Non-vanishing theorem (smooth version)

Let X be smooth projective, D nef divisor, A a $\mathbb{Q}$-divisor with $\{A\}$ nc and $\ulcorner A \urcorner \geq 0$.

If $aD + A - K_X$ is nef and big for some $a > 0$, then
$$H^0(X, mD + \ulcorner A \urcorner) \neq 0 \quad \text{for } m \gg 0$$

Proof. For simplicity $aD + A - K_X$ ample.
(The result has been used only in this case.)

1st step: Suppose D is numerically trivial.

Since Kleiman is a numerical criterion:
$A - K_X$ is ample and by the KV-vanishing (version 2b): $H^i(X, \ulcorner A \urcorner) = 0 \quad i > 0$.

Similarly, $mD + A - K_X$ ample for all m and again by KV-vanishing: $H^i(X, mD + \ulcorner A \urcorner) = 0$ for $i > 0$.

Thus,
$$h^0(X, mD + \ulcorner A \urcorner) = \chi(X, mD + \ulcorner A \urcorner) \overset{(*)}{=} \chi(X, \ulcorner A \urcorner) = h^0(X, \ulcorner A \urcorner) \neq 0,$$
since $\ulcorner A \urcorner$ is effective.

(In $(*)$ we use that D is numerically trivial.)

From now on: $D \not\equiv_{num} 0$ (not numerically trivial)

2nd step: Claim: Choose $\alpha \in \mathbb{N}$ with αA integral

Then there exists $q \in \mathbb{N}$ s.t.

$$h^0(X, \alpha k(qD + A - K_X)) > \frac{\alpha^d}{d!}(d+1)^d k^d \quad \text{for } k \gg 0$$

Proof: Use $H := aD + A - K_X$ ample. Then $H^i . D^{d-i} \geq 0 \ \forall i$

(D is nef and hence in the closure of the ample cone and

$H^i H_0^{d-i} > 0$ for H, H_0 ample.)

Moreover, since $D \underset{\text{num}}{\not\sim} 0$ one has $H^{d-1} . D > 0$.

(Either by Hodge index: $H^{d-1} . D = 0 \Rightarrow D$ "primitive"

$\Rightarrow H^{d-2} . D^2 \leq 0$ and "$= 0$" iff $D \underset{\text{num}}{\sim} 0$.

or by using that H is an element in the ample cone, which is open and the hyperplanes "$H_0^{d-1} = 0$" sweep out an open subset of $N_1(X)$.)

In particular, one finds $r \gg 0$ s.t.

$$(rD + aD + A - K_X)^d = (rD + H)^d > (d+1)^d$$

Write $q = r + a$. Then Serre vanishing yields ($qD + A - K_X$ is again ample!)

$$h^0(X, \alpha k(qD + A - K_X)) = \chi(X, \alpha k(qD + A - K_X)) \quad k \gg 0$$

$$= \frac{\alpha^d k^d}{d!}(qD + A - K_X)^d + \text{lower order terms in } k$$

$$> \frac{\alpha^d k^d}{d!}(d+1)^d \quad \text{for } k \gg 0.$$

<u>3rd step</u>: Let q be as above and pick $x \in X$. $k \gg 0$

<u>Claim</u>: $\exists\, 0 \neq s \in H^0(X, \alpha k(qD + A - K_X))$ with $\mathrm{mult}_x(s) \geq \alpha k(d+1)$, i.e. $s_x \in \mathfrak{m}_x^{\alpha k(d+1)}$

<u>Proof</u>: $0 \to \mathfrak{m}_x^{\alpha k(d+1)} \to \mathcal{O}_X \to \mathbb{C}_x^N \to 0$

with $N = \binom{\alpha k(d+1) + d - 1}{d}$ = # monomials of degree $< \alpha k(d+1)$

(Recall: $\dim \mathbb{C}[z_1, \dots z_d]_k = \binom{d-1+k}{k} = \binom{d-1+k}{d-1}$

$\Rightarrow \dim \mathbb{C}[z_1, \dots z_d]_{<k} = \sum_{i=0}^{k-1} \binom{d-1+i}{d-1} = \binom{d-1+k}{d}$.)

Now use long exact sequence

$0 \to H^0(X, \mathfrak{m}_x^{\alpha k(d+1)}(\alpha k(qD + A - K_X)))$

$\to H^0(X, \alpha k(qD + A - K_X)) \xrightarrow{\xi} \mathbb{C}^N$

Since $h^0(X, \alpha k(qD + A - K_X)) > \frac{\alpha^d k^d (d+1)^d}{d!}$ for $k \gg 0$

and $N = \frac{\alpha^d k^d (d+1)^d}{d!}$ + lower order terms,

the map ξ cannot be injective.

<u>4th step</u>. Choose $x \in X \setminus A \rightsquigarrow \mathrm{Bl}_x X \xrightarrow{f_1} X$

and $f_2 : Y \to \mathrm{Bl}_x X$ resolution of strict transform of $A + Z(s)$.

Let $f := f_1 \circ f_2 : Y \to X$

Then there exist nc divisor $\sum F_i$ s.t.

0) • $\{F_i\}$ = {exceptional divisors for $f: Y \to X$}

$\cup$ {strict transforms of components of A}

Wlog F_1 = strict transform of $f_1^{-1}(x)$

1) • $K_Y = f^* K_X + \sum a_i F_i$ with $a_i \geq 0$

(If F_i is f-exceptional, then $a_i > 0$)

2) • $f^*(\underbrace{q D + A - K_X}_{\text{ample}}) - (d+1) \sum \delta_i F_i$ $\quad 0 < \delta_i \ll 1 \in \mathbb{Q}$

is ample on Y

(For the exceptional F_i this is the usual argument and for the others, which would not be necessary, use openness.)

3) • $f^* Z(s) = \sum r_i F_i$, where s is as in 3.

$r_i \geq 0$

4) • Write $f^* A + \sum a_i F_i = \sum b_i F_i$. Then $b_i > -1$, for $\lceil A \rceil \geq 0$.

(There is something to prove here!)

Easy observations:

- $\mathrm{mult}_x(s) \geq \alpha k(d+1) \underset{3}{\Longrightarrow} r_1 \geq \alpha k(d+1)$.

- $x \notin A \underset{4)}{\Longrightarrow} a_1 = b_1$

- $K_{Bl_x X} = f_1^* K_X + (d-1)E \underset{1)}{\Longrightarrow} a_1 = b_1 = d-1$

Let $c := \min \left\{ \frac{b_i + 1 - \delta_i}{r_i} \,\middle|\, r_i \neq 0 \right\}$.

- By 4), 1): $b_i + 1 - \delta_i > 0 \quad \Longrightarrow c > 0$

$$\Longrightarrow c \leq \frac{b_1 + 1 - \delta_1}{r_1} = \frac{a_1 + 1 - \delta_1}{r_1} = \frac{(d-1)+1-\delta_1}{r_1} \lneqq \frac{d}{r_1} \leq \frac{d}{\alpha k(d+1)}$$

Wlog may assume c is only attained once:

$$c = \frac{b_0 + 1 - \delta_0}{r_0}$$

Set: $A' := \sum_{j \neq 0} (-c r_j + b_j - \delta_j) F_j$, $B := F_0$

$N := m f^* D + A' - B - K_Y$

$$N = m f^* D + A' - B - K_Y$$

$$= m f^* D + \sum_{j \neq 0} (-c r_j + b_j - \delta_j) F_j - F_0 - f^* K_X + \sum a_j F_j)$$

$$= m f^* D + \Big(-\sum_j c r_j F_j + c r_0 F_0\Big) + \Big(\sum_j b_j F_j - b_0 F_0\Big) + \Big(-\sum \delta_j F_j + \delta_0 F_0\Big) - F_0 - f^* K_X - \sum a_j F_j$$

$$= m f^* D + f^* A + \underbrace{(c r_0 - b_0 + \delta_0 - 1)}_{=0} F_0 - c \underbrace{f^* Z(s)}_{\alpha k (q f^* D + f^* A - f^* K_X)} - \sum \delta_j F_j - f^* K_X$$

$$= \underbrace{(m-q) f^* D}_{\substack{\text{nef} \\ m > q}} + \underbrace{\underbrace{(1-c\alpha k) q f^* D + (1-c\alpha k) f^* A - (1-c\alpha k) f^* K_X - \sum \delta_j F_j}_{(1-c\alpha k) f^*(qD + A - K_X) - \sum \delta_j F_j}}_{\text{ample}}, \quad \text{since} \quad 1 - c\alpha k > 1 - \frac{\alpha}{\alpha+1} = \frac{1}{\alpha+1} \quad (\text{Use } 2.)$$

(nef + ample = ample)

$\Rightarrow N$ is ample

Apply KV-vanishing (version 2.6):

$$H^1(Y, \ulcorner N \urcorner + K_Y) = 0 \text{ , i.e. } H^1(Y, mf^*D + \ulcorner A' \urcorner - B) = 0.$$

Using $0 \to \mathcal{O}(-B) \to \mathcal{O}_Y \to \mathcal{O}_B \to 0$ this implies

$$H^0(Y, mf^*D + \ulcorner A' \urcorner) \twoheadrightarrow H^0(B, (mf^*D + \ulcorner A' \urcorner)_{|B})$$

<u>5th step</u>

<u>Claim</u>: By induction hypothesis:

$$H^0(B, (mf^*D + \ulcorner A' \urcorner)_{|B}) \neq 0$$

<u>Proof</u>: Need to check the assumptions

- f^*D nef $\Rightarrow f^*D_{|B = F_0}$ nef
- $A' = \sum_{j \neq 0} c_j F_j$ nc $\Rightarrow A'_{|B}$ nc (and not only $\{A'_{|B}\}$ nc!)
- $\ulcorner A' \urcorner \geq 0$ by definition of c.
- $N = mf^*D + A' - B - K_Y$ ample $\Rightarrow N_{|B}$ ample.

Since $(-B - K_Y)_{|B} = K_B$, this shows $(mf^*D + A')_{|B} - K_B$ ample

$$\Rightarrow H^0(Y, mf^*D + \ulcorner A' \urcorner) \neq 0$$

<u>6th step</u>: <u>Claim</u>: $f^*\ulcorner A \urcorner + \sum a_j F_j \geq \ulcorner A' \urcorner$

Then

$$0 \neq H^0(Y, mf^*D + \ulcorner A' \urcorner) \subset H^0(Y, mf^*D + f^*\ulcorner A \urcorner + \sum a_j F_j)$$
$$= H^0(X, mD + \ulcorner A \urcorner) \text{ , for } a_j \geq 0.$$

Proof of Claim:

$$\lceil A \rceil = \sum_{j \neq 0} \lceil (-c r_j + b_j - \delta_j) \rceil F_j$$

$$\leq \sum_{j \neq 0} \lceil b_j \rceil F_j \qquad \text{since } -c r_j - \delta_j < 0$$

$$\leq \sum \lceil b_j \rceil F_j \qquad \text{since } b_0 > -1$$

$$= f^* \lceil A \rceil + \sum a_j F_j$$

$\square$

Matsuki's version of the non-vanishing:

X normal, proj, $\Delta = \sum d_i \Delta_i$, $0 \leq d_i \leq 1$, G effective Cartier, L nef Cartier s.t. • (X, Δ) $\mathbb{Q}$-factorial and log terminal

• $aL + G - (K_X + \Delta)$ ample for some $a \in \mathbb{N}$

$\Rightarrow H^0(X, mL + G) \neq 0$ for $m \gg 0$

Comparison: $A \hat{=} G - \Delta$ $\quad D \hat{=} L$ $\quad$ (at least if $d_i < 1$)