D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
8th week

38. *The image of a proper scheme is proper.* Let $f : X \longrightarrow Y$ be a morphism of S-schemes. Suppose that $Y \longrightarrow S$ is separated.

i) Show that the graph $\Gamma_f : X \longrightarrow X \times_S Y$ is a closed immersion.

ii) Let $Z \subset X$ be a closed subscheme that is proper over S. Show that $f(Z) \subset Y$ is closed.

39. *Finite morphisms are proper.* Recall that a ring homomorphism $f : A \longrightarrow B$ is *integral* if B is integral over $f(A)$, i.e. every $b \in B$ is a root of a monic polynomial $x^n + f(a_1)x^{n-1} + \ldots + f(a_n) \in f(A)[x]$. Recall that B is a finitely generated A-module if and only if f is integral and B is a finitely generated A-algebra.

Use the going-up theorem for integral ring extensions $A \subset B$ to prove that integral (and in particular finite) morphisms $X \longrightarrow Y$ of schemes are proper.

40. *Global regular functions on projective spaces.* Show that for a ring A one has $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) \cong A$. Deduce from this that $\mathbb{P}^n_A$ is affine if and only if $n = 0$. (*Hint*: Use the standard open cover.)

41. *Quasi-projective schemes.* i) Show that any affine morphism of finite type $X \longrightarrow Y = \mathrm{Spec}(A)$ is quasi-projective. (The assertion holds under much weaker assumptions on Y, though.)

ii) Suppose $f_i : X_i \longrightarrow Y$, $i = 1, 2$, are quasi-projective morphisms. Show that then also $X_1 \times_Y X_2 \longrightarrow Y$ is quasi-projective.

42. *Composition of projective morphisms.* Use the Segre embedding to prove that the composition of projective morphisms is projective.

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43. *Veronese embedding.* The following is analogous to the construction of the Segre embedding discussed in class. Consider all monomials $M_0, \dots, M_N$ of degree d in $n+1$ variables $x_0, \dots, x_n$, where $N = \binom{n+d}{n} - 1$. Let $\mathbb{Z}[y_0, \dots y_N] \longrightarrow \mathbb{Z}[x_0, \dots, x_n]$ be the homomorphism of graded $\mathbb{Z}$-algebras defined by $y_i \longmapsto M_i$. (Replace $\mathbb{Z}$ by some field k if you prefer.)

i) Show that this induces a morphism of schemes $\nu_d : \mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathbb{P}^N_{\mathbb{Z}}$, the Veronese embedding.

ii) Show that f is a closed immersion. For any other base scheme Y the Veronese embedding $\nu_d : \mathbb{P}^n_Y \longrightarrow \mathbb{P}^N_Y$ is obtained by base change and is also a closed embedding.

iii) For an algebraically closed field k, write out $\nu_3 : \mathbb{P}^1_k \longrightarrow \mathbb{P}^3_k$ on closed points. (The image is called the *twisted cubic.*)