Motivic integration

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Zusammenfassung

Maxim Kontsevich hat 1995 während eines Vortrages in Orsay die Theorie der motivischen Integration eingeführt. Diese Theorie wurde dann hauptsächlich von Denef, Loeser [9], Batyrev [2], und Looijenga [18] ausgearbeitet. Die zentralen Objekte sind zum einen der Raum der formalen Schleifen einer Varietät X über einem Körper k. Dies ist ein Schema, dessen k-rationale Punkte den $k[[t]]$-wertigen Punkten von X entsprechen. Zum anderen wird ein Maß definiert, welches nicht reellwertig ist, sondern Werte in einer Erweiterung des Grothendieck-Ringes der Varietäten annimmt. Dies ist einer der Gründe, warum diese Theorie auch häufig geometrische, motivische Integration genannt wird, da das Volumen der messbaren Mengen eine geometrische Interpretation besitzt.

Die motivische Integration hat sich seit ihrer Einführung als sehr nützlich erwiesen. Eines der Hauptresultate, das mit dieser Theorie bewiesen wurde, ist, dass zwei glatte, birational äquivalente Calabi–Yau Varietäten die gleichen Hodge-Zahlen besitzen. Diese Fragestellung hat Anwendungen in der Stringtheorie. Zuvor hatte Batyrev bereits mithilfe p-adischer Integration gezeigt, dass zwei solche Varietäten die gleichen Bettizahlen besitzen [3]. Kontsevich hat dann unter Verwendung der Transformationsformel der motivischen Integration das stärkere Resultat über die Hodge-Zahlen bewiesen [16].

Das Ziel dieser Arbeit ist es eine einfach zugängliche Einführung in die Theorie der motivischen Integration zu geben. Deshalb konzentrieren wir uns auf glatte, komplexe Varietäten. Das erste Kapitel dient dem Verständnis des Raumes der formalen Schleifen eines Schemas vom endlichem Typ über einem Körper. Dazu werden die so genannten Jet-Schemata untersucht, welche endlichdimensionale Approximationen der Schleifenräume sind. Das zweite Kapitel ist der Hauptteil der Arbeit. Zuerst werden die benötigten Bestandteile der Theorie eingeführt. Anschließend wird das motivische Integral definiert und der Spezialfall eines Divisors mit einfachen, normalen Überkreuzungen diskutiert. Danach wird die Transformationsformel, welche der Hauptbestandteil der Theorie ist, erläutert und bewiesen. Zum Schluss werden wir Anwendungen der Theorie erörtern.

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Contents

Introduction

In 1995, Kontsevich introduced the theory of motivic integration during a lecture in Orsay. The foundations then have been worked out by Denef, Loeser [9], Batyrev [2], and Looijenga [18]. The main ingredients of this theory are, on the one hand, the space of arcs of a k-variety X. This is a scheme, whose k-rational points are the $k[[t]]$-valued points of X. On the other hand, one introduces a measure, which does not take values in the real numbers, but in an extension of the Grothendieck ring of varieties. This is one of the reasons why the theory is also often called geometric motivic integration, as the values are geometric in nature.

Since its introduction, motivic integration has proven to be very fruitful. One of its main results is that two smooth birational equivalent Calabi–Yau varieties X and Y have the same Hodge numbers, $h^{p,q}(X) = h^{p,q}(Y)$. The motivation behind this problem came from string theory. Using methods of p-adic integration, Batyrev had already proven that two such varieties have the same Betti numbers, that is $\dim H^i(X, \mathbb{C}) = \dim H^i(Y, \mathbb{C})$ [3]. Kontsevich then used the transformation rule of motivic integration to prove the stronger result about the Hodge numbers [16].

This thesis aims to give an easily accessible introduction to the theory of motivic integration. Therefore we focus on smooth complex varieties. The first chapter is dedicated to understanding the arc space $\mathcal{L}(X)$ for a finite type k-scheme X. For this we will study the so called jet schemes associated to X. The second chapter is the principal part of this thesis. At first, the necessary ingredients for the theory will be introduced. Afterwards we define the motivic integral and discuss the special case of simple normal crossings divisors. Subsequently, we turn our attention to the transformation rule, which is the main tool of the theory. At last, we will draw some consequences of this theorem.

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1 Jet schemes and arc spaces

In this section we define the jet schemes for a scheme X of finite type over an algebraically closed field k of characteristic zero and we prove their existence as well as some of their basic properties. The jet schemes are finite-dimensional approximations of the space of arcs, which is the space on which we integrate in the next chapter. These spaces are not only used for the purpose of motivic integration but also their geometry yields several new invariants. This chapter closely follows [20].

1.1 Jet schemes

Definition 1.1. Let k be an algebraically closed field, X a scheme of finite type over k and $m \in \mathbb{Z}_{\geq 0}$. A scheme of finite type over k is called the *m-th jet scheme* $\mathcal{L}_m(X)$ if for every k-algebra A we have a functorial bijection

$$\operatorname{Hom}(\operatorname{Spec}(A), \mathcal{L}_m(X)) \simeq \operatorname{Hom}(\operatorname{Spec}(A[t]/(t^{m+1})), X).$$

In other words, the functor taking a scheme X to its m-th jet scheme $\mathcal{L}_m(X)$ is right adjoint to the base extension operation $\operatorname{Spec}(A) \mapsto \operatorname{Spec}(A) \times_k \operatorname{Spec}(k[t]/(t^{m+1}))$.

As $\mathcal{L}_m(X)$ is required to be of finite type over k, it is completely described by the restriction of its functor of points to affine schemes over k, cf. [11, Prop. VI-2]. Hence $\mathcal{L}_m(X)$ is unique up to unique isomorphism if it exists.

For $m > n$ we have the truncation morphism $A[t]/(t^{m+1}) \twoheadrightarrow A[t]/(t^{n+1})$ which induces a closed immersion $\operatorname{Spec}(A[t]/(t^{n+1})) \hookrightarrow \operatorname{Spec}(A[t]/(t^{m+1}))$. Hence, if $\mathcal{L}_m(X)$ and $\mathcal{L}_n(X)$ exist, we get a projection $\pi_{m,n} \colon \mathcal{L}_m(X) \to \mathcal{L}_n(X)$. As for $m > n > k$ the truncation morphisms commute, one has $\pi_{n,k} \circ \pi_{m,n} = \pi_{m,k}$.

Next we will show the existence of the m-th jet scheme in two steps.

Lemma 1.2. [20, Lem. 1.3] *Let X be a scheme of finite type over k and suppose that the m-th jet scheme $\mathcal{L}_m(X)$ exists. Then for an open $U \subset X$ we have $\mathcal{L}_m(U) = \pi_{m,0}^{-1}(U)$.*

Proof. Observe that as topological spaces we have $\operatorname{Spec}(A) = \operatorname{Spec}(A[t]/(t^{m+1}))$. For a morphism $Y \to X$ to factor through $U \subset X$ is a set-theoretic issue. Let $\operatorname{Spec}(A) \to \operatorname{Spec}(A[t]/(t^{m+1}))$ be induced by the truncation morphism, which is a homeomorphism on the underlying topological spaces. Thus we know that $\operatorname{Spec}(A[t]/(t^{m+1})) \to X$ factors through U if and only if $\operatorname{Spec}(A) \to \operatorname{Spec}(A[t]/(t^{m+1})) \to X$ factors through U.

Now giving a morphism $\operatorname{Spec}(A) \to \pi_{m,0}^{-1}(U)$ is equivalent to giving a morphism $\operatorname{Spec}(A[t]/(t^{m+1})) \to X$ such that the composition $\operatorname{Spec}(A) \to \operatorname{Spec}(A[t]/(t^{m+1})) \to X$ factors through U. By the previous discussion this is equivalent to $\operatorname{Spec}(A[t]/(t^{m+1})) \to X$ factoring through U which shows that $\pi_{m,0}^{-1}(U) = \mathcal{L}_m(U)$. □

Proposition 1.3. [20, Prop. 1.2] *For every scheme X of finite type over k and every $m \in \mathbb{Z}_{\geq 0}$ the m-th jet scheme $\mathcal{L}_m(X)$ exists.*

Proof. Suppose first that X is affine. Thus we can write $X = \mathrm{Spec}(k[x_1, \ldots, x_n]/I)$ for an ideal $I = (f_1, \ldots, f_r)$. Subsequently giving a morphism $\mathrm{Spec}(A[t]/(t^{m+1})) \to X$ is equivalent to giving a morphism $\phi \colon k[x_1, \ldots, x_n]/I \to A[t]/(t^{m+1})$. By the universal property of the polynomial ring this is equivalent to giving elements $g_i = \phi(x_i) = \sum_{j=0}^{m} a_{i,j} t^j$ fulfilling $f_l(g_1, \ldots, g_n) = 0$ for $l \in \{1, \ldots, r\}$. This last condition can be rewritten as

$$f_l(g_1, \ldots, g_n) = \sum_{s=0}^{m} h_{l,s}((a_{i,j})_{i,j}) t^s$$

for suitable polynomials $h_{l,s}$. This can be achieved by ordering with respect to t^s. We now define $\mathcal{L}_m(X)$ as the spectrum of the ring $k[x_{1,0}, \ldots, x_{n,m}]/((h_{l,s})_{l,s})$. By construction this scheme fulfills the conditions of Definition 1.1 and we have a natural projection $\pi_{m,p} \colon \mathcal{L}_m(X) \to \mathcal{L}_p(X)$, which is given on the level of rings by the homomorphism sending $x_{i,j}$ to $x_{i,j}$.

Consider now an arbitrary scheme X of finite type over k and cover it by finitely many open affines $U_1, \ldots, U_n$. We know that for every i the m-th jet scheme $\mathcal{L}_m(U_i)$ exists and it comes with the projection $\pi^i_{m,0} \colon \mathcal{L}_m(U_i) \to U_i$. To glue these schemes we use Lemma 1.2 to see that $(\pi^i_{m,0})^{-1}(U_i \cap U_j)$ and $(\pi^j_{m,0})^{-1}(U_i \cap U_j)$ are both isomorphic to $\mathcal{L}_m(U_i \cap U_j)$ over X. Due to the functorial description they are canonically isomorphic and we can glue the schemes $\mathcal{L}_m(U_i)$ together with the projections $\pi^i_{m,0}$ to obtain $\mathcal{L}_m(X)$ and $\pi_{m,0}$. The condition for $\mathcal{L}_m(X)$ to be an m-th jet schemes now follows directly from the affine case. □

This proof shows that if X is affine then $\mathcal{L}_m(X)$ is also affine. On top of that, the projections $\pi_{m,n}$ are all affine morphisms.

There are familiar descriptions for the first two jet schemes. It follows directly from the definitions that $\mathcal{L}_0(X) = X$ for every scheme X. The first jet scheme $\mathcal{L}_1(X)$ is isomorphic to the total tangent bundle $TX = \mathbf{Spec}(\mathrm{Sym}(\Omega_X))$, where $\Omega_X = \Omega_{X/k}$ is the cotangent sheaf. The proof of Proposition 1.3 shows that it is enough to prove this in the case $X = \mathrm{Spec}(R)$. For a k-algebra A giving a morphism $f \colon \mathrm{Spec}(A) \to \mathrm{Spec}(\mathrm{Sym}(\Omega_R))$ is the same as giving a k-algebra homomorphism $\varphi \colon R \to A$ and a k-derivation $d \colon R \to A$ viewing A as an R-module via φ. This is equivalent to giving a morphism $g \colon R \to A[t]/(t^2)$, where $g(r) = \varphi(r) + td(r)$.

Note that taking X to $\mathcal{L}_m(X)$ gives a functor from the category of schemes of finite type over k to itself. Namely if we have a morphism $f \colon X \to Y$ we get a corresponding map $f_m \colon \mathcal{L}_m(X) \to \mathcal{L}_m(Y)$. At the level of functors of points, this takes an $A[t]/(t^{m+1})$-valued point γ of X to $f \circ \gamma$. These morphisms are compatible with the projections such that

$$\begin{tikzcd} \mathcal{L}_m(X) \arrow[r, "f_m"] \arrow[d, "\pi_{m,m-1}"'] & \mathcal{L}_m(Y) \arrow[d, "\pi_{m,m-1}"] \\ \mathcal{L}_{m-1}(X) \arrow[r, "f_{m-1}"] & \mathcal{L}_{m-1}(Y) \end{tikzcd}$$

commutes.

Our next goal is to describe the maps $\pi_{m,m-1}$ in the case that X is nonsingular. This is our general setting in the next chapter. We will see that they are all locally $\mathbb{A}^n$-bundles. In order to prove this, we will need the notion of an *étale morphism*.

Definition 1.4. A morphism $f\colon X \to Y$ is called *formally étale* if for any affine Y-scheme Z and nilpotent ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$ the corresponding closed immersion $i\colon V \hookrightarrow Z$ has the following property:
Every Y-morphism $g\colon V \to X$ extends uniquely to a Y-morphism $\tilde{g}\colon Z \to X$ such that the diagram

$$\begin{tikzcd} Z \arrow[rr, dashed, "\tilde{g}"] & & X \\ & V \arrow[ul, hook, "i"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. The morphism f is called *étale* if it is finitely presented and formally étale.

For a morphism $f\colon X \to Y$ being étale is equivalent to being flat and unramified which is also equivalent to f being smooth of relative dimension zero [12, Cor. 17.6.2].

Lemma 1.5. [20, Lem. 1.8] *If $f\colon X \to Y$ is an étale morphism, then for every nonnegative integer m the commutative diagram*

$$\begin{tikzcd} \mathcal{L}_m(X) \arrow[r, "f_m"] \arrow[d, "\pi_{m,0}"'] & \mathcal{L}_m(Y) \arrow[d, "\pi_{m,0}"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is Cartesian.

Proof. We show that $\mathcal{L}_m(X)$ has the universal property of the fiber product. It suffices to prove that for all k-algebras A and morphisms $g\colon \operatorname{Spec}(A) \to X$ and $h\colon \operatorname{Spec}(A) \to \mathcal{L}_m(Y)$ with $f \circ g = \pi_{m,0} \circ h$ there exists a unique morphism $\phi\colon \operatorname{Spec}(A) \to \mathcal{L}_m(X)$ commuting with f_m and $\pi_{m,0}$ respectively. By the functorial description of $\mathcal{L}_m(Y)$ this corresponds to a commutative diagram of the form

$$\begin{tikzcd} \operatorname{Spec}(A) \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(A[t]/(t^{m+1})) \arrow[r] & Y. \end{tikzcd}$$

It remains for us to show that there exists a unique morphism $\operatorname{Spec}(A[t]/(t^{m+1})) \to X$ making both triangles commutative. However, as $\operatorname{Spec}(A) \to \operatorname{Spec}(A[t]/(t^{m+1}))$ is a closed immersion associated to the nilpotent ideal $(t) \subset A[t]/(t^{m+1})$, this follows immediately from the fact that f is formally étale. □

Recall that a morphism of schemes $f\colon X \to Y$ is called a Zariski locally trivial fibration with fiber F, if there exists an open cover $Y = \bigcup U_i$ with $f^{-1}(U_i) \simeq U_i \times F$ such that the restriction of f corresponds to the projection onto the first component.

Corollary 1.6. [20, Cor. 1.9] *For a nonsingular variety X of dimension n all projections $\pi_{m,m-1}\colon \mathcal{L}_m(X) \to \mathcal{L}_{m-1}(X)$ are locally trivial $\mathbb{A}^n$-bundles (i.e. locally trivial fibrations with fiber $\mathbb{A}^n$). In particular, $\mathcal{L}_m(X)$ is a nonsingular variety of dimension $(m+1)n$.*

Proof. Since X is nonsingular, we can take an affine open $U = \operatorname{Spec}(A) \subset X$ and elements $a_1, \dots, a_n \in A$ such that $da_1, \dots, da_n$ form a basis of Ω_A, as Ω_X is locally free of rank n. These elements yield a morphism $U \to \mathbb{A}^n$ which is étale [17, 6.2 Prop. 2.10]. Now we use Lemma 1.5 and the fact that $\mathcal{L}_m(\mathbb{A}^n) = \mathbb{A}^{(m+1)n}$ to conclude $\mathcal{L}_m(U) \simeq U \times \mathbb{A}^{mn}$. Note that under these isomorphisms the projections $\pi_{m,m-1}$ correspond to the projections $U \times \mathbb{A}^{mn} \to U \times \mathbb{A}^{(m-1)n}$ that forget the last n components. This concludes the proof. $\square$

1.2 Arc spaces

For X of finite type over k we have an inverse system,

$$\cdots \to \mathcal{L}_m(X) \to \mathcal{L}_{m-1}(X) \to \cdots \to \mathcal{L}_0(X) = X.$$

As all the occuring morphisms are affine, the inverse limit is again a scheme over k [21, Tag 01YV]. Hence we obtain a scheme

$$\mathcal{L}(X) \coloneqq \mathcal{L}_\infty(X) = \varprojlim \mathcal{L}_m(X),$$

which is called the *space of arcs* of X. The corresponding affine projections $\mathcal{L}(X) \to \mathcal{L}_m(X)$ will be denoted by π_m.

If X is an affine scheme of finite type over k we observe that

$$\begin{aligned}
\operatorname{Hom}(\operatorname{Spec}(A), \mathcal{L}(X)) &\simeq \varprojlim \operatorname{Hom}(\operatorname{Spec}(A), \mathcal{L}_m(X)) \\
&\simeq \varprojlim \operatorname{Hom}(\operatorname{Spec}(A[t]/(t^{m+1})), X) \\
&\simeq \varprojlim \operatorname{Hom}(\Gamma(X, \mathcal{O}_X), A[t]/(t^{m+1})) \\
&\simeq \operatorname{Hom}(\Gamma(X, \mathcal{O}_X), A[[t]]) \\
&\simeq \operatorname{Hom}(\operatorname{Spec}(A[[t]]), X).
\end{aligned}$$

For X not necessarily affine it still holds that

$$\operatorname{Hom}(\operatorname{Spec}(k), \mathcal{L}(X)) \simeq \operatorname{Hom}(\operatorname{Spec}(k[[t]]), X)$$

as

$$\varprojlim \operatorname{Hom}(\operatorname{Spec}(k[t]/(t^{m+1})), X) \simeq \operatorname{Hom}(\varinjlim \operatorname{Spec}(k[t]/(t^{m+1})), X)$$

and $\varinjlim \operatorname{Spec}(k[t]/(t^{m+1})) \simeq \operatorname{Spec}(k[[t]])$. In an analogous manner, for any field K containing k, we obtain that

$$\operatorname{Hom}(\operatorname{Spec}(K), \mathcal{L}(X)) \simeq \operatorname{Hom}(\operatorname{Spec}(K[[t]]), X).$$

Thus for an element $\gamma \in \mathcal{L}(X)$ we have a morphism $\operatorname{Spec}(K(\gamma)) \to \mathcal{L}(X)$ which corresponds to a unique morphism $\operatorname{Spec}(K(\gamma)[[t]]) \to X$. The latter morphism is called an *arc of* X. We use the notion of point in $\mathcal{L}(X)$ and the associated arc interchangeably.

If we have a morphism $f\colon X \to Y$, we get an induced map $f_\infty\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ via the maps f_m and the definition of the arc space as a projective limit. Observe that for an étale morphism $f\colon X \to Y$ we can use Lemma 1.5 to conclude that

$$\begin{tikzcd} \mathcal{L}(X) \arrow[r, "f_\infty"] \arrow[d, "\pi_0"'] & \mathcal{L}(Y) \arrow[d, "\pi_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is also Cartesian. This implies for nonsingular varieties X that locally $\mathcal{L}(X) \simeq X \times \mathbb{A}^\infty$ and the projections π_m are all surjective. Moreover, we can conclude that f_∞ is an open immersion if f is an open immersion. This is the corresponding version of Lemma 1.2 for the arc space. If the morphism f is a closed immersion, the induced morphism f_∞ again inherits the same property. This can be checked locally. If Y is an affine scheme, then it is defined by elements $g_1, \dots, g_r$ in some affine space. The closed immersion f then corresponds to elements $g_1, \dots, g_s$ in the same affine space with $s \geq r$. Proceeding in an analogous manner as in the proof of Proposition 1.3, these elements determine equations for the arc spaces $\mathcal{L}(Y)$ and $\mathcal{L}(X)$ respectively. Hence, $\mathcal{L}(X)$ is a closed subscheme of the arc space $\mathcal{L}(Y)$.

Now we prove a lemma for the induced map f_∞, which we will need in Chapter 2 for the proof of the transformation rule. The statement can be interpreted as saying that for nice maps between two varieties the induced map on the arc spaces is also nice away from a small subset.

Lemma 1.7. [20, Prop. 2.1] *Let $f \colon X \to Y$ be a proper birational morphism of varieties over k. If Z is a proper closed subset of Y such that f is an isomorphism over $Y \setminus Z$, then the induced map*

$$\mathcal{L}(X) \setminus \mathcal{L}(f^{-1}(Z)) \to \mathcal{L}(Y) \setminus \mathcal{L}(Z)$$

is bijective as a map of sets.

Proof. Let $U = Y \setminus Z$. An arc in Y corresponds to a morphism $\gamma\colon \mathrm{Spec}(K[[t]]) \to Y$. Since f is proper, we can use the valuative criterion to see that this arc lies in the image of f_∞ if and only if the induced map $\mathrm{Spec}(K((t))) \to Y$ can be lifted to X. We show that the generic point $\eta \in \mathrm{Spec}(K((t)))$ is mapped to U if and only if γ is not in $\mathcal{L}(Z)$. This proves the lemma, as $f|_{f^{-1}(U)}$ is an isomorphism. Thus we get a unique lift $\mathrm{Spec}(K((t))) \to X$ which in return yields a unique lift $\tilde{\gamma}\colon \mathrm{Spec}(K[[t]]) \to X$ by the valuative criterion

$$\begin{tikzcd} \mathrm{Spec}(K((t))) \arrow[r, dashed] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(K[[t]]) \arrow[r, "\gamma"] \arrow[ur, dashed, "\tilde{\gamma}"] & Y. \end{tikzcd}$$

Hence every arc of Y that is not contained in $\mathcal{L}(Z)$ has a unique lift to an arc in X.

It remains to prove that the image of η lies in U if and only if γ is not in $\mathcal{L}(Z)$. Therefore, we take an affine open neighborhood $V = \mathrm{Spec}(B)$ of the image of the closed point under γ. If $B = k[x_1, \dots, x_m]/(f_1, \dots, f_r)$ then $Z \cap V \subset V$ is isomorphic to $\mathrm{Spec}(A)$ for a ring $A = k[x_1, \dots, x_m]/(f_1, \dots, f_r, g_1, \dots, g_l)$. Now γ corresponds to a map of rings $\alpha\colon B \to K[[t]]$ whose kernel is the image of the generic point. This map is equivalent to power series $h_1, \dots h_m$ fulfilling $f_i(h_1, \dots, h_m) = 0$. Furthermore, γ lies in $\mathcal{L}(Z \cap V)$ if and only if also $g_j(h_1, \dots, h_m) = 0$ for all j. The latter is equivalent to $(g_1, \dots, g_l) \subset \mathrm{Ker}(\alpha)$ which corresponds to $\gamma(\eta) \in Z \cap V$, concluding the proof. □

Although the map $g\colon \mathcal{L}(X)\setminus\mathcal{L}(f^{-1}(Z))\to\mathcal{L}(Y)\setminus\mathcal{L}(Z)$ from the lemma is bijective, it is not always an isomorphism of schemes. For example, if X and Y are smooth varieties, the map g is an isomorphism if and only if the map f is an isomorphism. Smoothness implies that the projections π_0 are faithfully flat morphisms. The restrictions of these projections to the open sets $\mathcal{L}(X)\setminus\mathcal{L}(f^{-1}(Z))$ and $\mathcal{L}(Y)\setminus\mathcal{L}(Z)$ respectively are still faithfully flat, as the morphisms remain surjective. Thus, if g is an isomorphism, the proper birational morphism f between the smooth varieties is also flat. Hence, f is an isomorphism.

The lemma also shows that for a proper birational morphism f, the induced morphisms $f_m\colon \mathcal{L}_m(X)\to\mathcal{L}_m(Y)$ are all surjective. Namely, if we take $\gamma_m\in\mathcal{L}_m(Y)$, the set $\pi_m^{-1}(\gamma_m)$ is not contained in $\mathcal{L}(Z)$, where $Y\setminus Z$ is the biggest open set over which f is an isomorphism. Hence we can find $\gamma'\in\mathcal{L}(X)$ mapping to $\gamma\in\pi_m^{-1}(\gamma_m)$ by the lemma and $\pi_m(\gamma')$ maps to γ_m.

2 Motivic integration

In this chapter we concentrate on smooth complex varieties X of dimension n. We consider effective divisors on X without using linear equivalence. As X is smooth, all its local rings $\mathcal{O}_{X,x}$ are regular and hence factorial. Therefore, the group of Cartier divisors and the group of Weil divisors are isomorphic and we use both of these notions. The theory of motivic integration can be generalized to algebraic varieties of pure dimension over a field of characteristic zero as done in [9].

2.1 Preparations

We associate to every effective divisor D a function F_D which is measurable with respect to a measure μ on the space of arcs. Our goal is be to integrate F_D over the space of arcs $\mathcal{L}(X)$.

Definition 2.1. Let D be an effective divisor on X and f a local equation for D on an open $U \subset X$. For an arc $\gamma \in \mathcal{L}(X)$, for which we have $\pi_0(\gamma) \in U$, define a function

$$F_D \colon \mathcal{L}(X) \to \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

where γ is mapped to the order of vanishing of the formal power series $f(\gamma(z))$ at $z = 0$.

If we write $D = \sum_{i=1}^r a_i D_i$ for prime divisors D_i, we can consider for each D_i a local equation f_i. In this case f decomposes into $f = \prod_{i=1}^r f_i^{a_i}$. Thus, we can write $F_D = \sum_{i=1}^r a_i F_{D_i}$.

Definition 2.2. A set $C \subset \mathcal{L}(X)$ is called a *cylinder set* if there is an $m \in \mathbb{Z}_{\geq 0}$ and a constructible subset $B_m \subset \mathcal{L}_m(X)$ such that $C = \pi_m^{-1}(B_m)$.

Recall that a subset is called *constructible* if it is a finite disjoint union of locally closed subsets. If the topological space is Noetherian, we can drop the condition that the union has to be disjoint. Cylinder sets form a Boolean algebra, which means that they are closed under finite union and taking complements. We can think of the cylinder sets as nice subspaces of the arc space obtained by lifting a nice subspace of m-jets. We will later see, that these sets are measurable.

To be able to integrate we need to understand the function F_D. Let therefore $D = \sum_{i=1}^r a_i D_i$ be an effective divisor and $J \subseteq \{1, \ldots, r\}$ an arbitrary subset. We define

$$D_J := \begin{cases} \cap_{j \in J} D_J & J \neq \emptyset \\ Y & J = \emptyset \end{cases} \qquad\qquad D_J^\circ := D_J \setminus \bigcup_{i \in \{1,\ldots,r\} \setminus J} D_i.$$

It follows that

$$X = \bigsqcup_{J \subseteq \{1,\ldots,r\}} D_J^\circ \qquad \text{and} \qquad \mathcal{L}(X) = \bigsqcup_{J \subseteq \{1,\ldots,r\}} \pi_0^{-1}(D_J^\circ).$$

Hence we can divide the space of arcs into disjoint cylinder sets. Using the above, we define a partition of the set $F_D^{-1}(s)$. For any $s \in \mathbb{Z}_{\geq 0}$ and $J \subseteq \{1, \ldots, r\}$ consider

$$N_{J,s} \coloneqq \left\{ (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0} \mid \sum a_i n_i = s \text{ such that } n_j > 0 \text{ if and only if } j \in J \right\}.$$

As $F_{D_i}(\gamma) = 0$ if and only if $\pi_0(\gamma) \notin D_i$, we conclude that

$$\gamma \in \pi_0^{-1}(D_J^\circ) \cap F_D^{-1}(s) \text{ if and only if } (F_{D_1}(\gamma), \ldots, F_{D_r}(\gamma)) \in N_{J,s}.$$

As a result, we produce a finite partition of the level set

$$F_D^{-1}(s) = \bigsqcup_{J \subseteq \{1,\ldots,r\}} \bigsqcup_{(n_1,\ldots,n_r) \in N_{J,s}} \left(\bigcap_{i=1}^{r} F_{D_i}^{-1}(n_i) \right). \tag{2.1}$$

With these preparations we are able to prove an important property of the sets $F_D^{-1}(s)$.

Proposition 2.3. *If D is an effective divisor then $F_D^{-1}(s)$ is a cylinder set for every $s \in \mathbb{Z}_{\geq 0}$.*

Proof. As the collection of cylinder sets forms a Boolean algebra, it suffices to prove by (2.1) that $F_{D_i}^{-1}(n_i)$ is a cylinder set for arbitrary $J \subseteq \{1, \ldots, r\}$ and $(n_1, \ldots, n_r) \in N_{J,s}$. We cover X by finitely many open affines U such that D_i can be described by one equation on U. By the same argument as above we only need to prove that $F_{D_i}^{-1}(n_i) \cap \pi_0^{-1}(U)$ is cylinder. Thus we can set $D = D_i$, $m = n_i$ and assume without loss of generality that D is a hypersurface given by a polynomial f on an affine variety X.

For an arc $\gamma \in \mathcal{L}(X)$ consider its truncation to $\mathcal{L}_m(X)$. In the proof of Proposition 1.3 we showed that $\mathcal{L}_m(X) = \operatorname{Spec}(k[x_{1,0}, \ldots, x_{n,m}]/(h_{l,s})_{l,s})$ by establishing a correspondence between the functorial description and elements $g_i = \sum a_{i,j} t^j$ fulfilling certain equations. The subset of $\mathcal{L}_m(X)$ defined by the image of $F_D^{-1}(m) \subset \mathcal{L}(X)$ corresponds to additional requirements for the elements $a_{i,j}$. Namely, we require that

$$f(g_1, \ldots, g_n) = \sum_{k=0}^{m} h_k((a_{i,j})_{i,j}) t^k \in A[t]/(t^{m+1})$$

is a polynomial, whose terms of degree smaller than m vanish and the m-th term does not vanish. Hence, if we set $B_m = V(h_0, \ldots, h_{m-1}) \cap D(h_m) \subset \mathcal{L}_m(X)$, we obtain that $\pi_m^{-1}(B_m) = F_D^{-1}(m)$. This proves the assertion, as B_m is the intersection of an open and a closed subset. □

Note however that $F_D^{-1}(\infty)$ is not a cylinder set. Locally an arc $\gamma \in F_D^{-1}(\infty)$ can be viewed as an n-tuple of power series. For a constructible set $B_m \subset \mathcal{L}_m(X)$, arcs in $\pi_m^{-1}(B_m)$ can be viewed as n-tuples of power series for which the terms in degree higher than m can take any value. However, membership in $F_D^{-1}(\infty)$ requires conditions on the terms in every degree.

Our next aim is to define a measure μ on the space of arcs. As it is not real-valued, we begin by introducing the ring in which μ takes values.

Definition 2.4. Let $\mathcal{V}_\mathbb{C}$ denote the category of complex algebraic varieties. Consider the quotient of the free abelian group of isomorphism classes $[X]$ of complex algebraic varieties by the relation $[X] = [Y] + [X \setminus Y]$ for $Y \subset X$ a closed subset. This group has the structure of a ring by defining the product of two elements $[X]$ and $[Y]$ to be $[X \times Y]$. The resulting ring is called the *Grothendieck ring of complex algebraic varieties* and is denoted by $K_0(\mathcal{V}_\mathbb{C})$.

We obtain a map $\mathcal{V}_{\mathbb{C}} \to K_0(\mathcal{V}_{\mathbb{C}})$ by sending X to $[X]$. It is universal with respect to maps which are additive on disjoint unions of constructible subsets and which respect products. This means for example that $[X] = [Y]$ implies $\chi(X) = \chi(Y)$, where χ is the topological Euler characteristic. Moreover, if $X \to Y$ is a Zariski locally trivial fibration with fiber F, we have $[X] = [Y] \cdot [F]$.

We will write $\mathbb{L}$ for $[\mathbb{A}^1]$ and 1 for [point], which is the identity element in the ring. We denote by $\mathcal{M} := K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ the localization of $K_0(\mathcal{V}_{\mathbb{C}})$ with respect to the multiplicative system $\{1, \mathbb{L}, \mathbb{L}^2, \dots\}$. It is yet unknown whether $\mathcal{M}$ is a domain or not [22, 2.11].

Definition 2.5. Let X be a variety of dimension n. We define a function

$$\tilde{\mu} \colon \{C \subset \mathcal{L}(X) \text{ cylinder}\} \to \mathcal{M}$$

which sends $C = \pi_m^{-1}(B_m) \mapsto [B_m] \cdot \mathbb{L}^{-n(m+1)}$.

Observe that this definition is independent of the choice of m. If $r > m$, then the projection $\pi_{r,m} \colon \mathcal{L}_r(X) \to \mathcal{L}_m(X)$ is a locally trivial fibration with fiber $\mathbb{A}^{n(r-m)}$ and thus $[\pi_{r,m}^{-1}(B_m)] = [B_m] \cdot \mathbb{L}^{n(r-m)}$. The function $\tilde{\mu}$ assigns to every cylinder set a "volume" in $\mathcal{M}$. As the map $\mathcal{V}_{\mathbb{C}} \to K_0(\mathcal{V}_{\mathbb{C}})$ is additive on disjoint unions of constructible sets, we have

$$\tilde{\mu}\left(\bigsqcup_{i=1}^{k} B_i\right) = \sum_{i=1}^{k} \tilde{\mu}(B_i)$$

for cylinder sets $B_1, \dots, B_k$. For this reason we call $\tilde{\mu}$ a finitely additive measure. Note that there are different conventions in the definition of the measure among different authors. Craw [6] and Denef, Loeser [9] for example use the same definition as given above. Veys [22] and Looijenga [18] define the measure of the cylinder set $\pi_m^{-1}(B_m)$ as $[B_m] \cdot \mathbb{L}^{-nm}$. It is essentially a matter of taste which definition one decides to use.

By Proposition 2.3 the set $F_D^{-1}(s)$ is $\tilde{\mu}$-measurable for all $s \in \mathbb{Z}_{\geq 0}$. As $F_D^{-1}(\infty)$ is not cylinder, we have to extend $\tilde{\mu}$ to a measure μ such that $F_D^{-1}(\infty)$ is μ-measurable.

Definition 2.6. Let $F^m\mathcal{M} \subset \mathcal{M}$ be the subgroup generated by elements of the form $[V]/\mathbb{L}^i$ for $i - \dim V \geq m$. Consider the decreasing filtration

$$\cdots \supset F^{-1}\mathcal{M} \supset F^0\mathcal{M} \supset F^1\mathcal{M} \supset \cdots$$

and denote by $\hat{\mathcal{M}} = \varprojlim \mathcal{M}/F^m\mathcal{M}$ the completion of $\mathcal{M}$ with respect to this filtration.

By composing with the natural completion map $\phi \colon \mathcal{M} \to \hat{\mathcal{M}}$ we obtain a measure $\tilde{\mu}$ with values in $\hat{\mathcal{M}}$.

Definition 2.7. Denote by $\mathcal{C}$ the collection of all countable disjoint unions of cylinder sets $\sqcup_{i\in\mathbb{N}} C_i$ for which $\tilde{\mu}(C_i) \to 0$ as $i \to \infty$, together with their complements. Define the measure $\mu \colon \mathcal{C} \to \hat{\mathcal{M}}$ by

$$\bigsqcup_{i\in\mathbb{N}} C_i \mapsto \sum_{i\in\mathbb{N}} \tilde{\mu}(C_i).$$

This definition is independent of the choice of C_i, for details see [9, Def.-Prop. 3.2]. The key observation is that a cylinder set, which is contained in a countable union of disjoint cylinder sets, is already contained in a finite union of these sets, cf. [2, Thm. 6.6].

This extension μ of $\tilde{\mu}$ is the measure we were looking for. We now show that the function F_D is μ-measurable. With that knowledge we are able to define the motivic integral.

Lemma 2.8. *For an effective divisor D we have $\tilde{\mu}(\pi_k^{-1}(\pi_k(F_D^{-1}(\infty)))) \to 0 \in \hat{\mathcal{M}}$ as $k \to \infty$.*

Proof. We only prove the assertion in the case that D is smooth. First note that the set $\pi_k(F_D^{-1}(\infty)) \subset \mathcal{L}_k(X)$ is constructible. This can be proven locally by using the proof of Proposition 2.3 for $F_D^{-1}(k+1)$ where we have to make the small adaptation that also h_k has to vanish. Hence we obtain that $\pi_k(F_D^{-1}(k+1)) = V(h_0, \dots, h_k) = \mathcal{L}_k(D) \subset \mathcal{L}_k(X)$ is a closed subset. The claim subsequently follows by the fact that $\pi_k(F_D^{-1}(\infty)) = \pi_k(F_D^{-1}(k+1))$.

We have to study the sets $\mathcal{L}_k(D) \subset \mathcal{L}_k(X)$. Since D in nonsingular, we have by Corollary 1.6 that $\pi_k \colon \mathcal{L}_k(D) \to D$ is a locally trivial fibration with fiber $\mathbb{A}^{n-1}$. Thereby it follows that $\dim(\mathcal{L}_k(D)) = \dim(D) + k(n-1) = (n-1)(k+1)$. This yields

$$\tilde{\mu}(\pi_k^{-1}(\pi_k(F_D^{-1}(\infty)))) = [\mathcal{L}_k(D)] \cdot \mathbb{L}^{-n(k+1)}$$

which lies in $F^{k+1}\mathcal{M}$, as $n(k+1) - (n-1)(k+1) = k+1$. Thus, by the definition of the topology on $\hat{\mathcal{M}}$, $\tilde{\mu}(\pi_k^{-1}(\pi_k(F_D^{-1}(\infty))))$ tends to zero as k tends to infinity. This proves the assertion for smooth D. □

For the general case one needs statements about arc spaces of possibly singular varieties, cf. [9, Lem. 4.3, 4.4]. These imply that the dimension going from $\mathcal{L}_k(D)$ to $\mathcal{L}_{k+1}(D)$ grows eventually by $n-1$ such that one can use a similar argument as above. However, this theory goes beyond the extent of this thesis. For a proof of the lemma in the general case we refer to [4, Prop. 4.5].

Proposition 2.9. [6, Prop. 2.12] *For every effective divisor D, the set $F_D^{-1}(\infty)$ lies in $\mathcal{C}$. Thus, F_D is μ-measurable and moreover $\mu(F_D^{-1}(\infty)) = 0$.*

Proof. First observe that $F_D^{-1}(\infty) = \bigcap_{k \in \mathbb{Z}_{\geq 0}} \pi_k^{-1}(\pi_k(F_D^{-1}(\infty)))$. This equality can be checked locally where it corresponds to the fact that a power series vanishes if and only if all its truncations vanish. Note that the sets $\pi_k(F_D^{-1}(\infty)) \subset \mathcal{L}_k(X)$ are all constructible (see the proof of the lemma above). Hence by taking complements we can write

$$\mathcal{L}(X) \setminus F_D^{-1}(\infty) = \mathcal{L}(X) \setminus \pi_0^{-1}(\pi_0(F_D^{-1}(\infty))) \sqcup \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \pi_k^{-1}(\pi_k(F_D^{-1}(\infty))) \setminus \pi_{k+1}^{-1}(\pi_{k+1}(F_D^{-1}(\infty))) \tag{2.2}$$

where the right hand side is a countable disjoint union of cylinder sets.

Now $F_D^{-1}(\infty)$ lies in $\mathcal{C}$ if and only if its complement lies in $\mathcal{C}$. Thus it suffices to show that $\tilde{\mu}(\pi_k^{-1}(\pi_k(F_D^{-1}(\infty)))) \to 0$ as k approaches infinity, which is true by Lemma 2.8. This shows that F_D is μ-measurable.

For the second part of the proposition we use again (2.2) to compute

$$\begin{aligned}\mu(\mathcal{L}(X) \setminus F_D^{-1}(\infty)) &= \tilde{\mu}(\mathcal{L}(X) \setminus \pi_0^{-1}(\pi_0(F_D^{-1}(\infty)))) \\ &\quad + \sum_{k \in \mathbb{Z}_{\geq 0}} \tilde{\mu}(\pi_k^{-1}(\pi_k(F_D^{-1}(\infty))) \setminus \pi_{k+1}^{-1}(\pi_{k+1}(F_D^{-1}(\infty)))).\end{aligned}$$

As this is a telescoping series, it equals $\mu(\mathcal{L}(X)) - \lim_{k\to\infty} \tilde{\mu}(\pi_k^{-1}(\pi_k(F_D^{-1}(\infty))))$. By the above lemma, this equals $\mu(\mathcal{L}(X))$ implying $\mu(F_D^{-1}(\infty)) = 0$. □

2.2 The motivic integral

In this section we define the motivic integral for the function F_D associated to an effective divisor D. In the special case of D having only simple normal crossings we show that the integral can be computed by using only a finite sum.

Definition 2.10. Let X be a nonsingular complex variety and D an effective divisor on X. Then the *motivic integral* is given by

$$\int_{\mathcal{L}(X)} F_D \,\mathrm{d}\mu := \sum_{s\in\mathbb{Z}_{\geq 0}\cup\{\infty\}} \mu(F_D^{-1}(s)) \cdot \mathbb{L}^{-s} \in \hat{\mathcal{M}}.$$

Note that in the definition it would be sufficient to sum over all $s \in \mathbb{Z}_{\geq 0}$, as $F_D^{-1}(\infty) \subset \mathcal{L}(X)$ has measure zero.

Example 2.11. We now compute an easy example to acquire a better understanding of the definition. Suppose $Y \subset X$ is a smooth subvariety of codimension 1 and consider it as a divisor D. Then the cylinder set $F_D^{-1}(s)$ is the same as $\pi_{s-1}^{-1}(\mathcal{L}_{s-1}(Y)) \setminus \pi_s^{-1}(\mathcal{L}_s(Y))$, where $\mathcal{L}_m(Y) \subset \mathcal{L}_m(X)$ is the corresponding closed subvariety. Corollary 1.6 tells us that $\mathcal{L}_s(Y) \to Y$ is a locally trivial $\mathbb{A}^{n-1}$-bundle. This implies

$$\begin{aligned}\mu(F_D^{-1}(s)) &= \mu(\pi_{s-1}^{-1}(\mathcal{L}_{s-1}(Y))) - \mu(\pi_s^{-1}(\mathcal{L}_s(Y)))\\ &= [\mathcal{L}_{s-1}(Y)] \cdot \mathbb{L}^{-ns} - [\mathcal{L}_s(Y)] \cdot \mathbb{L}^{-n(s+1)} = [Y] \cdot (\mathbb{L}-1) \cdot \mathbb{L}^{-n-s}.\end{aligned}$$

Now we can compute the motivic integral

$$\begin{aligned}\int_{\mathcal{L}(X)} F_D \,\mathrm{d}\mu &= [X \setminus Y] \cdot \mathbb{L}^{-n} + \sum_{s=1}^{\infty} [Y] \cdot (\mathbb{L}-1) \cdot \mathbb{L}^{-n-s} \cdot \mathbb{L}^{-s}\\ &= [X \setminus Y] \cdot \mathbb{L}^{-n} + [Y] \cdot (\mathbb{L}-1) \cdot \mathbb{L}^{-n-2} \cdot \sum_{s=0}^{\infty} \mathbb{L}^{-2s}\\ &= [X \setminus Y] \cdot \mathbb{L}^{-n} + [Y] \cdot \frac{\mathbb{L}-1}{\mathbb{L}^2 \cdot (1-\mathbb{L}^{-2})} \cdot \mathbb{L}^{-n}\\ &= [X \setminus Y] \cdot \mathbb{L}^{-n} + [Y] \cdot \frac{\mathbb{L}-1}{\mathbb{L}^2 - 1} \cdot \mathbb{L}^{-n}.\end{aligned}$$

We now show that one can compute the motivic integral by using only a finite sum if the divisor D has only simple normal crossing. The above example is a special case of the following theorem.

Definition 2.12. An effective divisor $D = \sum_{i=1}^{r} a_i D_i$ on X is said to have *simple normal crossings* if for every point $x \in X$ there exists an affine neighborhood $x \in U \subset X$ with elements $z_1, \ldots, z_n \in \mathcal{O}_X(U)$ such that $dz_1, \ldots, dz_n$ form a basis of Ω_U and a local defining equation for D is given by

$$f = z_1^{a_1} \cdots z_{j_x}^{a_{j_x}} \qquad \text{for some } j_x \leq n.$$

Divisors with simple normal crossings are an important and particularly nice class of divisors. They often appear in the study of resolutions of singularities. An example is the union of several coordinate hyperplanes in the affine n-space $\mathbb{A}^n$.

Theorem 2.13. [6, Thm. 2.15] *Let X be a nonsingular complex variety and $D = \sum_{i=1}^r a_i D_i$ an effective divisor on X with simple normal crossings. Then*

$$\begin{aligned}\int_{\mathcal{L}(X)} F_D \,\mathrm{d}\mu &= \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} \frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1} \right) \cdot \mathbb{L}^{-n} \\ &= \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} \frac{1}{[\mathbb{P}^{a_j}]} \right) \cdot \mathbb{L}^{-n}.\end{aligned}$$

Proof. Recall that in Chapter 2.1 we introduced a partition of the set $F_D^{-1}(s)$. By the lemma below we can conclude for a subset $J \subseteq \{1,\dots,r\}$ that

$$\mu\Big(\bigcap_{i=1,\dots,r} F_{D_i}^{-1}(n_i) \Big) = [D_J^\circ] \cdot \mathbb{L}^{-\sum_{j\in J} n_j} \cdot (\mathbb{L}-1)^{|J|} \cdot \mathbb{L}^{-n}$$

for $(n_1,\dots,n_r) \in N_{J,s}$ and $s \in \mathbb{Z}_{\geq 0}$. Using this we compute

$$\begin{aligned}\int_{\mathcal{L}(X)} F_D \,\mathrm{d}\mu &= \sum_{s \in \mathbb{Z}_{\geq 0}} \mu(F_D^{-1}(s)) \cdot \mathbb{L}^{-s} \\ &= \sum_{s \in \mathbb{Z}_{\geq 0}} \sum_{J \subseteq \{1,\dots,r\}} \sum_{(n_1,\dots,n_r) \in N_{J,s}} \mu \left(\bigcap_{i=1,\dots r} F_{D_i}^{-1}(n_i) \right) \cdot \mathbb{L}^{-\sum_{j \in J} a_j n_j} \\ &= \sum_{s \in \mathbb{Z}_{\geq 0}} \sum_{J \subseteq \{1,\dots,r\}} \sum_{(n_1,\dots,n_r) \in N_{J,s}} [D_J^\circ] \cdot (\mathbb{L}-1)^{|J|} \cdot \mathbb{L}^{-n} \cdot \prod_{j \in J} \mathbb{L}^{-(a_j+1)n_j} \\ &= \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\sum_{s \in \mathbb{Z}_{\geq 0}} \sum_{(n_1,\dots,n_r) \in N_{J,s}} \left(\prod_{j \in J} (\mathbb{L}-1) \cdot \mathbb{L}^{-(a_j+1)n_j} \right) \right) \cdot \mathbb{L}^{-n} \\ &= \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} (\mathbb{L}-1) \cdot \sum_{n_j > 0} \mathbb{L}^{-(a_j+1)n_j} \right) \cdot \mathbb{L}^{-n} \\ &= \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} (\mathbb{L}-1) \cdot \left(\frac{1}{1-\mathbb{L}^{-(a_j+1)}} - 1 \right) \right) \cdot \mathbb{L}^{-n} \\ &= \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} \frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1} \right) \cdot \mathbb{L}^{-n} \\ &= \sum_{J \subseteq \{1,\dots,r\}} [D_J^\circ] \cdot \left(\prod_{j \in J} \frac{1}{[\mathbb{P}^{a_j}]} \right) \cdot \mathbb{L}^{-n}.\end{aligned}$$

□

Lemma 2.14. *Let $D = \sum_{i=1}^r a_i D_i$ be an effective divisor with simple normal crossings and $J \subseteq \{1,\dots,r\}$ an arbitrary subset. Assume $n_1,\dots,n_r$ are nonnegative integers with $n_i \neq 0$ if and only if $i \in J$. Then*

$$\bigcap_{i=1,\dots,r} F_{D_i}^{-1}(n_i) = \pi_m^{-1}(B_m),$$

where $m = \max_i\{n_i\}$ and $B_m \subset \mathcal{L}_m(X)$ is a constructible set with

$$[B_m] = [D_J^\circ] \cdot \mathbb{L}^{nm - \sum_{j\in J} n_j} \cdot (\mathbb{L}-1)^{|J|} \in \hat{\mathcal{M}}.$$

Proof. Let $J \subseteq \{1,\dots,r\}$ be a subset and $n_1,\dots,n_r$ arbitrary nonnegative integers such that $n_i \neq 0$ if and only if $i \in J$. We cover X by finitely many open affines $x \in U \subset X$ such that D is given on U by $f = z_1^{a_1} \cdots z_{j_x}^{a_{j_x}}$ as in Definition 2.12 and we consider

$$U_{n_1,\dots,n_r} := \bigcap_{i=1,\dots,r} F_{D_i}^{-1}(n_i) \cap \pi_0^{-1}(U).$$

If $J \nsubseteq \{1,\dots,j_x\}$, then $D_J^\circ \cap U = \emptyset$, as there exist a $j \in J$ with $j \notin \{1,\dots,j_x\}$ which forces $D_j \cap U = \emptyset$ by looking at the local equation f of D on U. Since $U_{n_1,\dots,n_r}$ is a subset of $\pi_0^{-1}(D_J^\circ \cap U)$ it has to be empty. Hence we only have to consider the case where $J \subseteq \{1,\dots,j_x\}$, which implies $|J| \leq n$ by the definition of simple normal crossings.

Let $m = \max_i\{n_i\}$ and consider the truncation $\pi_m(U_{n_1,\dots,n_r}) \subset \mathcal{L}_m(X)$. Recall that for a nonsingular variety X, the projection $\pi_{m,0}\colon \mathcal{L}_m(X) \to X$ is a locally trivial fibration with fiber $\mathbb{A}^{nm}$. Thus, we can assume that $\pi_m(U_{n_1,\dots,n_r}) \subset \mathcal{L}_m(U) \simeq U \times \mathbb{A}^{nm}$. Consider for an arc $\gamma \in \mathcal{L}(X)$ the condition $F_{D_i}(\gamma) = n_i$, where D_i is given by $z_i = 0$. Using the above isomorphism this is equivalent to its truncation $\pi_m(\gamma)$ lying in the locally closed set $V(z_i, x_{i,1}, \dots, x_{i,n_i-1}) \cap D(x_{i,n_i})$, where we use the same notation as in the proof of Proposition 1.3. Hence we obtain that $F_{D_i}^{-1}(n_i) \cap \pi_0^{-1}(U) = \pi_m^{-1}(C_i)$ for a constructible set $C_i \subset \mathcal{L}_m(U)$ being isomorphic to $(U \cap D_i) \times \mathbb{A}^{m-n_i} \times (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^{m(n-1)}$. It follows by considering all conditions $F_{D_j}(\gamma) = n_j$ for all $j \in \{1,\dots,r\}$ that $U_{n_1,\dots,n_r} = \pi_m^{-1}(B'_m)$, where

$$B'_m \simeq (U \cap D_J^\circ) \times \mathbb{A}^{m|J| - \sum_{j\in J} n_j} \times \left(\mathbb{A}^1 \setminus \{0\}\right)^{|J|} \times \mathbb{A}^{m(n-|J|)}.$$

If we take the union over the finite cover $\{U\}$ of X, we see that $\bigcap_{i=1,\dots,r} F_{D_i}^{-1}(n_i) = \pi_m^{-1}(B_m)$ with

$$[B_m] = [D_J^\circ \times \mathbb{A}^{m|J| - \sum_{j\in J} n_j} \times \left(\mathbb{A}^1 \setminus \{0\}\right)^{|J|} \times \mathbb{A}^{m(n-|J|)}] = [D_J^\circ] \cdot \mathbb{L}^{mn - \sum_{j\in J} n_j} \cdot (\mathbb{L}-1)^{|J|},$$

as the map sending a variety on its class in $\hat{\mathcal{M}}$ is additive on disjoint unions of constructible subsets. □

Corollary 2.15. *For an effective divisor D on X with simple normal crossings the motivic integral of the associated function F_D is an element of the subring*

$$\phi(\mathcal{M})\left[\left\{\frac{1}{\mathbb{L}^i - 1}\right\}_{i\in\mathbb{N}}\right]$$

of $\hat{\mathcal{M}}$. □

2.3 Transformation rule

The goal of this section is to prove the transformation rule for the motivic integral. It establishes a connection between the motivic integrals of a divisor D on Y and its pullback f^*D on X for a proper birational morphism $f\colon X \to Y$.

Before we state the theorem we need some preparations. Consider a proper birational morphism $f\colon X \to Y$ between smooth varieties. This morphism gives rise to an exact sequence

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0,$$

cf. [13, II, Prop. 8.11]. Note that the cotangent sheaves are locally free of the same rank as the varieties are smooth. In our situation this sequence is also left exact. To see this localize the sequence at the generic point η of X. As f is birational, we have $K(X) \simeq K(Y)$ which implies ${\Omega_{X/Y}}_\eta \simeq \Omega_{K(X)/K(Y)} = 0$. Thus the map $\alpha\colon f^*\Omega_Y \to \Omega_X$ localized at the generic point is a surjection between finite-dimensional $K(X)$-vector spaces of the same dimension. Hence it is also injective at the generic point. Since X is integral this implies injectivity of the sequence, as the cotangent sheaves are locally free.

By taking the n-th exterior power we get an injection $f^*\omega_Y \hookrightarrow \omega_X$. Tensoring with $(f^*\omega_Y)^*$ yields $\mathcal{O}_X \hookrightarrow \omega_X \otimes (f^*\omega_Y)^*$. Thus the divisor associated to the section of the line bundle $\omega_X \otimes (f^*\omega_Y)^*$ is an effective divisor supported on $Z = X \setminus U$ where U is the biggest open subset such that $f|_U$ is an isomorphism. Using Zariski's Main Theorem one can even show that its support is actually equal to Z, cf. [7, Lem. B.2.3]. Note that locally this divisor is given by the determinant of the morphism α between the free modules of rank n. We call it the *relative canonical divisor* and denote it by $K_{X/Y}$.

The next lemma is important for the transformation rule. It states that the associated map f_∞ sends under certain conditions cylinder sets to cylinder sets. This will allow us to connect partitions of the arc spaces of the domain and the target.

Lemma 2.16. *Let $f\colon X \to Y$ be a proper birational morphism, $C \subset \mathcal{L}(X)$ a cylinder set and denote by C' its image $f_\infty(C)$. Assume that there is a nonnegative integer e such that for $m \geq 2e$ every element in the fiber $f_m^{-1}(\gamma'_m)$ over a jet $\gamma'_m \in \pi_m(C')$ maps under the projection $\pi_{m,m-e}$ to the same jet $\gamma_{m-e} \in \pi_{m-e}(C)$. Then the set C' is again a cylinder set.*

Proof. Since C is a cylinder set, there is a constructible set $B_k \subset \mathcal{L}_k(X)$ such that $C = \pi_k^{-1}(B_k)$. We can assume without loss of generality that $k \geq e$. By Chevalley's Theorem, the image $B' = f_{k+e}(B)$ of the constructible set $B \coloneqq \pi_{k+e}(C) \subset L_{k+e}(X)$ is again constructible [19, p. 72, Cor. 2]. Hence, to prove this lemma it suffices to show the equality $C' = \pi_{k+e}^{-1}(B')$. Obviously C' is contained in $\pi_{k+e}^{-1}(B')$ so we only have to prove the other inclusion.

For an arc $\gamma' \in \pi_{k+e}^{-1}(B')$, consider its truncation $\gamma'_m = \pi_m(\gamma') \in \mathcal{L}_m(Y)$ for an integer $m \geq k+e$. We claim that γ'_m even lies in $\pi_m(C')$. From the definition of B' we know that after further truncation $\gamma'_{k+e} = \pi_{k+e}(\gamma') \in f_{k+e}(\pi_{k+e}(C))$. By the assumptions on e and k, the fiber over γ'_{k+e} under f_{k+e}, which is nonempty by the conclusion of Lemma 1.7, lies in $\pi_{k+e}(C)$. Again by the choice of k, the preimage of $\pi_{k+e}(C)$ under $\pi_{m,k+e}$ is exactly $\pi_m(C)$. This implies that the fiber over γ'_m under the morphism f_m is nonempty and lies in $\pi_m(C)$. Thus, we have proven that γ'_m is an element of $f_m(\pi_m(C)) = \pi_m(C')$.

Now, take an arbitrary arc $\gamma' \in \pi_{k+e}^{-1}(B')$ and denote by γ'_{k+e} its truncation in B'. Take a preimage $\tilde{\gamma}_{k+e} \in f_{k+e}^{-1}(\gamma'_{k+e})$ and define for $i \leq k$ the jets $\gamma_i = \pi_{k+e,i}(\tilde{\gamma}_{k+e})$. We repeat this procedure. Consider $\pi_{k+2e}(\gamma') \in \mathcal{L}_{k+2e}(Y)$ and take a preimage $\tilde{\gamma}_{k+2e}$ of the jet under f_{k+2e}. Define once more γ_i as the truncation of $\tilde{\gamma}_{k+2e}$ for $k < i \leq k+e$. We have shown that $\gamma'_m \in \pi_m(C')$ for $m \geq k+e$. This implies by the assumption of the lemma that $\pi_{i,j}(\gamma_i) = \gamma_j$ for $i > k \geq j$, as $f_{k+e}(\pi_{k+2e,k+e}(\tilde{\gamma}_{k+2e})) = f_{k+e}(\tilde{\gamma}_{k+e}) = \gamma'_{k+e}$. Inductively, we obtain a sequence of jets γ_m satisfying $\pi_{m,m-1}(\gamma_m) = \gamma_{m-1}$ and $f_m(\gamma_m) = \pi_m(\gamma')$ by construction.

These elements determine an arc $\gamma \in \mathcal{L}(X)$ with $f_\infty(\gamma) = \gamma'$. As $\pi_{k+e}(\gamma) = \gamma_{k+e} \in \pi_{k+e}(C)$ we conclude that γ lies in the cylinder C, which proves the assertion. □

Let us now proceed with the main result of this thesis. The transformation rule states how the motivic integral changes under a proper birational morphism. It is the key ingredient when proving results using motivic integration. As the relative canonical divisor $K_{X/Y}$ is locally given by the Jacobian determinant, one can perceive an analogy to the change of variables formula in multivariable calculus. Its proof will keep us occupied for the next pages.

Theorem 2.17 (Transformation rule [6, Thm. 2.18])**.** *Let $f\colon X \to Y$ be a proper birational morphism and $K_{X/Y}$ the relative canonical divisor. Then we have the following transformation rule for an effective divisor D on Y*

$$\int_{\mathcal{L}(Y)} F_D \,\mathrm{d}\mu = \int_{\mathcal{L}(X)} F_{f^*D+K_{X/Y}} \,\mathrm{d}\mu.$$

Proof. Let $U \subset Y$ be the biggest open subset over which f is an isomorphism and denote by Z its complement. Recall that we proved in Lemma 1.7 that the associated map f_∞ is bijective away from the set $\mathcal{L}(f^{-1}(Z)) \subset \mathcal{L}(X)$. This subset has measure zero, since $\mathcal{L}(f^{-1}(Z)) = F_E^{-1}(\infty)$ where E is the exceptional divisor of the birational morphism f.

Consider the cylinder sets $C_{k,s} := F^{-1}_{K_{X/Y}}(k) \cap F^{-1}_{f^*D}(s) \subset \mathcal{L}(X)$ for $k, s \in \mathbb{Z}_{\geq 0}$. These sets form a partition of $\mathcal{L}(X)$ modulo a subset of measure zero. So by Lemma 1.7 we obtain the equality

$$F_D^{-1}(s) = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} f_\infty(C_{k,s})$$

modulo a subset of measure zero, as $F_{f^*D}(\gamma) = F_D(f_\infty(\gamma))$. Part (i) of the proposition below allows us to apply Lemma 2.16 to the sets $C_{k,s}$. Hence, the above subdivision of $F_D^{-1}(s)$ consists of cylinder sets. By using again the proposition below, we see that $\mu(C_{k,s}) = \mu(f_\infty(C_{k,s})) \cdot \mathbb{L}^k$. Next, we use these facts to compute

$$\begin{aligned}
\int_{\mathcal{L}(Y)} F_D \,\mathrm{d}\mu &= \sum_{s \in \mathbb{Z}_{\geq 0}} \mu(F_D^{-1}(s)) \cdot \mathbb{L}^{-s} \\
&= \sum_{k,s \in \mathbb{Z}_{\geq 0}} \mu(f_\infty(C_{k,s})) \cdot \mathbb{L}^{-s} \\
&= \sum_{k,s \in \mathbb{Z}_{\geq 0}} \mu(C_{k,s}) \cdot \mathbb{L}^{-(s+k)} \\
&= \sum_{0 \leq k \leq s'} \mu(C_{k,s'-k}) \cdot \mathbb{L}^{-s'} \\
&= \sum_{s' \in \mathbb{Z}_{\geq 0}} \mu(F^{-1}_{f^*D+K_{X/Y}}(s')) \cdot \mathbb{L}^{-s'} \\
&= \int_{\mathcal{L}(X)} F_{f^*D+K_{X/Y}} \,\mathrm{d}\mu,
\end{aligned}$$

where we set $s' = s+k$ and used in the penultimate line that $\sqcup_{0 \leq k \leq s'} C_{k,s'-k}$ forms a partition of $F^{-1}_{f^*D+K_{X/Y}}(s')$. This finishes the proof of the theorem. □

Proposition 2.18. *Let $f\colon X \to Y$ be a proper birational morphism and $K_{X/Y}$ the relative canonical divisor. Set $C_e = F^{-1}_{K_{X/Y}}(e)$ and $C'_e = f_\infty(C_e)$. Then, for $m \geq 2e$*

(i) *For a jet $\gamma'_m \in \pi_m(C'_e)$ the fiber $f_m^{-1}(\gamma'_m)$ lies inside a fiber of the projection $\pi_{m,m-e}\colon \pi_m(C_e) \to \pi_{m-e}(C_e)$.*

(ii) *The set $\pi_m(C'_e)$ can be covered by finitely many disjoint constructible subsets B_i such that the preimage $f_m^{-1}(B_i) \subset \pi_m(C_e)$ is isomorphic to $B_i \times \mathbb{A}^e$.*

The proof of the transformation rule shows, why this proposition is so important. The first part implies that the image of the cylinder sets C_e will again be cylinder sets using Lemma 2.16. The second part explains the relationship between the motivic volume of C_e and its image.

This is a modification of Lemma 3.4 in [9] where one also finds a proof of these two assertions. In this thesis, we take a different approach and follow [4]. Namely, we prove the proposition in the case of a blowup $f\colon X' = \mathrm{Bl}_Y X \to X$ along a smooth subvariety $Y \subset X$ and we show how one can deduce from this the transformation rule for a general proper birational morphism.

Proof. Using Lemma 2.19 below we only have to concentrate on a point $\gamma'_m \in \pi_m(C'_e)$ to show the assertions.

Let $f\colon X' = \mathrm{Bl}_Y X \to X$ be the blowup along a smooth subvariety $Y \subset X$. The assertions of the lemma can be checked locally. Hence, as in the proof of Corollary 1.6, we can assume that there is an étale morphism $\varphi\colon X \to \mathbb{A}^n$ such that $\varphi(Y) \subset \mathbb{A}^n$ is given by the vanishing of the first $n-c$ coordinates and $\varphi^{-1}(\varphi(Y)) = Y$. Now we want to reduce to the case $Y = \mathbb{A}^{n-c} \subset \mathbb{A}^n = X$. We therefore use that $\mathrm{Bl}_Y X \simeq X \times_{\mathbb{A}^n} \mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n$ as blowing-up commutes with flat base change [21, Tag 0805]. As being étale is stable under base change, this in particular implies that $\mathrm{Bl}_Y X$ is étale over $\mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n$. Hence, by using Lemma 1.5, we obtain isomorphisms

$$\mathcal{L}_m(\mathrm{Bl}_Y X) \simeq \mathrm{Bl}_Y X \times_{\mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n} \mathcal{L}_m(\mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n) \simeq X \times_{\mathbb{A}^n} \mathcal{L}_m(\mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n).$$

Note that under these isomorphisms the projection $\pi_{m,m-e}\colon \mathcal{L}_m(\mathrm{Bl}_Y X) \to \mathcal{L}_{m-e}(\mathrm{Bl}_Y X)$ corresponds to

$$\mathrm{id}_X \times \pi_{m,m-e}\colon X \times_{\mathbb{A}^n} \mathcal{L}_m(\mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n) \to X \times_{\mathbb{A}^n} \mathcal{L}_{m-e}(\mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n).$$

Analogously, $f_m\colon \mathcal{L}_m(\mathrm{Bl}_Y X) \to \mathcal{L}_m(X)$ corresponds to

$$\mathrm{id}_X \times f_m\colon X \times_{\mathbb{A}^n} \mathcal{L}_m(\mathrm{Bl}_{\varphi(Y)}\mathbb{A}^n) \to X \times_{\mathbb{A}^n} \mathcal{L}_m(\mathbb{A}^n).$$

Thus it suffices to show the affine case described above.

Purely to simplify the notation we assume that $n = 3$ and $c = 3$. Thus f is the blow-up of the origin in $\mathbb{A}^3$. The other cases are all similar computations. Hence we consider $f\colon X' \to X = \mathbb{A}^3$ where $X' = \mathrm{Bl}_0\mathbb{A}^3 \subset \mathbb{A}^3 \times \mathbb{P}^2$ is given by the relations $x_i y_j - x_j y_i$ with (x_1, x_2, x_3) and (y_1, y_2, y_3) being the coordinates of $\mathbb{A}^3$ respectively the homogeneous coordinates of $\mathbb{P}^2$. It suffices to consider an affine open $U \subset X'$ due to the local nature of our question. So let U be the affine patch where $y_1 = 1$. Then the relations above simplify to $x_2 = y_2 x_1$ and $x_3 = y_3 x_1$. Hence, $U \simeq \operatorname{Spec}(k[x_1, y_2, y_3]) = \mathbb{A}^3$ and f is given on the level of rings by

$$k[z_1, z_2, z_3] \to k[x_1, y_2, y_3]$$

sending z_1 to x_1 and z_i to $x_1 y_i$ for $i = 2, 3$. The exceptional divisor E of this birational map is given by the vanishing of x_1. Recall that the relative canonical divisor $K_{X'/X}$ is locally given by the determinant of the associated cotangent map. This implies that $K_{X'/X} = 2E$, since

$$\det \begin{pmatrix} 1 & y_2 & y_3 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix} = x_1^2.$$

Take an arc $\gamma \in C_e$ and consider its truncation $\gamma_m = \pi_m(\gamma)$. In our local situation γ_m is given by the three polynomials

$$\begin{aligned} \gamma_m(x_1) &= t^{e/2} \sum_{i=0}^{m-e/2} a_i t^i, \\ \gamma_m(y_2) &= \sum_{i=0}^{m} b_i t^i, \\ \gamma_m(y_3) &= \sum_{i=0}^{m} c_i t^i, \end{aligned}$$

where we require $a_0 \neq 0$ such that $F_{K_{X'/X}}(\gamma)$ is equal to e. If e is not divisible by 2, the cylinder C_e is empty (or in the general case e being divisible by $c-1$). The image $\gamma'_m = f_m(\gamma_m)$ is determined by the three truncated power series

$$\begin{aligned} \gamma'_m(z_1) &= \gamma_m(x_1) = t^{e/2} \sum_{i=0}^{m-e/2} a_i t^i, \\ \gamma'_m(z_2) &= \gamma_m(x_1)\gamma_m(y_2) = t^{e/2} \sum_{i=0}^{m-e/2} a_i t^i \sum_{i=0}^{m} b_i t^i && \mod t^{m+1}, \\ \gamma'_m(z_3) &= \gamma_m(x_1)\gamma_m(y_3) = t^{e/2} \sum_{i=0}^{m-e/2} a_i t^i \sum_{i=0}^{m} c_i t^i && \mod t^{m+1}. \end{aligned}$$

Due to the occurrence of $t^{e/2}$ in the last two equations, the coefficients $b_{m-e/2+1}, \ldots, b_m$ do not occur if one expands the equation for $\gamma'_m(z_2)$. In an analogous manner, $\gamma'_m(z_3)$ does not depend on $c_{m-e/2+1}, \ldots, c_m$. Conversely, if we are given the three truncated power series of the image γ'_m of a jet $\gamma_m \in \pi_m(C_e)$ we can recover all other coefficients. Namely, given the truncated power series

$$\gamma'_m(z_2) = t^{e/2} \sum_{i=0}^{m-e/2} \beta_i t^i$$

and knowing all a_i's, we can use the fact that $a_0 \neq 0$ to see inductively

$$
\begin{aligned}
b_0 &= \beta_0/a_0 \\
b_1 &= (\beta_1 - a_1 b_0)/a_0 \\
&\ \vdots \\
b_s &= (\beta_s - (a_s b_0 + \cdots + a_1 b_{s-1}))/a_0.
\end{aligned}
$$

This can be done as long as $s \leq m - e/2$, since a_s is known for these s. Naturally we can use the same argument to recover $c_0, \ldots, c_{m-e/2}$.

Summing up all these observations, we see that the fiber over a point $\gamma'_m \in f_m(\pi_m(C_e))$ under f_m is isomorphic to an affine space of dimension $e = 2 \cdot e/2$. The isomorphism is given via the coefficients b_i and c_i for $i \geq m - e/2 + 1$. Applying Lemma 2.19 yields assertion (ii). Moreover we have shown that two jets γ_m and $\tilde{\gamma}_m$, which are both mapped to γ'_m by the morphism f_m, can only differ in these last $e/2$ coefficients. Thus their image in $\mathcal{L}_{m-e}$ under the projection $\pi_{m,m-e}$ is the same. Note that in our case it is sufficient to consider the projection $\pi_{m,m-e/2}$. For a blowup along a smooth subvariety of codimension c, the same computation shows that γ_m and $\tilde{\gamma}_m$ become equal in $\mathcal{L}_{m-e/(c-1)}$. Hence, in all cases they are equal after projecting to $\mathcal{L}_{m-e}$. This proves (i) and finishes the proof of the proposition in the case of blowups along smooth subvarieties. □

Lemma 2.19. [4, Lem. 3.5] *Let $f\colon X \to Y$ be a morphism of reduced schemes of finite type over k. If for all points $y \in Y$ the fiber satisfies $Spec(k(y)) \times_Y X \simeq Spec(k(y)) \times \mathbb{A}^n$, then there exists a finite partition $Y = \sqcup B_i$ into constructible sets with $f^{-1}(B_i) \simeq B_i \times \mathbb{A}^n$.*

Proof. This is a local assertion so we can assume Y to be irreducible. By assumption the fiber over the generic point $\eta \in Y$ is isomorphic to $\mathbb{A}^n$. The isomorphism $X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,\eta}) \xrightarrow{\simeq} \mathbb{A}^n_Y \times_Y \operatorname{Spec}(\mathcal{O}_{Y,\eta})$ over Y can be extended to an open set $\eta \in U \subset Y$, cf. [17, Exc. 3.2.5]. Since $X \times_Y U \simeq f^{-1}(U)$, we have that f is an $\mathbb{A}^n$-bundle over U. Observe that the morphism f restricted to the preimage of $Y \setminus U$ is a morphism with smaller dimensional base which still has the property that all its fibers are affine spaces. Hence the assertion of the lemma follows by induction. □

So far, we have proven the transformation rule for blowups of smooth varieties along smooth subvarieties. We will now use a decomposition for proper birational morphisms to deduce the result for all such morphisms.

Theorem 2.20 (Weak Factorization, [1])**.** *Let $\varphi\colon X \dashrightarrow Y$ be a birational map between smooth complete varieties over an algebraically closed field k of characteristic zero. Then there exists a factorization of φ*

$$
\begin{tikzcd}
 & Z_1 \arrow[dl] \arrow[dr] & & Z_3 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Z_{n-2} \arrow[dl] \arrow[dr] & & Z_n \arrow[dl] \arrow[dr] & \\
X & & Z_2 & & Z_4 & \cdots & Z_{n-3} & & Z_{n-1} & & Y
\end{tikzcd}
$$

such that all occurring maps are blowups along a smooth irreducible center. On top of that, this factorization can be chosen in such a way that there is an index i such that for $j \leq i$ the rational maps $X \dashleftarrow Z_j$ to the left and for $j \geq i$ the rational maps $Z_j \dashrightarrow Y$ to the right are in fact projective morphisms.

This is a famous result in birational geometry. As in our case the occurring varieties are not assumed to be complete, we use the generalization that such a factorization also exists for proper birational morphisms between smooth varieties over an algebraically closed field of characteristic zero [5, 1.2].

Proof of Theorem 2.17. Let $f\colon X \to Y$ be a proper birational morphism between smooth varieties. Then we can factor it into a chain as in the theorem above. Let i be as in the second part of the theorem. For $i \geq 2$ we use that we have proven the transformation rule for blowups along a smooth subvariety together with part (ii) of the lemma below. Hence we conclude that the transformation rule holds for $Z_2 \to X$. Applying part (i) of the lemma below yields the assertion for the morphism $Z_3 \to X$. This can be done inductively until one reaches $Z_i \to X$. In an analogous manner, we conclude that the transformation rule holds for the morphism $Z_i \to Y$. Using the fact that f is a proper birational morphism and again part (ii) of the lemma finishes the proof. □

Lemma 2.21. *Let*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "h"] & & Z \end{tikzcd}$$

be a commutative diagram of proper birational morphisms.

(i) *If the transformation rule holds for f and h, then it also holds for g.*

(ii) *If the transformation rule holds for f and g, then it also holds for h.*

Proof. Note that we have $K_{X/Z} = K_{X/Y} + f^*K_{Y/Z}$. This is even true without using linear equivalence, as locally this boils down to the determinant being a homomorphism. Hence, we can compute for an effective divisor D on Z

$$\begin{aligned}\int_{\mathcal{L}(Z)} F_D \,\mathrm{d}\mu &= \int_{\mathcal{L}(Y)} F_{h^*D+K_{Y/Z}} \,\mathrm{d}\mu \\ &= \int_{\mathcal{L}(X)} F_{f^*(h^*D)+f^*K_{Y/Z}+K_{X/Y}} \,\mathrm{d}\mu = \int_{\mathcal{L}(X)} F_{g^*D+K_{X/Z}} \,\mathrm{d}\mu.\end{aligned}$$

This proves (i). The proof of (ii) is done similarly. □

2.4 Applications

We want to draw some consequences of the transformation rule. Therefore we start by introducing the notion of K-equivalent varieties.

Definition 2.22. Two nonsingular projective varieties are called *K-equivalent*, if there exists a nonsingular projective variety Z and birational morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$.

It is obvious that two K-equivalent varieties are also birationally equivalent. Observe that for a variety Z with morphisms f and g as in the definition above, one has an equality of the relative canonical divisors $K_{Z/X} = K_{Z/Y}$. To see this consider the differences $K_{Z/X} - K_{Z/Y}$ and $K_{Z/Y} - K_{Z/X}$. They are both numerically trivial, as the relative canonical divisors are linearly equivalent. We now use a statement for a proper birational morphism $\alpha\colon S \to T$ between normal varieties which is sometimes called Negativity Lemma. It says that a divisor D on S is effective if and only if $\alpha_* D$ is effective, if the divisor $-D$ is α-nef, cf. [15, Lem. 3.39]. We apply the lemma to the divisor $D = K_{Z/X} - K_{Z/Y}$ and the morphism g. Hence we can conclude that D is an effective divisor, as $K_{Z/X}$ is effective and $g_* K_{Z/Y} = 0$, since $K_{Z/Y}$ is supported on the exceptional locus of g. The same argument shows that $K_{Z/Y} - K_{Z/X}$ is an effective divisor. This implies the equality $K_{Z/X} = K_{Z/Y}$.

Corollary 2.23. [22, 4.1] *Let X and Y be K-equivalent varieties. Then $[X] = [Y]$ in $\hat{\mathcal{M}}$.*

Proof. Let Z, f, g be as in Definition 2.22, D the trivial divisor on X and $K_{Z/X}$ the relative canonical divisor. The transformation rule yields

$$[X] \cdot \mathbb{L}^{-n} = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(X)} F_D \, \mathrm{d}\mu = \int_{\mathcal{L}(Z)} F_{f^*D + K_{Z/X}} \, \mathrm{d}\mu.$$

As X and Y are K-equivalent, the same computation for Y yields the same right hand side. $\square$

We will now loosely touch on Hodge numbers to conclude the result stated in the introduction of this thesis. These are important birational invariants of a variety.

Let X be a smooth complex projective variety. Then one defines the *Hodge numbers* $h^{p,q}(X)$ of X as the $\mathbb{C}$-dimension of $H^q(X, \Omega_X^p)$. These numbers can be put together to form the *Hodge polynomial* of X

$$\sum_{p,q=0}^{n} (-1)^{p+q} h^{p,q}(X) u^p v^q \in \mathbb{Z}[u, v].$$

Deligne [8] has proven that one can extend this construction to all complex varieties by using the mixed Hodge structure on the cohomology with compact support. More precisely, he showed that there is a function

$$E : \mathcal{V}_{\mathbb{C}} \to \mathbb{Z}[u, v]$$

satisfying the following conditions:

(i) If X is smooth and projective, the polynomial $E(X)$ is the Hodge polynomial defined above.

(ii) If $Z \subset X$ is a closed subvariety, then $E(X) = E(Z) + E(X \setminus Z)$.

(iii) For two varieties X and Y, one has $E(X \times Y) = E(X) \cdot E(Y)$.

The polynomial $E(X)$ is called the *Hodge–Deligne polynomial* of X. The properties (ii) and (iii) imply that the function E factors through the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$. It is easy to compute that $E(\mathbb{P}^1) = 1 + uv$. This implies that $E(\mathbb{A}^1) = 1 + uv - 1 = uv$. Thus, if we localize the ring $\mathbb{Z}[u, v]$ in uv, we obtain a map

$$\alpha\colon \mathcal{M} \to \mathbb{Z}[u, v]_{uv}$$

extending E. The completion map $\phi\colon \mathcal{M} \to \hat{\mathcal{M}}$ has as its kernel $\bigcap_{m\in\mathbb{Z}} F^m\mathcal{M}$. The degree of the polynomial associated to $[V] \cdot \mathbb{L}^{-i} \in F^m\mathcal{M}$ is bounded by $2\dim(V) - 2i \leq -2m$. Hence, every element in the kernel of ϕ is mapped to the zero polynomial and the map α factors through the image $\phi(\mathcal{M}) \subset \hat{\mathcal{M}}$.

We can now draw a direct consequence of Corollary 2.23.

Theorem 2.24. [16] *Two K-equivalent varieties have the same Hodge numbers.* □

A nonsingular complex projective variety is called a *Calabi–Yau variety* if its canonical bundle is linear equivalent to the structure sheaf. Sometimes it is also required, that the i-th cohomology for $0 < i < \dim X$ vanishes. It is immediate from the definitions that two birational Calabi–Yau varieties are K-equivalent. We can take a birational map between the varieties and use a resolution of indeterminacies. The theorem then implies that they have the same Hodge numbers. This result was one of the reasons for Kontsevich to invent motivic integration. It has been conjectured by Batyrev, who used p-adic integration to prove that two such varieties have the same Betti numbers [3]. Nowadays, there exists a proof of the theorem using p-adic integration [14].

Another application of the corollary is used in the study of resolutions of singularities. A resolution $\varphi\colon X \to Y$ with X a smooth complex projective variety is called *crepant*, if $\varphi^*\omega_Y \simeq \omega_X$. One directly sees that two different choices of crepant resolutions are K-equivalent. We can again conclude that the Hodge numbers do not depend on the chosen resolution.

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