D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
9th week

44. *Locality of 'separated'.* Use the valuative criterion to prove the following assertion: A morphism $f : X \longrightarrow Y$ is separated if and only if there exists an open cover $Y = \bigcup U_i$ such that the restriction $f^{-1}(U_i) \longrightarrow U_i$ is separated. State and prove the analogous result for 'proper'.

45. *Globally generated sheaves.* Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on a ringed space. Show that $\mathcal{F}$ is globally generated if and only if the natural map

$$\Gamma(X, \mathcal{F}) \otimes_{\Gamma(X, \mathcal{O}_X(X))} \mathcal{O}_{X,x} \longrightarrow \mathcal{F}_x$$

is surjective for every $x \in X$.

46. *Projection formula.* Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Show that for a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules and a locally free sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules of finite rank there exists a natural isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G}.$$

47. *$M \longmapsto \tilde{M}$ and adjunction.* Let X be the affine scheme $\mathrm{Spec}(A)$ and consider an A-module M and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Show that $M \longmapsto \tilde{M}$ is left adjoint to $\mathcal{F} \longmapsto \Gamma(X, \mathcal{F})$, i.e. that there exist functorial (in M and $\mathcal{F}$) isomorphisms

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

48. *Fibre dimension.* Let X be a noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We will consider the function

$$\varphi(x) := \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x),$$

where $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of the point $x \in X$. Use Nakayama's lemma to prove the following statements.

i) The function φ is upper semi-continuous, i.e. for any $n \in \mathbb{Z}$ the set $\{x \in X \mid \varphi(x) \geq n\}$ is closed.

ii) If $\mathcal{F}$ is locally free and X is connected, then φ is a constant function.

iii) Conversely, if X is reduced and φ is constant, then $\mathcal{F}$ is locally free.