D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
12th week

57. *Ample invertible sheaves.* Let X be a noetherian scheme.

i) Show that if $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves on X such that $\mathcal{L}$ is ample, then $\mathcal{L}^n \otimes \mathcal{M}$ is ample for $n \gg 0$. Conclude that any invertible sheaf $\mathcal{M}$ is isomorphic to some $\mathcal{L}_1 \otimes \mathcal{L}_2^*$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ ample.

ii) Is the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ of two ample (resp. very ample) invertible sheaves again ample (resp. very ample)?

58. *Invertible sheaves via cocycles.* Let $X = \bigcup U_i$ be an open covering of a scheme (or, more generally, a ringed space). Suppose $\mathcal{L} \in \mathrm{Pic}(X)$ and $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$. Consider the associated cocycle

$$\{\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} \in H^0(U_{ij}, \mathcal{O}_X^*)\} \in \check{H}^1(X = \bigcup U_i, \mathcal{O}_X^*).$$

i) Determine the cocycle for the dual invertible sheaf $\mathcal{L}^*$.

ii) A global section $s \in H^0(X, \mathcal{L})$ determines sections $\varphi_i(s|_{U_i}) \in H^0(U_i, \mathcal{O}_X)$. Show that they satisfy a cocycle relation with respect to φ_{ij} and prove, conversely, that local sections satisfying this relation can be glued to a global section.

iii) Use that $\check{H}^1(X, \mathcal{F}) \cong H^1(X, \mathcal{F})$ for sheaves of abelian groups (on an arbitrary topological space X) to prove that there exist natural isomorphisms

$$\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*) \cong H^1(X, \mathcal{O}_X^*).$$

59. *Projection from linear subspaces.* Consider the linear subspace $\mathbb{P}^{m-1} \cong V_+(x_0, \ldots, x_n) \subset \mathbb{P}^{n+m}$. Describe the linear projection from the linear space $\mathbb{P}^{m-1}$ to $\mathbb{P}^n$ in terms of an invertible sheaf $\mathcal{L}$ on $X = \mathbb{P}^{n+m}$ and sections $s_0, \ldots, s_n$ of $\mathcal{L}$.

60. *Morphisms from projective spaces.* Let $\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$ be a morphism.

i) Show that $\varphi(\mathbb{P}^n_k)$ is a point or that φ is a finite morphism.

ii) Give an example of a finite morphism $\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$ which is not constant and has fibres that contain more than one point.

iii) Give an example of a morphism $\varphi : \mathbb{P}^n_k \longrightarrow \mathbb{P}^m_k$ that is injective but not a closed immersion.

61. *Ample invertible sheaves on the quadric.* Consider the quadric $Q = \mathbb{P}^1_k \times \mathbb{P}^1_k$ and use that $\mathrm{Pic}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$, i.e. every invertible sheaf on Q is isomorphic to a unique $\mathcal{O}(a,b) := p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$.

i) Determine all ample invertible sheaves on Q. Are they all very ample?

ii) Compute the cohomology groups $H^1(Q, \mathcal{O}(a,b))$.

Final Examen: July 13th (Friday). Kleiner Hörsaal 8-10 am.