D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
4th week

17. *Morphisms to affine schemes.* If $(f, f^\sharp) : X \longrightarrow \mathrm{Spec}(A)$ is a morphism of schemes, then taking global sections of $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \longrightarrow f_*\mathcal{O}_X$ yields a homomorphism of rings $A \longrightarrow \Gamma(X, \mathcal{O}_X)$. Show that this map

$$\mathrm{Hom}_{Sch}(X, \mathrm{Spec}(A)) \longrightarrow \mathrm{Hom}_{Ring}(A, \Gamma(X, \mathcal{O}_X))$$

is bijective.

18. *Flasque sheaves.*

i) Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be an exact sequence with $\mathcal{F}$ and $\mathcal{F}'$ flasque. Show that $\mathcal{F}''$ is flasque, too.

ii) Show that for any sheaf $\mathcal{F}$ the sheaf $\mathcal{G}$ given by

$$\mathcal{G}(U) := \{s : U \longrightarrow \bigcup_{x \in U} \mathcal{F}_x \mid s(x) \in \mathcal{F}_x\}$$

is flasque and that the natural morphism $\mathcal{F} \longrightarrow \mathcal{G}$ is injective. (This is the beginning of the natural *flasque resolution* of the sheaf $\mathcal{F}$.)

19. *Abelian categories.* Recall the abstract notion of an abelian category.

i) Show that the categories of abelian groups Ab and of sheaves of abelian groups $Sh_{Ab}(X)$ are abelian categories.

ii) Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf $\mathcal{F}$ of abelian groups is a *a sheaf of $\mathcal{O}_X$-modules*, if for all open subsets $U \subset X$ the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module such that the restriction maps are compatible with these module structures. Show that the category $Mod(X, \mathcal{O}_X)$ of $\mathcal{O}_X$-modules is an abelian category.

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20. *Extending a sheaf by zero.* Consider the inclusion of an open subset $j : U \hookrightarrow X$ and a sheaf $\mathcal{F}$ on U. Then $j_!\mathcal{F}$ is defined as the sheaf on X that associates $\mathcal{F}(V)$ to an open subset $V \subset X$ if $V \subset U$ and 0 otherwise.

i) Show that there is natural injection of sheaves $j_!(\mathcal{F}|_U) \longrightarrow \mathcal{F}$ for all sheaves $\mathcal{F}$ on X.

ii) Show that there is a natural bijection $\mathrm{Hom}(j_!\mathbb{Z}_U, \mathcal{G}) = \mathcal{G}(U)$ for all sheaves $\mathcal{G}$. Here, $\mathbb{Z}_U$ is the constant sheaf on U associated to $\mathbb{Z}$.

iii) State and prove an analogous statement for ringed spaces $(X, \mathcal{O}_X)$, i.e. replace $\mathbb{Z}_U$ by $\mathcal{O}_U = \mathcal{O}_X|_U$ and consider sheaves $\mathcal{G}$ of $\mathcal{O}_X$-modules.

iv) Show that a sheaf $\mathcal{F}$ that is an injective object in $Sh_{Ab}(X)$ is flasque.

21. *Quasi-isomorphisms.* Find explicit examples of:

i) quasi-isomorphic complexes (e.g. in $\mathrm{Kom}(Ab)$) that are not isomorphic;

ii) of a quasi-isomorphism that has no inverse in the homotopy category $\mathrm{K}(\mathcal{A})$;

iii) of a complex morphism $f : A^\bullet \longrightarrow B^\bullet$ with $H^i(f) = 0$ for all i but without f being homotopic to zero.