D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
2nd week

6. *Sheafification of injective morphisms.* Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves on X such that $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective for all open sets $U \subset X$. Using the universality of the sheafification $\mathcal{F} \longrightarrow \mathcal{F}^\dagger$, the composition of φ with the natural morphism $\mathcal{G} \longrightarrow \mathcal{G}^\dagger$ induces a natural morphism of sheaves $\varphi^\dagger : \mathcal{F}^\dagger \longrightarrow \mathcal{G}^\dagger$. Show that this morphism yields injective maps $\varphi_U^\dagger : \mathcal{F}^\dagger(U) \longrightarrow \mathcal{G}^\dagger(U)$ for all open subsets $U \subset X$. Is the same true for 'injective' replaced by 'surjective'?

7. *Locally constant sheaves.* A sheaf $\mathcal{F}$ on a topological space X is called *locally constant* if every point $x \in X$ admits an open neighbourhood $x \in U \subset X$ such that the restriction $\mathcal{F}|_U$ is a constant sheaf.
Show that for an irreducible topological space X (i.e. all open sets are connected) the following conditions are equivalent: (i) $\mathcal{F}$ is a constant sheaf; (ii) $\mathcal{F}$ is a locally constant sheaf; (iii) For all non-empty open subsets $U \subset X$ the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(U)$ is bijective.

8. *Direct sum.* Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of abelian groups on a topological space X. The direct sum $\mathcal{F} \oplus \mathcal{G}$ is the presheaf $U \longmapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$.
i) Show that this defines a direct sum and a direct product in the category $Sh_{Ab}(X)$ of presheaves of abelian groups on X.
ii) Show that $\mathcal{F} \oplus \mathcal{G}$ is a sheaf if $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

9. *Subsheaf with support.* Let $Z \subset X$ be a closed subset. For any sheaf $\mathcal{F}$ of abelian groups on X one defines for any open $U \subset X$ the subgroup $\Gamma_{Z \cap U}(U, \mathcal{F})$ of all sections $s \in \Gamma(U, \mathcal{F})$ with $\mathrm{supp}(s) \subset Z \cap U$.
Show that this defines a sheaf (which is denoted by $\mathcal{H}^0_Z(\mathcal{F})$).
A sheaf $\mathcal{F}$ is said to be *supported on* Z if $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{F}$.

10. *Sheaves of modules on* $\mathrm{Spec}(A)$. Recall the definitions of $\mathrm{Spec}(A)$ with its Zariski topology and of the localizations A_f, $A_\mathfrak{p}$ as used in the first lecture. As shown there, $\mathrm{Spec}(A)$ comes with a natural sheaf (of commutative rings): The *sheaf of regular functions* $\mathcal{O}$.
(i) Imitate the construction of $\mathcal{O}$ to associate to any A-module M a sheaf $\tilde{M}$ on $\mathrm{Spec}(A)$. For this you will need to recall the localization of a module $M_\mathfrak{p}$.
(ii) After the lecture on Friday April 13 you should be able to show that there are natural isomorphisms $(\tilde{M})_\mathfrak{p} \cong M_\mathfrak{p}$, $\Gamma(D(f), \tilde{M}) \cong M_f$, and $\Gamma(\mathrm{Spec}(A), \tilde{M}) \cong M$.

Continued on next page.

The last exercise is not necessary for the understanding of the lectures at this point.

11. *Grothendieck topology.* The notion of a (pre)sheaf on a topological space can be formalized as follows.
A Grothendieck topology $(\mathcal{C}, \mathrm{Cov}_{\mathcal{C}})$ consists of a category $\mathcal{C}$ with a set $\mathrm{Cov}_{\mathcal{C}}$ of collections $\{\pi_i : U_i \longrightarrow U\}_i$ of morphisms in $\mathcal{C}$ (called *coverings* of U) subject to the following conditions:
(1) Any isomorphism $\varphi : U \xrightarrow{\sim} U$ defines a covering $\{U \longrightarrow U\} \in \mathrm{Cov}_{\mathcal{C}}$.
(2) Suppose we are given $\{\pi_i : U_i \longrightarrow U\}_i \in \mathrm{Cov}_{\mathcal{C}}$ and for each i a covering $\{\pi_{ij} : U_{ij} \longrightarrow U_i\}_j \in \mathrm{Cov}_{\mathcal{C}}$. Then $\{\pi_i \circ \pi_{ij} : U_{ij} \longrightarrow U\}_{ij} \in \mathrm{Cov}_{\mathcal{C}}$ is a covering.
(3) If $\{\pi_i : U_i \longrightarrow U\}_i$ is a covering and $V \longrightarrow U$ is a morphism in $\mathcal{C}$, then $\{\tilde{\pi}_i : U_i \times_U V \longrightarrow V\}_i$ is a covering.
(In particular, one assumes that the fibre products in (2) and (3) exist. Recall the abstract notion of a fibre product.)

(i) Show that for a topological space X the category of open sets Ouv_X comes with a natural Grothendieck topology given by the usual open coverings $U = \bigcup U_i$. Show that the notions presheaf, sheaf, stalk, morphism of (pre)sheaves, etc., can be phrased entirely in terms of this Grothendieck topology.

Here is another example of a Grothendieck topology: For a finite group G consider the category G-*Sets* of sets S with a left G-action $G \times S \longrightarrow S$. Morphisms in this category are maps that commute with the G-action.
(ii) Show that the collections of $\{S_i \longrightarrow S\}_i$ with $\bigcup S_i \longrightarrow S$ surjective define a Grothendieck topology on G-*Sets*.

The group G itself comes with a natural left G-action (by multiplication). The corresponding object is denoted $\langle G \rangle \in G$-*Sets*.
(iii) Show that any sheaf $\mathcal{F}$ on G-*Sets* yields a set $\mathcal{F}(\langle G \rangle)$ that is endowed with a natural left G-action. (In fact, $\mathcal{F}$ is determined by this set $\mathcal{F}(S) = \mathrm{Hom}_G(S, \mathcal{F}(\langle G \rangle))$.)

The final object in G-*Sets* consists of a set $\{*\}$ of one element.
(iv) One can show that for a sheaf $\mathcal{F}$ the space of sections $\mathcal{F}(\{*\})$ is the fixed point set $\mathcal{F}(\langle G \rangle)^G$.

The lecture on Fridays will henceforth take place in the Großer Hörsaal.