DERIVED CATEGORIES OF HYPER-KÄHLER VARIETIES VIA THE LLV ALGEBRA

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Abstract. We review work of Taelman [Tae81] on derived categories of hyper-Kähler varieties. More precisely, we study the LLV algebra from a different perspective to prove that it is a derived invariant. Applications to the action of derived equivalences on cohomology and the study of their Hodge structures are given.

1. Introduction

We present the first part of Taelman's paper [Tae81]. The main incentive and goal is to understand better the bounded derived category $\mathrm{D}^b(X) := \mathrm{D}^b(\mathrm{Coh}(X))$ and its group of autoequivalences $\mathrm{Aut}(\mathrm{D}^b(X))$.

These notes are, however, light on derived categories and focus more on a different persepective of the Looijenga–Lunts–Verbitsky (LLV) Lie algebra $\mathfrak{g}(X)$ [Ver96, LL97] which will allow us to show the following.

Theorem 1.1. *A derived equivalence $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ between projective hyper-Kähler manifolds induces naturally a Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \simeq \mathfrak{g}(Y)$$

which is equivariant for the induced isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X,\mathbb{Q}) \simeq \mathrm{H}^*(Y,\mathbb{Q}).$$

Notation. Throughout these notes X and Y will be hyper-Kähler manifolds of dimension $2n$. All functors will be implicitely derived.

2. Quick review of derived categories

Let us recall one of the most important results in the study of derived equivalences due to Orlov [Orl97].

Theorem 2.1. *Let X and Y be smooth projective varieties and $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ an exact derived equivalence. Then Φ is isomorphic to a Fourier–Mukai functor, i.e. there exists $\mathcal{E} \in \mathrm{D}^b(X \times Y)$ such that*

$$\Phi \simeq \mathrm{FM}_{\mathcal{E}} := p_{Y*} \circ \mathcal{E} \otimes \_ \circ p_X^*.$$

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Moreover, a derived equivalence as in the theorem naturally induces isomorphisms of several invariants associated to the varieties such as (topological) K-theory. For us the most important invariant will be singular cohomology. Namely, every equivalence $\mathrm{FM}_{\mathcal{E}}$ induces a cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ given by the correspondence $v(\mathcal{E}) \in \mathrm{H}^*(X \times Y)$ where $v = \mathrm{ch}(\_)\mathrm{td}$ is the Mukai vector. These are compatible via the Mukai vector, i.e. the following diagram commutes

$$\begin{tikzcd} \mathrm{D}^b(X) \arrow[r, "\Phi"] \arrow[d, "v"] & \mathrm{D}^b(Y) \arrow[d, "v"] \\ \mathrm{H}^*(X, \mathbb{Q}) \arrow[r, "\Phi^{\mathrm{H}}"] & \mathrm{H}^*(Y, \mathbb{Q}). \end{tikzcd} \tag{2.1}$$

Hence, the study of derived categories leads naturally to cycles on hyper-Kähler varieties.

Remark 2.2. Let us mention properties of the cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$.

- Since $v(\mathcal{E}) \in \oplus_p \mathrm{H}^{p,p}(X \times Y)$, the isomorphism $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects the weight zero Hodge structure on $\mathrm{H}^*(X \times Y)$ for which a class x is in $\mathrm{H}^{-i,i}(X)$ if and only if in the decomposition given by the usual Hodge decomposition every component of x lies in $\mathrm{H}^{p,q}(X)$ with $q - p = -i$.
- The isomorphism $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ respects the generalized Mukai pairing, see [Căl03].
- The cohomological Fourier–Mukai transform $\mathrm{FM}_{\mathcal{E}}^{\mathrm{H}}$ does not respect neither the cup product structure on cohomology nor the cohomological grading.

3. Recollection of the LLV Lie algebra

We quickly recall the definition of the LLV Lie algebra. For a more thorough discussion we refer to the Bottini's notes.

Let X be a hyper-Kähler manifold and $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ a cohomology class. We attach to it the operator

$$e_\lambda := \lambda \cup \_ \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$$

given by cup product with the class λ. We say that λ has the *Hard Lefschetz property*, if for all i the maps

$$e_\lambda^i \colon \mathrm{H}^{2n-i}(X, \mathbb{Q}) \longrightarrow \mathrm{H}^{2n+i}(X, \mathbb{Q})$$

are isomorphisms. We denote by $h \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ the grading operator acting on $\mathrm{H}^i(X, \mathbb{Q})$ via $(i - 2n)\mathrm{id}$.

For a Hard Lefschetz class $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$, the triple

$$(e_\lambda, h, f_\lambda),$$

where f_λ is the dual Lefschetz operator, is isomorphic to the Lie algebra $\mathfrak{sl}_2$.

Definition 3.1. The LLV Lie algebra $\mathfrak{g}(X)$ is the Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X, \mathbb{Q}))$ generated by all $\mathfrak{sl}_2$-triples $(e_\lambda, h, f_\lambda)$ for $\lambda \in \mathrm{H}^2(X, \mathbb{Q})$ Hard Lefschetz.

As said in the beginning, we refer to Bottini's notes or [LL97] for more details and properties of $\mathfrak{g}(X)$. Our main goal is to relate the Lie algebra $\mathfrak{g}$ with $\mathrm{D}^b(X)$. Note that since a cohomological Fourier–Mukai functor does not respect cup product nor grading, which are the defining properties of the LLV algebra, it is a priori not clear how this can be done. The main ingredient for it is the ring of polyvector fields, to be introduced now.

4. Polyvector fields

Definition 4.1. The *ring of polyvector fields* $\mathrm{HT}^*(X)$ is the graded $\mathbb{C}$-algebra whose degree k part is

$$\mathrm{HT}^k := \oplus_{p+q=k} \mathrm{H}^q(X, \Lambda^p \mathcal{T}_X).$$

The ring structure is induced by the one from the exterior algebra.

For X a hyper-Kähler variety we can choose a symplectic form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ which induces isomorphisms

$$\Lambda^p \mathcal{T}_X \simeq \Omega_X^p$$

which induce a $\mathbb{C}$-algebra isomorphism

$$\mathrm{HT}^*(X) = \mathrm{H}^*(X, \Lambda^* \mathcal{T}_X) \simeq \mathrm{H}^*(X, \Omega_X^*) \simeq \mathrm{H}^*(X, \mathbb{C}). \tag{4.1}$$

Thus, as a $\mathbb{C}$-algebra, the ring of polyvectors is isomorphic to the de Rham cohomology.

In this note, we are more interested in another viewpoint of the polyvector fields. Namely, the ring of polyvectors acts on the de Rham cohomology by contraction, i.e. given $v \in \mathrm{H}^q(X, \Lambda^p \mathcal{T}_X)$ and $x \in \mathrm{H}^{q'}(X, \Omega_X^{p'})$ the action is

$$v \lrcorner x \in \mathrm{H}^{q+q'}(X, \Omega_X^{p'-p}).$$

The following is immediate, see also [Tae81, Lem. 2.4]

Lemma 4.2. *For X a hyper-Kähler manifold the de Rham cohomology is a free module of rank one over the polyvector fields generated by a Calabi–Yau form $\sigma^n \in \mathrm{H}^0(X, \Omega_X^{2n})$.*

Hence, one may identify the ring of polyvector fields $\mathrm{HT}^*(X)$ with the de Rham cohomology $\mathrm{H}^*(X, \mathbb{C})$. Via this identification the grading on $\mathrm{HT}^*(X)$ corresponds to the Hodge grading on $\mathrm{H}(X, \mathbb{C})$. In this way, the de Rham cohomology obtains a new ring structure, which this time has horizontal grading in contrast to the vertical grading of the usual cup product in the standard representation of the Hodge diamond. We will make this more precise and formal using Lie algebras and operators in the next section.

The reason why the ring of polyvectors are of interest to us is the following crucial result. It relies on the modified Hochschild–Konstant–Rosenberg isomorphism identifying Hochschild (co)homology with polyvectors and the de Rham cohomology [CRVdB12].

Theorem 4.3. *A derived equivalence* $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ *induces naturally a* $\mathbb{C}$*-algebra isomorphism* $\Phi^{\mathrm{HT}}\colon \mathrm{HT}^*(X) \simeq \mathrm{HT}^*(Y)$ *such that the action of the polyvector fields is equivariant for the induced isomorphism* $\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X\mathbb{C}) \simeq \mathrm{H}^*(Y,\mathbb{C})$.

Spelling this out, for $v \in \mathrm{HT}^*(X)$ and $x \in \mathrm{H}^*(X,\mathbb{C})$ we have

$$\Phi^{\mathrm{H}}(v \lrcorner x) = \Phi^{\mathrm{HT}}(v) \lrcorner \Phi^{\mathrm{H}}(x) \in \mathrm{H}^*(Y,\mathbb{C}).$$

5. Reinventing the LLV Lie algebra

We make more precise the observations from the last section and define a new Lie algebra, which will turn out to be isomorphic to $\mathfrak{g}(X)$ with extended scalars.

We consider the holomorphic grading operator h_p and the antihomolorphic grading operator h_q defined by acting on $\mathrm{H}^{k,l}(X)$ via

$$h_p = (k-n)\mathrm{id}, \quad h_q = (l-n)\mathrm{id}.$$

With these definitions the usual grading operator h for the cohomological grading is just $h = h_p + h_q$. We define the Hodge grading operator $h' := h_q - h_p$.

With this definition the action of the polyvector fields $\mathrm{HT}^*(X)$ on the de Rham cohomology $\mathrm{H}^*(X,\mathbb{C})$ has degree two with respect to the grading h'.

For $\mu \in \mathrm{HT}^2(X)$ we define the operator $e_\mu := \mu \lrcorner \_ \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$. We say that μ is Hard Lefschetz is the operator e_μ satisfies the Hard Lefschetz isomorphisms with respect to the grading operator h'. The Jacobson–Morozov theorem asserts that this is equivalent to the existence of an operator $f_\mu \in \mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$ such that

$$(e_\mu, h', f_\mu)$$

forms an $\mathfrak{sl}_2$-triple.

Definition 5.1. The complex Lie algebra $\mathfrak{g}'(X)$ is defined to be the smallest Lie subalgebra of $\mathrm{End}(\mathrm{H}^*(X,\mathbb{C}))$ containing all $\mathfrak{sl}_2$-triples (e_μ, h', f_μ) for all Hard Lefschetz $\mu \in \mathrm{HT}^2(X)$.

Equivalently, one could have defined the Lie algebra $\mathfrak{g}'$ as the Lie subalgebra of $\mathrm{End}(\mathrm{HT}^*(X))$ containing all $\mathfrak{sl}_2$-triples with μ Hard Lefschetz. Through the isomorphism

$$\mathrm{HT}^*(X) \lrcorner \sigma^n \simeq \mathrm{H}^*(X,\mathbb{C})$$

these two definitions are identified.

Recall from (4.1) that the choice of a symplectic form yields an abstract graded $\mathbb{C}$-algebra isomorphism

$$\mathrm{HT}^*(X) \simeq \mathrm{H}^*(X,\Omega_X^*) \simeq \mathrm{H}^*(X,\mathbb{C}).$$

Thus, the choice of a symplectic form leads to the following isomorphism.

Lemma 5.2. *There is an isomorphism of Lie algebras*

$$\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathfrak{g}'(X).$$

We also deduce the following consequence from Theorem 4.3.

Proposition 5.3. *For derived equivalent hyper-Kähler varieties* $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ *the isomorphism*

$$\Phi^{\mathrm{HT}}\colon \mathrm{HT}^2(X) \simeq \mathrm{HT}^2(Y)$$

induces naturally a Lie algebra isomorphism

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}'(X) \simeq \mathfrak{g}'(Y)$$

which is equivariant for the induced isomorphism Φ^{H}.

Spelling this again out means that for $j \in \mathfrak{g}(X)$ and $x \in \mathrm{H}^*(X, \mathbb{C})$ we have

$$\Phi^{\mathrm{H}}(j.x) = \Phi^{\mathfrak{g}}(j).\Phi^{\mathrm{H}}(x) \in \mathrm{H}^*(Y, \mathbb{C}).$$

The connection between all that has been said so far and the main tool for all the applications we will present is the following main theorem of [Tae81] which was also implicitely proven (but not stated in the form below) by Verbitsky [Ver99].

Theorem 5.4. *The Lie algebras* $\mathfrak{g}(X)\otimes_{\mathbb{Q}}\mathbb{C}$ *and* $\mathfrak{g}'(X)$ *are equal as Lie subalgebras of* $\mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$.

Proof. We will sketch Taelman's approach.

From Lemma 5.2 we infer that it is enough to show only the inclusion

$$\mathfrak{g}'(X) \subset \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

A straight-forward calculation shows that

$$(e_\sigma, h_p, e_{\check{\sigma}})$$

is an $\mathfrak{sl}_2$-triple, where $\check{\sigma} \in \mathrm{H}^0(\Lambda^2(\mathcal{T}_X))$ is the dual symplectic form (note that the Lefschetz operator e_σ acts via cup product, whereas $e_{\check{\sigma}}$ acts by contraction of polyvector fields).

Analogously or by Hodge symmetry, for the complex conjugate form $\bar{\sigma} \in \mathrm{H}^2(X, \mathcal{O}_X)$ the operator $e_{\bar{\sigma}}$ has the Hard Lefschetz property for the grading operator h_q. The Jacobson–Morozov Theorem grants the existence of an operator $g \in \mathrm{End}(\mathrm{H}^*(X, \mathbb{C}))$ such that

$$(e_{\bar{\sigma}}, h_q, g)$$

forms an $\mathfrak{sl}_2$-triple. An easy check shows that all elements from the $\mathfrak{sl}_2$-triple $(e_\sigma, h_p, e_{\check{\sigma}})$ commute with all elements from the $\mathfrak{sl}_2$-triple $(e_{\bar{\sigma}}, h_q, g)$. Thus we obtain two new $\mathfrak{sl}_2$-triples

$$(e_\sigma + e_{\bar{\sigma}}, h, e_{\check{\sigma}} + g), \quad (e_\sigma - e_{\bar{\sigma}}, h, e_{\check{\sigma}} - g).$$

This gives that $e_{\check{\sigma}} \in \mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. Since $[e_\sigma, e_{\check{\sigma}}] = h_p$, we deduce furthermore that h_p, h_q and therefore h' are all contained inside $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$.

Since evidently $e_{\bar{\sigma}}$ is also contained in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$ (the contraction action agrees with the cup product), it is left to show that for almost all $\mu \in \mathrm{H}^1(X, \mathcal{T}_X)$ the operator e_μ lies in $\mathfrak{g}(X) \otimes_{\mathbb{Q}} \mathbb{C}$. This follows from the identity

$$[e_{\check{\sigma}}, e_\eta] = e_\mu$$

for $\eta \in \mathrm{H}^1(X, \Omega_X)$ satisfying

$$\mu = \check{\sigma} \lrcorner \eta \in \mathrm{H}^1(X, \mathcal{T}_X)$$

which follows from a calculation, see [Tae81, Lem. 2.13]. □

The theorem implies that the isomorphism from Proposition 5.3 is already defined over $\mathbb{Q}$, since the same holds for the induced isomorphism on singular cohomology.

Corollary 5.5. *A derived equivalence $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ between hyper-Kähler varieties induces naturally a Lie algebra isomorphism*

$$\Phi^{\mathfrak{g}}\colon \mathfrak{g}(X) \simeq \mathfrak{g}(Y)$$

which is equivariant for the induced isomorphism

$$\Phi^{\mathrm{H}}\colon \mathrm{H}^*(X, \mathbb{Q}) \simeq \mathrm{H}^*(Y, \mathbb{Q}).$$

6. Verbitsky component and extended Mukai lattice

We want to draw consequences from Theorem 5.4 for the study of derived equivalences of hyper-Kähler manifolds and their induced actions on cohomology.

Definition 6.1. The *Verbitsky component* $\mathrm{SH}(X, \mathbb{Q}) \subset \mathrm{H}^*(X, \mathbb{Q})$ is the subalgebra generated by $\mathrm{H}^2(X, \mathbb{Q})$.

It is known that the Verbitsky component is an irreducible representation of the LLV Lie algebra and it is characterized as such as the irreducible representation whose Hodge strucutre attains the maximal possible width.

Corollary 6.2. *For a derived equivalence $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ between hyper-Kähler manifolds the induced isomorphism Φ^{H} restricts to a Hodge isometry*

$$\Phi^{\mathrm{SH}}\colon \mathrm{SH}(X, \mathbb{Q}) \simeq \mathrm{SH}(Y, \mathbb{Q}).$$

The pairing on the Verbitsky component is the *Mukai pairing* b_{SH} defined via

$$b_{\mathrm{SH}}(\lambda_1 \ldots \lambda_m, \mu_1 \ldots \mu_{2n-m}) := \int_X \lambda_1 \ldots \lambda_m \mu_1 \ldots \mu_{2n-m}$$

for classes $\lambda_i, \mu_j \in \mathrm{H}^2(X, \mathbb{Q})$.

We want to study the Verbitsky component and the LLV Lie algebra more closely to further refine the study of $\mathrm{Aut}(\mathrm{D}^b(X))$.

Definition 6.3. The *extended Mukai lattice* $(\tilde{\mathrm{H}}(X,\mathbb{Q}),\tilde{b})$ is the rational quadratic vector space defined by

$$\mathbb{Q}\alpha \oplus \mathrm{H}^2(X,\mathbb{Q}) \oplus \mathbb{Q}\beta.$$

The quadratic form $\tilde{b}$ on $\tilde{\mathrm{H}}(X,\mathbb{Q})$ restricts to the BBF form b on $\mathrm{H}^2(X,\mathbb{Q})$ and the two classes α and β are orthogonal to $\mathrm{H}^2(X,\mathbb{Q})$ and satisfy $\tilde{b}(\alpha,\beta) = -1$ as well as $\tilde{b}(\alpha,\alpha) = \tilde{b}(\beta,\beta) = 0$.

Furthermore, we define on $\tilde{\mathrm{H}}(X,\mathbb{Q})$ a grading by declaring α to be of degree -2, $\mathrm{H}^2(X,\mathbb{Q})$ sits in degree zero and β is of degree two. Finally, the extended Mukai lattice is equipped with a weight-two Hodge structure

$$\begin{aligned}
(\tilde{\mathrm{H}}(X,\mathbb{Q})\otimes\mathbb{C})^{2,0} &:= \mathrm{H}^{2,0}(X)\\
(\tilde{\mathrm{H}}(X,\mathbb{Q})\otimes\mathbb{C})^{0,2} &:= \mathrm{H}^{0,2}(X)\\
(\tilde{\mathrm{H}}(X,\mathbb{Q})\otimes\mathbb{C})^{1,1} &:= \mathrm{H}^{1,1}(X)\oplus\mathbb{C}\alpha\oplus\mathbb{C}\beta.
\end{aligned}$$

Verbitsky [Ver96] proved the existence of a graded morphism $\psi\colon \mathrm{SH}(X,\mathbb{Q}) \longrightarrow \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ sitting in a short exact sequence

$$0 \longrightarrow \mathrm{SH}(X,\mathbb{Q}) \xrightarrow{\psi} \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) \xrightarrow{\Delta} \mathrm{Sym}^{n-2}(\tilde{\mathrm{H}}(X,\mathbb{Q})) \longrightarrow 0.$$

Here, the map Δ is the Laplacian operator defined on pure tensors via

$$v_1\cdots v_n \longmapsto \sum_{i<j} \tilde{b}(v_i,v_j) v_1\cdots\hat{v}_i\cdots\hat{v}_j\cdots v_n.$$

The map ψ is uniquely determined (up to scaling) by the condition that it is a morphism of $\mathfrak{g}(X)$-modules.

The $\mathfrak{g}(X)$-structure of $\tilde{\mathrm{H}}(X,\mathbb{Q})$ is defined by the conditions $e_\omega(\alpha) = \omega$, $e_\omega(\mu) = b(\omega,\mu)\beta$ and $e_\omega(\beta) = 0$ for all classes $\omega,\mu \in \mathrm{H}^2(X,\mathbb{Q})$. The n-th symmetric power $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$ then inherits the structure of a $\mathfrak{g}(X)$-module by letting $\mathfrak{g}(X)$ act by derivations. We fix once and for all a choice of ψ by setting $\psi(1) = \alpha^n/n!$.

Taelman [Tae81, Sec. 3] showed that the map ψ is an isometry with respect to the Mukai pairing on $\mathrm{SH}(X,\mathbb{Q})$ and the pairing

$$b_{[n]}(x_1\cdots x_n, y_1\cdots y_n) = (-1)^n c_X \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n \tilde{b}(x_i, y_{\sigma(i)})$$

on $\mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q}))$, where c_X is the Fujiki constant characterized by the property

$$\int_X \omega^{2n} = c_X \frac{(2n)!}{2^n n!} b(\omega,\omega)^n$$

for all $\omega \in \mathrm{H}^2(X,\mathbb{Q})$. Note that our definition of $b_{[n]}$ differs from Taelman's definition by the Fujiki constant. Ours has the advantage that ψ is always an isometry.

Summing up, the inclusion ψ respects the

- $\mathfrak{g}(X)$-structure,

- quadratic forms,
- Hodge structures,
- gradings.

7. Action of derived equivalences on the extended Mukai lattice

Recall that we have deduced the existence of a representation

$$\rho^{\mathrm{SH}}\colon \mathrm{Aut}(\mathrm{D}^b(X)) \longrightarrow \mathrm{O}(\mathrm{SH}(X,\mathbb{Q}))$$

and the isometries in the image of this representation normalize the action of the LLV algebra $\mathfrak{g}(X)$, i.e. for these $g \in \mathrm{O}(\mathrm{SH}(X,\mathbb{Q}))$ we have

$$g\mathfrak{g}(X)g^{-1} = \mathfrak{g}(X) \subset \mathrm{End}(\mathrm{H}^*(X,\mathbb{Q})).$$

Let us study these automorphisms a bit further.

Definition 7.1. The group $\mathrm{Aut}(\mathrm{SH}(X,\mathbb{Q}), b_{\mathrm{SH}}, \mathfrak{g}(X))$ is the group of all isometries of the Verbitsky component that normalize the action of the LLV algebra.

The main representation-theoretic input for our discussion is the following result [Tae81, Sec. 4].

Proposition 7.2. *If n is odd or the second Betti number is odd, then*

$$\mathrm{Aut}(\mathrm{SH}(X,\mathbb{Q}), b_{\mathrm{SH}}, \mathfrak{g}(X)) \simeq \mathrm{O}(\tilde{\mathrm{H}}(X,\mathbb{Q})).$$

We make this isomorphism more explicit. Let X and Y be deformation-equivalent hyper-Kähler manifolds together with $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$. Then there exists a unique Hodge isometry

$$\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X,\mathbb{Q}) \simeq \tilde{\mathrm{H}}(Y,\mathbb{Q})$$

inducing the following commutative diagram

$$\begin{array}{ccc} \mathrm{SH}(X,\mathbb{Q}) & \xrightarrow{\det((\Phi^{\tilde{\mathrm{H}}}))^{n+1}\Phi^{\mathrm{SH}}} & \mathrm{SH}(Y,\mathbb{Q}) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathrm{Sym}^n(\tilde{\mathrm{H}}(X,\mathbb{Q})) & \xrightarrow{\mathrm{Sym}^n\Phi^{\tilde{\mathrm{H}}}} & \mathrm{Sym}^n(\tilde{\mathrm{H}}(Y,\mathbb{Q})). \end{array} \tag{7.1}$$

Remark 7.3. For all the known examples, derived equivalent hyper-Kähler varieties are deformation-equivalent. In general, however, it is not known whether derived equivalent hyper-Kähler varieties are deformation-equivalent. Without this assumption, the above proposition has to weakened as we shall demonstrate.

The (linear algebra) problem is that if there exists an abstract isometry between Verbitsky components of hyper-Kähler varieties which normalizes the LLV algebra, it is not true that the corresponding Mukai lattices must be isometric. As an example, take a rational quadratic vector space (V, q) and consider the quadratic space $(V, \lambda q)$ for $\lambda \in \mathbb{Q} \backslash \mathbb{Q}^2$. These spaces are not isometric, but the kernels of the respective Laplacians

$$\mathrm{Sym}^2(V) \xrightarrow{\Delta} \mathbb{Q}$$

are isometric.

One can, using similitudes, still formulate a version of the above in the general case. This will be needed in the next section for the application to Hodge structures.

Theorem 7.4. *Let X and Y be arbitrary hyper-Kähler varieties and $\Phi\colon \mathrm{D}^b(X) \simeq \mathrm{D}^b(Y)$ a derived equivalence. Then there exists a Hodge similitude $\Phi^{\tilde{\mathrm{H}}}\colon \tilde{\mathrm{H}}(X, \mathbb{Q}) \longrightarrow \tilde{\mathrm{H}}(Y, \mathbb{Q})$ and a scalar $\lambda \in \mathbb{Q}$ such that*

$$\begin{tikzcd} \mathrm{SH}(X, \mathbb{Q}) \arrow[r, "\Phi^{\mathrm{SH}}"] \arrow[d, "\psi"] & \mathrm{SH}(Y, \mathbb{Q}) \arrow[d, "\psi"] \\ \mathrm{Sym}^n(\tilde{\mathrm{H}}(X, \mathbb{Q})) \arrow[r, "\lambda \mathrm{Sym}^n \Phi^{\tilde{\mathrm{H}}}"] & \mathrm{Sym}^n(\tilde{\mathrm{H}}(Y, \mathbb{Q})) \end{tikzcd} \tag{7.2}$$

commutes.

8. Hodge structures

In this last section we want to give one application of the results presented so far regarding derived equivalent hyper-Kähler varieties and their Hodge structures.

We first want to recall a recent result of Soldatenkov [Sol21][1], whose statement and proof are similiar in flavour to what we will discuss afterwards for derived equivalences.

Theorem 8.1. *Let X and Y be arbitrary hyper-Kähler varieties and $\varphi\colon \mathrm{H}^2(X, \mathbb{Q}) \simeq \mathrm{H}^2(Y, \mathbb{Q})$ and isomorphism of $\mathbb{Q}$-Hodge structures, which is the restriction of a global algebra automorphism $\phi\colon \mathrm{H}^*(X, \mathbb{Q}) \simeq \mathrm{H}^*(Y, \mathbb{Q})$. Then for all $i \in \mathbb{Z}$ the restrictions*

$$\phi\colon \mathrm{H}^i(X, \mathbb{Q}) \simeq \mathrm{H}^i(Y, \mathbb{Q})$$

are isomorphisms of $\mathbb{Q}$-Hodge structures.

Proof. We briefly sketch the argument. Since ϕ is a graded algebra automorphism, the adjoint action yields an isomorphism

$$\mathrm{ad}(\phi)\colon \mathfrak{g}(X) \simeq \mathfrak{g}(Y).$$

The fact that ϕ is graded implies that $\mathrm{ad}(\phi)(h) = h$. Moreover, the restriction of ϕ to $\mathrm{H}^2(X, \mathbb{Q})$ respects the Hodge structures. This implies that $\mathrm{ad}(\phi)(h') = h'$. Since $h + h' = 2h_q$ and $h - h' = 2h_p$ we deduce $\mathrm{ad}(\phi)(h_p) = h_p$ and $\mathrm{ad}(\phi)(h_q) = h_q$. This is equivalent to ϕ being a morphism of $\mathbb{Q}$-Hodge structures. ☐

[1]We thank Andrey Soldatenkov for a stimulating conversation about his results.

The assertion that the isomorphism of Hodge structures is the restriction of a global algebra automorphism is frequently met. For example, Hodge isometries with positive determinant can be extended to algebra automorphisms of the even cohomology by integrating the LLV action. For more details and examples we refer to [Sol21].

With this in mind, we can now prove the following result of Taelman [Tae81, Sec. 5].

Theorem 8.2. *Let X and Y be derived equivalent hyper-Kähler varieties. Then for all $i \in \mathbb{Z}$ we have an isomorphism*

$$\mathrm{H}^i(X, \mathbb{Q}) \simeq \mathrm{H}^i(Y, \mathbb{Q})$$

of $\mathbb{Q}$-Hodge structures.

Proof. Let us denote by Φ a derived equivalence between X and Y. From Theorem 7.4 we know that there exists a Hodge similitude $\phi\colon \tilde{\mathrm{H}}(X, \mathbb{Q}) \longrightarrow \tilde{\mathrm{H}}(Y, \mathbb{Q})$ such that under the isomorphism $\mathfrak{g}(X) \simeq \mathfrak{so}(\tilde{\mathrm{H}}(X, \mathbb{Q}))$ the Lie algebra isomorphism $\Phi^{\mathfrak{g}}$ corresponds to $\mathrm{ad}(\phi)$.

By Witt cancelation for vector spaces one easily shows that there exists a Hodge isometry $\gamma \in \mathrm{SO}(\tilde{\mathrm{H}}(Y, \mathbb{Q}))$ such that the composition $\gamma \circ \phi$ is now a graded Hodge similitude, i.e. α and β are mapped to multiplies of themselves. By definition, this implies that the adjoint morphism of $\gamma \circ \phi$ satisfies

$$\mathrm{ad}(\gamma \circ \phi)(h) = h, \quad \mathrm{ad}(\gamma \circ \phi)(h') = h'. \tag{8.1}$$

Let us for the moment assume that we can find a global algebra isomorphism $\eta\colon \mathrm{H}^*(Y, \mathbb{Q}) \simeq \mathrm{H}^*(Y, \mathbb{Q})$ whose adjoint action equals γ as isomorphisms of the LLV Lie algebra $\mathfrak{g}(Y)$. Then we can consider the composition

$$\eta \circ \Phi^{\mathrm{H}}\colon \mathrm{H}^*(X, \mathbb{Q}) \simeq \mathrm{H}^*(Y, \mathbb{Q}).$$

From (8.1) we infer again that $\mathrm{ad}(\eta \circ \Phi^{\mathrm{H}})(h) = h$ and $\mathrm{ad}(\eta \circ \Phi^{\mathrm{H}})(h') = h'$. As in the proof of Theorem8.1 this implies that $\eta \circ \Phi^{\mathrm{H}}$ induces in each degree the desired isomorphism of Hodge structures.

It is left to prove the existence of the global algebra isomorphism η. In general, the integrated action of the LLV Lie algebra can only be extended to the even cohomology. This is circumvented by using the $\mathbb{Q}$-algebraic group QSpin. For details we refer to [Tae81, Sec. 5]. $\square$

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