D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
6th week

27. *Noetherian topological spaces.* Suppose X is a noetherian topological space.
i) Show that every subset $Y \subset X$ with the induced topology is noetherian.
ii) Show that any open subset of X (and in particular X itself) is quasi-compact.

27. *Schemes are T_0-spaces.* Let X be a scheme.
i) If X is irreducible and consists of more than two points, then X is not Hausdorff.
ii) Show that X is a T_0-space, i.e. for any two distinct points $x, y \in X$ there exists an open subset $U \subset X$ containing exactly one of the two points.

29. *Reduced schemes and reduction of schemes.* Let X be a scheme.
i) Show that X is reduced if and only if all local rings $\mathcal{O}_{X,x}$ are reduced.
ii) Construct the *reduction* $X_{\rm red}$ of X. Its topological space is the same as that of X and its structure sheaf is given by $\mathcal{O}_{X_{\rm red}}(U) = \mathcal{O}_X(U)/\mathfrak{N}(\mathcal{O}_X(U))$, where $\mathfrak{N}(A)$ of a ring A is its nilradical. Show that this defines a scheme and a natural morphism of schemes $X_{\rm red} \longrightarrow X$ which is a homeomorphism of topological spaces.
iii) Show that $X_{\rm red}$ has the following universal property: If $Y \longrightarrow X$ is a morphism of schemes with Y reduced, then it factors uniquely over $X_{\rm red} \longrightarrow X$.

30. *Normalization.* Let X be an integral scheme and $\eta \in X$ its generic point. For any open $\mathrm{Spec}(A) \subset X$ consider the integral closure $A \subset \tilde{A} \subset Q(A) = k(\eta) = K(X)$ and the associated affine scheme $\widetilde{U} := \mathrm{Spec}(\widetilde{A})$.
i) Show that the schemes $\widetilde{U}$ can be glued to a scheme $\widetilde{X}$ (the *normalization* of X) that comes with a natural morphism $\nu : \widetilde{X} \longrightarrow X$ extending $\widetilde{U} \longrightarrow U$. The normalization $\widetilde{X}$ is normal.
ii) Prove the following universality property: Every dominant morphism $Z \longrightarrow X$ from a normal scheme Z factors uniquely through $\nu : \widetilde{X} \longrightarrow X$.

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31. *Points under base change.* Consider the natural morphism $\mathbb{A}^2_{\bar{\mathbb{Q}}} \longrightarrow \mathbb{A}^2_{\mathbb{Q}}$ and determine the images of the following points: i) $(x-\sqrt{2}, y-\sqrt{2})$; ii) $(x-\sqrt{2}, y-\sqrt{3})$; iii) $(\sqrt{2}x - \sqrt{3}y)$.

32. $\mathbb{P}^n$ *by glueing.* One can construct the *projective space* $\mathbb{P}^n_A$ over some ring A by glueing $(n+1)$ affine spaces $\mathbb{A}^n_A$ as follows:

i) Consider $B := A[x_0, \ldots, x_n]_{x_0 \cdot \ldots \cdot x_n} = A[x_0, x_0^{-1}, \ldots, x_n, x_n^{-1}]$ with its subrings

$$B_i := A[\frac{x_0}{x_i}, \ldots \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}].$$

Show that $B_i \cong A[y_1, \ldots, y_n]$ and $B_i[\frac{x_i}{x_j}] = B_j[\frac{x_j}{x_i}]$ as subrings of B.

ii) Glue $(n+1)$ copies of $\mathbb{A}^n_A = \operatorname{Spec}(A[y_1, \ldots, y_n])$ over the open sets $\operatorname{Spec}(B_i[\frac{x_i}{x_j}]) \subset \operatorname{Spec}(B_i) \cong \operatorname{Spec}(A[y_1, \ldots, y_n])$ over the identities $B_i[\frac{x_i}{x_j}] = B_j[\frac{x_j}{x_i}]$.

iii) Study the points of $\mathbb{P}^1_k$ for an algebraically closed field k.