D. Huybrechts Summer 2012

Exercises Algebraic Geometry I
1st week

1. *Basis of topology.* Let X be a topological space and let $\mathcal{B}$ be a basis of open sets. Show that for any sheaf $\mathcal{F}$ on X and any open set $V \subset X$ the space of sections $\mathcal{F}(V)$ can naturally be identified with

$$\{(s_U) \in \prod_{U \in \mathcal{B}, U \subset V} \mathcal{F}(U) \mid s_{U_1}|_{U_2} = s_{U_2}\},$$

which can also be written as the inverse limit

$$\varprojlim_{V \supset U \in \mathcal{B}} \mathcal{F}(U).$$

Recall the universal property of the inverse limit and the fact that it is only left exact in general.

2. *Support of a section.* Let $\mathcal{F}$ be a sheaf on a topological space X and let $s, t \in \mathcal{F}(U)$ be two sections over an open set $U \subset X$. Show that the set of points $x \in U$ with $s_x = t_x$ in $\mathcal{F}_x$ is an open subset of U.

If $\mathcal{F}$ is a sheaf of abelian groups, one defines the *support* $\mathrm{supp}(s)$ of a section $s \in \mathcal{F}(U)$ as the set of points $x \in U$ such that $0 \neq s_x \in \mathcal{F}_x$. Show that $\mathrm{supp}(s)$ is a closed subset of U.

3. *Espace étale of a presheaf.* Let $\mathcal{F}$ be a presheaf on a topological space X. Define

$$|\mathcal{F}| := \bigsqcup_{x \in X} \mathcal{F}_x,$$

which comes with a natural projection $\pi : |\mathcal{F}| \longrightarrow X$, $(s \in \mathcal{F}_x) \longmapsto x$. Then any $s \in \mathcal{F}(U)$ defines a section of π over U by $x \longmapsto s_x$. One endows $|\mathcal{F}|$ with the strongest topology such that all $s \in \mathcal{F}(U)$ define continuous sections $x \longmapsto s_x$. Show that the sheafification $\mathcal{F}^\dagger$ can be described as the sheaf of continuous sections of $|\mathcal{F}| \longrightarrow X$.

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4. *Gluing of sheaves.* Let X be a topological space and let $X = \bigcup U_i$ be an open covering. We use the shorthand $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$.

Consider sheaves $\mathcal{F}_i$ on U_i and *gluings* $\varphi_{ij} : \mathcal{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_i|_{U_{ij}}$. Show that if the *cocycle condition* $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on U_{ijk} is satisified, then there exists a sheaf $\mathcal{F}$ on X together with isomorphisms $\varphi_i : \mathcal{F}|_{U_i} \cong \mathcal{F}_i$ such that $\varphi_{ij} \circ \varphi_i = \varphi_j$ on U_{ij}. The $(\mathcal{F}, \varphi_i)$ is unique up to unique isomorphism.

5. *Exponential map.* Consider $X = \mathbb{C} \setminus \{0\}$ with its usual topology and let $\mathcal{O}_X$ be the sheaf of holomorphic functions, i.e. $\mathcal{O}_X(U) = \{f : U \longrightarrow \mathbb{C} \mid \text{holomorphic}\}$. Similarly, let $\mathcal{O}_X^*$ be the sheaf of holomorphic functions without zeroes. (Throughout, you may work with differentiable function instead of holomorphic ones if you prefer.)

Show that the exponential map defines a morphism of sheaves (of abelian groups)

$$\exp : \mathcal{O}_X \longrightarrow \mathcal{O}_X^*, f \in \mathcal{O}_X(U) \longmapsto \exp(f) \in \mathcal{O}_X^*(U).$$

Find a basis $\mathcal{B}$ of the topology such that $\exp_U : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X^*(U)$ is surjective for all $U \in \mathcal{B}$. Note that $\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X^*(X)$ is not surjective and compare this with the fact that $\varprojlim$ is only left exact in general. Describe the kernel of $\exp_U$.