D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
7th week

33. *Zariski tangent space via functor of points.* Consider the category of schemes (Sch) and a contravariant functor $\Phi : (Sch)^o \longrightarrow (Sets)$. A k-rational point of Φ is by definition an element $x \in \Phi(\mathrm{Spec}(k))$ and the Zariski tangent space $T_x\Phi$ of Φ at $x \in \Phi(\mathrm{Spec}(k))$ is the fibre over x of $\Phi(\mathrm{Spec}(k[\varepsilon]) \longrightarrow \Phi(\mathrm{Spec}(k))$ (which is induced by the natural $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k[\varepsilon])$ corresponding to the projection $k[\varepsilon] \longrightarrow k$).

i) Show that elements in $T_x\Phi$ can naturally be multiplied by elements in k. (*Hint*: Use $\varepsilon \longmapsto \lambda\varepsilon$.)

ii) Show that $T_x\Phi$ can naturally be endowed with the structure of a k-vector space if Φ satisfies

$$\Phi(\mathrm{Spec}(k[X]/(X^2))) \times_{\Phi(\mathrm{Spec}(k))} \Phi(\mathrm{Spec}(k[Y]/(Y^2))) = \Phi(\mathrm{Spec}(k[X,Y]/(X,Y)^2)).$$

(*Hint*: Use the morphism $k[X,Y] \longrightarrow k[\varepsilon]$, $X, Y \longmapsto \varepsilon$).

iii) Compare the notion of $T_x\Phi$ for $\Phi = h_X$ with the notion of the Zariski tangent space T_x as introduced in Exercise 16.

34. *Properties of morphisms under base change.*

i) Show that 'quasi-finite' (i.e. finite fibres) and 'injective' are not preserved under base change.

ii) Show that 'finite', 'surjective', and 'closed immersion' are preserved under base change. (Here, a morphism $f : X \longrightarrow Y$ is called surjective if it is surjective on the underlying sets.)

35. *Fibres.* Consider the subscheme $X \subset \mathbb{A}^2_{\mathbb{Z}}$ given by $XY^2 - m$, for some $m \in \mathbb{Z}$. Study the fibres of $X \longrightarrow \mathrm{Spec}(\mathbb{Z})$. Which ones are irreducible?

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36. *Affine schemes under base field extensions.* Let k be an algebraically closed field, let $k_0 \subset k$ be a subfield, and let X_0 be a scheme over k_0. In this exercise we study the relation between X_0 and $X := X_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k)$ in examples.

i) Let $k = \mathbb{C}$ and $k_0 = \mathbb{R}$. Consider $X_0 = \mathrm{Spec}(k_0[x, y|/(y^2 - x(x^2 - 1)))$. Which residue fields are possible for points in X_0 and how does the set $X_0(\mathbb{R})$ of $\mathbb{R}$-rational points look like? The Galois group $G := \mathrm{Gal}(k/k_0) \cong \mathbb{Z}/2\mathbb{Z}$ acts on X and $X(\mathbb{C})$. Study the fibres of $X(\mathbb{C}) \longrightarrow X_0$ in terms of G-orbits.

ii) Let again $k = \mathbb{C}$ and $k_0 = \mathbb{R}$. Let $X_0 = \mathrm{Spec}(k_0[x, y]/(x^2 + y^2))$. Check that X_0 is irreducible while X is not. Describe X geometrically.

iii) Let k_0 be an imperfect field of characteristic $p > 0$ and let $a \in k_0 \setminus k_0^p$. Let ℓ be a non-trivial homogeneous linear polynomial in X, Y and $X_0 = \mathrm{Spec}(k_0[X, Y]/(\ell^p - a))$. Prove that X_0 is reduced. Prove further that $X \cong \mathrm{Spec}(k[X, Y]/((\ell - a^{1/p})^p))$ and thus not reduced.

37. *Morphisms into separated schemes.* Consider schemes X and Y over a base scheme S. Assume that X is reduced (or even stronger integral) and that $Y \longrightarrow S$ is separated. Show that two morphisms $f, g : X \longrightarrow Y$ over S that coincide on a dense open subset $U \subset X$ are actually equal. (*Hint*: Consider the composition of the graph $X \longrightarrow X \times_S Y$ of f with $(g, \mathrm{id}) : X \times_S Y \longrightarrow Y \times_S Y$.) Give counterexamples if one of the hypotheses is dropped.