D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
3rd week

12. *Sheaf image.* Consider a morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ of abelian groups.

i) Find an example that shows that the presheaf image $\mathrm{Im}^{pr}(\varphi)$ does not in general coincide with the sheaf image $\mathrm{Im}(\varphi)$.

ii) Show that $(\mathrm{Im}^{pr}(\varphi))_x = (\mathrm{Im}(\varphi))_x \cong \mathrm{Im}(\varphi_x)$ for all $x \in X$.

iii) Show that $\mathrm{Im}(\varphi)$ is naturally a subsheaf of $\mathcal{G}$, i.e. $\mathrm{Im}(\varphi)(U) \subset \mathcal{G}(U)$ for all open $U \subset X$.

iv) Show that $\mathrm{Im}(\varphi) \cong \mathcal{F}/\mathrm{Ker}(\varphi)$ (sheaf quotient).

v) Show that $\mathrm{Im}(\varphi)$ has the categorical property of an image in $Sh_{Ab}(X)$, i.e.

$$\mathrm{Im}(\varphi) \cong \mathrm{Ker}(\mathcal{G} \longrightarrow \mathrm{Coker}(\varphi)).$$

In other words, $\mathrm{Coker}(\varphi) \cong \mathcal{G}/\mathrm{Im}(\varphi)$.

13. *Sheaf $\mathcal{H}om$.* For two sheaves $\mathcal{F}$, $\mathcal{G}$ of abelian groups on X and any open set $U \subset X$ the set of morphisms $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is an abelian group. Show that

$$U \longmapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

defines a sheaf of abelian groups on X. It will be denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

14. *Direct and inverse image.* Complete the proof that for a continuous map $f : X \longrightarrow Y$ the two functors $f_* : Sh(X) \longrightarrow Sh(Y)$ and $f^{-1} : Sh(Y) \longrightarrow Sh(X)$ are adjoint to each other $f^{-1} \dashv f_*$. More precisely:

i) There exist isomorphisms $\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{G})$ which are functorial in $\mathcal{G} \in Sh(Y)$ and $\mathcal{F} \in Sh(X)$.

ii) Use i) to obtain natural morphisms $\mathcal{G} \longrightarrow f_* f^{-1}\mathcal{G}$ and $f^{-1} f_* \mathcal{F} \longrightarrow \mathcal{F}$ and describe them explicitly.

Verify also that for the composition of two continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ one has $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

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15. *Rational points.* Let $(X, \mathcal{O}_X)$ be a scheme (for simplicity you may assume that it is an affine scheme) and $x \in X$ with residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.

i) Show that for a field K to give a morphism of schemes $\mathrm{Spec}(K) \longrightarrow (X, \mathcal{O}_X)$ with image x is equivalent to give a field inclusion $k(x) \hookrightarrow K$.

ii) If $(X, \mathcal{O}_X)$ is a k-scheme for some field k, i.e. a morphism of schemes

$$(X, \mathcal{O}_X) \longrightarrow \mathrm{Spec}(k)$$

is fixed, then every residue field $k(x)$ is naturally a field extension $k \subset k(x)$. A point $x \in X$ is *rational* if this extension is bijective, i.e. $k = k(x)$. The *set of rational points* is denoted by $X(k)$ and can by i) also be described as the set of k-morphisms $\mathrm{Spec}(k) \longrightarrow (X, \mathcal{O}_X)$, i.e. morphisms such that the composition

$$\mathrm{Spec}(k) \longrightarrow (X, \mathcal{O}_X) \longrightarrow \mathrm{Spec}(k)$$

is the identity of sheaves.

16. *Zariski tangent space.* Let $(X, \mathcal{O}_X)$ be a scheme. For a point $x \in X$ the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$ is considered as vector space over the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. The Zariski tangent space T_x of X at $x \in X$ is defined as the dual of this vector space, i.e.

$$T_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

Assume $(X, \mathcal{O}_X)$ is a k-scheme (see Exercise 15) and denote the *ring of dual numbers* $k[T]/(T^2)$ by $k[\varepsilon]$.

Show that to give a k-morphism $\mathrm{Spec}(k[\varepsilon]) \longrightarrow (X, \mathcal{O}_X)$ is equivalent to give a rational point $x \in X$ (see Exercise 15) and an element $v \in T_x$. (For simplicity you may again assume that X is an affine scheme.)

Tutorials:

– Tue 4-6 pm - Seminarraum 0.007 (Galinat).

– Wed 4-6 pm - Seminarraum 0.008 (Neupert).

– Thu 8-10 am - Seminarraum 0.003 (Kulke). **NEW!**