D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
5th week

22. *Injective resolutions of groups and modules.* Let A be a ring and I be an A-module.

i) Show that I is injective if for any ideal $\mathfrak{a} \subset A$ the induced map

$$\mathrm{Hom}_A(A, I) \longrightarrow \mathrm{Hom}_A(\mathfrak{a}, I)$$

is surjective.

ii) Show that any divisible group G (i.e. $g \longmapsto ng$ is surjective for all $n > 0$) is an injective object in Ab. In particular, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

iii) Show that $I(G) = \prod_{J(G)} \mathbb{Q}/\mathbb{Z}$ is a divisible (and hence injective) group. Here, the index set $J(G)$ is the set $\mathrm{Hom}_{Ab}(G, \mathbb{Q}/\mathbb{Z})$.

iv) Show that the natural map $G \longrightarrow I(G)$, $g \longmapsto (f(g))_f$ is injective. (Pick for $g \in G$ a non-trivial homomorphism $\langle g \rangle \longrightarrow \mathbb{Q}/\mathbb{Z}$ and use the injectivity of $\mathbb{Q}/\mathbb{Z}$ to extend it to a homomorphism $G \longrightarrow \mathbb{Q}/\mathbb{Z}$.) Conclude that the category Ab of abelian groups has enough injectives.

v) Check that the same argument shows that $Mod(A)$ has enough injectives for any ring.

23. *Injective objects in $Sh_{Ab}(X)$ and $Mod(X, \mathcal{O}_X)$.* Use the previous exercise to show that both categories have enough injectives. *Hint*: First, embed a sheaf $\mathcal{F}$ into $\prod_{x \in X} i_{x*}\mathcal{F}_x$ (where $i_x : \{x\} \hookrightarrow X$ is the inclusion) and use then the injective resolution for each $\mathcal{F}_x$.

24. *Splitting off cohomology.* Let $\mathcal{A}$ be an abelian category (think of Ab or $Sh_{Ab}(X)$) and let $A^\bullet$ be a complex with $H^i(A^\bullet) = 0$ for $i > k$. Show that there exists a distinguished triangle in the derived category $D(\mathcal{A})$

$$B^\bullet \longrightarrow A^\bullet \longrightarrow H^k(A^\bullet)[-k] \longrightarrow B^\bullet[1]$$

such that the induced maps $H^i(B^\bullet) \longrightarrow H^i(A^\bullet)$ are isomorphisms for all $i < k$ and $H^i(B^\bullet) = 0$ for all $i \geq k$.

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25. *Uniqueness of injective resolutions.* Let $\mathcal{A}$ be an abelian category and $f : A \longrightarrow B$ be a morphism in $\mathcal{A}$. Let $0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \ldots$ be an injective resolution and $0 \longrightarrow A \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \ldots$ be an arbitrary resolution. Show that there exists a morphism of complexes $F : I^\bullet \longrightarrow M^\bullet$ extending f. Moreover, prove that F is unique up to homotopy.

26. *Distinguished open sets.* Recall that the open sets $D(f) \subset \mathrm{Spec}(A)$ of all prime ideals not containing $f \in A$ form a basis of the topology. Define similar sets X_f for any scheme X and any $f \in \Gamma(X, \mathcal{O}_X)$. More precisely, let $X_f \subset X$ be the set of points $x \in X$ such that the stalk $f_x \in \mathcal{O}_{X,x}$ is not contained in the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ (or, equivalently, that the image of f in the residue field $k(x)$ is non-trivial). Prove that X_f is an open subset. (*Warning*: For general schemes X the ring $\Gamma(X, \mathcal{O}_X)$ is too small for the sets X_f to form a basis of the topology.)