EFFECTIVITY OF SEMI-POSITIVE LINE BUNDLES

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Abstract. We review work by Campana–Oguiso–Peternell [COP10] and Verbitsky [Ver10] showing that a semi-positive line bundle on a hyperkähler manifold admits at least one non-trivial section. This is modest but tangible evidence towards the SYZ conjecture for hyperkähler manifolds.

1. Main theorem and motivation

1.1. **Main theorem.** The following result was first proved in the non-algebraic setting by Campana–Oguiso–Peternell [COP10] and later, applying similar techniques, by Verbitsky [Ver10].

Theorem 1.1. *Any semi-positive line bundle L on a compact hyperkähler manifold is $\mathbb{Q}$-effective, i.e. $H^0(X, L^m) \neq 0$ for some $m > 0$.*

A line bundle L, say on a compact Kähler manifold, is *semi-positive* if it admits a smooth hermitian metric with semi-positive curvature. *Warning:* The term semi-positive is used with different meaning in other contexts.

Clearly, any ample line bundle is semi-positive, as due to Kodaira's theorem being ample is equivalent to admitting a hermitian metric with positive curvature. Also, semi-positive line bundles are nef. However, the converse is not true. There exist line bundles on projective manifolds which are nef but not semi-positive, e.g. one finds in [Har70, Thm. I.10.5] Mumford's example of a nef line bundle that is not semi-ample and in [DPS01, Sec. 2.5] an example of nef line bundles that is not semi-positive. However, the situation is expected to be better on compact hyperkähler manifolds (or, more generally, on Calabi–Yau manifolds).

Conjecture 1.2. *Any nef line bundle on a compact hyperkähler manifold is semi-ample, i.e. some positive power L^m is globally generated, and, in particular, semi-positive.*

There are two cases to be considered here: For a nef line bundle on a compact hyperkähler manifold either $q(L) > 0$ or $q(L) = 0$, where q is the Beauville–Bogomolov quadratic pairing on $H^2(X, \mathbb{Z})$. In the first case, L is nef and big, X is projective [Huy99, Thm. 3.11], and, therefore, L is semi-ample by the base-point free theorem [CKM88]. Hence, only the case of a nef line bundle L with $q(L) = 0$ needs to be dealt with and we shall restrict to this case in what follows.

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Remark 1.3. The naive idea to approach Theorem 1.1 and Conjecture 1.2 is of course to apply the hyperkähler Riemann–Roch formula $\chi(L) = \sum a_i q(L)^i$, see [Huy99, Huy03a], which for $q(L) = 0$ reduces to $\chi(L) = n + 1$. The problem now is that we have a priori no control over the higher cohomology groups $H^q(X, L)$ as the usual vanishing results do not apply. In fact, by applying a result of Matsushita [Mat05, Thm. 1.3] showing that $R^j f_* \mathcal{O}_X \simeq \Omega^j_{\mathbb{P}^n}$ for a Lagrangian fibration $f\colon X \longrightarrow \mathbb{P}^n$, we know that in this case $H^q(X, f^*\mathcal{O}(k)) \simeq H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k))$ for $k > 0$ by Bott vanishing and Leray spectral sequence. Hence, $H^q(X, f^*\mathcal{O}(k)) \neq 0$ for $k > q \leqslant n$.

1.2. **SYZ conjecture.** Assume that L is a non-trivial semi-ample line bundle with $q(L) = 0$ on a compact hyperkähler manifold X of dimension $2n$. Then the linear system $|L^m|$, $m \gg 0$, defines a Lagrangian fibration $X \twoheadrightarrow B$ over a normal base B of dimension n.

As any compact hyperkähler manifold X with $b_2(X) \geqslant 5$ deforms to a compact hyperkähler manifold X' that admits a nef line bundle of square zero, Conjecture 1.2 would thus confirm the following version of the Strominger–Yau–Zaslov (SYZ) conjecture for Calabi–Yau threefolds.

Conjecture 1.4. *Every compact hyperkähler manifold is deformation equivalent to a compact hyperkähler manifold with a Lagrangian fibration.*

The conjecture has been verified for all known deformation types of compact hyperkähler manifolds. This is obvious for those deformation equivalent to Hilbert schemes of K3 surfaces or to generalized Kummer varieties. See [Rap07, Cor. 1.1.10], for those deformation equivalent to the examples of O'Grady in dimension six and ten.

1.3. **Notation.** Typically X will denote a compact hyperkähler manifold X of dimension $2n$, i.e. X is a simply-connected, compact Kähler manifold with $H^0(X, \Omega_X^2)$ spanned by an everywhere non-degenerate holomorphic two-form σ. The second cohomology $H^2(X, \mathbb{Z})$ is endowed with the Beauville–Bogomolov form q which is of signature $(3, b_2(X)-3)$ and which satisfies $q(\alpha)^n = c_X \cdot \int \alpha^{2n}$ for all classes $\alpha \in H^2(X, \mathbb{Z})$ and some positive rational number $c_X \in \mathbb{Q}$, the Fujiki constant. The square of a class $\alpha \in H^{1,1}(X)$ can alternatively be computed as $q(\alpha) = \int \alpha^2 (\sigma\bar{\sigma})^{n-1}$ (up to a positive scaling factor not depending on α).

2. Preparation

We shall prepare the ground for the actual proof by recalling the main results and techniques that go into it.

2.1. **Hard Lefschetz theorem.** The following result is due to Mourougane [Mou99, Thm. 2.6].[1]

[1] In [Mou99] one finds the dual statement, namely that $H^q(X, \Omega_X^p \otimes F) \hookrightarrow H^{d-p}(X, \Omega_X^{d-q} \otimes F)$ is injective. Is there a typo in his result? Should he not assume F to be semi-negative?

Proposition 2.1. *Let L be a semi-positive line bundle on a compact Kähler manifold X of dimension d with Kähler class ω. Then the product with ω^q defines surjective maps*

$$H^0(X, \Omega_X^{d-q} \otimes L) \twoheadrightarrow H^q(X, \Omega_X^d \otimes L).$$

For $L \simeq \mathcal{O}_X$, this is the content of the Hard Lefschetz theorem which, in fact, asserts the bijectivity of the map. The techniques to prove the more general statement are similar.

If X is a hyperkähler manifold of dimension $d = 2n$ and so $\Omega_X^{2n} \simeq \mathcal{O}_X$, one obtains surjections

$$H^0(X, \Omega^{2n-q} \otimes L) \twoheadrightarrow H^q(X, L).$$

This allows one to turn the non-vanishing of higher cohomology groups of L into the existence of global sections of powers of L.

Remark 2.2. (i) The result fails if L is only assumed to be nef without having a semi-positive metric, see [DPS01, Sec. 2.5] for an example. In this case, there is a variant of the above due to Takegoshi [Tak97, Thm. 1] for nef line bundles and to Demailly–Peternell–Schneider [DPS01, Thm. 2.1] for pseudo-effective line bundles: For a line bundle L on a compact Kähler manifold X of dimension d with a singular hermitian metric h with semi-positive curvature current the product with ω^q defines a surjection

$$H^0(X, \Omega_X^{d-q} \otimes L \otimes \mathcal{I}(h)) \twoheadrightarrow H^q(X, \Omega_X^d \otimes L \otimes \mathcal{I}(h)), \tag{2.1}$$

where $\mathcal{I}(h)$ denotes the multiplier ideal sheaf.

(ii) Due to [Tak97, Thm. 2], for $q > n$ the morphism $H^q(X, L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, L)$ induced by the inclusion $L \otimes \mathcal{I}(h) \subset L$ is the zero map for any nef line bundle L on a compact hyperkähler manifold X of dimension $2n$. In fact, according to another result of Verbitsky [Ver07, Thm. 1.6], one has $H^q(X, L) = 0$, $q > n$, for any nef and, more generally, for any pseudo-effective line bundle L.

2.2. **Finiteness of non-polar hypersurfaces.** An integral hypersurface $Y \subset X$ of a compact complex manifold is called *polar* if there exists a meromorphic function $f \in K(X)$ that has a pole along Y, i.e. Y is contained in the pole divisor $(f)_\infty$ of f. On a projective manifold, every integral hypersurface is polar. However, for general non-projective manifolds this fails, but the following result was proved by Fischer–Forster [FF79] and in the case that $K(X) = \mathbb{C}$ by Krasnov [Kra75].

Proposition 2.3. *A compact connected complex manifold X contains at most finitely many integral hypersurfaces $Y \subset X$ that are not polar. More precisely, the number of non-polar hypersurfaces is bounded by $h^{1,1}(X) + \dim(X) - h^{1,0}(X)$.*

For the proof one needs the following elementary but useful observation, see [Kra75, Prop. 1].

Lemma 2.4. *Let E be a vector bundle of rank r on X. Then the space of meromorphic sections of E is of dimension at most r, i.e.*

$$\dim_{K(X)} H^0(X, E \otimes \mathcal{K}_X) \leqslant \mathrm{rk}(E),$$

where $\mathcal{K}_X$ is the sheaf or rational (meromorphic) functions and $K(X) = H^0(X, \mathcal{K}_X)$ is the function field of X. In particular, if $K(X) = \mathbb{C}$, then for any vector bundle E one has $h^0(X, E) \leqslant \mathrm{rk}(E)$.

Proof. Suppose there exist sections $s_1, \ldots, s_{r+1} \in H^0(X, E \otimes \mathcal{K}_X)$ linearly independent over $K(X)$. Then there is a proper closed analytic subset such that all sections s_i are holomorphic on its open complement $U \subset X$ and such that (after renumbering) the sections $s_1, \ldots, s_{r'}$ span the subspace $\langle s_1(x), \ldots, s_r(x) \rangle \subset E(x)$ of constant (maximal) dimension r' at every point $x \in U$. In particular, on U we can write $(*)$ $s_r = \sum_{i=1}^{r'} a_i \cdot s_i$ for certain holomorphic functions $a_i \in \mathcal{O}_X(U)$.

It suffices to check that the a_i are meromorphic functions which is a local question. Thus, we may think of the s_i as vectors $s_i = (s_j^i)_{j=1,\ldots,r}$ of meromorphic functions and view (a_i) as a solution of the system of linear equations $(*)$. Expressing (a_i) in terms of the adjoint matrix and the vector (s_j^r) proves that all a_i are indeed meromorphic. □

Proof of proposition. We shall follow [Kra75] and assume $K(X) = \mathbb{C}$. This is the only case that will be needed for Corollary 2.5 and its application later on. For the general case we refer to [FF79].

Applying $d\log$, the sheaf of complexified Cartier divisors $\mathcal{K}_X^*/\mathcal{O}_X^* \otimes_{\mathbb{Z}} \mathbb{C}$ is identified with the the quotient of $\Omega_X^1 \subset \Omega_{X,\log}^1$, where the latter sheaf is by definition locally generated by all holomorphic one-forms and logarithmic one-forms $d\log f$ with f a local section of $\mathcal{K}_X^*$. Taking cohomology yields a long exact sequence

$$0 \longrightarrow H^0(X, \Omega_X^1) \longrightarrow H^0(X, \Omega_{X,\log}^1) \longrightarrow \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^1(X, \Omega_X) \longrightarrow \cdots .$$

Since $H^0(X, \Omega_{X,\log}^1) \subset H^0(X, \Omega_X^1 \otimes \mathcal{K}_X)$ and since we assume $K(X) = \mathbb{C}$, the lemma implies $\dim_{\mathbb{C}}(\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{C}) \leqslant h^1(X, \Omega_X^1) + \dim(X) - h^0(X, \Omega_X^1)$. □

Corollary 2.5. *If a compact complex manifold X contains infinitely many integral hypersurfaces, then its algebraic dimension satisfies $a(X) > 0$, i.e. $K(X) \neq \mathbb{C}$.* □

2.3. **Sections of twists of vector bundles.** Proposition 2.1 does not directly produce sections of powers of L. For this one needs the following result due to Demailly–Peternell–Schneider [DPS01, Prop. 2.15]

Proposition 2.6. *Let L be a line bundle and E a vector bundle (or, more generally, a torsion free sheaf) on a compact complex manifold X. Assume m_i is an unbounded sequence of positive integers such that $H^0(X, E \otimes L^{m_i}) \neq 0$.*

(i) *Then there exists a line bundle M on X and an unbounded sequence m_i' of positive integers such that $H^0(X, M\otimes L^{m_i'})\neq 0$.*

(ii) *There exist infinitely many integral hypersurfaces $Y\subset X$, and in particular $K(X)\neq\mathbb{C}$, or L is $\mathbb{Q}$-effective.*

Proof. Let $F\subset E$ be the coherent subsheaf of E generated by all homomorphisms $L^{-m_i}\longrightarrow E$. By assumption, F is non-trivial, so F is a torsion free sheaf of positive rank $r := \mathrm{rk}(F) > 0$. Furthermore, we may assume that there exist integers $n_1,\ldots,n_{r-1}$ (among the m_i) and an unbounded subsequence m_j of m_i for which there exist injections $L^{-n_1}\oplus\cdots\oplus L^{-n_{r-1}}\oplus L^{-m_j}\hookrightarrow F$.

Taking determinants yields non-trivial global sections $s_j\in H^0(X, M\otimes L^{m_j'})$, where $M := \det(F)$ and $m_j' := \sum n_i + m_j$. This proves (i).

Let us turn to (ii). There is nothing to prove in the case that X is projective or weaker that $K(X)\neq\mathbb{C}$. So we assume that X contains only finitely many integral hypersurfaces $Y_1,\ldots,Y_N\subset X$. Then the zero loci $Z(s_j)\subset X$ as Weil divisors are in the convex hull of the finitely many Y_i, i.e. $M\otimes L^{m_j'}\in\sum\mathbb{Z}_{\geqslant 0}\cdot\mathcal{O}(Y_i)$. As the sequence m_j' is unbounded, also $L^{m_j''}\in\sum\mathbb{Z}_{\geqslant 0}\cdot\mathcal{O}(Y_i)$ for an unbounded sequence m_j'' and, in particular, L is $\mathbb{Q}$-effective. $\square$

A priori, X could contain only one integral hypersurface $Y\subset X$ and all sections of powers of L^m are of the form s^m for some $s\in H^0(X,L)$ with $Z(s)=Y$. In other words, the above result only ensures the existence of one non-trivial section of the line bundles L^m up to passing to powers, which is not enough to make progress on Conjecture 1.2.

2.4. **Cones on hyperkähler manifolds.** We recall some basic notations and facts concerning the various cones relevant for the arguments below.

The *positive cone* $\mathcal{C}_X\subset H^{1,1}(X,\mathbb{R})$ of a compact hyperkähler manifold X is the connected component of the open set of all classes $\alpha\in H^{1,1}(X,\mathbb{R})$ with $q(\alpha)>0$. It contains the *Kähler cone* $\mathcal{K}_X\subset\mathcal{C}_X$ of all Kähler classes as an open subcone. The closure of the Kähler cone $\overline{\mathcal{K}}_X\subset\overline{\mathcal{C}}_X$, the *nef cone*, is the set of all classes $\alpha\in\overline{\mathcal{C}}_X$ with $\int_C\alpha\geqslant 0$ for all rational curves $C\subset X$, cf. [Huy03b, Prop. 3.2], and the open Kähler cone $\mathcal{K}_X\subset\mathcal{C}_X$ is the set of all classes $\alpha\in\mathcal{C}_X$ with $\int_C\alpha>0$ for all rational curves $C\subset X$, cf. [Bou01, Thm. 1.2].

The *birational Kähler cone* $\mathcal{BK}_X$ is by definition the union $\bigcup\mathcal{K}_{X'}$ of all Kähler cones of birational compact hyperkähler manifolds X'. Here, we use that any birational correspondence $X\sim X'$ induces a natural Hodge isometry $H^2(X,\mathbb{Z})\simeq H^2(X',\mathbb{Z})$, cf. [Huy03a, Prop. 25.14]. Clearly, $\mathcal{BK}_X\subset\mathcal{C}_X$ and according to [Huy03a, Prop. 28.7] its closure $\overline{\mathcal{BK}}_X\subset\overline{\mathcal{C}}_X$, the *modified nef cone*, is the set of all classes $\alpha\in\overline{\mathcal{C}}_X$ with $q(\alpha,D)\geqslant 0$ for all uniruled divisors. Of course, it suffices to test this for *prime exceptional divisors*, i.e. irreducible divisors $D\subset X$ with $q(D)<0$. Alternatively, $\overline{\mathcal{BK}}_X$ can be described as the dual of the pseudo-effective cone $\mathcal{E}_X$ of all classes $\alpha\in H^{1,1}(X,\mathbb{R})$ that can be represented by a positive current, see [Huy03b, Cor. 4.6]. In particular, all effective divisors $D \subset X$ define classes in $\mathcal{E}_X$. Note that in particular $q(L, D) \geqslant 0$ for any nef line bundle L and any effective divisor $D \subset X$.

According to Boucksom [Bou04, Thm. 4.8], any pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ admits a *Zariski decomposition* $\alpha = P(\alpha) + N(\alpha)$, where $P(\alpha) \in \overline{\mathcal{BK}}_X$ and $N(\alpha)$ is the class of an exceptional $\mathbb{R}$-divisor, i.e. $N(\alpha) = \sum a_i D_i$ with $D_i \subset X$ irreducible divisors such that the matrix $(q(D_i, D_j))$ is negative definite. Furthermore, $P(\alpha)$ and $N(\alpha)$ are orthogonal, i.e. $q(P(\alpha), N(\alpha)) = 0$. We shall need the Zariski decomposition only for divisor classes $\alpha \in H^{1,1}(X, \mathbb{Z})$ and in this case $P(\alpha)$ and $N(\alpha)$ are in fact rational.

2.5. **Stability of the tangent bundle.** Due to existence of a Kähler–Einstein metric in each Kähler class, the tangent bundle $\mathcal{T}_X$ of a compact hyperkähler manifold X is μ-stable with respect to any Kähler class $\omega \in \mathcal{K}_X$. In fact, stability holds with respect to all ω in the interior of $\overline{\mathcal{BK}}_X$, see also Section 5.1.

Proposition 2.7. *Let X be a compact hyperkähler manifold and $M \subset \Omega_X^{\otimes N}$ a line bundle in some tensor power of its cotangent bundle. Then the dual M^* is pseudo-effective.*

Proof. In the projective case, the assertion is a consequence of a general result due to Campana–Peternell [CP11, Thm. 0.1] showing that any torsion free quotient $(\Omega_X^1)^{\otimes q} \twoheadrightarrow \mathcal{F}$ has a pseudo-effective determinant $\det(\mathcal{F})$ unless X is uniruled.

Verbitsky [Ver10] gives an alternative argument relying on the observation that all tensor powers $\Omega_X^{\otimes N}$ of the cotangent bundle are μ-semistable with respect to any class in the birational Kähler cone. More precisely, let $\alpha \in \mathcal{BK}_X$ be a class corresponding to Kähler class $\omega' \in \mathcal{K}_{X'}$ on some birational model X' of X. Since X and X' are isomorphic in codimension one, the inclusion $M \subset \Omega_X^{\otimes N}$ carries over to an inclusion $M' \subset \Omega_{X'}^{\otimes N}$. To conclude use the stability of $\Omega_{X'}$, which proves $q(\alpha, M) = q(\omega', M') \leqslant 0$. Hence, $q(\alpha, M^*) \geqslant 0$ for all $\alpha \in \overline{\mathcal{BK}}_X$, i.e. M^* is pseudo-effective. □

3. Proofs

In this section we present two proofs of the main theorem. The original of Campana–Oguiso–Peternell [COP10] applies only to the case that the hyperkähler manifold is non-projective. Verbitsky [Ver10] showed how to combine the original approach with Boucksom's Zariski decomposition to also cover the algebraic case. In the next section we will sketch a different approach that reduces the projective case to the non-projective one.

3.1. **Non-algebraic case.** We follow the arguments in [COP10].

Proof. Assume L is a non-trivial nef line bundle on a non-projective compact hyperkähler manifold X of dimension $2n$ and assume $q(L) = 0$. Suppose $H^0(X, L^m) = 0$ for all $m > 0$. The Riemann–Roch formula [Huy99, Huy03a] simply states $\chi(X, L^m) = n + 1$. Thus, there exists an even number $q > 0$ and an unbounded sequence m_i of positive integers such that $H^q(X, L^{m_i}) \neq 0$. By virtue of Proposition 2.1 this implies $H^0(X, \Omega_X^{2n-q} \otimes L^{m_i}) \neq 0$.

Combining Corollay 2.5 and Proposition 2.6, we conclude that L is $\mathbb{Q}$-effective, in which case we are done, or $K(X) \neq \mathbb{C}$. For example, the former case holds if $\rho(X) = 1$. In the latter case, the algebraic reduction [Uen75, Ch. 3] provides us with a diagram

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\pi}\swarrow & & \searrow^{\tilde{f}} \\ X & \overset{}{\underset{f}{\dashrightarrow}} & B. \end{array}$$

Here, $\tilde{X}$ is a compact complex manifold, B is smooth and projective of dimension at least one, and π is birational. The pull-back $\tilde{f}^*H$ of a very ample line bundle H on B can be written as $\tilde{f}^*H \simeq \pi^*M \otimes \mathcal{O}(-E)$ with $E \subset \tilde{X}$ effective, in fact π-exceptional but possibly trivial, and $M \in \mathrm{Pic}(X)$. This yields inclusions

$$H^0(B, H) \hookrightarrow H^0(\tilde{X}, \tilde{f}^*H) \hookrightarrow H^0(\tilde{X}, \pi^*M) \simeq H^0(X, M).$$

Since $\dim(B) \geqslant 1$, this shows that M is non-trivial and effective. In fact, as H is very ample, we may assume that M admits two linearly independent sections with distinct zero divisors $D_1, D_2 \subset X$ without irreducible components.

According to [Bou04, Prop. 4.2], for any two such divisors $D_1, D_2 \subset X$ we have $q(D_1, D_2) \geqslant 0$. Indeed, up to a positive scalar $q(D_1, D_2) = \int_{D_1 \cap D_2} (\sigma\bar{\sigma})^{n-1} \geqslant 0$, since $(\sigma\bar{\sigma})^{n-1}$ is a positive form. Applied to our situation this yields $q(M) \geqslant 0$. On the other hand, since X is assumed to be non-projective, the projectivity criterion [Huy99, Thm. 3.11] implies $q(M) \leqslant 0$. Therefore, $q(M) = 0$. However, as the form q restricted to $H^{1,1}(X, \mathbb{R})$ satisfies the Hodge index theorem, every line bundle M on X with $q(M) = 0$ is a rational multiple of L. As L was assumed semi-positive (hence, nef) and M is effective, M is a positive rational multiple of L. Therefore, L is $\mathbb{Q}$-effective. □

3.2. **Algebraic case.** In fact, the following arguments taken from [Ver10] apply also to non-algebraic hyperkähler manifolds and thus subsume the original proof in [COP10]

Proof. The first part of the proof is identical to the one in the non-algebraic case. From Proposition 2.6 we deduce the existence of a line bundle M with $H^0(X, M \otimes L^{m_i}) \neq 0$ for an unbounded sequence of positive integers. Since L is nef with $q(L) = 0$, this implies $q(M \otimes L^{m_i}, L) \geqslant 0$ and $q(M, L) \geqslant 0$. On the other hand, the line bundle M in the proof of Proposition 2.6 was constructed as the determinant $M = \det(F)$ of a subsheaf $F \subset E = \Omega_X^{2n-q}$ and, therefore, $M \subset \Omega_X^{\otimes N}$ for some N. Then, Proposition 2.7 shows that M^* is pseudo-effective and, hence, $q(M, L) \leqslant 0$, see Section 2.4. Therefore, $q(M, L) = 0$. In the case that $\rho(X) = 2$, we can conclude already that M is a rational multiple of L and that, therefore, L is $\mathbb{Q}$-effective.

For $\rho(X) > 2$, we consider the Zariski decomposition of the pseudo-effective line bundle M^* as $P + N$ with P contained in the closure of the birational Kähler cone and N exceptional effective. In particular, $q(P) \geqslant 0$ with $q(P, L) > 0$ unless P is a rational multiple of L and $q(N) < 0$ unless $N = 0$. Then, $0 = q(L, M) = q(L, P) + q(L, N)$ with both summands non-negative and, therefore, both zero. Thus, the Zariski decomposition of M^* is of the form $\lambda L + N$, with $\lambda \in \mathbb{Q}_{\geqslant 0}$. On the other hand, $M \otimes L^{m_i}$ is effective for an unbounded sequence of positive integers m_i. Hence, $(m_i - \lambda)L$ can be written as the sum of the two effective divisors $M \otimes L^{m_i}$ and N. Therefore, L itself is $\mathbb{Q}$-effective. □

4. Semi-positivity under deformations

We will now show that alternatively the proof in the algebraic case can be reduced via deformation to the non-algebraic case. The techniques are potentially relevant to make progress on Conjecture 1.2.

First recall that for a smooth proper family $\mathcal{X} \longrightarrow \Delta$ is of complex manifolds with central fibre $X = \mathcal{X}_0$ of Kähler type, all nearby fibres $\mathcal{X}_t$ are of Kähler type as well, i.e. for all t after shrinking Δ to an open neighbourhood of $0 \in \Delta$. More precisely, if the Kähler class on X stays of type $(1, 1)$ on the nearby fibres, then it is Kähler there as well. This classical result is due to Kodaira and Spencer [KS60]. Note that since the Kähler property is a combination of the open condition that a real $(1, 1)$-form ω is positive and the closed condition $d\omega = 0$, this is a priori not clear. In the case of closed semi-positive forms ω, the corresponding statement fails. Similarly, if $\alpha \in H^{1,1}(X, \mathbb{R})$ is a nef class that stays of type $(1, 1)$ on all the fibres $\mathcal{X}_t$, as a class on $\mathcal{X}_t$ it need not be nef, see [Mor92] for an example.

4.1. **Degenerate twistor lines.** We shall describe a one-parameter deformation of a compact hyperkähler manifold endowed with a semi-positive isotropic $(1, 1)$-form, see [Ver15]. In the following, we let X be a compact hyperkähler manifold of dimension $2n$ with a fixed holomorphic two-form σ and a Ricci-flat Kähler form ω. Furthermore, we assume that η is a semi-positive closed real $(1, 1)$-form with isotropic cohomology class $[\eta] \in H^{1,1}(X, \mathbb{R})$, i.e. $q([\eta]) = 0$. Note that we also know

$$q([\sigma]) = 0,\ q([\sigma], [\eta]) = 0 \text{ and } q([\sigma] + [\eta]) = 0.$$

Now, by virtue of Verbitsky's description of the cohomology generated by $H^2(X, \mathbb{R})$, see [Bog96, Ver96], these equalities imply

$$[\eta]^{n+1} = 0 \text{ and } ([\sigma] + [\eta])^{n+1} = 0$$

in $H^{2n+2}(X, \mathbb{R})$. The semi-positivity of η implies that the same equalities hold on the level of forms, see [Ver15, Sec. 3].

Lemma 4.1. *Under the above assumptions, the following assertions hold true:*

(i) $\eta^{n+1} = 0$.

(ii) $(\sigma+\eta)^{n+1}=0$.

(iii) $(\sigma+\eta)^n\wedge(\bar\sigma+\eta)^n$ *is a volume form.*

Proof. We skip the proof. This is a point-wise statement which boils down to linear algebra. □

Of course, the same results hold for all positive multiples $t\eta$ which will be used to define a family of complex structures on X.

First recall that the kernel of the map $\sigma\colon T_{\mathbb{C}}X\longrightarrow T_{\mathbb{C}}X^*$ naturally induced by the complex two-form σ is the bundle of $(0,1)$-vector fields $T^{0,1}\subset T_{\mathbb{C}}X$. Copying this, one defines for the closed two-forms

$$\sigma_t := \sigma + t\cdot\eta$$

the bundle $T_t^{0,1}\subset T_{\mathbb{C}}X$ as the kernel of the map induced by the complex two-form σ_t. Lemma 4.1, (iii) implies that $T_t^{0,1}$ really is a complex vector bundle of dimension $2n$ and that $T_{\mathbb{C}}X=T_t^{1,0}\oplus T_t^{0,1}$, where $T_t^{1,0}$ is the complex conjugate of $T_t^{0,1}$. This direct sum decomposition describes an almost complex structure I_t on the differentiable manifold M underlying X.

Lemma 4.2. *The almost complex structure I_t is integrable and σ_t is a holomorphic symplectic form on (M,I_t). Furthermore, for all t the form η is of type $(1,1)$ with respect to I_t and η is semi-positive for small t.*

Proof. Due to the Newlander–Nirenberg theorem, it suffices to show that $T_t^{0,1}$ is preserved under the Lie bracket, i.e. $[T_t^{0,1},T_t^{0,1}]\subset T_t^{0,1}$, cf. [Huy05, Sec. 2.6]. The standard formula for the derivative of differential forms applied to $v_1,v_2\in T_t^{0,1}$ and arbitrary $v\in T_{\mathbb{C}}X$ shows

$$\begin{aligned}0=(d\sigma_t)(v_1,v_2,v) \;&=\; v_1(\sigma_t(v_2,v))-v_2(\sigma_t(v_1,v))+v(\sigma_t(v_1,v_2))\\ &\quad -\sigma_t([v_1,v_2],v)+\sigma_t([v_1,v],v_2)-\sigma_t([v_2,v],v),\end{aligned}$$

which by definition of $T_t^{0,1}$ yields $\sigma([v_1,v_2],v)=0$ for all $v\in T_{\mathbb{C}}X$ and, therefore, $[v_1,v_2]\in T_t^{0,1}$.

By construction, σ_t is of type $(2,0)$ on (M,I_t), closed since σ and η are closed, and non-degenerate by Lemma 4.1, (iii). Hence, σ_t is a holomorphic symplectic form on (M,I_t). The form η is of type $(1,1)$ with respect to I_t, for $\sigma_t^n\wedge\eta=(\sigma+t\cdot\eta)^n\wedge\eta=0$ by Lemma 4.1, (ii).

The semipositivity of a smooth form can be checked point-wise and it is suffices to verify it at a general point $x\in M$, where we can assume η to be of maximal rank n.[2] Now, choose a family of forms $\alpha_i(t)$, $i=1,\dots,2n$, that are of type $(1,0)$ with respect to I_t, vary smoothly with t, and form a basis of $(T_t^{1,0})^*$ at x with respect to which η is diagonal, i.e. $\eta(x)=i\sum a_i(t)\cdot(\alpha_i(t)\wedge\overline{\alpha}_i(t))(x)$. By Lemma 4.1, (i) and the semipositivity of η with respect to I_0, the coefficients satisfy $a_i(0)\geqslant 0$ and exactly n of them, say $a_1(0),\dots,a_n(0)$, are strictly

[2] This is oversimplifying things a little: There is such an open subset, but it may be open only in the analytic topology. To make this rigorous one has to work with an analytically dense union of open subsets $\bigcup U_i\subset M$ such that on each U_i the rank n_i of η is constant. Then $\eta^{n_i+1}|_{U_i}\neq 0$ but $\eta^{n_i}|_{U_i}=0$. The rest of the argument remains unchanged.

positive. By continuity, $a_1(t), \ldots, a_n(t) > 0$ for all small t and $\eta^{n+1} = 0$ then implies that we still must have $a_{n+1}(t) = \cdots = a_{2n}(t) = 0$ for those t. Hence, η is semi-positive for small t. □

Remark 4.3. It is possible to show that in fact η is semi-positive with respect to all I_t, $t \in \mathbb{C}$, but we will not need this stronger statement.

Altogether the above describes a smooth family $\mathcal{X} \longrightarrow \mathbb{C}$ of compact hyperkähler manifolds $\mathcal{X}_t := (M, I_t)$, called *degenerate twistor family*, together with a constant closed real $(1,1)$-form η that is semi-positive for small $t \in \Delta \subset \mathbb{C}$.[3] The central fibre $\mathcal{X}_0$ is the original compact hyperkähler manifold X.

4.2. **Semi-continuity.** We apply the above construction to the case of a non-trivial semi-positive line bundle L with $q(L) = 0$. By assumption, $c_1(L) = [\eta]$ for some semi-positive closed real $(1,1)$-form η. For example, if $f \colon X \longrightarrow \mathbb{P}^n$ is a Lagrangian fibration and $L \simeq f^*\mathcal{O}(1)$, then any positive form η_0 on $\mathbb{P}^n$ representing the hyperplane class $c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}^n, \mathbb{Z})$ induces a semi-positive form $\eta = f^*\eta_0$ that represents $c_1(L)$. The properties in Lemma 4.1, for example $\eta^{n+1} = 0$, are obvious in this case. Note that in contrast to the uniqueness of Ricci-flat Kähler forms in any given Kähler class, the form η satisfying the degenerate Monge–Ampère equation $\eta^{2n} = 0$, or even $\eta^{n+1} = 0$, is certainly not unique.

Lemma 4.4. *The degenerate twistor line $\mathcal{X} \longrightarrow \mathbb{C}$ associated to a semi-positive form η representing $c_1(L)$ has the property that for very general $t \in \mathbb{C}$, the fibre $\mathcal{X}_t = (M, I_t)$ is non-projective. Furthermore, the complex line bundle L is holomorphic with respect to all $t \in \mathbb{C}$ and semi-positive for small $t \in \Delta$.*

Proof. As the first Chern class of the complex line bundle L satisfies $q(c_1(L), \sigma_t) = q([\eta], [\sigma]) + t \cdot q([\eta], [\eta]) = 0$, the line bundle L is holomorphic on all fibres $\mathcal{X}_T$. Since the non-trivial nef class $c_1(L)$ is not orthogonal to any class in the positive cone $\mathcal{C}_X$, the very general fibre is not projective.

On every fibre the class $c_1(L)$ is represented by the closed real $(1,1)$-form η and any such form is the curvature of a uniquely determined hermitian structure on L. Since for small t the form η is still semi-positive, one finds that L is a semi-positive holomorphic line bundle on all fibres $\mathcal{X}_t$ for small t. □

This allows one to show that the original result of Campana–Oguiso–Peternell [COP10] for non-projective hyperkähler manifolds is enough to conclude the result for all hyperkähler manifolds, which gives an alternative proof of Verbitsky's result [Ver10].

Corollary 4.5. *Assume any semi-positive line bundle on a non-projective hyperkähler manifold X is $\mathbb{Q}$-effective. Then the same also holds for projective hyperkähler manifolds.*

[3] There is a minor technical issue here. The parameter t above was assumed to be real and positive. Either, Lemma 4.1 has to be adapted in (iii) to say that $(\sigma + t\eta)^n \wedge (\bar{\sigma} + \bar{t}\eta)^n$ is a volume form or the family is first constructed just over $\mathbb{R}_{>0} \cap \Delta \subset \mathbb{C}$ and then extended from there.

Proof. Indeed, as explained above, any compact hyperkähler manifold X with a semi-positive line bundle L with $q(L) = 0$ can be realized as the central fibre of a degenerate twistor family $\mathcal{X} \longrightarrow \mathbb{C}$. The very general fibre $\mathcal{X}_t$ is non-projective and [COP10] thus applies to L considered as a semi-positive line bundle on $\mathcal{X}_t$. Therefore, L is $\mathbb{Q}$-effective on the very general fibre $\mathcal{X}_t$, $t \in \Delta$, and, by semi-continuity, the same holds for the central fibre. □

5. Open questions

Besides the two conjectures stated in the introduction, there are a number of related questions that seem approachable.

5.1. **Stability of the tangent bundle.** Due to the existence of a hyperkähler (and hence Kähler–Einstein) metric on a hyperkähler manifold X, the tangent bundle $\mathcal{T}_X$ is μ-stable. In fact, $\mathcal{T}_X$ is μ-stable with respect to every Kähler class and, as explained in Section 2.5, with respect to the generic class in the birational Kähler cone.

Question 5.1. *Is the tangent bundle $\mathcal{T}_X$ of a hyperkähler manifold μ-stable with respect to any class in the positive cone?*

This would subsume Proposition 2.7 and would allow one to conclude the stronger statement that the line bundle M constructed in the proof of Proposition 2.6 and used in the two proofs in Section 3 is contained in the closure of the positive cone.

5.2. **Elliptic and parabolic hyperkähler manifolds.** The paper [COP10] discusses the possibilities for the algebraic dimension $a(X) = \mathrm{trdeg} K(X)$ of a compact hyperkähler manifold and how the algebraic dimension is related to the intersection form on the Néron–Severi group. We only touch upon one aspect here.

Question 5.2. *Assume X is a compact hyperkähler manifold of algebraic dimension zero, i.e. $K(X) = \mathbb{C}$. Is the Beauville–Bogomolov form q on $\mathrm{NS}(X) \simeq H^{1,1}(X, \mathbb{Z})$ negative definite?*

Following [COP10], X is called *elliptic* if q is negative definite on $\mathrm{NS}(X)$. It is known that elliptic hyperkähler manifolds satisfy $K(X) = \mathbb{C}$. The above question is the converse.

Similarly, X is called *parabolic* if q on $\mathrm{NS}(X)$ is semi-negative definite with one isotropic direction and *hyperbolic* if it has signature $(1, \rho(X) - 1)$. By the Hodge index theorem and the projectivity criterion for hyperkähler manifolds, the latter is equivalent to X being projective. The analogue of Question 5.2 in the parabolic case is the conjecture that X is parabolic if and only if $a(X) = n$. According to [COP10, Thm. 3.6], any non-algebraic compact hyperkähler manifold satisfies $a(X) \leqslant n$, so that the cases $0 < a(X) < n$ would need to be excluded.

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