D. Huybrechts Summer 2012

**Exercises Algebraic Geometry I**
**10th week**

**49.** *Quasi-coherence under sheaf operations.* Prove (at least some of) the following assertions.

i) For (quasi-)coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a scheme $(X, \mathcal{O}_X)$ the sheaf $\mathcal{F} \otimes \mathcal{G}$ is (quasi-)coherent. (*Warning*: One has to be careful with $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ for quasi-coherent sheaves.)

ii) If $\mathcal{F}$ is a quasi-coherent sheaf on a scheme $X$ and $f : Y \longrightarrow X$ is a morphism of schemes, then the pull-back $f^*\mathcal{F}$ is again quasi-coherent.

iii) Prove ii) for 'quasi-coherent' replaced by 'coherent' for $X$ and $Y$ locally Noetherian. (If $A \longrightarrow B$ is a ring homomorphism and $M$ is an $A$-module, then $M$ finitely generated implies $M \otimes_A B$ finitely generated, but $M$ coherent does not imply $M \otimes_A B$ coherent.)

iv) Let $f : X \longrightarrow Y$ be a morphism. Assume that $f$ is quasi-compact and separated or that $X$ is Noetherian. Then $f_*\mathcal{F}$ of a quasi-coherent sheaf $\mathcal{F}$ is again quasi-coherent.

v) Give an example that iv) does not hold for 'quasi-coherent' replaced by 'coherent'.

**50.** *Restriction of sections on affine schemes.* Let $X$ be an affine scheme $\mathrm{Spec}(A)$ and $\mathcal{F}$ a quasi-coherent sheaf on $X$. Use the equivalence $\mathrm{Qcoh}(X) \cong A$-mod to prove the following assertions.

i) If for $s \in \Gamma(X, \mathcal{F})$ and $a \in A$ one knows that $s|_{D(a)} \in \Gamma(D(a), \mathcal{F})$ is trivial, then there exists $n > 0$ such that $a^n \cdot s = 0$ in $\Gamma(X, \mathcal{F})$.

ii) Let $t \in \Gamma(D(a), \mathcal{F})$. Then there exists $n > 0$ and $s \in \Gamma(X, \mathcal{F})$ such that $s|_{D(a)} = a^n \cdot t$ in $\Gamma(D(a), \mathcal{F})$.

**51.** *Products of* Proj. Let $B = \bigoplus_{n \geq 0} B_n$ and $C = \bigoplus_{n \geq 0} C_n$ be two graded rings with $A := B_0 \cong C_0$. Consider $B \times_A C := \bigoplus_{n \geq 0} B_n \otimes_A C_n$ and the schemes $X := \mathrm{Proj}(B)$ and $Y := \mathrm{Proj}(C)$.

i) Show that $X \times_{\mathrm{Spec}(A)} Y \cong \mathrm{Proj}(B \times_A C)$ and that

ii) under this isomorphism $\mathcal{O}(1)$ on $\mathrm{Proj}(B \times_A C)$ is isomorphic to $p_1^*\mathcal{O}_X(1) \otimes p_2^*\mathcal{O}_Y(1)$, where $p_1$ and $p_2$ are the two projections from $X \times_{\mathrm{Spec}(A)} Y$.

**52.** *Segre embedding (again).* Consider in the previous exercise $B = k[x_0, \ldots, x_n]$ and $C = k[y_0, \ldots, y_m]$.

i) Show that $B \times_A C$ is naturally a graded quotient of $k[z_0, \ldots, z_N]$ with $N = mn + m + n$ and use this to

ii) describe the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$.