D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
13th week

62. *Divisors on* $\mathbb{P}^n_k$. Use the isomorphism $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ (for X a factorial, noetherian, separated, integral scheme) and the fact that $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$ generated by a hyperplane $H \cong \mathbb{P}^{n-1}_k \subset \mathbb{P}^n_k$ to show that $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ generated by $\mathcal{O}(1)$. In particular, show that under the isomorphism H is mapped to $\mathcal{O}(1)$ (and not e.g. to $\mathcal{O}(-1)$).

63. *Veronese surface in* $\mathbb{P}^4_k$. The complete linear system $|\mathcal{O}(2)|$ on $\mathbb{P}^2_k$ defines the Veronese embedding $\mathbb{P}^2_k \hookrightarrow \mathbb{P}^5_k$. Show that the linear system given by $\langle x_0^2, x_1^2, x_2^2, x_1(x_0 - x_2), x_2(x_0 - x_1)\rangle \subset H^0(\mathbb{P}^2_k, \mathcal{O}(2))$ defines a closed embedding

$$\mathbb{P}^2_k \hookrightarrow \mathbb{P}^4_k,$$

which is called the Veronese surface in $\mathbb{P}^4$.

65. *Maximum principle in algebraic geometry.* Let X be an integral projective (proper is enough) scheme over an algebraically closed field k. The aim of this exercise is to prove that then $H^0(X, \mathcal{O}_X) \cong k$.

i) Recall that on any scheme X over an arbitrary field k giving a section $s \in H^0(X, \mathcal{O}_X)$ is equivalent to giving a morphism of k-schemes $X \longrightarrow \mathbb{A}^1_k$. If X is proper over k, then the image of $X \longrightarrow \mathbb{A}^1_k$ is closed.

ii) Suppose that X is integral and proper over k. Show that any $s \in H^0(X, \mathcal{O}_X)$ is either zero or invertible.

iii) If now k is algebraically closed and X integral and projective over k, then $H^0(X, \mathcal{O}_X) = k$. (Use that $H^0(X, \mathcal{O}_X)$ is a finite dimensional k-vector space.)

64. *Trivial and torsion invertible sheaves.* Let X be an integral projective scheme over an algebraically closed field k.

i) Assume $H^0(X, \mathcal{L}) \neq 0$ and $H^0(X, \mathcal{L}^*) \neq 0$ for some invertible sheaf $\mathcal{L}$. Show that then $\mathcal{L} \cong \mathcal{O}_X$.

ii) Let $\mathcal{L} \in \mathrm{Pic}(X)$ of order n. Show $H^0(X, \mathcal{L}^m) = k$ for $n|m$ and $= 0$ otherwise.

66. *Base locus.* Let X be a projective integral scheme over $k = \bar{k}$. A point $x \in X$ is a base point of a linear system $\mathbb{P}(V) \subset |\mathcal{L}|$ if $s_x \in \mathfrak{m}_x \mathcal{L}$ for all $s \in V$. Thus, $\mathcal{L}$ is globally generated if and only if $|\mathcal{L}|$ has no base points.

i) Prove that the base locus $\mathrm{Bs} \subset X$, i.e. the set of all base points, is closed.

ii) Show that for any $\mathcal{L}$ there exists an effective (Cartier) divisor D such that the base locus of the complete linear system given by $\mathcal{L}(-D) := \mathcal{L} \otimes \mathcal{O}(-D)$ is of codimension ≥ 2.

Final examen: July 13th (Friday). Kleiner Hörsaal 8-10 am.
and again: August 31st (Friday).