D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
11th week

53. *Additivity of the Euler characteristic.* Let X be a projective scheme over a field k. For any coherent sheaf $\mathcal{F}$ on X one defines the Euler characteristic

$$\chi(\mathcal{F}) := \sum(-1)^i \dim_k H^i(X, \mathcal{F}).$$

Show that $\chi(\ )$ is additive for short exact sequences, i.e. for any short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of coherent sheaves one has $\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$.

In other words, $\chi(\ )$ defines additive homomorphism

$$\chi : K(X) \longrightarrow \mathbb{Z}$$

from the Grothendieck group $K(X) = K(\mathrm{Coh}(X))$.

54. *Cohomology of hypersurfaces.* Let $X \subset \mathbb{P}^n_k$, $n > 1$, be a hypersurface, i.e. X is the closed subscheme defined by a homogenous polynomial $0 \neq f \in k[x_0, \ldots, x_n]$. Prove the following assertions:

i) The restriction map $H^0(\mathbb{P}^n_k, \mathcal{O}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d))$ is surjective for all d.

ii) X is connected.

iii) $H^i(X, \mathcal{O}_X(d)) = 0$ for all $0 < i < n-1$ and all d.

(Similar results hold for complete intersections.)

55. *Arithmetic genus.* For a projective scheme X of dimension d over a field k the arithmetic genus is defined as

$$p_a(X) := (-1)^d(\chi(\mathcal{O}_X) - 1).$$

i) Compute $p_a(\mathbb{P}^n_k)$.

ii) Show that the arithmetic genus of a plane curve $C \subset \mathbb{P}^2_k$ defined by a polynomial of degree d is $(d-1)(d-2)/2$.

iii) Conclude that a plane curve $C \subset \mathbb{P}^2_k$ of degree $d \neq 1, 2$ cannot be isomorphic to $\mathbb{P}^1_k$.

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56. *Hilbert polynomial.* This exercise requires familiarity with Hilbert polynomials of graded modules as in my class on commutative algebra last term or in [Hartshorne, Chapter I.7] or in ....

Let X be a projective scheme over a field k and let $\mathcal{F}$ be a coherent sheaf on X. Depending on a very ample line bundle $\mathcal{O}_X(1)$ on X, one defines the Hilbert polynomial $P(X, \mathcal{F})(n) = \chi(\mathcal{F}(n))$.

i) More precisely: Show that there exists a numerical polynomial $P(X, \mathcal{F}) \in \mathbb{Q}[z]$ such that $P(X, \mathcal{F})(n) = \chi(\mathcal{F}(n))$ for all $n \in \mathbb{Z}$. (Use induction over the dimension of the support of $\mathcal{F}$.)

ii) Show that for $\mathcal{F}$ on $X = \mathbb{P}^n_k$ the Hilbert polynomial $P(X, \mathcal{F})$ equals the Hilbert polynomial of the graded module $\Gamma_* \mathcal{F}$ over $k[x_0, \ldots, x_n]$.