ARITHMETIC FINITENESS FOR VERY IRREGULAR VARIETIES

Bonn WiSe 2024/2025

Introduction

The seminar consists of three parts

(1) The Lawrence–Venkatesh theorem and strategy (Talks 1-5).
(2) The extension to very irregular varieties (Talks 6-7).
(3) Questions of big monodromy via Tannakian groups (Talks 8-12).

In the first we will outline the approach of Lawrence–Venkatesh to the Shafarevich conjecture for hypersurfaces of sufficiently high degree and dimension. The key ideas are to some extent similar to Faltings' proof of the Mordell conjecture, but certain difficulties are circumvented by the use of Hodge theory (p-adic Torelli theorems, but also in the form of monodromy of variations of Hodge structures).

We will then shift focus to more recent papers by Javanpeykar–Krämer–Maculan–Lehn and Krämer–Maculan (and implicitly Lawrence–Sawin) who extend the techniques to subvarieties of abelian varieties. First we will see what adjustments need to be made to the method of Lawrence–Venkatesh, and conclude the proof of the main theorem of Krämer–Maculan, assuming the monodromy criterion. The final 5 talks will be on Tannakian criteria for bigness of monodromy.

Schedule

Date	**Speaker**
18.10 (Talk 1)	Frank
25.10 (Talk 2)	Giacomo
8.11 (Talk 3)	Moritz
15.11 (Talk 4)	Daniel
22.11 (Talk 5)	Frank
29.11 (Talk 6)	Jiexiang
6.12 (Talk 7)	Gebhard
13.12 (Talk 8)	Riccardo
17.1 (Talk 9)	Hannah
24.1 (Talk 10)	Alessio
31.1 (Talk 11)	TBD

Talks

1. **The Shafarevich conjecture.** Briefly outline Faltings' proof of the Mordell conjecture, following e.g. [Mac22, sect. 1.1]: Explain briefly the Kodaira-Parshin reduction to the Shafarevich conjecture for abelian varieties, and state Faltings' finiteness for pure semisimple representations. Cite other known instances of the Shafarevich conjecture [Mac22, sect. 1.3]; explain that they all rely on the case of abelian varieties or on simple finiteness results in Galois cohomology, and that recently a new method was introduced by Lawrence-Venkatesh [LV20]. Discuss the philosophy of the Lang-Vojta conjecture; as an example, admitting the Lang-Vojta conjecture, show the Shafarevich conjecture for complete intersections [JL17].

2. **The Lawrence–Venkatesh strategy.** Prove the Hermit–Minkowski finiteness of semisimple representations argument [LV20, Lemma 2.3] used by Faltings and also by Lawrence–Venkatesh. Outline the key steps of the proof [LV20]: big monodromy, density of fibres. Highlight how the Tate conjecture implies semisimplicity in great generality, and how much of the work goes into avoiding this.

3. **Transcendence properties of period maps.** The goal of this talk is to present the Ax-Schanuel theorem of Bakker and Tsimerman and the p-adic analogue we will be using. Briefly explain the classical transcendence theory as in [BT20, sect. 1], then state and explain the main theorem [BT19, th. 1.1]. Time permitting you could prove the Ax-Lindemann-Weierstrass theorem for uniformizations of algebraic tori [BT20, th. 4.1.3] where one can already see many arguments of the period domain case at work. Next, transfer the Bakker-Tsimerman theorem to a p-adic setting from [LV20, sect. 9.2].

4. **Monodromy for hypersurfaces.** Prove the main results needed from [Bea86]. See also Huybrechts' cubics book for an account of this.

5. **Conclusion of the Lawrence-Venkatesh method.** Formulate the nondensity criterion for integral points by Lawrence-Venkatesh [LV20, thm. 10.1], and sketch how to deduce the nondensity of projective hypersurfaces with good reduction as in prop. 10.2 of loc. cit. Then outline the proof of the nondensity criterion [LV20, sect. 10.3], but omit the proof of lemma 10.4 and prop. 10.6. Instead, stress the importance of big monodromy, the role of p-adic period mappings and how to circumvent semisimplicity of the arising Galois representations. In the next talks these ideas will be generalized to prove the nondensity criterion in [KM23, th. D].

6. **Very irregular varieties.** Introduce very irregular varieties, and state some generalities on subvarieties of abelian varieties, ampleness of the normal bundle, degeneracy. State the main theorems of [KM23] and deduce Theorems A, B, C from D.

7. **Finiteness from big monodromy.** Give an idea of the proof of Theorem D from [KM23] ([KM23, sect. 1.5, sect. 8]), highlighting what needs to be changed from the Lawrence–Venkatesh method.

8. **Tannaka groups of perverse sheaves on abelian varieties.** Recall the notion of a Tannakian category and simple examples [DM82]. Then introduce the Tannakian categories arising from the convolution of perverse sheaves on abelian varieties, following [JKLM23, sect. 3] and the reference therein. Explain how to get fiber functors from generic vanishing for perverse sheaves on abelian varieties, why the dimension of the corresponding representation is the Euler characteristic of the perverse sheaf, and when this representation is self-dual. Then briefly discuss

some examples of Tannaka groups of subvarieties such as points, curves in their Jacobian [KW15, th. 6.1], and Fano surfaces of cubic threefolds [Krä16].

9. **Big monodromy from sheaf convolution.** Recall that as an input for the Lawrence-Venkatesh approach to the Shafarevich conjecture we need a criterion for big monodromy [JKLM23, main theorem (monodromy version)]. Explain how this reduces to [JKLM23, main theorem (Tannaka version)]: Relate the monodromy of a family of subvarieties to the Tannaka group of a general fiber by stating and proving [JKLM23, th. 4.10], an analog of the theorem of the fixed part [And92]. As in the case of hypersurfaces [LS20], the key is to look at the Galois sequence for the Tannaka groups of perverse sheaves over the function field of the base. Here one should mention the analogy to the sequence relating the geometric and arithmetic étale fundamental group of a variety; for details see [JKLM23, sect. 4].

10. **Characteristic cycles.** Briefly recall the notion of holonomic $\mathscr{D}$-modules and the de Rham functor which relates them to perverse sheaves, see e.g. [HTT08]. To any holonomic $\mathscr{D}$-module on a smooth variety one attaches a Lagrangian cycle on the cotangent bundle by taking the associated graded for a good filtration. Explain this construction and mention that every conic Lagrangian cycle on the cotangent bundle is a linear combination of conormal varieties [HTT08, sect. 2.2 and E.3]. The main goal of this talk is then to discuss the case of holonomic $\mathscr{D}$-modules on abelian varieties: Explain the notion of Gauss maps and their relation with representation theory following [Krä22, sect. 1.d - 1.e] or [JKLM23, sect. 5.4]; the takehome should be that for any smooth subvariety the corresponding representation is minuscule.

11. **The simplicity criterion.** The goal of this talk is to prove the simplicity criterion in [JKLM23, th. 6.1]. Explain how the results about characteristic cycles from the previous talk allow to draw geometric consequences from the nonsimplicity of the Tannaka group. Give the proof in the case of smooth ample divisors as in [LS20, lemma 4.6]. Then explain how to do the general case by using the Pontryagin product on Segre classes to control the dimension [JKLM23, cor. 5.7 and sect. 6.5]; the same tools will be used in the next talk.

12. **Wedge powers (*).** Once we know that the Tannaka group is simple and acts via a minuscule representation, it only remains to rule out wedge powers and spin representations [JKLM23, sect. 1.3]. The goal of this talk is to prove theorem 7.3 of loc. cit. which says that wedge powers only arise from sums of curves. Introduce the notion of subvariety being a wedge power, and explain why this happens for sums of a curve. Then discuss the converse direction following section 7.2, and explain how the Pontryagin product on Segre classes gives the crucial dimension bound [JKLM23, th. 7.5]. If time permits, explain briefly how the argument can be modified to rule out spin representations.

References

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