_1/2}\,\mathrm{td}$, in our case they coincide since $c_1 = 0$.)
We know that, for any hyperkähler manifold, $\int_X (\sigma + \bar{\sigma})^{2n} = c_X q(\sigma + \bar{\sigma})^n > 0$. Hence we can write

$$N(c_2) = \frac{\int_X c_2(\sigma + \bar{\sigma})^{2n-2}}{\left(\int_X (\sigma + \bar{\sigma})^{2n}\right)^{\frac{n-1}{n}}}.$$

The result is a rearrangement[1] of [3, (1)],

$$\frac{1}{(192\pi^2 n)^n}\frac{\|R\|^{2n}}{\mathrm{Vol}(X)^{n-1}} = \int_X \mathrm{td}^{\frac{1}{2}}(X) \tag{6}$$

where

- $\mathrm{Vol}(X)$ is the volume form on X, $\mathrm{Vol}(X) = \frac{1}{2^{2n}(2n)!}\int_X (\sigma+\bar\sigma)^{2n}$,
- $\|R\|$ the $\mathcal{L}^2$ norm of the Riemann curvature tensor,

$$\|R\|^2 = \frac{8\pi^2}{2^{2n-2}(2n-2)!}\int_X c_2(\sigma+\bar\sigma)^{2n-2}.$$

Note that $\int_X c_2(\sigma+\bar\sigma)^{2n-2} \geq 0$ since it is, up to a positive constant, equal to $\|R\|^2$. If $\int_X c_2(\sigma+\bar\sigma)^{2n-2} = 0$ then X would be flat, hence a torus by Bieberbach Theorem, absurd.

□

We write $q(\cdot,\cdot)$ for the bilinear form associated to q; there is a non-degenerate scalar product on $\mathrm{Sym}^2 H^2(X,\mathbb{C})$ given by

$$\langle [v_1\otimes v_2],[w_1\otimes w_2]\rangle \mapsto \frac{1}{2}(q(v_1,w_1)q(v_2,w_2)+q(v_1,w_2)q(v_2,w_1)). \tag{7}$$

We denote by $q^\vee$ the unique element of $\mathrm{Sym}^2 H^2(X,\mathbb{C})$ such that $\langle [v_1\otimes v_2], q^\vee\rangle = q(v_1,v_2)$ for every $v_1,v_2\in H^2(X,\mathbb{C})$. We call $q^\vee$ also the corresponding element of $H^{(4)}$.

Let $\alpha \in H^{(4)}$ be of type $(2,2)$ on all small deformations of X. Then α is a multiple of $q^\vee$. To prove it, consider the quadratic form on $H^2(X,\mathbb{C})$ given by $\beta \mapsto \int_X \alpha\beta^{2(n-1)}$: by (4) the form is a multiple of q, so up to a multiple α satisfies the condition defining $q^\vee$.
To show that $q^\vee$ is of type $(2,2)$: the coefficient of $q^\vee$ on $H^{4,0}$ (resp. $H^{0,4}$) is zero since it is generated by $\mathrm{Sym}^2 H^{2,0}$ (resp. $\mathrm{Sym}^2 H^{0,2}$) and $q(\sigma)=0$ (resp. $q(\bar\sigma)=0$). The coefficients of $q^\vee$ on $H^{3,1}$ (resp. $H^{1,3}$) are zero since $q(\sigma,x)=0$ (resp. $q(\bar\sigma,x)=0$) for every $x\in h^{1,1}$.

Proposition 2.4. [2, Lemma 3] *Let X be an irreducible compact hyperkähler manifold of dimension 4. Then*

$$3b_2 N(c_2(X))^2 \leq (b_2+2)c_2^2. \tag{8}$$

The equality holds if and only if $c_2(X)\in H^{(4)}$.

Proof. The orthogonal complement of $H^{(4)}$ with respect to the intersection form is the primitive chomology $H^4_{\mathrm{prim}}(X,\mathbb{C})$ i.e. $H^4(X,\mathbb{C}) = H^{(4)}\oplus H^4_{\mathrm{prim}}(X,\mathbb{C})$. We can write $c_2(X) = \lambda q^\vee + r$ for some $\lambda\in\mathbb{C}^*$, $r\in H^4_{\mathrm{prim}}(X,\mathbb{C})$. By the second Hodge-Riemann bilinear relations, the intersection form is positive on r and vanishes if and only if $r=0$, since r is still of type $(2,2)$ and is primitive. So we have

$$c_2^2 = \lambda^2\int_X (q^\vee)^2 + \int_X r^2 \geq \lambda^2 \int_X (q^\vee)^2.$$

[1]Hitchin and Sawon use a different convention for exterior products of differential forms. The two conventions differ by a binomial coefficient: if we use Hitchin and Sawon's formulation for $\mathrm{Vol}(X)$ and $\|R\|$, in terms of $\sigma^n\bar\sigma^n$ and $\sigma^{n-1}\bar\sigma^{n-1}$ respectively, then (5) becomes $\frac{((2n)!)^{n-1}N(c_2(X))^n}{(24n(2n-2)!)^n}\cdot\frac{\binom{2(n-1)}{n-1}^n}{\binom{2n}{n}^{n-1}} = \int_X \mathrm{td}^{\frac{1}{2}}(X)$.

But $\lambda^2 \int_X (q^\vee)^2 = \lambda \int_X (\lambda q^\vee + r) q^\vee = \lambda \int_X c_2(X) q^\vee$ and so

$$c_2^2 \geq \lambda^2 \int_X (q^\vee)^2 = \frac{(\lambda \int_X c_2(X) q^\vee)^2}{\lambda^2 \int_X (q^\vee)^2} = \frac{(\int_X c_2(X) q^\vee)^2}{\int_X (q^\vee)^2}.$$

Now let $\{e_i\}_{i=1,\dots,b_2}$ be a orthonormal basis of $H^2(X,\mathbb{C})$ with respect to q. Then we use (7) to compute the coefficients of $q^\vee$ in the basis $\{[e_i \otimes e_j]\}_{i,j}$ and we obtain $\langle [e_i \otimes e_j], q^\vee \rangle = \delta_{i,j}$ i.e. $q^\vee = \sum_{i=1,\dots,b_2} e_i^2$ in $\operatorname{Sym}^2 H^2(X,\mathbb{C})$. So we have

$$q(e_i) = 1 \Longrightarrow \int_X e_i^4 = c_X.$$

For $i \neq j$

$$4c_X = q(e_i + e_j)^2 c_X = \int_X (e_i + e_j)^4 = \int_X (e_i^4 + 4e_i^3 e_j + 6e_i^2 e_j^2 + 4e_i e_j^3 + e_j^4),$$

$$4c_X = \int_X (e_i - e_j)^4 = \int_X (e_i^4 - 4e_i^3 e_j + 6e_i^2 e_j^2 - 4e_i e_j^3 + e_j^4)$$

In turn the two equalities above implies $c_X - 3\int_X e_i^2 e_j^2 = \int_X e_i^3 e_j + e_i e_j^3 = -(c_X - 3\int_X e_i^2 e_j^2)$. So

$$c_X - 3\int_X e_i^2 e_j^2 = 0 \Longleftrightarrow \int_X e_i^2 e_j^2 = \frac{c_X}{3}.$$

Now we can compute

$$\int_X (q^\vee)^2 = \int_X \left(\sum_{i=1,\dots,b_2} e_i^2 \right)^2 = b_2(b_2-1)\frac{c_X}{3} + b_2 c_X = \frac{b_2(b_2+2)}{3} c_X,$$

$$\int_X c_2(X) q^\vee = \sum_{i=1,\dots,b_2} \int_X c_2(X) e_i^2 = \sum_{i=1,\dots,b_2} \frac{\int_X c_2(X) e_i^2}{(\int_X e_i^4)^{\frac{1}{2}}} \cdot \left(\int_X e_i^4 \right)^{\frac{1}{2}} = b_2 N(c_2(X)) (c_X)^{\frac{1}{2}}.$$

Putting all together we obtain

$$c_2^2 \geq \frac{(\int_X c_2(X) q^\vee)^2}{\int_X (q^\vee)^2} = 3b_2 N(c_2(X))^2/(b_2+2).$$

Finally, the equality holds if and only if $\int_X r^2 = 0$ if and only if $r = 0$, $c_2(X) = \lambda q^\vee \in H^{(4)}$. $\square$

3. Bounds on b_3

Consider again X of dimension 4. A formal computation shows

$$\int_X \operatorname{td}^{\frac{1}{2}}(X) = \operatorname{td}^{\frac{1}{2}}(X)_4 = \frac{1}{5760}(7c_2^2 - 4c_4). \tag{9}$$

Moreover, by Hirzebruch-Riemann-Roch we have

$$3 = \chi(X, \mathcal{O}_X) = \int_X \operatorname{td}(X) = \operatorname{td}(X)_4 = \frac{1}{720}(3c_2^2 - c_4). \tag{10}$$

Theorem 3.1. [2, Theorem 2] *Let X be an irreducible compact hyperkähler manifold of dimension 4. Then*

$$b_3 \leq \frac{4(23 - b_2)(8 - b_2)}{b_2 + 1}. \tag{11}$$

If $b_2 > 7$, then $(b_2, b_3) \in \{(8,0),(23,0)\}$.

Proof. We substitute Lemma 2.3, with $n = 2$, in Proposition 2.4 to obtain

$$3b_2 \frac{(24 \cdot 4)^2}{4!} \int_X \mathrm{td}^{\frac{1}{2}}(X) \leq (b_2 + 2)c_2^2.$$

By substituting in (9) the expression for $e(X) = c_4$ given by (10) we get

$$\int_X \mathrm{td}^{\frac{1}{2}}(X) = \frac{1}{5760}(7c_2^2 - 4(3c_2^2 - 720 \cdot 3)) = \frac{3}{2} - \frac{c_2^2}{1152}.$$

Hence

$$(b_2 + 2)c_2^2 \geq 2 \cdot 24^2 b_2 \int_X \mathrm{td}^{\frac{1}{2}}(X) = 2 \cdot 24^2 b_2 (\frac{3}{2} - \frac{c_2^2}{1152}) = b_2(3 \cdot 24^2 - c_2^2). \tag{12}$$

We have $h^{1,1} - 2h^{2,1} = \chi^1 = 12 - \frac{c_4}{6}$, see Olivier's talk; using

$$b_2 = 2 + h^{1,1} \qquad b_3 = 2h^{1,2}$$

a simple computation gives $c_4 = 3(16 + 4b_2 - b_3)$. We use this in (10) to have $c_2^2 = 736 + 4b_2 - b_3$. Then (12) becomes $(b_2 + 1)b_3 \leq 4(23 - b_2)(8 - b_2)$ as in the statement,

If $b_2 > 7$ then the RHS of (11) is at most zero, since $b_2 \leq 23$, so it has to be zero. □

Corollary 3.2. [2, Corollary 1] *If $b_2 \leq 7$, then one of the following holds:*

- $b_2 = 3$ *and* $b_3 = 4\ell$ *with* $\ell \leq 17$*;*
- $b_2 = 4$ *and* $b_3 = 4\ell$ *with* $\ell \leq 15$*;*
- $b_2 = 5$ *and* $b_3 = 4\ell$ *with* $\ell \leq 9$*;*
- $b_2 = 6$ *and* $b_3 = 4\ell$ *with* $\ell \leq 4$*;*
- $b_2 = 7$ *and* $b_3 = 4\ell$ *with* $\ell \in \{0, 2\}$*.*

Proof. By [1, Lemma 1.2], $4|b_k$ for k odd. Then the bounds are obtained using either (2) or (11). Guan proved in [2] that the case $(b_2, b_3) = (7, 4)$ cannot occur. □

Remark 3.3. When $b_2 = 7$, either $b_3 = 0$ or the Hodge diamond of X is the same of the Hodge diamond of a Kummer variety.

Remark 3.4. Given a couple (b_2, b_3), it is possible to compute $N(c_2)$ using Lemma 2.3, since the Chern numbers of X are computed in the proof of Theorem 3.1. Then it is possible to check which couples give an equality in (8). Hence, using Proposition 2.4, one can check that $c_2 \in H^{(4)}$ if and only if $(b_2, b_3) \in \{(5, 36), (7, 8), (8, 0), (23, 0)\}$.

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