Picard group of $\mathcal{K}_g$, $6 \le g \le 10$, after Greer, Li, Tian

Notations: $\mathcal{K}_g$ moduli space of quasi-polarized K3 surfaces of genus g.
$D^g_{d,n} = \{ S \in \mathcal{K}_g : \mathrm{Pic}\, S \supset \text{primitive rk2 sublattice } \{L, \beta\},\ L^2 = 2g-2,\ L.\beta = d,\ \beta^2 = n \}$

For $2 \le g \le 5$, general K3 surface $\rightsquigarrow$ complete intersection in $\mathbb{P}^g$ (or double cover of $\mathbb{P}^2$ when $g=2$) $\rightsquigarrow$ proof = explicit construction of the moduli space of such complete intersections + GIT.

For $g \ge 5$, not complete intersections in $\mathbb{P}^g$.

(PART OF) WHAT THEY PROVE: $6 \le g \le 10$

$\mathrm{Pic}_{\mathbb{Q}}(\mathcal{K}_g)$ spanned by NL divisors.
Basis:

- $g=6$, $\{D^6_{0,-2}, D^6_{2,2}, D^6_{k,0}, k=1,\dots,4\}$
- $g=7$, $\{D^7_{0,-2}, D^7_{2,2}, D^7_{3,2}, D^7_{k,0}, k=1,\dots,4\}$
- $g=8$, $\{D^8_{0,-2}, D^8_{2,2}, D^8_{3,2}, D^8_{k,0}, k=1,\dots,4\}$
- $g=9$, $\{D^9_{0,-2}, D^9_{2,2}, D^9_{3,2}, D^9_{k,0}, k=1,\dots,5\}$
- $g=10$, $\{D^{10}_{0,-2}, D^{10}_{2,2}, D^{10}_{3,2}, D^{10}_{4,2}, D^{10}_{k,0}, k=1,\dots,6\}$

OUTLINE OF PROOF

1. Brill Noether general K3 surface (S, L) of genus g $\rightsquigarrow$ (smooth) complete intersection with respect to a vector bundle (E_g) over a homogeneous space (X_g) (following Mukai)
2. Non-BN general K3 surfaces $\subset$ NL divisors (small exception: $D^6_{1,0}$ contains BN general curves!)
3. Construct (using step 1) birational models of $\mathcal{K}_g$ to show $\mathrm{Pic}_{\mathbb{Q}}\, \mathcal{K}^{BN}_g = 0$

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1. BN general K3 and their projective models (or Mukai models)

Def: A polarized K3 surface (S, L) is Brill-Noether general if $h^0(M)h^0(N) < h^0(L)$ for all nontrivial line bundles M, N satisfying $M \otimes N \cong L$.

Idea: Basically like BN general curves. In fact, more precisely, if a smooth curve $C \in |L|$ is BN general, then so is (S, L).

Recipe for Mukai models:
When (S, L) is BN general of genus g, and $r s = g$, there exists a semi-stable vector bundle V_r on S such that:
(i) V_r is globally generated, $H^0(S, V_r) \otimes \mathcal{O}_S \twoheadrightarrow V_r$
(ii) rank $V_r = r$
(iii) $h^0(S, V_r) = r+s$ and $h^i(S, V_r) = 0, i > 0$
(iv) $\bigwedge^r V_r \cong L$
Then we can define

$$\Phi_{V_r} : S \to Gr(r, H^0(S, V_r)^\vee), \quad x \mapsto [V_{r,x}]^\vee$$

Furthermore, we have $\bigwedge^r H^0(V_r) \otimes \mathcal{O}_S \to \bigwedge^r V_r$ induce linear maps

$$\lambda_r : \bigwedge^r H^0(V_r) \to H^0(\bigwedge^r V_r).$$

Observe:

$$\begin{tikzcd} S \arrow[r, "\Phi_{V_r}"] \arrow[d, "\psi_L"'] \arrow[dr, phantom, "\circlearrowleft"] & Gr(r, H^0(S,V_r)^\vee) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}H^0(L) \arrow[r, "\alpha"] & \mathbb{P}(\bigwedge^r H^0(S,V_r)^\vee) \end{tikzcd}$$

λ_r surjective $\Rightarrow$ α embedding

6	8	9	10
2.3	2.4	3.3	2.5

$\boxed{\text{Genus 6}}$ Mukai shows that

(a)
$$\begin{tikzcd} S \arrow[r, "\Phi_{V_2}"] \arrow[d, "\psi_L"'] \arrow[dr, phantom, "\circlearrowleft"] & Gr(2,5) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}^6 \arrow[r, hook, "\alpha"] & \mathbb{P}^9 \end{tikzcd}$$

(b) There is a quadric Q in $\mathbb{P}^9$ such that $\alpha(\mathbb{P}^6) \cap \text{Plücker}(Gr(2,5)) \cap Q \cong \psi_L(S)$.

Paraphrasing Mukai, Greer, Li, Tian:
$X_6 = Gr(2,5)$, $E_6 = \mathcal{O}_{X_6}(1)^{\oplus 3} \oplus \mathcal{O}_{X_6}(2)$,
$\psi_L(S)$ is a complete intersection in X_6 with respect to E_6 i.e. S is a complete intersection of a quadric and a codim. 3 linear section in X_6.

$\boxed{\text{Genus 7}}$ Mukai shows that

$X_7 = \mathrm{OGr}(5,10)$, $E_7 = \mathcal{O}_{X_7}(1)^{\oplus 8}$,
$\psi_L(S)$ is cut out by a codim. 8 linear section in $\mathrm{OGr}(5,10)$.

7 being prime, genus 7 demands a slightly different recipe (Mukai):

$$V_7 = N^*_{\ldots} \otimes L^{\otimes 2}, \quad \bigwedge^5 V_7 \cong L^{\otimes 2}$$
conormal bundle

$\boxed{\text{Genus 8}}$ $X_8 = Gr(2,6)$, $E_8 = \mathcal{O}_{X_8}(1)^{\oplus 6}$, $\psi_L(S)$ codim 6 lin. section in $Gr(2,6)$.

$$\begin{tikzcd} S \arrow[r, "\Phi_{V_2}"] \arrow[d, "\psi_L"'] \arrow[dr, phantom, "\circlearrowleft"] & Gr(2,6) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}^8 \arrow[r, hook] & \mathbb{P}^{14} \end{tikzcd}$$

$S = Gr(2,6) \cap \alpha(\mathbb{P}^8)$
What's left? Check that the intersection is indeed of dimension 2.

$\boxed{\text{Genus 9}}$ $X_9 = \mathrm{LGr}(3,6)$, $E_9 = \mathcal{O}_{X_9}(1)^{\oplus 4}$,
$\psi_L(S)$ is a codim. 4 linear section in X_9.

Rigid bundle V_3 of rank 3
(a) $\lambda_2 : \bigwedge^2 H^0(S, V_3) \to H^0(S, \bigwedge^2 V_3)$
(b) $\lambda_3 : \bigwedge^3 H^0(S, V_3) \to H^0(S, \bigwedge^3 V_3)$
Mukai: λ_3 not injective!

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[d, "\psi_L"'] & \mathrm{LGr}(3,6) \arrow[r, phantom, "\subset"] \arrow[d] & Gr(3,6) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}^9 \arrow[r, hook] & \mathbb{P}^{13} \arrow[r, hook] & \mathbb{P}^{19} \end{tikzcd}$$

$\ker \lambda_3$ cuts out $\mathrm{LGr}(3,6)$

$\boxed{\text{Genus 10}}$ X_{10}: flag variety of dimension 5 associated with the adjoint representation of G_2 in $\mathbb{P}^{13}$.
$E_{10} = \mathcal{O}_{X_{10}}(1)^{\oplus 3}$
$\psi_L(S)$ is a codimension 3 linear section of X_{10}.

Mukai:
$\lambda_2 : \bigwedge^2 H^0(S, V_2) \to H^0(S, \bigwedge^2 V_2)$ has non-trivial kernel, that cuts out X_{10} in $Gr(2,7)$.

2. Non BN general K3 surfaces

Greer, Li, Tian (Lemma 2.8)

Locus of non-BN general K3 surfaces in $\mathcal{K}_g$ is a union of NL divisors $D^g_{d,n}$ satisfying [illegible]

For $6 \le g \le 10$, K3 surfaces are non-BN general iff they lie in the following NL divisors:
(1) $D^g_{0,-2}$, $D^g_{k,0}$, $k = 1, 2, \dots$
(2)
- $g=6$, $D^6_{2,2}$
- $g=7$, $D^7_{2,2}$, $D^7_{3,2}$
- $g=8$, $D^8_{2,2}$, $D^8_{3,2}$
- $g=9$, $D^9_{2,2}$, $D^9_{3,2}$
- $g=10$, $D^{10}_{2,2}$, $D^{10}_{3,2}$, $D^{10}_{4,2}$

Idea of proof: Construct M, N such that $M \otimes N \cong L$, but $\underbrace{h^0(M)h^0(N) \ge h^0(L)}_{\text{This inequality}}$
puts restrictions on permissible values of d, g, n.

3. GIT construction

g	X_g	E_g	W_g	G_g
6	$Gr(2,5)$	$\mathcal{O}_{X_6}(1)^{\oplus 3} \oplus \mathcal{O}_{X_6}(2)$	$\mathbb{P}(H^0(\mathbb{P}_6, \mathcal{O}_{\mathbb{P}_6}(2)))$	$PSL(5)$
7	$\mathrm{OGr}(5,10)$	$\mathcal{O}_{X_7}(1)^{\oplus 8}$	$Gr(8, \widetilde{W}_7)$	$\mathrm{Spin}(10)$
8	$Gr(2,6)$	$\mathcal{O}_{X_8}(1)^{\oplus 6}$	$Gr(6, \widetilde{W}_8)$	$SL(6)$
9	$\mathrm{LGr}(3,6)$	$\mathcal{O}_{X_9}(1)^{\oplus 4}$	$Gr(4, \widetilde{W}_9)$	$Sp(6)$
10	"that flag variety"	$\mathcal{O}_{X_{10}}(1)^{\oplus 3}$	$Gr(3, \widetilde{W}_{10})$	$G_2/Z(G_2)$

where $\widetilde{W}_g = H^0(X_g, \mathcal{O}_{X_g}(1))$

Mukai proved $\Phi_g : W_g^{ss} /\!/ G_g \dashrightarrow \mathcal{K}_g$ is birational.

Greer, Li, Tian: describe the smooth complete intersections in $\mathcal{K}_g$ via Φ_g.

X smooth projective variety, E globally generated vector bundle on X.
$\Delta_E := \{ s \in \mathbb{P}(H^0(X, E)) : (s)_0 \text{ is singular, or not a complete intersection w.r.t } E \}$
discriminant locus

Greer Li Tian: Δ_E irreducible in all our examples (happens when complete intersections have isolated singularities)

Greer Li Tian (Theorem 3.6)
For $6 \le g \le 10$, if $S \in W_g$ is a smooth complete intersection, then it is GIT stable in W_g.
(uses the fact that $\mathrm{Aut}_L(S)$ is finite, Chapter 5, Prop 3.3, Daniel's book).

Sketch of proof the main theorem
$U_g \subseteq W_g$ open subset parametrizing smooth complete intersections i.e. $U_g = W_g - \Delta_g$.
$\Phi_g' : U_g /\!/ G_g \to \mathcal{K}_g$ open immersion

$\mathcal{K}_g^{BN} := \mathcal{K}_g$ − divisors listed in the main theorem
Then $\Phi_g'(U_g) \supset \mathcal{K}_g^{BN}$ (by the work of Mukai in 1.)

$W_g - U_g = \Delta_g$ is an irreducible divisor, and $\mathrm{Pic}(W_g) \cong \mathbb{Z}$.

$\mathrm{Pic}(U_g /\!/ G_g) \hookrightarrow \mathrm{Pic}(U_g)_{G_g}$ (G_g linearized line bundles on U_g).

$\mathrm{Ker}(\mathrm{Pic}_{\mathbb{Q}}(U_g)_{G_g} \to \mathrm{Pic}_{\mathbb{Q}}(U_g)) \cong \chi(G_g) = 0$ (group of rat'l characters of G_g, G_g simple)

Therefore, $\mathrm{Pic}_{\mathbb{Q}}(U_g /\!/ G_g) = 0$.

$\Rightarrow \mathrm{Pic}_{\mathbb{Q}}(\mathcal{K}_g^{BN}) = 0$ (localization sequence for Picard)

Conclusion: $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{K}_g)$ spanned by the NL divisors in "the list".
That they form a basis comes from Li-Tian's dimension count on $\mathrm{Pic}^{NL}_{\mathbb{Q}}(\mathcal{K}_g)$.