CHARACTERISTIC FOLIATIONS

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Abstract. The restriction of the holomorphic symplectic form on a hyperkähler manifold X to a smooth hypersurface $D \subset X$ leads to a regular foliation $\mathcal{F} \subset \mathcal{T}_D$ of rank one, the characteristic foliation. We survey, with essentially complete proofs, a series of recent results concerning the geometry of the characteristic foliation, starting with work by Hwang–Viehweg [HV10], but also covering articles by Amerik–Campana [AC14] and Abugaliev [Ab19, Ab21]. The picture is complete in dimension four and shows that the behavior of the leaves of $\mathcal{F}$ on D is determined by the Beauville–Bogomolov square $q(D)$ of D. In higher dimensions, some of the results depend on the abundance conjecture in dimension $\dim(X) - 1$.

1. Main theorem and motivation

Throughout, $D \subset X$ denotes a smooth connected hypersurface in a compact hyperkähler manifold X of complex dimension $2n$, i.e. X is a simply connected, compact Kähler manifold such that $H^0(X, \Omega_X^2)$ is spanned by a holomorphic symplectic form σ. Usually X will be in addition assumed to be projective, although one expects all results to hold in general.

The symplectic form σ induces a regular foliation of rank one on D, i.e. a line sub-bundle $\mathcal{F} \subset \mathcal{T}_D$. We shall denote a generic leaf of the foliation by L and its Zariski closure by $\bar{L}$. The space of leaves will be denoted $D/\mathcal{F}$. These notions will all be recalled in Sections 2 and 3.

Our goal to discuss the following table and establish equivalence of all assertions in each row. We will throughout assume $n > 1$, but see Remark 1.1.

	(i)	(ii)	(iii)	(iv)
(1)	$\dim \bar{L} = 1$	$L = \bar{L} \simeq \mathbb{P}^1$	$q(D) < 0$	D is uniruled
(2)	$\dim \bar{L} = n$	$\bar{L}$ Lagr. torus	$q(D) = 0$	$D = f^{-1}H \subset X \xrightarrow{f} B$ Lagr. fibration
(3)	$\dim \bar{L} = 2n - 1$	$\bar{L} = D$	$q(D) > 0$	D is of general type

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Essentially, in each row the four conditions are known or at least expected to be equivalent to each other. The assumption on D to be smooth is essential, see Section 8.2. The following serves as a guide for what will be discussed in the subsequent sections, where precise references will be provided.

Case (1): Closed leaves.

$$(1): \quad \text{(i)} \xRightarrow{\S\,4.1} \text{(iv)} \xRightarrow{\S\,4.2} \text{(iii)} \xRightarrow{\S\,4.3} \text{(iv)} \xRightarrow{\S\,4.4} \text{(i)}$$

Additionally, we observe the easy equivalence $\text{(i)} \xLeftrightarrow{\S\,4.5} \text{(ii)}$.

Case (2): Lagrangian fibrations.

$$(2): \quad \text{(i)} \xRightarrow{\S\,5.1} \text{(iii)} \underset{\S\,5.3}{\overset{\S\,5.2}{\rightleftarrows}} \text{(iv)} \xRightarrow{\S\,5.4} \text{(i)}$$

The green color of the arrows indicates that this direction is currently only proved assuming the abundance conjecture in dimension $2n-1$, so the proof is only complete for $n = 2$.

We also address $\text{(iv)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}$ in Section 5.5.

Case (3): Dense leaves

$$(3): \quad \text{(i)} \xRightarrow{\S\,6.1} \text{(iii)} \xRightarrow{\S\,6.2} \text{(iv)} \xRightarrow{\S\,6.3} \text{(iii)} \xRightarrow{\S\,6.4} \text{(i)}$$

Note that the equivalence (i) $\Leftrightarrow$ (ii) is clear in this case.

Additionally, we will also provide a direct argument for $\text{(iv)} \xRightarrow{\S\,6.5} \text{(i)}$.

Remark 1.1. Let us consider the case $n = 1$, i.e. X a K3 surface. Then, a smooth hypersurface $D \subset X$ is just a smooth curve. Clearly, in this case (i) holds in all three cases (1), (2), and (3). The equivalence of the other conditions (ii), (iii), and (iv) in (1) and (2) and of (iii) and (iv) in (3) is well known.

2. Preparations I: Linear algebra of the characteristic foliation

We collect some linear algebra results and discuss applications to the geometry of the leaves of a foliation.

2.1. We begin with discussing some easy linear algebra results that will be used throughout the later sections.

Let W be a vector space together with a symplectic structure σ, i.e. $\sigma \in \bigwedge^2 W^*$ such that the induced map $\sigma\colon W \xrightarrow{\sim} W^*$ is an isomorphism. In this situation, the dimension of W is even, so $\dim W = 2n$.

Lemma 2.1. *Assume $V \subset W$ is a subspace of codimension one. Then the subspace*

$$F := \ker\left(\sigma|_V\colon\ V \hookrightarrow W \xrightarrow{\sigma} W^* \twoheadrightarrow V^*\right) \subset V$$

is of dimension one.

Similarly, if $U \subset W$ is of codimension two, then either $\dim\ker(\sigma|_U\colon U \to U^*) = 2$ *or* $\sigma|_U \in \bigwedge^2 U^*$ *is non-degenerate, i.e.* $\ker(\sigma|_U) = 0$.

Proof. Since $W \xrightarrow{\sim} W^* \twoheadrightarrow V$ has a one-dimensional kernel, we have $\dim F \leqslant 1$. Furthermore, since $\dim V = 2n-1$ is odd, the alternating form $\sigma|_V \in \bigwedge^2 V^*$ cannot be non-degenerate, i.e. $\ker(\sigma|_V) \neq 0$. Hence, $\dim F = 1$. The proof of the second assertion is analogous. □

Lemma 2.2. *Assume $V \subset W$ is of codimension one and let $F = \ker(\sigma|_V) \subset V$. Then σ naturally induces a symplectic structure $\bar\sigma$ on V/F.*

Proof. By definition of F, the restriction $\sigma|_V\colon V \to V^*$ factorizes as $V \twoheadrightarrow V/F \hookrightarrow V^*$. By definition of F, the image takes value in $(V/F)^* \subset V^*$, which for dimension reasons gives $\bar\sigma\colon V/F \xrightarrow{\sim} (V/F)^*$. □

Here are a few more concepts from linear algebra: A subspace $U \subset W$ of codimension c is called *isotropic* if $q|_U \in \bigwedge^2 U^*$ is trivial or, equivalently, if

$$U \subset U^\perp := \ker(W \xrightarrow{\sim} W^* \twoheadrightarrow U^*) = \{w \in W \mid q(U,w) = 0\}.$$

If $U^\perp \subset U$, the subspace is called *coisotropic*. Since by definition $U^\perp$ is of dimension c, a subspace $U \subset W$ is coisotropic if and only if $\dim\ker(\sigma|_U\colon U \to U^*) = c$. Finally, $U \subset W$ is *Lagrangian* if U is simultaneously isotropic and coisotropic, i.e. $U = U^\perp$.

Lemma 2.3. *Assume $V \subset W$ is of codimension one and let $F = \ker(\sigma|_V) \subset V$.*

(i) *Then for any Lagrangian subspace $U \subset W$ that is contained in V one has $F \subset U$.*

(ii) *If a subspace of codimension two $U \subset W$ is contained in V and $F \subset U$, then U is coisotropic.*

Proof. The first claim follows from the commutativity of the diagram

$$\begin{array}{ccccc} & & F & & \\ & & \downarrow & & \\ U & \hookrightarrow & V & \hookrightarrow & W \\ {\scriptstyle 0}\downarrow & & {\scriptstyle \sigma|_V}\downarrow & & {\scriptstyle\sigma}\downarrow\wr \\ U^* & \twoheadleftarrow & V^* & \twoheadleftarrow & W^* \end{array}$$

and the assumption that U is Lagrangian, which implies that $U = \ker(W \xrightarrow{\sim} W^* \twoheadrightarrow U^*)$.

For the second assertion apply Lemma 2.1. Since $F \subset U \subset V$, the restriction $\sigma|_U$ is degenerate and, therefore, $\dim \ker(\sigma_U) = 2$. □

2.2. A *regular foliation* of a smooth variety (or complex manifold) D is a locally free subsheaf $\mathcal{F} \subset \mathcal{T}_D$ with locally free quotient and such that $\mathcal{F}$ is integrable, i.e. $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$. Note that the integrability condition is automatically satisfied if $\mathrm{rk}(\mathcal{F}) = 1$, which is the case of interest to us.

A *leaf* of a foliation is a maximal connected and immersed complex submanifold $L \subset D$ with $\mathcal{F}|_L = \mathcal{T}_L$ as subsheaves of $\mathcal{T}_D|_L$. The integrability condition ensures that there exists a (unique) leaf through any point of D. A submanifold $Z \subset D$ is *invariant* under the foliation if $\mathcal{F}|_Z \subset \mathcal{T}_Z$ as subsheaves of $\mathcal{T}_D|_Z$. If Z is a singular subvariety of D, then we call Z invariant if its smooth locus is invariant. It is not hard to see that the Zariski closure of an invariant complex submanifold is invariant. Also note that every leaf L intersecting an invariant submanifold $Z \subset D$ is contained in its closure.

A leaf $L \subset D$ is typically not closed. Its Zariski closure $\bar{L} \subset D$ can be identified with the smallest subvariety containing L that is invariant under the foliation.

Consider now the case of a smooth hypersurface $D \subset X$ of a compact hyperkähler manifold. By virtue of Lemma 2.1, the kernel

$$\mathcal{F} := \ker\left(\sigma|_D \colon\ \mathcal{T}_D \hookrightarrow \mathcal{T}_X|_D \xrightarrow{\ \sigma\ } \Omega_X^*|_D \twoheadrightarrow \Omega_D^*\right) \subset \mathcal{T}_D$$

is a sub-line bundle with locally free kernel. It is called the *characteristic foliation* of the hypersurface $D \subset X$ and was first studied by Hwang and Viehweg [HV10].

Lemma 2.4. *The normal bundle of the characteristic foliation $\mathcal{N}_{\mathcal{F}} := \mathcal{T}_D/\mathcal{F}$ is naturally endowed with a symplectic structure and*

$$\mathcal{F} \simeq \omega_D^*.$$

In particular, any local transverse section Σ of a leaf $L \subset D$ has a natural symplectic structure.

Proof. The first assertion follows from Lemma 2.2. It gives $\det(\mathcal{N}_{\mathcal{F}}) \simeq \mathcal{O}_D$ and hence $\mathcal{F} \simeq \det(\mathcal{T}_D) \simeq \omega_D^*$. □

Remark 2.5. For foliations in general $\det(\mathcal{F})^*$ is often called the canonical bundle $\omega_{\mathcal{F}}$ of the foliation. For the characteristic foliation we thus have $\omega_{\mathcal{F}} \simeq \omega_D$. Note that for $n = 1$, $\mathcal{F} \simeq \mathcal{T}_D$ which is not interesting so that we usually assume $n > 1$. Then, according to [Dr17, Thm. 1.1], $\det(\mathcal{F}) \simeq \omega_D^*$ cannot be big and nef.

The geometric versions of isotropic, coisotropic, and Lagrangian for subspaces of a symplectic vector space are readily defined: For example, a subvariety $Z \subset X$ is coisotropic if the rank of $\sigma|_Z \colon \mathcal{T}_Z \longrightarrow \Omega_Z$ (say over the smooth locus of Z) is $2\dim(Z) - \dim(X)$ or, equivalently, if $\mathrm{rk}(\ker(\sigma|_Z)) = \mathrm{codim}(Z \subset X)$.

The geometric analogue of Lemma 2.3 is the following.

Corollary 2.6. *Assume $D \subset X$ is a smooth hypersurface of a compact hyperkähler manifold.*

(i) *If $T \subset X$ is a smooth Lagrangian submanifold that is contained in D, then T is covered by leaves or, equivalently, every leaf $L \subset D$ intersecting T is contained in T.*

(ii) *Furthermore, any invariant subvariety $Z \subset X$ of codimension two that is contained in $D \subset X$ is coisotropic.* □

3. Preparations II: Space of leaves

There is no standard text on foliations on complex manifolds or algebraic varieties, but see e.g. [CN85]. The arguments typically rely very much on the differentiable theory. The holomorphic version of Reeb's classical theorem, cf. [HV10, KCT07], is one example.

3.1. Consider a foliation $\mathcal{F}$ (of rank one) on a compact complex manifold D. The space of leaves is the quotient $D/\mathcal{F}$ by the equivalence relation that identifies two points if they are contained in the same leaf. The quotient topology is often complicated and frequently non-Hausdorff, but the projection $\pi \colon D \longrightarrow D/\mathcal{F}$ is open, i.e. for any open set $U \subset D$ the union of all leaves intersecting U (its saturation) is again an open subset. For more information see [CN85, Ch. III]. A typical example is that of a $\mathbb{P}^1$-bundle $\pi \colon D = \mathbb{P}(\mathcal{E}) \longrightarrow Z$ with $\mathcal{F} = \mathcal{T}_\pi$. In this situation, $D/\mathcal{F} = Z$. We will come back to the local structure of $D/\mathcal{F}$ and the map $\pi \colon D \longrightarrow D/\mathcal{F}$ in the case that the foliation is *algebraically integrable*, i.e. when every leaf is compact.[1]

3.2. Let $L = \bar{L} \subset D$ be a compact leaf. For a fixed point $x \in L$ we pick a small transversal section $x \in \Sigma_x \subset D$ (think of it as a germ of a transversal section). Consider a closed loop $\gamma \colon [0,1] \longrightarrow L$ with $\gamma(0) = \gamma(1) = x$ and pick a point $y \in \Sigma_x$ close to x. Then there exists a differentiable map $\Phi \colon \Sigma_x \times S^1 \longrightarrow X$ such that $\Phi(0,t) = \gamma(t)$ and $\Phi(y,0) = y$. The pull-back of the foliation $\mathcal{F}$ defines a real foliation of rank one on $\Sigma_x \times S^1$.

[1] or, equivalently if admits one compact leaf with finite holonomy [Pe01, Thm. 1].

Starting with a point $(y,0)$ and integrating defines a path $\gamma_y\colon [0,1] \longrightarrow \Sigma_x \times S^1$ satisfying $\gamma_y(t) = (\rho_{\gamma,y}(t), t)$ and $\gamma_y(0) = (y,0)$. Note, however, that this path is not necessarily closed, so possibly $\gamma_y(1) \neq (y,0)$.

It turns out that the map $y \longmapsto \rho_{\gamma,y}(1)$ only depends on the homotopy class of γ, which gives rise to the following.

Definition 3.1. The *holonomy* of a compact leaf $L \subset D$ is the group homomorphism

$$\rho\colon \pi_1(L,x) \longrightarrow \mathrm{Diff}(\Sigma_x),\ \gamma \longmapsto (\rho_\gamma\colon y \longmapsto \rho_{\gamma,y}(1)).$$

The leaf has *finite holonomy* if the image of ρ, the *holonomy group*

$$G_L \coloneqq \mathrm{Im}(\rho) \subset \mathrm{Diff}(\Sigma_x),$$

is finite.

Note that the image of ρ only depends on x up to conjugation. In particular, the property of a leaf to have finite holonomy does not depend on the base point.

Since we are interested in foliations of rank one, a compact leaf will be a compact complex curve. If this curve is rational, i.e. $L = \bar{L} \simeq \mathbb{P}^1$, then it has automatically finite (and in fact trivial) holonomy. Also note that due to a result of Holmann [Ho89, Thm. 3.1], one knows that if all leaves of a foliation on a Kähler manifold are compact, then they all have finite holonomy.[2]

Theorem 3.2. *Assume that a foliation $\mathcal{F}$ of rank one on a smooth projective variety D has one leaf isomorphic to $\mathbb{P}^1$. Then $\mathcal{F}$ is algebraically integrable and all leaves are curves isomorphic to $\mathbb{P}^1$.*

Proof. According to a result of Pereira [Pe01, Thm. 1], for a foliation on a compact Kähler manifold the existence of one compact leaf with finite holonomy implies that all leaves are compact with finite holonomy. This proves that $\mathcal{F}$ is algebraically integrable. Reeb stability [HV10, Prop. 2.5] then implies that all leaves are isomorphic to each other.[3] □

3.3. Let us come back to the space of leaves $D/\mathcal{F}$ and the map $\pi\colon D \longrightarrow D/\mathcal{F}$ for an algebraically integrable foliation (of rank one) on a smooth projective variety D. We collect the facts that will be used at various places later:[4]

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[2] In fact, Holman [Ho89, Prop. 1.4] proved that for a holomorphic foliation with only compact leaves stability is equivalent to finite holonomy.

[3] A word on the name 'Reeb stability': A leaf L is stable if every open neighbourhood $L \subset U$ of it contains an invariant open neighbourhood $L \subset U' \subset U$. The foliation is stable if all leaves are stable. Reeb stability in the holomorphic context as in [HV10, KCT07] can be viewed as saying that compact leaves with finite holonomy are stable.

[4] These results seem well known to the experts but we could not find a source with complete proofs. Thanks to J.-B. Bost for an instructive email exchange.

• The map $\pi\colon x \longmapsto |G_{L_x}| \cdot [L_x]$ defines a holomorphic map from D to the Chow variety (or Barlet space). Here, L_x is the unique leaf through x and G_{L_x} is its holonomy group.

• The quotient $D/\mathcal{F}$ can be identified with (the normalization of) the image of π. In particular, $D/\mathcal{F}$ is an algebraic variety and $\pi\colon D \longrightarrow D/\mathcal{F}$ is a proper morphism.[5]

• Assume $x \in \Sigma_x \subset D$ is a transversal section of a leaf $x \in L$ as in Section 3.2. Then locally Σ_x/G_x is a chart of $D/\mathcal{F}$ at the point corresponding to the leaf L_x. In particular, $D/\mathcal{F}$ has quotient singularities.

• If the fibres of $D \longrightarrow D/\mathcal{F}$ are rational, i.e. we are in the situation of Theorem 3.2, then $D/\mathcal{F}$ is a smooth projective variety. Furthermore, for the characteristic foliation, $D/\mathcal{F}$ comes with a natural symplectic structure.

• For a dense open subset of points $x \in D$ the leaf L_x through x has trivial holonomy $|G_{L_x}| = 1$.

• The scheme-theoretic fibre of $\pi\colon D \longrightarrow D/\mathcal{F}$ over a point $[L] \in D/\mathcal{F}$ corresponding to a leaf with trivial holonomy $|G_L| = 1$ is the leaf L. The fibre is non-reduced over points with non-trivial holonomy; more precisely, it is a multiple fibre with multiplicity $|G_L| \neq 1$.

3.4. It is easy to prove that a smooth curve $C \subset S$ in a K3 surface with $(C.C) \geqslant 0$ is nef. The following is the hyperkähler analogue of this fact.[6]

Proposition 3.3. *Let $D \subset X$ be a smooth hypersurface in a projective hyperkähler manifold X. If $q(D) \geqslant 0$, then D is nef.*

Proof. Assume D is not nef. Then there exists an irreducible curve $C \subset X$ with $D \cdot C < 0$. The latter implies $C \subset D$ and $\deg(\omega_D|_C) < 0$, which by Lemma 2.4 shows $\deg(\mathcal{F}|_C) > 0$, i.e. $\mathcal{F}|_C$ is ample.

However, the latter implies that the foliation $\mathcal{F}$ is algebraic, i.e. all leaves are compact or, equivalently, algebraic curves. This is a consequence of a much more general result that was original proved by Bogomolov–McQuillan [BMcQ16] and Bost [Bo01] with details provided by Kebekus, Solá Conde, and Toma [KCT07, Thm. 1 & 2]: If the restriction of a foliation to some complete curve C is an ample vector bundle, then the leaf through any point of C is algebraic. Moreover, the leaf through a general point of C is rationally connected and, in fact, all leaves are rationally connected. In fact, according to Theorem 3.2, all we need is one compact rational leaf.[7]

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[5] We are not quite sure whether this map should expected to be flat, but it is open.

[6] We wish to thank R. Abugaliev for communicating this result to us. It seems known to some experts, but has not been written down anywhere.

[7] Note that in this sense Reeb stability shows that the ampleness along C determines the behavior of the foliation globally.

In our situation the result means that all leaves of $\mathcal{F}$ are smooth rational curves and, in particular, D is uniruled. Then, the discussion in Section 4.2, which is independent of this proposition, leads to the contradiction $q(D) < 0$. □

4. Case (1): Closed leaves

4.1. (i) ⇒ (iv): We assume that the leaves of the foliation have one-dimensional closures and want to show that D is then uniruled.[8]

First of all, since the boundary $\bar{L}\backslash L$ is invariant, it is a union of leaves. However, under our assumption all leaves are one-dimensional and, therefore, all leaves are in fact closed $\bar{L} = L$. Thus, all leaves are algebraic curves, i.e. $\mathcal{F}$ is algebraically integrable, and the natural projection

$$\pi\colon D \longrightarrow D/\mathcal{F}$$

is a proper morphism between algebraic varieties.

The proof proceeds in three steps.

(1) Prove that π has no multiple fibres in codimension one and that the canonical divisor of $D/\mathcal{F}$ is trivial.
(2) Deduce the isotriviality of π, combining results of Miyaoka and Hwang–Viehweg, and consider a finite quasi-étale cover of D that splits into a product.
(3) Reach a contradiction by considering the numerical dimension of K_D.

(1) Intuitively, the morphism $\pi\colon D \longrightarrow D/\mathcal{F}$ induced by the algebraically integrable foliation $\mathcal{F}$ contracts all curves with tangent space contained in the kernel of $\sigma|_D$. Therefore $\sigma|_D$ should descend to a non-degenerate two-form on the space $D/\mathcal{F}$ and in particular $D/\mathcal{F}$ should have trivial canonical bundle.

To make this rigorous take a locally transverse section Σ of the foliation, which serves as a local model for $D/\mathcal{F}$. Then, by Lemma 2.4, $\sigma|_\Sigma$ is symplectic and glues to a global symplectic form on $D/\mathcal{F}$. The details can be found in [Sa09, Lem. 6].

Looking at the local behavior of the symplectic form around the multiple fibres, Amerik and Campana [AC14] proved the following:

Lemma 4.1 (Amerik–Campana)**.** *The map π has no multiple fibre in codimension one. Moreover some multiple of the canonical bundle of $D/\mathcal{F}$ is trivial.*

Proof. Suppose by contradiction that there exists a divisor $V \subset D/\mathcal{F}$ such that the fibres over V are multiple of order $m > 1$. The statement is local around a generic point $0 \in V$. We can take a local multisection W over 0 that meets transversally the non-reduced fibres. We choose coordinates (z, t) around V such that z are coordinates for V and t parametrizes the normal direction.

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[8] This part is the most technical one of all of this survey and we will have to be sketchy at points.

Then we can choose coordinates (u, s, w) around W in such a way that W is given by the equation $w = 0$ and $\pi(u, s, w) = (u, s^m)$.

By the discussion in the previous section we know that $\sigma^{n-1} = \pi^*\alpha$ for some form of top degree on the base at least over the smooth locus of the space of leaves. We will show that α extends to a holomorphic form in codimension one. Locally we write $\alpha = G(z,t) \cdot dz \wedge dt$ where dz is a $n-2$ form.

We write $|G(z,t)| = e^{g(z,t)} t^{-c}$ where g is a real-valued bounded function and we claim that $c = 1 - 1/m$. Roughly speaking we are saying that α, more precisely its coefficient $G(z,t)$, has poles of multiplicity exactly $1 - 1/m$ over V. Assuming for the moment that this is true, then this means that the holomorphic function $G(z,t)$ has poles of order strictly less than one around $t = 0$, that is not possible. Hence there exists a subset of $D/\mathcal{F}$ whose complement has codimension at least two where α is non-zero, hence trivializes the canonical bundle of this big open subset.

To prove the claim we denote by π_0 the restriction of $\pi\colon D \longrightarrow D/\mathcal{F}$ to W. In coordinates $\pi_0(u, s, w) = (u, s^m)$. By the base change formula we see that the restriction of σ^{n-1} to W is $\pi_0^*\alpha = G(u, s^m) \cdot m \cdot s^{m-1} \cdot du \wedge ds = \sigma^{n-1}|_W = h(u,s) \cdot du \wedge ds$ for some function $h(u,s)$ that does not vanish when $t = 0$. Thus, we can write

$$|G(z,t)| = |G(u, s^m)| = \frac{|h(u,s)|}{m} |s|^{1-m} = e^{g(u,s)} |s|^{1-m} = e^{g(z,t)} |t|^{-1+1/m}$$

which proves the claim. □

The singular fibres of $\pi\colon D \longrightarrow D/\mathcal{F}$ are simply multiples of smooth curves. By the above lemma we can assume π is smooth over the complement $D^o/\mathcal{F} \subset D/\mathcal{F}$ of a closed set of codimension two and we may assume that $D^o/\mathcal{F}$ is smooth. Denote by $\pi^o\colon D^o \longrightarrow D^o/\mathcal{F}$ the restriction of π. If one leaf is rational, then by Reeb stability, see Theorem 3.2, all the leaves are rational curves and we are done.[9] So we can assume all the leaves have positive genus and singular, i.e. multiple, fibres appear in codimension at least two.

(2) We want to prove that $\pi^o\colon D^o \longrightarrow D^o/\mathcal{F}$ is isotrivial. There are two possibilities depending on the genus g of the general leaf: If $g = 1$, then the fibration has to be isotrivial, for otherwise one of the fibres would be rational, in which case we are done already

The isotriviality is less trivial for $g > 1$. It follows from the observation that the following results of Miyaoka–Mori and Hwang–Viehweg contradict each other.

- For any coherent subsheaf $\mathcal{H} \subset \Omega_{D^o/\mathcal{F}}$ one has $\kappa(D^o/\mathcal{F}, \det(\mathcal{H})) \leqslant 0$. Indeed, according to [Miy87, Cor. 8.6] or [MM86, Thm. 1], the restriction of $\Omega_{D^o/\mathcal{F}}$ to a generic complete intersection

[9] One can avoid using Reeb stability here: Instead of showing (i) ⇒ (iv) one shows (i) ⇒ (iii) and then uses Section 4.3 to complete by (iii) ⇒ (iv). Indeed, $q(D) < 0$ is equivalent to $\int_D c_1(\mathcal{F}) H^{2n-2} > 0$ for some ample divisor H on D. The latter follows if one can show $\int_D c_1(\mathcal{F}) \pi^* H_0^{2n-2} > 0$ for some ample divisor H_0 on $D/\mathcal{F}$, which in turn would follow from $\int_L c_1(\mathcal{F})|_L > 0$, i.e. $g(L) = 0$.

curve is semi-positive. At the same time, its determinant is trivial. Hence, all sub-sheaves of $\Omega_{D^o/\mathcal{F}}$ have non-positive degree, which leads to the assertion.

• Assuming $g > 1$, there exists a coherent subsheaf $\mathcal{H} \subset \Omega_{D^o/\mathcal{F}}$ such that $\mathrm{var}(\pi^o) = \kappa(D^o/\mathcal{F}, \mathcal{H})$, cf. [HV10, Thm. 3.2 & Prop. 4.4]. Roughly, the relative cotangent sheaf of the natural map $D^o \longrightarrow \mathcal{M}_g$ provides this sub-sheaf. (Strictly speaking, this is only true after passing to a finite cover of D^o which does not affect the argument.)

Once isotriviality for $g = 1$ and $g > 1$ has been established, one can use the fact that the moduli space of curves with sufficiently many marked points is fine. Hence, there exists a finite étale cover $\Delta \longrightarrow D^o/\mathcal{F}$ such that the pull-back $\tilde{D} := D^o \times_{D^o/\mathcal{F}} \Delta$ splits as $\tilde{D} \simeq \Delta \times C$, where C is the generic fibre of π.

(3) Since $D^o/\mathcal{F}$ has trivial canonical bundle, the same holds for Δ. Hence,

$$\nu(\tilde{D}, \omega_D) = \kappa(\tilde{D}) = \begin{cases} 0 & \text{if } g = 1 \\ 1 & \text{if } g > 1. \end{cases}$$

As the numerical (and also the Kodaira) dimension is preserved under étale maps, one finds

$$\nu(D, \omega_D) = \begin{cases} 0 & \text{if } g = 1 \\ 1 & \text{if } g > 1. \end{cases}$$

Since $\omega_D = \mathcal{O}_X(D)|_D$, we have $\nu(D, \omega_D) = \nu(X, D) - 1$. However, the numerical dimension of a nef divisor in a hyperkähler manifold can be $0, n$ or $2n$. Since $n > 1$, the only possibility is that $n = 2$ and $g > 1$, which is excluded as follows: A fibre S of the canonical map is equivalent as a cycle, up to a multiple, to $D \cdot D$. This means that S is Lagrangian, for $\int_S \sigma\bar{\sigma} = q(D) = 0$. Hence, by Corollary 2.6, the leaves of the characteristic foliation must be contained in S and induce a fibration on S of curves of genus at least two. This contradicts the fact that the canonical bundle of S is trivial.

4.2. (iv) ⇒ (iii): We assume that D is uniruled and will show $q(D) < 0$ by excluding $q(D) > 0$ and $q(D) = 0$.

Suppose $q(D) > 0$. Then D is contained in the interior of the positive cone and, therefore, also in the interior of the pseudo-effective cone. Hence, D is big [La04, Lem. 2.2.3], i.e. $h^0(X, \mathcal{O}(kD)) \sim k^{2n}$, which implies $h^0(D, \omega_D^k) \sim k^{2n-1}$ contradicting the assumption that D is uniruled.

Suppose $q(D) = 0$. If D is nef, then ω_D is nef too, which again would contradict the assumption that D uniruled. If D is not contained in the closure of the movable cone, then it is contained in the interior of the pseudo-effective cone and one argues as above. If D is contained in the boundary of the movable cone, D is the limit of movable divisors and hence its restriction to D is still a limit of effective divisors. However, this implies that $\omega_D \simeq \mathcal{O}(D)|_D$ is pseudo-effective which contradicts D uniruled.[10]

The discussion should be compared to the result [LPT18, Thm. 3.7] asserting in the general setting that D is uniruled if and only if $\omega_{\mathcal{F}}$ is not pseudo-effective. The above discussion can be interpreted as saying that any smooth divisor $D \subset X$ with $q(D) \geqslant 0$ has a pseudo-effective ω_D, thus D cannot be uniruled.

4.3. (iii) $\Rightarrow$ (iv): We assume $q(D) < 0$ and want to show that D is then uniruled. (The smoothness of D is not essential.) We offer two proofs.

First, it is known that prime exceptional divisors are uniruled [Hu03, Prop. 5.4] or [Bo04, Prop. 4.7 & Thm. 4.3], but in fact the arguments there prove that any irreducible D with $q(D) < 0$ is uniruled. Indeed, since the positive cone is self-dual, it contains a class α such that $q(\alpha, D) < 0$. Hence, there exists a bimeromorphic map between hyperkähler manifolds $f\colon X \dashrightarrow X'$ such that $f_*\alpha = \omega' + \sum a_i D_i'$ for some prime exceptional and hence uniruled divisors D_i', positive real number a_i, and a Kähler class ω', cf. [Hu03] or [Bo04, Thm. 4.3 (ii)]. Since the quadratic form is preserved by f, we have $0 > q(\alpha, D) = q(\omega' + \sum a_i D_i', f_*D) > \sum a_i q(D_i', f_*D)$ and hence for some i we have $q(D_i', f_*D) < 0$. This implies that f_*D and D_i coincide and that in particular D is uniruled since its push-forward in X' is.

Here is a more direct proof relying on the criterion for uniruledness by Miyaoka and Mori [MM86, Miy87]: A smooth projective variety Z of dimension d is uniruled if $\int_Z \mathrm{c}_1(\omega_Z) \cdot H_Z^{d-1} < 0$ for an ample divisor H_Z on Z. Applied to $Z = D$ and observing that $q(D) < 0$ implies $\int_D \mathrm{c}_1(D) \cdot H|_D^{2n-2} = \int_X [D]^2 \cdot H^{2n-2} < 0$ for any ample divisor H on X, it implies the result.

4.4. (iv) $\Rightarrow$ (i): We assume that D is uniruled and want to prove that the leaves are closed.

By assumption, there exists a dominant morphism $\varphi\colon \mathbb{P}^1 \times V \longrightarrow D$ with $\dim(V) = 2n-2$. Since $\mathbb{P}^1$ admits no non-trivial forms of degree one or two, the pull-back of σ to $\mathbb{P}^1 \times V$ is the pull-back of a holomorphic form on V. This readily shows that all $\varphi_t(\mathbb{P}^1) \subset D$ are invariant with respect to the foliation. Hence, the generic leaf is of this form, which proves the claim. See [Dr11, Prop. 4.5] for a generalization to singular uniruled divisors.

4.5. (i) $\Leftrightarrow$ (ii): Clearly, (ii) implies (i). The converse is part of the discussion in Section 4.1.

5. Case (2): Lagrangian fibrations

5.1. (i) $\Rightarrow$ (iii): We assume $\dim \bar{L} = n$ and want to exclude $q(D) < 0$ and $q(D) > 0$.

First, according to Section 4, if $q(D) < 0$, then $\dim \bar{L} = 1$. To exclude $q(D) > 0$, use that according to Section 6.4[11] it would imply that the leaves are dense.

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[10] We wish to thank R. Abugaliev for his help with this argument.

[11] We leave it to the reader to check that the argument is not circular.

5.2. (iii) ⇒ (iv): We assume $q(D) = 0$. Then, by Proposition 3.3, D and hence $\omega_D \simeq \mathcal{O}(D)|_D$ are nef. Assuming the abundance conjecture in dimension $2n-1$, we know that ω_D is semi-ample. Hence, by [DHP13, Cor. 1.8] also D is semi-ample,[12] i.e. some power $\mathcal{O}(kD)$ defines a Lagrangian fibration $f\colon X \twoheadrightarrow B$ and, therefore, kD is the pull-back of a divisor in B. Hence, D is vertical.

Remark 5.1. The implication (i) ⇒ (iv) in dimension four was first proved by Amerik–Guseva [AG16].

5.3. (iv) ⇒ (iii): We assume now that there exists a Lagrangian fibration $X \twoheadrightarrow B$ such that D is the pre-image (as a set) of a hypersurface $H \subset B$. Then $[D]$ and $f^*[H]$ are proportional, since $[H]^{n+1} = 0$, also $[D]^{n+1} = 0$ and, therefore, $q(D) = 0$.

5.4. (iv) ⇒ (i): We assume that X comes with a Lagrangian fibration $f\colon X \twoheadrightarrow B$ such that $D = f^{-1}(H)$ (as sets) for some hypersurface $H \subset B$ and want to show that the closure of the generic leaf is of dimension n (and in fact a torus).

Assume first that H is contained in the discriminant locus $\Delta \subset B$. Then D is algebraically integrable by a result of Hwang and Oguiso [HO09, Thm. 1.2]. By the results of Section 4, the latter implies that D is uniruled and hence $q(D) < 0$, which contradicts (iii) that we proved already in Section 5.3.[13]

Assume now that H is not contained in the discriminant divisor. Then, since D is smooth, the generic fibre of $f|_D\colon D \twoheadrightarrow H = f(D)$ is a smooth Lagrangian torus. By Corollary 2.6, the generic leaf is contained in a fibre of $D \twoheadrightarrow f(D)$. We have to show that it is dense in there. Note that for $n = 2$ the result is immediate. Indeed, if the generic leaf is not dense in the fibre, then by Section 4 the foliation is algebraically integrable and the leaves are rational curves, which however do not exist in a torus.

Let $T \coloneqq f^{-1}(t)$, $t \in f(D)$, be a generic fibre. The foliation $\mathcal{F} \subset \mathcal{T}_D$ induces a foliation $\mathcal{F}|_T \subset \mathcal{T}_T$ of the abelian variety T. It is well known that the closure of a leaf of a foliation on an abelian variety is a translate of an abelian subvariety. Indeed, observing $\mathcal{O}_F \simeq \mathcal{F}|_T \subset \mathcal{T}_T \simeq \mathcal{O}_F^{\oplus n}$ and writing $T = \mathbb{C}^n/\Gamma$, one finds that the leaves of the foliation $\mathcal{F}$ are given by the images under the natural projection $\mathbb{C}^n \twoheadrightarrow T$ of the translates of the line $\mathbb{C} \subset \mathbb{C}^n$ corresponding to $\mathcal{F}|_T \subset \mathcal{T}_T$. The closure of the leaf through the origin then corresponds to the smallest linear subspace $\mathbb{C}^m \subset \mathbb{C}^n$ containing the given line and such that $\Gamma \cap \mathbb{C}^m \subset \mathbb{C}^m$ is a lattice.

Thus, if the abelian variety T is known to be simple, which is frequently the case, then the assertion is immediate.

[12] This is a highly non-trivial statement asserting that $H^0(X, \mathcal{O}(kD)) \twoheadrightarrow H^0(D, \mathcal{O}(kD)|_D)$ is surjective for sufficiently divisible k. For an alternative, algebraic argument see [AC14, Cor. 5.2].

[13] The argument shows that for any component of the discriminant divisor $H \subset B$ the reduction of $f^{-1}(H)$ cannot be smooth. Either it consists of more than one component, with possibly each component individually smooth, or it is irreducible but singular.

If T is not simple, then Abugaliev proceeds in two steps. The first is a result of general interest [Ab19, Thm. 0.5].[14]

Lemma 5.2 (Abugaliev)**.** *Let $f\colon X \twoheadrightarrow B$ be a Lagrangian fibration of a projective hyperkähler manifold over a smooth base B and let $H \subset B$ be a hypersurface not contained in the discriminant divisor of f.*

If $D = f^{-1}(H) \subset X$ is smooth, then for the generic fibre $T = f^{-1}(t)$, $t \in H$:

$$\mathrm{Im}\left(H^*(X,\mathbb{Q}) \xrightarrow{\mathrm{res}_{X,T}} H^*(T,\mathbb{Q})\right) = \mathrm{Im}\left(H^*(D,\mathbb{Q}) \xrightarrow{\mathrm{res}_{D,T}} H^*(T,\mathbb{Q})\right).$$

Note that the left hand side is known to be isomorphic to $H^*(\mathbb{P}^n,\mathbb{Q})$ according to results by Matsushita, Oguiso, Voisin, and Shen–Yin, see the survey [HM21, Thm. 2.1] for references. In particular, it is of dimension one in each even degree.

Proof. The assertion is invariant under small deformations of H, which preserve the smoothness of D. Hence, one may assume that the intersection $H \cap \Delta$ with the discriminant locus is sufficiently generic such that $\pi_1(H\backslash\Delta) \twoheadrightarrow \pi_1(B\backslash\Delta)$ is surjective (and in fact an isomorphism for $n > 2$) by [De81, Lem. 1.4] applied to $B\backslash\Delta$. In particular, the monodromy invariant parts of $H^*(T,\mathbb{Q})$ for the two families $X \twoheadrightarrow B$ and $D \twoheadrightarrow H$ coincide. Thus, Deligne's invariant cycle theorem implies

$$\mathrm{Im}(\mathrm{res}_{X,T}) = H^*(T,\mathbb{Q})^{\pi_1(B\backslash\Delta)} = H^*(T,\mathbb{Q})^{\pi_1(H\backslash\Delta)} = \mathrm{Im}(\mathrm{res}_{D,T}),$$

which concludes the proof. □

The idea of the second step is that the family of tori obtained as closures of leaves $L \subset \bar{L} \subset T$ contained in a fixed generic fibre T is distinguished and hence invariant under monodromy. This yields a cohomology class in $H^{2k}(T,\mathbb{Q})$ that is invariant under monodromy of the family $D \twoheadrightarrow H$. However, classes that are invariant under the monodromy of the family $X \twoheadrightarrow B$ are all powers of the polarization and, therefore, cannot be realized by proper subtori.

5.5. (iv) ⇒ (ii) ⇒ (i): Assume first (iv) holds. By Corollary 2.6, the generic leaf L is contained in a fibre of $D \twoheadrightarrow f(D) \subset B$. If we allow ourselves to use (iv) ⇒ (i) in Section 5.4, then the closure $\bar{L}$ is the generic fibre which is a torus. The second implication (ii) ⇒ (i) is clear.

6. Case (3): Dense leaves

6.1. (i) ⇒ (iii): We assume that $\dim \bar{L} = 2n-1$ and want to exclude that $q(D) < 0$ or $q(D) = 0$.

First, by the results of Section 4, we know that the three conditions $q(D) < 0$, D uniruled, and $\dim \bar{L} = 1$ are all equivalent. Hence, $q(D) < 0$ is excluded for $\dim \bar{L} > 1$.

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[14] The reader will observe that the result actually holds without assuming that X is hyperkähler or that f is a Lagrangian fibration.

Next suppose $q(D) = 0$. Then by Proposition 3.3 D is nef and hence also $\omega_D \simeq \mathcal{O}(D)|_D$ is. Assuming the abundance conjecture in dimension $2n-1$, we conclude that ω_D and, therefore, $\mathcal{O}(D)$ are semi-ample, cf. the argument in Section 5.2. Hence, X comes with a Lagrangian fibration $X \twoheadrightarrow B$ such that some multiple of D is the pre-image of a divisor in B. But then the leaves are not dense by Section 5.4.

6.2. (iii) $\Rightarrow$ (iv): We assume $q(D) > 0$. Clearly, if D is ample, then by adjunction $\omega_D \simeq \mathcal{O}(D)|_D$ is ample as well and, therefore, D is of general type. If D is only nef, then a priori also ω_D is only nef. However, $q(D) > 0$ implies $\int_D \mathrm{c}_1(\omega_D)^{2n-1} > 0$, i.e. ω_D is big and nef. By Kawamata–Viehweg vanishing $H^i(D, \omega_D^k) = 0$ for $k > 1$ and $i > 0$ and by the Hirzebruch–Riemann–Roch formula $h^0(D, \omega_D^k) \sim k^{2n-1}$ and so D is of general type. Since by Proposition 3.3, any smooth hypersurface $D \subset X$ with $q(D) \geqslant 0$ is nef, this concludes the proof.

Here is an alternative argument not relying on Proposition 3.3: Since D is contained in the interior of the positive cone, it is also contained in the interior of the pseudo-effective cone and, therefore, big [La04, Prop. 2.2.6], i.e. $h^0(X, \mathcal{O}(kD)) \sim k^{2n}$. Using $\omega_D \simeq \mathcal{O}(D)|_D$ eventually shows that ω_D is big, i.e. that D is of general type.

6.3. (iv) $\Rightarrow$ (iii): We assume that D is of general type and want to prove $q(D) > 0$ by excluding the other two possibilities $q(D) < 0$ and $q(D) = 0$.

Suppose $q(D) < 0$. Then by virtue of Section 4.3 (e.g. by applying the Miyaoka–Mori numerical criterion for uniruledness [MM86, Miy87]) we know that D is uniruled, so in particular not of general type.

Next, suppose that $q(D) = 0$, which implies $\int_D \mathrm{c}_1(\omega_D)^{2n-1} = (D)^{2n} = 0$. Now use again Proposition 3.3 to conclude that $\mathcal{O}(D)$ and hence $\omega_D \simeq \mathcal{O}(D)|_D$ are nef. However, a nef divisor E on a projective variety Z of dimension m is big, i.e. $h^0(Z, \mathcal{O}(kE)) \sim k^m$, if and only if $(E)^m > 0$, see [La04, Thm. 2.2.16]. Since D is assumed to be of general type and so $h^0(D, \mathcal{O}(kD)|_D) \sim k^{2n-1}$, this is a contradiction.

6.4. (iii) $\Rightarrow$ (i): We assume $q(D) > 0$ and want to show that the leaves are dense. We have seen already that $q(D) > 0$ implies that D is big and nef. The first step is to prove a version of the Lefschetz hyperplane theorem [Ab21, Prop. 3.1].

Lemma 6.1. *Assume $D \subset X$ is a smooth hypersurface of a hyperkähler manifold that is big and nef. Then the restriction*

$$H^i(X, \mathbb{Q}) \xrightarrow{\sim} H^i(D, \mathbb{Q})$$

is an isomorphism for $i < \dim(D) = 2n-1$.

Proof. For an ample hypersurface this is the content of the Lefschetz hyperplane theorem [La04, Thm. 3.1.17]. If D is just big and nef, Kawamata–Viehweg vanishing still shows that all higher cohomology groups $H^i(X, \mathcal{O}(kD))$, $i > 0$, are trivial. Hence, D deforms sideways with X in any family $\mathcal{X} \longrightarrow \Delta$ for which the line bundle $\mathcal{O}(D)$ deforms. However, the very general fibre $\mathcal{X}_t$

of the universal such deformation has Picard number one and, therefore, a generic deformation $D_t \subset \mathcal{X}_t$ of $D \subset X$ is ample [Hu99, Thm. 3.11]. Hence, $H^i(\mathcal{X}_t, \mathbb{Q}) \xrightarrow{\sim} H^i(D_t, \mathbb{Q})$ for $i < \dim(D_t)$ by the classical Lefschetz hyperplane theorem. Since the assertion is topological, this suffices to conclude.[15] □

The key step is the following result [Ab21, Prop. 4.1].

Proposition 6.2 (Abugaliev). *A smooth hypersurface $D \subset X$ which is big and nef (or, equivalently, $q(D) > 0$) cannot be covered by coisotropic varieties of codimension two.*

Proof. Recall that a subvariety $Z \subset X$ of codimension two is called coisotropic if the kernel of $\sigma|_Z \colon \mathcal{T}_Z \longrightarrow \Omega_Z$ (over the smooth locus) is a sheaf of rank two, see Section 2.2.

First observe that by Lemma 6.1, there exists a class $\alpha \in H^2(X, \mathbb{Q})$ with $\alpha|_D = [Z] \in H^2(D, \mathbb{Q})$. Clearly, the class α is of type $(1,1)$. On the other hand, if $Z \subset X$ is a coisotropic subvariety of codimension two, then $0 = [Z] \wedge \sigma^{n-1} \in H^{2n+2}(X, \mathbb{C})$. So, if $Z \subset D$ is coisotropic and we write $[Z] = \alpha|_D \in H^2(D, \mathbb{Q})$, then $0 = [D] \wedge \alpha \wedge \sigma^{n-1} \in H^{2n+2}(X, \mathbb{C})$, which implies $\int_X [D] \wedge \alpha \wedge \sigma^{n-1} \wedge \bar{\sigma}^{n-1} = 0$ and, therefore, $q(D, \alpha) = 0$. Now use the well known formula

$$q(\gamma_1, \gamma_2) \cdot \int_X \gamma_1^{2n} = 2q(\gamma_1) \cdot \int_X \gamma_1^{2n-1} \wedge \gamma_2, \tag{6.1}$$

cf. [Hu02, Exer. 23.2] to deduce from $q(D) > 0$, $\int_X [D]^{2n} > 0$, and $q(D, \alpha) = 0$, that $0 = \int_X [D]^{2n-1} \wedge \alpha = \int_Z [D]|_Z^{2n-2}$. If D is ample, this is absurd. So we proved the stronger statement that an ample, smooth hypersurface $D \subset X$ does not contain any coisotropic subvariety of codimension two.[16]

If D is only big and nef, then $q(D, \alpha) = 0$ still implies $q(\alpha) < 0$ by the Hodge index theorem, which in turn, by a formula [Ab21, Lem. 4.2] similar to (6.1), gives $\int_X [D]^{2n-2} \wedge \alpha \wedge \alpha < 0$. On the other hand, if $\alpha|_D = [Z_1] = [Z_2]$ with $Y \coloneqq Z_1 \cap Z_2 \subset D$ of codimension two in D or empty, then one obtains the contradiction $0 \leqslant \int_Y D|_Y^{2n-3} = \int_X [D]^{2n-2} \wedge \alpha \wedge \alpha < 0$. □

Recall that a subvariety $Z \subset D$ is called *invariant* under the characteristic foliation of the smooth hypersurface D if the leaf through any $x \in Z$ is contained in Z or, equivalently, if $\mathcal{F}|_Z \subset \mathcal{T}_D|_Z$ is contained in $\mathcal{T}_Z$ (over the smooth locus of Z).

The following result [Ab21, Thm. 0.5] is now a consequence of the above discussion. It concludes the proof of (iii) $\Rightarrow$ (i) in Case (3).

Theorem 6.3 (Abugaliev). *Assume $D \subset X$ is a smooth hypersurface of a hyperkähler manifold X satisfying $q(D) > 0$. Then the generic leaf of the characteristic foliation on D is Zariski dense.*

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[15] The original proof in [Ab21] uses the Kodaira–Akizuki–Nakano vanishing theorem. The above argument is quicker, but uses deformation theory and the projectivity criterion for hyperkähler manifolds.

[16] In dimension four this says that a smooth ample hypersurface does not contain any smooth Lagrangian surface, see Section 8.2.

Proof. If the generic leaf $L \subset D$ is not Zariski dense, then its Zariski closure $\bar{L} \subset D$ defines a proper closed subvariety $Z \subset D$. The family of all such leaves gives a covering family $\{Z_t\}$ of D. Assume first that $Z_t \subset D$ is of codimension two in X. Since $Z_t = \bar{L}$ is clearly invariant under the characteristic foliation and hence coisotropic by Corollary 2.6, (ii), this is a contradiction to Proposition 6.2.

If the subvarieties Z_t are of higher codimension, taking unions produces a covering family $\{Z'_s\}$ of D consisting of subvarieties of codimension two in X and such that each Z'_s is a union of $Z_t = \bar{L}$. In particular, again to Lemma 6.3, each Z'_s is coisotropic and one concludes as before. □

6.5. (iv) ⇒ (i): Of course, this direction is a consequence of the implications proved before, but we wish to mention a weaker statement due to Hwang–Viehweg that motivated much of the later work on characteristic foliations. They proved [HV10, Thm. 1.2] that the characteristic foliation of a smooth hypersurface $D \subset X$ cannot be algebraic or, in other words, that $\dim \bar{L} > 1$.

7. Alternative summary

We think it is instructive to present the discussion structured in a slightly different way which makes it more evident where and how foliations are used.

7.1. (iii) ⇒ (iv): This part only involves more or less classical results and Proposition 3.3, i.e. the nefness of D if $q(D) \geqslant 0$.

Assume $q(D) < 0$. Then $\int_D K_D \cdot H^{2n-2} < 0$ and by [MM86, Miy87] D is uniruled.

Assume $q(D) = 0$. Then D and hence ω are nef by Proposition 3.3. Using abundance conjecture in dimension $2n-1$ combined with [DHP13], one finds that D is semi-ample. Therefore, $\mathcal{O}(kD)$ defines a Lagrangian fibration $f\colon X \twoheadrightarrow B$ for some $k > 0$ and hence $D = f^{-1}(f(D))$ (as sets).

Assume $q(D) > 0$. In this case D is of general type for which we gave two proofs: The one not using Proposition 3.3 just observed that under these assumptions D is in the interior of the pseudo-effective cone and hence big.

7.2. (iv) ⇒ (iii): Again, only Proposition 3.3 is used.

Assume D is uniruled. Then $q(D) < 0$ is proved by excluding $q(D) > 0$ and $q(D) = 0$. If $q(D) > 0$, then D is of general type as explained above. To exclude $q(D) = 0$, one distinguishes two cases: First, D is in the boundary of the movable cone, then $\omega_D = \mathcal{O}(D)|_D$ is a limit of effective divisor and, therefore, pseudo-effective which contradicts D uniruled. Second, if D is not contained in the boundary of the movable cone, then D is in the interior of the pseudo-effective cone. Hence, D and ω_D are big, contradicting D uniruled. Alternatively, one could again apply Proposition 3.3 to see that D and hence ω_D are nef, but the later clearly contradicts D uniruled.

Assume $D = f^{-1}(H)$ is the set theoretic pre-image of a hypersurface $H \subset B$ in the base of a Lagrangian fibration $f\colon X \twoheadrightarrow B$. Then the classes $[D], f^*[H] \in H^2(X,\mathbb{Z})$ are proportional. Since $[H]^{n+1} = 0$, also $[D]^{n+1} = 0$ in $H^{2n+2}(X,\mathbb{Z})$ and, therefore, $q(D) = 0$

Assume D is of general type. Then $q(D) > 0$ is proved by excluding $q(D) < 0$ and $q(D) = 0$. Indeed, the later implies D uniruled as explained before. The former is excluded by observing that ω_D is nef by Proposition 3.3 and big, as D is of general type. However, this implies $\int_D K_D^{2n-1} > 0$.

8. Examples

8.1. We provide examples of divisors for the three situations when $X = S^{[2]}$ for a K3 surface S.

(i) The natural example for Case (1) is the exceptional divisor $D = E$ of the Hilbert–Chow morphism $\pi\colon S^{[2]} \twoheadrightarrow S^{(2)}$. It is well known that $q(E) = -2$ and it is a $\mathbb{P}^1$ bundle over the diagonal of $S^{(2)}$. More explicitly:

- The divisor E is naturally isomorphic to $\mathbb{P}(\Omega_S)$.
- The restriction of the symplectic form of $S^{[2]}$ to E is the pullback of the symplectic form on S via the projection $\mathbb{P}(\Omega_S) \twoheadrightarrow S$.
- The characteristic foliation $\mathcal{F}$ is the relative tangent bundle $\mathcal{T}_\pi$ of the map $\pi\colon \mathbb{P}(\Omega_S) \twoheadrightarrow S$.
- The leaves are the $\mathbb{P}^1$ contracted by the Hilbert–Chow morphism, that via the identification with $\mathbb{P}(\Omega_S)$ is just the projection to S. Hence, S is the space of leaves $D/\mathcal{F}$.

(ii) Assume that S admits a genus one fibration. Then its Hilbert scheme $S^{[2]}$ of two points admits a natural Lagrangian fibration over the Hilbert scheme of two points on $\mathbb{P}^1$, i.e. $\mathbb{P}^2$. Denote by S_{t_0} a smooth fibre of $S \twoheadrightarrow \mathbb{P}^1$ over a point t. Then the subvariety of $S^{[2]}$ given by $D = \{\{p,q\} \mid p \in S_{t_0}\}$ satisfies:

- It is a smooth divisor with $q(D) = 0$.
- Its image in $\mathbb{P}^2$ is the line described by $\{t_0, t\}$, $t \in \mathbb{P}^1$.
- The leaves of the characteristic foliation on D are contained in the fibres S_t but are not compact, i.e. they are dense in the fibres.

8.2. If the hypersurface $D \subset X$ is not smooth, then typically the conditions (i)-(iv) are not equivalent.

(i) Let us first discuss this in Case (2). Consider the discriminant divisor $\Delta \subset X$ of a Lagrangian fibration $f\colon X \twoheadrightarrow B$. In general it is not an irreducible divisor. By [HO09, Thm. 1.2] the characteristic foliation of any irreducible component of Δ is algebraically integrable. Assume that there is a component of D of Δ such that $D = f^{-1}(\pi(D))$. This happens for instance if X is general among the hyperkähler with a Lagrangian fibration. Then, $q(D) = 0$

but its characteristic foliation is algebraically integrable. This divisor satisfies (iii) and (iv) of Case (2) but not (i).

For a divisor such that $D \neq f^{-1}(f(D))$ we see that in Case (2) also (iv) $\Rightarrow$ (iii) does not hold for singular divisors.

(ii) We turn to Case (3). Consider a smooth cubic fourfold $Y \subset \mathbb{P}^5$ and its Fano variety of lines $X := F(Y)$, which is a hyperkähler fourfold. The set of lines contained in a hyperplane section $Y \cap H$ is a Lagrangian surface $F(Y \cap H) \subset X$ which for generic H is smooth and of general type. For a one-dimensional family $\{Y \cap H_t\}$ of hyperplane sections these Lagrangian surfaces sweep out a hypersurface $D \subset X$. Since for a generic cubic fourfold, $X = F(Y)$ has Picard number one, one has $q(D) > 0$.

According to Corollary 2.6, (i), any leaf that intersects a generic $F(Y \cap H_t)$ is contained in it. However, if D were smooth, then the results if Section 6.4 would imply that the generic leaf is dense. Contradiction. Hence, D cannot be smooth (for whatever one-dimensional family of hyperplane section $\{Y \cap H_t\}$).

In particular this is an example of a singular divisor that satisfies (iii) and (iv) of Case (3) but does not satisfy (i).

More abstractly, a smooth ample hypersurface $D \subset X$ in a hyperkähler fourfold does not contain any Lagrangian surface, cf. proof of Proposition 6.2. In particular, for a general cubic fourfold Y, a smooth Lagrangian surface $F(Y \cap H_t)$ cannot be contained in any smooth divisor of $X = F(Y)$.

References

[Ab19] R. Abugaliev *Characteristic foliation on vertical hypersurfaces on holomorphic symplectic manifolds with Lagrangian fibration.* arXiv:1909.07260. 1, 13

[Ab21] R. Abugaliev *Characteristic foliation on hypersurfaces with positive Beauville–Bogomolov–Fujiki square.* arXiv:2102.02799. 1, 14, 15

[AC14] E. Amerik and F. Campana *Characteristic foliation on non-uniruled smooth divisors on projective hyperkähler manifolds.* J. Lond. Math. Soc. 95 (2014), 115–127. 1, 8, 12

[AG16] E. Amerik and L. Guseva *On the characteristic foliation on a smooth hypersurface in a holomorphic symplectic fourfold.* arxiv:1611.00416 (2016). 12

[BMcQ16] F. Bogomolov, M. McQuillan *Rational curves on foliated varieties.* Foliation theory in algebraic geometry, 21–51, Simons Symp., Springer, Cham, 2016. 7

[Bo01] J.-B. Bost *Algebraic leaves of algebraic foliations over number fields.* Publ. Math. Inst. Hautes Études Sci. 93 (2001), 161–221. 7

[Bo04] S. Boucksom *Divisorial Zariski decompositions on compact complex manifolds.* Ann. Sci. Éc. Norm. Supér. 37 (2004), 45–76. 11

[CN85] C. Camacho, A. Nito *Geometric Theory of Foliations.* Springer (1985). 5

[De81] P. Deligne *Le groupe fondamental du complément d'une courbe plane n'ayant que des point doubles ordinaires est abélien (d'après W. Fulton).* Bourbaki Seminar, Vol. 1979/80, Springer, Berlin (1981), 1–10. 13

[DHP13] J.-P. Demailly, C. Hacon, M. Păun *Extension theorems, non-vanishing and the existence of good minimal models.* Acta Math. 210 (2013), 203–259. 12, 16

[Dr11] S. Druel *Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la première classe de Chern est nulle.* Math. Z. 267 (2011), 413–423. 11

[Dr17] S. Druel *On foliations with nef anti-canonical bundle.* Trans. Amer. Math. Soc. 369 (2017), 7765–7787. 5

[Ho89] H. Holmann *On the stability of holomorphic foliations.* Springer LNM 798 (1980), 192–202. 6

[Hu99] D. Huybrechts *Compact hyperkähler manifolds: Basic results.* Invent. Math. 135 (1999), 63–113. 15

[Hu03] D. Huybrechts *The Kähler cone of a compact hyperkähler manifold.* Math. Ann. 326 (2003), 499–513. 11

[Hu02] D. Huybrechts *Compact hyperkähler manifolds.* in Calabi–Yau manifolds and related geometries. Springer (2002). 15

[HM21] D. Huybrechts, M. Mauri *Lagrangian fibrations.* arXiv:2108.10193. 13

[HO09] J.M. Hwang, K. Oguiso *Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration.* Amer. J. Math. 134 (2009), 981–1007. 12, 17

[HV10] J.-M. Hwang and E. Viehweg *Characteristic foliation on a hypersurface of general type in a projective symplectic manifold.* 146 (2010), 497–506. 1, 4, 5, 6, 10, 16

[KCT07] S. Kebekus, L. Solá Conde, and T. Matei *Rationally connected foliations after Bogomolov and McQuillan.* J. Alg. Geom. 16 (2007), 65–81. 5, 6, 7

[La04] R. Lazarsfeld *Positivity in Algebraic Geometry I.* Springer 2004. 10, 14

[LPT18] F. Loray, J. Pereira, F. Touzet, *Singular foliations with trivial canonical class.* Invent. Math. 213 (2018), 1327–1380. 11

[Miy87] Y. Miyaoka, *Deformation of a morphism along a foliation and applications.* Proc. of Symp. Pure Math. 46 (1987), 245–268. 9, 11, 14, 16

[MM86] Y. Miyaoka, S. Mori *A numerical criterion for uniruledness.* Ann. Math. 124 (1986), 65–69. 9, 11, 14, 16

[Pe01] J. Pereira *Global stability for holomorphic foliations on Kähler manifolds.* Qual. Theory Dyn. Syst. 2 (2001), no. 2, 381–384. 5, 6

[Sa09] J. Sawon *Foliations on hypersurfaces in holomorphic symplectic manifolds.* IMRN 23 (2009), 4496–4545. 8

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