A SPLITTING FORMULA FOR SPECTRAL FLOW ON CLOSED 3-MANIFOLDS

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To my parents, Florian and Tabassum

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Abstract

In 1989 Edward Witten defined a topological quantum field theory using certain 3-manifold invariants involving the Feynman path integral. This integral is not mathematically rigorous, but Witten's invariants have two different interpretations. A comparison of these predicts a relationship between an entirely combinatorial description in terms of the axioms of topological quantum field theory and gauge theoretic quantities. In 1992 Lisa Jeffrey verified that the TQFT and the asymptotic expansion are consistent for lens spaces. She also confirmed this for torus bundles over the circle but left some details involving spectral flow unresolved. Based on her analysis she made a conjecture involving the spectral flow of the odd signature operator between flat $SU(2)$ connections on torus bundles over the circle.

This thesis establishes a splitting formula for spectral flow of the odd signature operator between flat $SU(2)$ connections on a closed 3-manifold M. It describes spectral flow on $M = S \cup X$ in terms of spectral flow on the solid torus S, spectral flow on its complement X (with certain Atiyah-Patodi-Singer boundary conditions), and two correction terms which depend only on the endpoints. The central ingredient to the proof of this theorem is a result by Liviu Nicolaescu about the relationship of spectral flow and the Maslov index of Cauchy data spaces, as well as extensions by Hans Boden, Chris Herald, Mark Daniel, Paul Kirk, Erik Klassen and Matthias Lesch.

The main application of this splitting formula is the proof of the above conjecture by Lisa Jeffrey. A major part of this dissertation lies in the technical issues involving

the Atiyah-Patodi-Singer boundary conditions and in the detailed analysis of spectral flow on S with these boundary conditions. This is used in this thesis for the study of spectral flow on torus bundles M over the circle, which also relies on computations of the twisted cohomology of $X = M - S$, where S is a certain embedded torus.

Contents

CHAPTER 1

Introduction

1.1. Gauge Theory

Gauge theory is an exciting recent development in mathematics which combines ideas from mathematics and physics to gain new insights in both fields. It has received a lot of attention by topologists after Donaldson's work in the 1980's. Simon Donaldson had found a deep, but mysterious link between Yang-Mills theory and 4-dimensional differential topology. Donaldson's use of gauge theory showed that the differentiable classification of smooth 4-manifolds is very different from their classification up to homeomorphism [**Do, DoK, FU, L**].

In gauge theory, we study connections on *principal G-bundles*, that is, certain bundles over manifolds with fiber a Lie group G. A *G-connection* is a way to lift paths in the underlying manifold to a path in the principal G-bundle. The space of connections turns out to be an infinite dimensional affine space, namely any connection is the sum of a fixed connection and a vector in an infinite dimensional vector space. Gauge transformations provide an equivalence relation on the space of connections. We may form the quotient space of the connections modulo gauge equivalence. We can gain topological information about a manifold by studying the collection of all connections, a submanifold of solutions to a differential equation or their respective quotient spaces.

This dissertation lies in the realm of *Chern-Simons gauge theory* which analyzes the *Chern-Simons function* on the space of connections. It is a 3 dimensional theory, meaning the underlying manifold has dimension 3. For an excellent introduction to this subject see [**Hu**].

1.2. Topological Quantum Field Theory

In the past 30 years, several topological arguments have been used systematically in quantum field theory.

In 1988, Michael Atiyah [**At1**] posed two problems for quantum field theorists.

(1) To give a physical interpretation to Donaldson theory. This was done by Edward Witten in the same year [**Wi1**]. Witten interpreted Donaldson-Floer theory as a topological quantum field theory in dimension $3+1$ (space and time).

(2) To find an intrinsical 3-dimensional definition of the Jones polynomial in knot theory as a topological quantum field theory in dimension $2+1$. This was covered by Witten in 1989 [**Wi2**].

Witten's work is the starting point of what is now known as topological quantum field theory. His 3-manifold invariants led to a lot of interesting work by mathematicians trying to interpret, axiomatize, analyze and compute them. A nice introduction is given in [**At3**].

1.3. Witten's Feynman Path Integral Invariant

In 1989, Edward Witten [**Wi2**] defined 3-manifold invariants (one for every integer k) by an integral over the gauge orbits of all connections involving the Chern-Simons function. This integral, called the *Feynman path integral*, does not have a rigorous mathematical meaning because there is (in general) no measure on the orbit space. Nevertheless Witten's physical insights led mathematicians to give interpretations of his new invariants by formally manipulating the integral in analogy to the finite dimensional case. The two main interpretations are the following.

(1) Witten argued that his invariants have certain nice properties, particularly their behavior under cutting and pasting of manifolds. His invariants were the prototypes of a *topological quantum field theory* (TQFT). Mathematicians, notably Atiyah [**At2, At1**], axiomatized Witten's invariants and used these axioms to compute the invariants.

(2) Witten proposed that one also has an *asymptotic expansion* of his invariants (as k approaches infinity) using the *method of stationary phase* by analogy with the finite dimensional setting, in which we have an expansion about the critical points of a Morse function f.

Daniel Freed and Robert Gompf computed the invariants for lens spaces and certain Seifert fibered 3-manifolds using the TQFT axioms and a computer [**FG, Go**].

Physicists assume that both interpretations are valid, which predicts a relationship between an entirely combinatorial description in terms of the axioms of TQFT and gauge theoretic quantities. A confirmation of this relationship for all or a large family of 3-manifolds is very interesting at the least and was the main objective of Lisa Jeffrey's thesis, which is contained in [**J**].

For more information on Chern-Simons field theory and Witten's 3-manifold invariants as well as their relation to other quantum invariants see [**Wi2, Gu, J, O, RT**].

1.4. Lisa Jeffrey's Thesis

In 1992, Lisa Jeffrey [**J**] attempted to relate the two interpretations of Witten's 3-manifold invariants for two families of manifolds. She verified that the TQFT and the asymptotic expansion are consistent for lens spaces. She also confirmed this for torus bundles M over the circle (or equivalently mapping tori over the torus), but left some details involving spectral flow unresolved. The *spectral flow* $\mathrm{SF}(D_t) \in \mathbf{Z}$ is roughly defined to be the algebraic intersection number in $[0,1] \times \mathbf{R}$ of the track of all eigenvalues (with multiplicities) of the path of self-adjoint differential operators D_t and 0. She considered a one-parameter family of certain differential operators, namely the *odd signature operator* D_{A_t} twisted by a path of $SU(2)$-connections A_t, and made her Conjecture 5.8 in [**J**] about spectral flow between flat connections. A connection is *flat* if it lifts homotopic loops in X to paths in the principal $SU(2)$-bundle with the same endpoints.

1.5. Work of P. Kirk and E. Klassen

In 1994, Paul Kirk and Erik Klassen [**KK2**] analyzed the spectral flow on a torus bundle M over the circle. For a certain pair of irreducible, flat $SU(2)$-connections A_0 and A_1 they considered a specific splitting of $M = S \cup_T X$ into a solid torus S and its complement X along a torus T, so that there is a path A_t from A_0 to A_1 which is flat on X.

Kirk and Klassen computed the spectral flow on X and S and related it to the spectral flow on M. They showed that the spectral flow on M is $0 \bmod 4$, provided that the dimension of the cohomology of T twisted by the holonomy $\mathrm{hol}(A_t|_T)$ does not jump up as t varies. The *holonomy* $\mathrm{hol}(a)$ of an $SU(2)$-connection a on T assigns to a path $\gamma : [0,1] \to T$ with $\gamma(0) = \gamma(1) = p \in T$ the difference $g \in SU(2)$ between the endpoints of a lift to the principal $SU(2)$-bundle. If a is flat, $\mathrm{hol}(a)$ is a representation of the fundamental group of T with values in $SU(2)$. With some more work this might prove Conjecture 5.8 in [**J**] for irreducible connections, provided that the twisted cohomology of T has constant dimension as t varies. In any case, Kirk and Klassen proved a topological property of torus bundles over S^1 predicted by physical reasoning.

The twisted cohomology of T being constant as t varies is equivalent to the kernel of the *tangential* or *boundary Dirac operator* of D_{A_t} having constant dimension. This restriction is well-known to people who work with paths of Dirac operators on manifolds with boundary.

Kirk and Klassen proposed that it might be possible to always arrange it so that the dimension of the twisted cohomology of T does not change. Unfortunately, this is not always the case. An example of a torus bundle over S^1 is provided in Lemma 6.2.1 where we will always run into the problem that the dimension of the twisted cohomology jumps. More examples have been found, where we will always run into this problem. Some have been listed in Section 6.2. Thus we are forced to deal with the case that the dimension changes, which is analyzed in this thesis. The core of this thesis deals with these dimension jumps.

We will also make extensive use of some further results in the work on spectral flow of the twisted odd signature operator by Kirk and Lesch [**KL**] and Boden, Herald, Kirk and Klassen [**BHKK**].

1.6. Motivation and Application

The motivation for this thesis is Lisa Jeffrey's Conjecture 5.8 in [**J**] involving spectral flow of the odd-signature operator on torus bundles over S^1. We develop the necessary tools for dealing with the case that the dimension of the twisted cohomology of the torus changes. When we apply them to torus bundles over S^1, we take the original approach of Kirk and Klassen in [**KK2**].

In order to compute spectral flow between two flat connections on a torus bundle M over S^1, the strategy is to split M into the solid torus S and the complement X, so that there is a connecting path of connections which is flat on X. In order to relate the spectral flow on S and X to the spectral flow on M we develop a splitting formula for spectral flow on closed 3-manifolds. The main challenge lies in the fact that the dimension of the twisted cohomology of the boundary torus is allowed to change. For the spectral flow on manifolds with boundary we need to introduce boundary conditions, which may vary with the connection.

Once the splitting formula is in place, we can use topology to try to compute spectral flow along the path of flat connections on X, because the kernel of the odd-signature operator can be identified with twisted cohomology.

In order to compute spectral flow on S we consider certain lens spaces, that is, two solid tori glued along the boundary, and apply our splitting formula for spectral flow to deduce information about the spectral flow on the solid torus with our chosen family boundary conditions.

1.7. Summary of Results

In Chapter 3 we define a family of Atiyah-Patodi-Singer boundary conditions, which forms the technical heart of this work. We first choose a family of connections $a_{\alpha,\beta}$ on the torus T parametrized by $(\alpha, \beta) \in \mathbf{R}^2$, which covers all gauge equivalence classes of flat connections on T. Then we introduce a space $\widetilde{\mathbf{R}}^2$ and a continuous smooth surjection $\pi : \widetilde{\mathbf{R}}^2 \to \mathbf{R}^2$, where $\pi^{-1}((\frac{1}{2}\mathbf{Z})^2)$ is a collection of circles and π is a homeomorphism on $\pi^{-1}(\mathbf{R}^2 - (\frac{1}{2}\mathbf{Z})^2)$, and we construct a nice family $\mathcal{P}$ of boundary conditions parametrized by $\varphi \in \widetilde{\mathbf{R}}^2$. In particular we show that given a family of connections A_φ on a manifold with boundary T parametrized by $\varphi \in U \subset \widetilde{\mathbf{R}}^2$, the odd signature operator D_{A_φ} with boundary conditions $\mathcal{P}_\varphi$ is self-adjoint, elliptic and varies continuously in $\varphi \in U$. This provides a convenient framework for working with the odd-signature on a manifold with torus boundary.

In Chapter 4 we develop a splitting formula for spectral flow, which expresses spectral flow between flat connections on a closed three manifold in terms of spectral flow on the solid torus and its complement with the boundary conditions from Chapter 3. For a precise statement see Theorem 4.5.5. Its main feature is that it allows the dimension of the twisted cohomology of the torus to change along the path of connections.

The application to keep in mind is the computation of spectral flow of the twisted odd signature operator between flat connections, whenever it is possible to find a path which is flat in the complement of a solid torus.

Chapter 5 presents an explicit way to compute spectral flow on the solid torus S with the boundary conditions from Chapter 3. We extend the family $a_{\alpha,\beta}$ of connections on T to a family of connections $A_{\alpha,\beta}$ on S. In Theorem 5.3.9 we give a cycle in $\tilde{\mathbf{R}}^2$ so that its algebraic intersection number with a path $\tilde{\rho}$ in $\tilde{\mathbf{R}}^2$ gives the spectral flow of $D_{A_{\pi\circ\tilde{\rho}}}$ with boundary conditions $\mathcal{P}_{\tilde{\rho}}$, where π is the projection of $\tilde{\mathbf{R}}^2$ to $\mathbf{R}^2$.

In Chapter 6 we analyze spectral flow on torus bundles over S^1. Given two flat connections on M we find a thickened knot S in M, so that there exists a path of connections on M which is flat on the complement of S. We investigate twisted cohomology to conclude that the spectral flow of the odd signature operator between two irreducible flat connections on a torus bundle over the circle is $0 \bmod 4$. This generalizes Theorem 7.5 in [**KK2**], removing the technical assumption that a "good pair" can be found.

CHAPTER 2

Preliminaries

The dissertation requires some knowledge in functional analysis, low-dimensional topology, gauge theory, harmonic analysis, Hodge theory, Lie groups and group theory. This chapter contains the necessary background material. Please note that most of our notational conventions are collected in the appendix.

2.1. Linear Operators

Linear operators are the usual suspects in functional analysis. The facts summarized in this section are only necessary to rigorously define *spectral flow* in the following section. For a detailed discussion of linear operators see [**K**] and [**BLP**].

We first need to set up some basic notation for linear operators. Let H and H' be Hilbert spaces. A *(linear) operator* D from H to H' is a linear function on a subspace $\mathscr{D}(D)$ of H. $\mathscr{D}(D)$ is called the *domain* of D. An operator $D : H \to H'$ is *densely defined* if $\mathscr{D}(D)$ is dense in H. An operator $D : H \to H'$ is *bounded* if there is a constant $C > 0$ such that $\|D\varphi\|_{H'} \leqslant C\|\varphi\|_H$ for all $\varphi \in \mathscr{D}(D)$. A bounded operator $D : H \to H'$ is *compact*, if $\overline{D(B)}$ is compact in H' for any bounded set $B \subset H$. Denote by $\mathcal{B}(H, H')$ the set of bounded operators from H to H'. An operator $D : H \to H'$ is *closed* if $\{(h, Dh) \mid h \in \mathscr{D}\}$ is closed in $H \times H'$.

Definition 2.1.1. Let $D : H \to H'$ be a closed operator. Then the *resolvent set* of D is

$$R(D) = \{\zeta \in \mathbf{C} \mid (D - \zeta)^{-1} \in \mathcal{B}(H', H) \text{ exists}\}$$

and the *spectrum* of D is $\Sigma(D) = \mathbf{C} - R(D)$. The operator-valued function

$$P(\zeta) = (D - \zeta)^{-1} : R(D) \to \mathcal{B}(H', H)$$

is called the *resolvent* of D.

It is shown on page 187 in [**K**], that $P(\zeta_0)$ being compact for $\zeta_0 \in R(D)$ implies that $P(\zeta)$ is compact for all $\zeta \in R(D)$. Thus, if such a $\zeta_0 \in R(D)$ exists, we simply say that D has *compact resolvent*. We have the following.

THEOREM 2.1.2 (Theorem 6.29, [**K**]). *Let $D : H \to H$ be a closed operator with compact resolvent. Then the spectrum of D consists entirely of isolated eigenvalues with finite multiplicities.*

Two operators $D : H \to H'$ and $D' : H' \to H$ are called *adjoint* if $\langle Dh, h' \rangle = \langle h, D'h' \rangle$ for all $h \in \mathscr{D}(D)$ and $h' \in \mathscr{D}(D')$. If $D : H \to H'$ is densely defined, then there is a unique operator $D^* : H' \to H$, such that if D and $D' : H' \to H$ are adjoint, then D' is a restriction of D^*, that is, $\mathscr{D}(D') \subset \mathscr{D}(D^*)$ and $D'(h') = D^*(h')$ for all $h' \in \mathscr{D}(D')$. The operator D^* is called *the adjoint* of D. The domain of the adjoint is

$$\mathscr{D}(D^*) = \{h' \in H' \mid \text{there is an } h \in H \text{ such that } \langle h', Dg \rangle = \langle h, g \rangle \text{ for all } g \in \mathscr{D}(D)\}$$

and for $h' \in \mathscr{D}(D^*)$ one defines $D^*(h')$ to be the unique $h \in H$ with

$$\langle h', Dg \rangle = \langle h, g \rangle \text{ for all } g \in \mathscr{D}(D).$$

For a proof see page 167 in [**K**].

A densely defined operator $D : H \to H$ is called *self-adjoint* if $D = D^*$ and $\mathscr{D}(D) = \mathscr{D}(D^*)$. Such an operator has the following important property.

THEOREM 2.1.3 (Theorem XI.8.1, [**Y**]). *A self-adjoint operator $D : H \to H$ has real spectrum.*

A densely defined operator $D : H \to H$ is *symmetric* if $\langle Dh, h' \rangle = \langle h, Dh' \rangle$ for all $h, h' \in \mathscr{D}(D)$, or equivalently if $\mathscr{D}(D) \subset \mathscr{D}(D^*)$ and $D^*|_{\mathscr{D}(D)} = D$. Note that a self-adjoint operator is symmetric, but not vice versa. The following shows when the spectrum $\Sigma(T)$ is continuous in T.

THEOREM 2.1.4 (Theorem 4.10, [**K**]). *Let $D : H \to H$ be self-adjoint and $A : H \to H$ be a bounded, symmetric operator. Then $D' = D + A$ is self-adjoint with $\mathscr{D}(D') = \mathscr{D}(D)$ and $\mathrm{dist}(\Sigma(D), \Sigma(D')) \leqslant \|A\|_H$, that is,*

$$\sup_{\zeta \in \Sigma(D)} \mathrm{dist}(\zeta, \Sigma(D')) \leqslant \|A\|_H \quad \textit{and} \quad \sup_{\zeta \in \Sigma(D')} \mathrm{dist}(\zeta, \Sigma(D)) \leqslant \|A\|_H.$$

The *gap* between subspaces K and L of H is defined to be

$$g(K, L) := \max[\hat{g}(K, L), \hat{g}(L, K)] \quad \text{where} \quad \hat{g}(K, L) = \sup_{||u||=1} \mathrm{dist}(u, L).$$

Let $\mathscr{H}$ be the set of subspaces of H. The *gap topology* is the topology on $\mathscr{H}$ induced by this function, that is, the open sets of $\mathscr{H}$ are unions of the open balls

$$B(K, r) := \{L \in \mathscr{H} \mid g(K, L) < r\},$$

where $K \in \mathscr{H}$ and $r > 0$.

The function g is not a metric. But there is a metric d called *gap metric* on $\mathscr{H}$, such that $g(K, L) \leqslant d(K, L) \leqslant 2g(K, L)$ for all $K, L \in \mathscr{H}$. Notice that d and g induce the same topology. See pages 197-199 in [**K**] for details.

The product space $H \times H$ is again a Hilbert space in the obvious way. Then the *graph* metric on the space of operators is given by $d(D_1, D_2) = \delta(P_1, P_2)$, where $P_i = \{(h, D_i h) \mid h \in \mathscr{D}(D_i)\} \subset H \times H$ is the graph of D_i.

2.2. Spectral Flow

Even though spectral flow is a very intuitive concept, several difficulties can arise when working with unbounded operators. For detailed information see [**BLP**].

Let D_t, $t \in [0,1]$, be a 1-parameter family of (possibly unbounded) self-adjoint operators with compact resolvent, continuous in the graph metric. Then Theorems 2.1.2, 2.1.4 and 2.1.3 imply that the spectrum $\Sigma(D_t)$ is a discrete subset of $\mathbf{R}$ for all $t \in [0,1]$ and changes continuously in t. Note that we also allow the domains to vary continuously in the gap metric. In the case where our operators act on smooth sections of a vector bundle over a manifold M with boundary, we simply call these domains boundary conditions.

A rigorous definition of *spectral flow* can be found in [**BLP**] in the case where M is a compact Riemannian manifold with boundary Σ and D_t is a family of symmetric elliptic differential operators of first order and of *Dirac type* on M acting on sections of a fixed Hermitian vector bundle E with coefficients depending coninuously on t, and a norm-continuous family $\{P_t\}$ of orthogonal projections of $L^2(\Sigma; E_\Sigma)$ defining well-posed boundary problems.

For the results in this paper, however, it is sufficient to confine ourselves to Atiyah-Patodi-Singer boundary conditions (see Section 2.9 for a definition) on a manifold with a collar. It is shown in [**KK3**], that by choosing a continuous family of Atiyah-Patodi-Singer boundary conditions the spectrum of D_t varies continuously in t (see [**KK3**, Theorem 4.1 and Section 5]).

So we can assume that D_0 and D_1 have discrete spectra. We may choose $\varepsilon > 0$ smaller than the modulus of the largest negative eigenvalue of D_0 and D_1. Then the *spectral flow* $\mathrm{SF}(D_t) \in \mathbf{Z}$ is roughly defined to be the algebraic intersection number in $[0,1] \times \mathbf{R}$ of the track of the spectrum

$$\{(t,\lambda) \mid t \in [0,1], \lambda \in \mathrm{Spec}(D_t)\}$$

and the line segment from $(0,-\varepsilon)$ to $(1,-\varepsilon)$. We call this the $(-\varepsilon,-\varepsilon)$-*convention*. The orientations are chosen so that if D_t has spectrum $\{n+t \mid n \in Z\}$ then $\mathrm{SF}(D_t) = 1$.

EXAMPLE 2.2.1. Define a 1-parameter family of (unbounded) operators by $D_t\varphi := i\frac{\partial}{\partial\theta}\varphi + t\varphi$ for $\varphi \in L^2(S^1, \mathbf{C})$, $t \in [0,1]$, with $\mathscr{D}(D_t) = H^1(S^1;\mathbf{C})$. Integration by parts yields for φ, $\psi \in H^1(S^1;\mathbf{C})$

$$\int_0^{2\pi} (i\frac{\partial}{\partial\theta}\varphi + t\varphi)\cdot\bar\psi\, d\theta = -\int_0^{2\pi} \varphi i\cdot\frac{\partial}{\partial\theta}\bar\psi\, d\theta + \int_0^{2\pi} \varphi\cdot t\bar\psi\, d\theta = \int_0^{2\pi} \varphi\cdot\overline{(i\frac{\partial}{\partial\theta}\psi + t\psi)}\, d\theta$$

and thus $\langle D_t\varphi, \psi\rangle = \langle\varphi, D_t\psi\rangle$. Furthermore D_t is self-adjoint, because ψ must be at least once (weak) differentiable. The set $\{e^{m\pi i}\}_{m\in\mathbf{Z}}$ is a (Hilbert space) basis of $H^1(S^1;\mathbf{C})$ with $D_te^{m\pi i} = (t-m)e^{m\pi i}$. Then $\mathrm{SF}(D_t) = 1$.

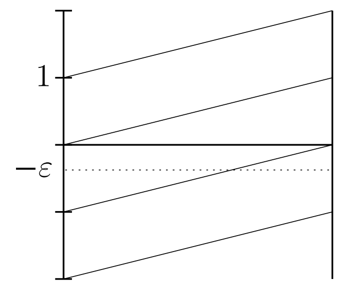

FIGURE 2.2.1. The spectrum of D_t as a function of t

REMARK 2.2.2. There are other conventions for spectral flow. In particular L. Jeffrey, P. Kirk and E. Klassen [**J, KK2**] define it to be the algebraic intersection number of the eigenvalues with the straight line from $(0,-\varepsilon)$ to $(1,\varepsilon)$. We relate to this as the $(-\varepsilon,\varepsilon)$-convention. However, the $(-\varepsilon,-\varepsilon)$-convention has the advantage of being additive. When we relate the spectral flow to results of Jeffrey or Kirk and Klassen, we need to subtract the dimension of the kernel of the differential operator at the endpoint.

2.3. The Lie group $SU(2)$ and its Lie Algebra

Lie groups and Lie algebras as well as their nice properties are being exploited in most areas of mathematics and physics.

We will identify the Lie group $SU(2) \approx S^3$ with the unit quaternions

$$\{v \in \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k \mid |v| = 1\}.$$

The group multiplication will be the usual multiplication in the unit quaternions, where $ij = k$, $jk = i$, $ki = j$ and $i^2 = j^2 = k^2 = -1$. Its Lie algebra $su(2)$ is the tangent space of $SU(2)$ at the identity element and can be identified with the vector space $\mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k = \mathbf{R}i \oplus \mathbf{C}j$ of imaginary quaternions. The adjoint action ad of $SU(2)$ on $su(2)$ corresponds to the conjugation of imaginary quaternions by unit quaternions. The inner product on $su(2) = \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$ is given by $< v, w >= v\bar{w}$.

2.4. Connections and Representations

Connections are the objects of interest in gauge theory. Representations are intimately related to flat connections.

Let P be a principal bundle with structure group G over a manifold M, that is, a smooth manifold P with a free (right) G action with orbit space M such that the orbit map $\pi : P \to M$ is a locally trivial fiber bundle. We denote the right action of $g \in G$ on $p \in P$ by $R_g p = i_p g = p \cdot g$. Let $\mathfrak{g}$ denote the Lie algebra $T_e G$ of G. The *fundamental vector field on P determined by $X \in \mathfrak{g}$* is $X^{\#}(p) = (i_p)_*(X)$.

There are several equivalent definitions for connections, but the following seems the most convenient for our purpose. However we will introduce several different ways of thinking about connections and discuss their relationship to representations. A *G-connection* on P is a $\mathfrak{g}$-valued 1-form A on P, such that

(1) $A(X^{\#}) = X$ for all $X \in \mathfrak{g}$.

(2) A is equivariant, that is, $R_g^*(A) = \mathrm{ad}(g^{-1})A$ for all $g \in G$.

Let $\mathcal{A}_P$ denote the space of connections on P. If $P = M \times G$ is trivial, we write $\mathcal{A}_M = \mathcal{A}_P$. If P is trivializable, we have a nice way of describing $\mathcal{A}_P$, which we will discuss in Proposition 2.4.2.

A (principal bundle) *automorphism* g of P is a fiber bundle automorphism of P, which preserves the right action of G on P, that is, for $p \in P$ and $h \in G$ we have

$$g(p \cdot h) = (g(p)) \cdot h.$$

Denote the group of automorphisms of P by $\mathcal{G}_P$. Notice that for $g \in \mathcal{G}_P$ and $h \in G$ we have $g \circ R_h = R_h \circ g$ and $g \circ i_p = i_{g(p)}$.

LEMMA 2.4.1. *There is a left action of* $\mathcal{G}_P$ *on* $\mathcal{A}_P$ *defined via pulling back by*

$$\mathcal{G}_P \times \mathcal{A}_P \ni (g, A) \mapsto g \cdot A = (g^{-1})^* A,$$

where $(g^* A)(v_p) = A(g_*(v_p))$ *for* $g \in \mathcal{G}_P$, $v_p \in T_p P$ *and* $p \in P$.

PROOF. We have the defining property (1)

$$(g^{-1} \cdot A)_p(X^{\#}) = A_{gp}(g_*(i_p)_* X) = A_{gp}((g \circ i_p)_* X) = A_{gp}((i_{gp})_* X) = X$$

and property (2)

$$R_h^*(g^{-1} \cdot A) = R_h^* g^*(A) = g^* R_h^*(A) = g^* \mathrm{ad}(h^{-1}) A = \mathrm{ad}(h^{-1}) g^* A = \mathrm{ad}(h^{-1})(g^{-1} \cdot A).$$

Thus $g \cdot A$ is a connection on P. We also have $g \cdot (h \cdot A) = (g \circ h) \cdot A$, because for $v_p \in T_p P$

$$\begin{aligned} (g \cdot (h \cdot A))(v_p) &= (h \cdot A)((g^{-1})_* v_p) = A((h^{-1})_* (g^{-1})_* v_p) \\ &= A((h^{-1} g^{-1})_* v_p) = A(((g \circ h)^{-1})_* (v_p)) = ((g \circ h) \cdot A)(v_p) \end{aligned}$$

□

We call $\mathcal{G}_P$ the group of *gauge transformations*.

PROPOSITION 2.4.2. *Let* P *be a principal* G*-bundle over a manifold* M.

(1) $\mathcal{G}_P$ *can be identified with* $C^\infty(M, G)$.

(2) *Suppose* P *is trivializable. A trivialization of* P *induces an identification of* $\mathcal{A}_P$ *with Lie-algebra-valued* 1*-forms* $\Omega^1(M) \otimes \mathfrak{g}$ *on* M.

PROOF. Consider (1). To see that $\mathcal{G}_P \cong C^\infty(M,G)$ as groups, let g be a (smooth) automorphism of P. Let p be in the fiber P_m of P over $m \in M$ and choose an isomorphism $\Phi : P_m \to G$. Let k_m be the unique element in G such that $k_m\Phi(p) = \Phi(g(p))$. Then k_m is independent of p since

$$k_m\Phi(p \cdot h) = k_m\Phi(p)h = \Phi(g(p))h = \Phi(g(p) \cdot h) = \Phi(g(ph)).$$

Furthermore if we choose $p \in P_m$ with $\Phi(p) = 1$, then for a different isomorphism $\Psi : P_m \to G$ we have $k_m\Phi(p) = k_m\Psi(p) = k_m$. Thus k_m is independent of the trivialization of the fiber, and $k : M \to G$ is a map. Since g is smooth, k is smooth.

Let us turn to (2). Let $A \in \mathcal{A}_P$. By trivializing $P = M \times G$, we can split the tangent space of P at $p = (m,g) \in P$ as $T_pP = T_mM \times T_gG$.

Let $j : M \to P = M \times G$ be the embedding $j(m) := (m,1)$. Then we can define a homomorphism

$$\begin{aligned} \Psi : \mathcal{A}_P &\to \Omega^1(M) \otimes \mathfrak{g} \\ A &\to j^*(A). \end{aligned}$$

On the other hand given $\tilde{A} \in \Omega^1(M;\mathfrak{g})$, define a 1-form $\Phi(\tilde{A}) := A$ on $P = M \times G$ where

$$A_{(m,g)}(v_m, w_g) = \tilde{A}_m(v_m) + (L_{g^{-1}})_* w_g.$$

Then for $X \in \mathfrak{g}$ and $p = (m,g) \in P$ we have $X^\# = (i_p)_*X = (0_m, (L_g)_*X)$ and thus

$$A_p(X^\#) = \tilde{A}_m(0_m) + X = X.$$

Since $\mathrm{ad}(h^{-1})_* = \mathrm{ad}(h^{-1})$ we have

$$R_h^*A(v_m, w_g) = A(0_m, (R_h)_*w_g) = \tilde{A}(0_m) + (L_{(gh)^{-1}})_*(R_h)_*w_g = \mathrm{ad}(h^{-1})(L_{g^{-1}})_*w_g.$$

This way we have a homomorphism $\Psi : \Omega^1(M;\mathfrak{g}) \to \mathcal{A}_P$.

Furthermore, the maps Φ and Ψ are inverses of each other, because

$$(\Psi \circ \Phi(\tilde{A}))v_m = j^*(A_{(m,g)})(v_m) = A_{(m,g)}(v_m, 0_g) = \tilde{A}_m v_m$$

and

$$(\Phi \circ \Psi(A))(v_m, w_g) = \Psi A(v_m) + (L_{g^{-1}})_* w_g = A(v_m, 0_g) + A(0_m, w_g) = A(v_m, w_g).$$

That is, $\mathcal{A}_P \cong \Omega^1(M) \otimes \mathfrak{g}$ as (affine) vector spaces. □

In this dissertation we are only interested in manifolds M of dimension $\dim M \leqslant 3$. In this case we have the following result.

LEMMA 2.4.3. *Let G be a simply connected Lie group. Every principal G-bundle P over a manifold M with $\dim M \leqslant 3$ is trivializable.*

PROOF. Isomorphism classes of principal G-bundles over a manifold M are in 1-1 correspondence with homotopy classes of maps from M to the classifying space BG of G, $[M, BG]$ (see [**DavK**]). Thus all we need to show is $|[M, BG]| = 1$. Let EG be the universal cover of BG. The long exact sequence of homotopy groups associated to the fibration $G \to EG \to BG$ yields

$$0 = \pi_n EG \to \pi_n BG \xrightarrow{\cong} \pi_{n-1} G \to \pi_{n-1} EG = 0$$

for $n \geqslant 1$. Since any Lie group has $\pi_2 = 0$ (see [**C, Browd**]), G is 2-connected and thus $\pi_i BG = 0$ for $1 \leqslant i \leqslant 3$. Since EG is contractible, BG is connected, that is, $\pi_0 BG = 0$. Thus BG is 3-connected. By obstruction theory any map from an n-dimensional CW-complex to an n-connected space is null-homotopic. Thus any map from M to BG is null-homotopic. □

We are only interested in principal $SU(2)$-bundles P over M. $SU(2)$ is 2-connected. Thus, after fixing a trivialization once and for all, we have $\mathcal{A}_M = \Omega^1(M) \otimes su(2)$ and $\mathcal{G}_M = C^\infty(M, SU(2))$.

In this trivialization we have a nice description of the action of a gauge transformation $g \in C^\infty(M, SU(2))$ on an $SU(2)$-connection A on M

$$g \cdot A = gAg^{-1} + g\, dg^{-1} = gAg^{-1} - dg\, g^{-1}.$$

Notice that $d_m g^{-1} : T_m M \to T_{g^{-1}(m)} SU(2)$, thus $g\,dg^{-1}$ is an $su(2)$-valued 1-form. gAg^{-1} is conjugation on the coefficients, so is an $su(2)$-valued 1-form as well.

We will also frequently think about connections in terms of covariant derivatives. A *covariant derivative* $\nabla : \Omega^0(E) \to \Omega^1(E)$ on a vector bundle E is a $\mathbf{R}$-linear homomorphism satisfying the Leibniz-rule

$$\nabla(f \wedge \alpha) = df \otimes \alpha + f \wedge \nabla\alpha$$

where $f \in \Omega^0(M)$, $\alpha \in \Omega^0(E)$ and d is the usual exterior derivative. We extend ∇ to an $\mathbf{R}$-linear map

$$\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$$

by tensoring with the de Rham complex: For $\omega \in \Omega^k(M)$ and $\alpha \in \Omega^0(E)$ we let

$$\nabla(\omega \wedge \alpha) := d\omega \otimes \alpha + (-1)^k \omega \wedge \nabla\alpha.$$

We are going to work with the adjoint vector bundle. Let $\mathrm{ad} : SU(2) \to \mathrm{Aut}(su(2))$ be the adjoint representation of $SU(2)$. Then the adjoint bundle associated to the principal $SU(2)$-bundle P over a manifold M is

$$\mathrm{ad}P = P \times_{\mathrm{ad}} su(2) = P \times su(2)/(pg, \mathrm{ad}(g^{-1})u) \sim (p, u).$$

In our case, $\dim M \leqslant 3$, we have $P \times_{\mathrm{ad}} su(2) = M \times su(2)$ induced by the choice of trivialization $P = M \times SU(2)$. (The identification is given by $(M \times SU(2)) \times_{\mathrm{ad}} su(2) \ni [((m, g), v)] \mapsto (m, \mathrm{ad}(g)v) \in M \times su(2)$.) Thus a choice of trivialization gives an identification

$$\Omega^*(\mathrm{ad}P) = \Omega^*(M) \otimes su(2).$$

Now we can associate to a connection A on P a covariant derivative or twisted exterior derivative on $\mathrm{ad}P = M \times su(2)$ by defining

$$\begin{aligned} d_A : \Omega^0(M) \otimes su(2) &\to \Omega^1(M) \otimes su(2) \\ a &\mapsto da + [A, a], \end{aligned}$$

where $[\cdot,\cdot]$ means taking the Lie bracket on the coefficients and wedging the form part. As before, extend the covariant derivative to $\Omega^*(M)\otimes su(2)$ by the Leibniz rule.

DEFINITION 2.4.4. The *Hodge star operator* $*$ on differential p-forms on M is defined via $\alpha\wedge *\beta=\langle\alpha,\beta\rangle\mathrm{vol}$, where $\langle\alpha,\beta\rangle$ is the inner product on p-forms induced by the Riemannian metric on M and vol denotes the volume form. We extend the Hodge star operator to $\Omega^p(M)\otimes su(2)$ by applying it to each matrix entry. For example in dimension 3 we have locally $*v\,dx = v\,dy\wedge dz$, where $v\in su(2)$ and dx, dy and dz are orthonormal. If $\dim M = n$, then for the Hodge star operator acting on p-forms we have $*^2=(-1)^{p(n-p)}$ and $d^*=(-1)^{n(p+1)+1}*d*$. Notice that $*^2 d=(-1)^{n+1}d*^2$. Then similarly $d_a^*=(-1)^{n(p+1)+1}*d_a*$. Notice that the Hodge star operator is self-adjoint.

We also need to frequently use the following terms.

(1) The *curvature* F_A of a connection $A\in\Omega^1(M)\otimes su(2)$ is the 2-form in $\Omega^2(M)\otimes su(2)$ defined by $[F_A,\varphi]:=d_A^2\varphi$, where $\varphi\in\Omega^0(M)\otimes su(2)$. We have $F_A = dA+A\wedge A$.
(2) We call A *flat*, if $F_A=0$. Let $\mathcal{F}_P\subset\mathcal{A}_P$ be the set of all flat connections on M. If P is trivial we write $\mathcal{F}_M$ instead of $\mathcal{F}_P$. If A is flat, then the *holonomy* $\mathrm{hol}(A):\pi_1 M\to SU(2)$ is a representation.
(3) $A\in\mathcal{A}_M$ is in *cylindrical form in a collar of* $\Sigma=\partial M$ if $A=i_s^*a$, where $i_s:\Sigma\to\Sigma\times[0,1]\subset M$ and $a\in\mathcal{A}_\Sigma$.

Let $\chi_G(M)$ be the variety of conjugacy classes of G-representations of the fundamental group $\pi_1(M)$. The following relationship between conjugacy classes of representations and gauge equivalence classes of connections will be exploited in the following chapters.

PROPOSITION 2.4.5 (Proposition 2.2.3, [**DoK**]). *Suppose*

(1) *G is a simply connected, compact Lie group,*

(2) *M is a 3-manifold,*

(3) *P is a principal G-bundle over M.*

Then the holonomy map induces a homeomorphism between $\mathcal{F}_M/\mathcal{G}$ and $\chi_G(M)$.

2.5. The Odd Signature Operator and the Tangential Operator

The odd signature operator is related to the Chern-Simons function. For a connection $A \in \Omega^1(M; su(2))$ on a 3-manifold M the Chern-Simons function is given by

$$cs(A) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A).$$

At a flat connection it is an extension of the Hessian of the Chern-Simons function by a "ghost" operator with symmetric spectrum. Thus the spectral flow of the two operators between flat connections is equal when the ghost operator has nonzero kernel at the endpoints. With our $(-\varepsilon, -\varepsilon)$ convention, the spectral flow of the two operators is the same up to the dimension of the kernel of the ghost operator at the endpoints.

Let Σ a surface and X a 3-manifold with collar $[0, 1] \times \Sigma$, where $\partial X = \Sigma$ corresponds to $\{0\} \times \Sigma$. We set $\Omega^i(X; su(2)) := \Omega^i(X) \otimes su(2)$ and $\Omega^{0+1}(X; su(2)) := \Omega^0(X; su(2)) \oplus \Omega^1(X; su(2))$.

DEFINITION 2.5.1. We write the restriction of $\Omega^{0+1}([0, 1] \times \Sigma; su(2))$ to $\Sigma = \partial X$ as

$$\begin{aligned} r : \Omega^{0+1}([0,1] \times \Sigma; su(2)) &\to \Omega^{0+1+2}(\Sigma; su(2)) \\ (\sigma, \tau) &\mapsto (i_0^*(\sigma), i_0^*(\tau), *i_0^*(\tau \dashv \frac{\partial}{\partial u})), \end{aligned}$$

where $i_u : \Sigma \hookrightarrow [0,1] \times \Sigma$ is the inclusion at u, $\dashv$ denotes contraction and $*$ is the Hodge star on differential forms on the 2 manifold Σ. This also gives us a restriction map of $\Omega^{0+1}(X; su(2))$ to $\Omega^{0+1+2}(\Sigma; su(2))$.

If we write $\tau = \beta + \omega\, du$, where u is the coordinate in $[0,1]$ and β does not have a du component, then a more intuitive way to write the restriction map is $r(\sigma, \beta + \omega du) = (\sigma|_\Sigma, \beta|_\Sigma, *(\omega|_\Sigma))$. At first glance it might seem odd that we get an extra 2-form when we restrict, but observe that the differential forms are sections of the cotangent bundle over the manifold. The restriction map is induced by the restriction of bundles, which must preserve the dimension of the fiber. The fiber in the case of 0 and 1-forms on a 3 manifold is $1+3$ dimensional. The fiber in the case of 0 and 1-forms on a 2 manifold is $1+2$ dimensional. Thus we need to have an extra dimension, which the 2-forms on the 2-manifold provide. This way of describing the restriction map is unique to dimension 3.

Definition 2.5.2. For an $SU(2)$-connection $A \in \Omega^1(M; su(2))$ the *odd signature operator twisted by* A is defined to be

$$\begin{aligned} D_A : \Omega^{0+1}(M; su(2)) &\to \Omega^{0+1}(M; su(2)) \\ (\alpha, \beta) &\mapsto (d_A^*\beta, *d_A\beta + d_A\alpha). \end{aligned}$$

Definition 2.5.3. For an $SU(2)$-connection $a \in \Omega^1(\Sigma; su(2))$ the *tangential operator* S_a of D_A (also called *de Rham operator*) is defined to be

$$\begin{aligned} S_a : \Omega^{0+1+2}(\Sigma; su(2)) &\to \Omega^{0+1+2}(\Sigma; su(2)) \\ (\alpha, \beta, \gamma) &\mapsto (*d_a\beta, - * d_a\alpha - d_a * \gamma, d_a * \beta). \end{aligned}$$

The operator $D_A \oplus D_A$ is the tangential operator of the signature operator from the positive eigenspace of the Hodge star operator to the negative eigenspace

$$d_A + d_A^* : \Omega^+(\mathrm{ad}P) \to \Omega^-(\mathrm{ad}P).$$

LEMMA 2.5.4. *Let Σ and M be closed 2 and 3-manifolds respectively.*

(1) *S_a is a first-order self-adjoint elliptic differential operator with respect to the L^2 inner product $\langle \alpha, \beta \rangle = -\int_\Sigma \mathrm{tr}(\alpha \wedge *\beta)$.*

(2) *D_A is a first-order self-adjoint elliptic differential operator with respect to the L^2 inner product $\langle \alpha, \beta \rangle = -\int_M \mathrm{tr}(\alpha \wedge *\beta)$.*

SKETCH OF PROOF. The self-adjointness is proven using integration by parts and the fact that the trace is invariant under conjugation. Also S_a is a Dirac operator, because S_a^2 differs from Δ_a and thus from Δ by a 0-degree differential operator. Thus S_a is elliptic. The same argument works for D_A. □

The following observation will come in handy when studying eigenspaces of S_a and harmonic forms.

LEMMA 2.5.5. *If $a \in \Omega^1(\Sigma; \mathbf{R}i)$, then the operator S_a preserves the splitting of*

$$\Omega^{0+1+2}(\Sigma; su(2)) = \Omega^{0+1+2}(\Sigma; \mathbf{R}i) \oplus \Omega^{0+1+2}(\Sigma; \mathbf{C}j),$$

and the action of S_a on the $\mathbf{R}i$ part is independent of a.

PROOF. The proof can be reduced to analyzing d_a, because $*$ does not change the coefficients. Let $\alpha \in \Omega^{0+1+2}(\Sigma; \mathbf{R}i) \subset \Omega^{0+1+2}(\Sigma; su(2))$. We certainly have $d\alpha \in \Omega^{0+1+2}(\Sigma; \mathbf{R}i)$. Since coefficients in $\mathbf{R}i$ commute with each other, we also have $[a, \alpha] = 0$ for all $\alpha \in \Omega^{0+1+2}(\Sigma; \mathbf{R}i)$. Thus $d_a\alpha \in \Omega^{0+1+2}(\Sigma; \mathbf{R}i)$. Consider $\alpha \in \Omega^{0+1+2}(\Sigma; \mathbf{C}j)$. Again one has $d\alpha \in \Omega^{0+1+2}(\Sigma; \mathbf{C}j)$, and $[\beta i, \alpha_1 j + \alpha_2 k] = 2(\beta\alpha_1 k - \beta\alpha_2 j) \in \mathbf{C}j$ implies $[a, \alpha] \in \Omega^{0+1+2}(\Sigma; \mathbf{C}j)$ □

2.6. The Twisted Laplacian and Harmonic Forms

The tangential operator S_a and the twisted Laplacian share some important properties. Harmonic forms turn out to be the kernel of S_a. This puts Lagrangians in the twisted cohomology and the scattering Lagrangian into perspective and shows the importance of harmonic forms for this work.

Let Σ be a surface. Define $\Delta_a := d_a d_a^* + d_a^* d_a$ to be the Laplacian on $\Omega^{0+1+2}(\Sigma; su(2))$ twisted by $a \in \mathcal{A}_\Sigma$. The following important identity is an immediate consequence of $d_a^2 = 0$ if a is flat.

LEMMA 2.6.1. *$\Delta_a = S_a^2$ if a is flat.* □

Denote by $\mathcal{H}^{0+1+2}(\Sigma; su(2)) := \mathrm{Ker}\Delta$ and $\mathcal{H}_a^{0+1+2}(\Sigma; su(2)) := \mathrm{Ker}\Delta_a$ the harmonic 0, 1 and 2-forms of Δ and Δ_a respectively. As an immediate consequence of Lemma 2.6.1 a λ-eigenvector φ of S_a is a λ^2-eigenvector of Δ_a. But a slightly modified reversed statement also holds.

LEMMA 2.6.2. *If a is flat and φ is a λ^2-eigenvector ($\lambda > 0$) for Δ_a, then $\varphi \pm \frac{1}{\lambda} S_a \varphi$ is a $\pm\lambda$-eigenvector for S_a. Furthermore $\mathrm{Ker} S_a = \mathrm{Ker}\Delta_a$.*

PROOF. Observe, that each eigenvector of Δ_a with nonzero (that is, positive) eigenvalue yields exactly two eigenvectors of S_a, for if $\Delta_a \varphi = \lambda^2 \varphi$ ($\lambda \neq 0$) and if we take $\psi = \varphi + \frac{1}{\lambda} S_a \varphi$, then

$$S_a \psi = S_a \varphi + \frac{1}{\lambda} \Delta_a \varphi = S_a \varphi + \frac{1}{\lambda} \lambda^2 \varphi = \lambda \left(\varphi + \frac{1}{\lambda} S_a \varphi \right) = \lambda \psi,$$

that is, ψ is a λ-eigenvector of S_a.

If $S_a \varphi = 0$, then certainly $\Delta_a \varphi = S_a^2 \varphi = 0$. On the other hand, since S_a is self-adjoint, we get

$$\|S_a \varphi\|^2 = \langle S_a \varphi, S_a \varphi \rangle = \langle S_a^2 \varphi, \varphi \rangle = 0,$$

if $S_a^2 \varphi = \Delta_a \varphi = 0$. □

LEMMA 2.6.3. *If $a \in \Omega^1(\Sigma; su(2))$ is flat, then*

$$\mathcal{H}_a^{0+1+2}(\Sigma; su(2)) \cong H^{0+1+2}(\Sigma; su(2)_{\mathrm{hol}(a)}),$$

where $H^{0+1+2}(\Sigma; su(2)_{\mathrm{hol}(a)})$ *is the cohomology of* Σ *with* $su(2)$*-coefficients twisted by the holonomy* $\mathrm{hol}(a)$.

PROOF. Hodge theory with d_a gives us the isomorphism between harmonic forms and twisted De Rham cohomology. □

Denote by $E_{a,\mu}$ the μ-eigenspace of S_a and let

$$\begin{aligned}
P^+_{a,\nu} &:= \mathrm{span}_{L^2}\{\psi \mid S_a\psi = \mu\psi, \mu > \nu\} = \overline{\bigoplus_{\mu>\nu} E_{a,\mu}}^{L^2}, \\
P^-_{a,\nu} &:= \mathrm{span}_{L^2}\{\psi \mid S_a\psi = \mu\psi, \mu < -\nu\} = \overline{\bigoplus_{\mu<-\nu} E_{a,\mu}}^{L^2}, \\
E^+_{a,\nu} &:= \bigoplus_{0<\mu\leqslant\nu} E_{a,\mu} \quad \text{and} \\
E^-_{a,\nu} &:= \bigoplus_{-\nu\leqslant\mu<0} E_{a,\mu}.
\end{aligned}$$

If $a \in \Omega^1(\Sigma; \mathbf{R}i)$, abbreviate

$$P^\pm_a := P^\pm_{a,0},\ P^\pm_{\mathbf{R}i} := P^\pm_a \cap L^2(\Omega^{0+1+2}(\Sigma; \mathbf{R}i)) \text{ and } P^\pm_{a,\mathbf{C}j} := P^\pm_a \cap L^2(\Omega^{0+1+2}(\Sigma; \mathbf{C}j)).$$

Notice, that the $\mathbf{R}i$-part of the eigenspaces of S_a do not depend on a, because $S_a\varphi = S_\theta\varphi + [a, \varphi] = S_\theta\varphi$. Thus we write $P^\pm_{\mathbf{R}i}$ instead of $P^\pm_{a,\mathbf{R}i}$ and $\mathcal{H}^{0+1+2}(T; \mathbf{R}i)$ instead of $\mathcal{H}^{0+1+2}_a(T; \mathbf{R}i)$.

By the spectral theorem for self-adjoint elliptic operators we have the splitting

$$L^2(\Omega^{0+1+2}(\Sigma, su(2))) = P^+_a \oplus \mathrm{Ker} S_a \oplus P^-_a. \tag{2.6.1}$$

Naturally S_a preserves the splitting (2.6.1) restricted to its domain into eigenspaces. We may combine the splitting (2.6.1) with Lemma 2.5.5 to get a finer splitting:

LEMMA 2.6.4. *Let* $a \in \Omega^1(\Sigma; \mathbf{R}i)$. *Then* S_a *preserves the splitting*

$$\begin{aligned}
&L^2(\Omega^{0+1+2}(\Sigma; su(2))) \cap \mathscr{D}(S_a) \\
&\quad = \left\{(P^+_{\mathbf{R}i} \oplus \mathcal{H}^{0+1+2}(\Sigma; \mathbf{R}i) \oplus P^-_{\mathbf{R}i}) \oplus (P^+_{a,\mathbf{C}j} \oplus \mathcal{H}^{0+1+2}_a(\Sigma; \mathbf{C}j) \oplus P^-_{a,\mathbf{C}j})\right\} \cap \mathscr{D}(S_a),
\end{aligned}$$

where $\mathcal{H}^{0+1+2}_a(T; \mathbf{C}j) = \mathcal{H}^{0+1+2}_a(\Sigma; su(2)) \cap \Omega^{0+1+2}(\Sigma; \mathbf{C}j)$. □

2.7. Symplectic Vector Spaces and Lagrangian Subspaces

Symplectic vector spaces and Lagrangian subspaces provide a convenient framework when dealing with differential operators on a manifold with boundary. Particularly a boundary condition is a Lagrangian subspace of the bundle restriction of domain to the boundary.

An *almost complex structure* on a real Hilbert space is an isometry J such that $J^2 = -\mathrm{Id}$. We can define an almost complex structure on $L^2(\Omega^{0+1+2}(\Sigma; su(2)))$ by

$$J(\alpha, \beta, \gamma) := (- * \gamma, *\beta, *\alpha).$$

It is easy to check using the properties of the Hodge star operator in Definition 2.4.4 that $J^2 = -\mathrm{Id}$ and that J is an isometry of $L^2(\Omega^{0+1+2}(\Sigma; su(2)))$. Furthermore J preserves the splitting of $L^2(\Omega^{0+1+2}(\Sigma; su(2)))$ into the direct sum $L^2(\Omega^{0+1+2}(\Sigma; \mathbf{R}i))$ and $L^2(\Omega^{0+1+2}(\Sigma; \mathbf{C}j))$.

LEMMA 2.7.1. *For an $SU(2)$-connection $a \in \Omega^1(\Sigma; su(2))$ we have*

$$JS_a = -S_aJ.$$

PROOF. We have

$$\begin{aligned}
JS_a(\alpha, \beta, \gamma) &= J(*d_a\beta, - * d_a\alpha - d_a * \gamma, d_a * \beta) \\
&= (- * d_a * \beta, - *^2 d_a\alpha - *d_a * \gamma, *^2 d_a\beta) \\
&= (- * d_a * \beta, - * d_a * \gamma + d_a *^2 \alpha, -d_a *^2 \beta) \\
&= -S_a(- * \gamma, *\beta, *\alpha) = -S_aJ(\alpha, \beta, \gamma).
\end{aligned}$$

□

A subspace Λ in a real Hilbert space with almost complex structure is *Lagrangian* if $J\Lambda = \Lambda^\perp$. Lemma 2.7.1 implies the following.

COROLLARY 2.7.2. *Let $a \in \mathcal{F}_\Sigma$. Then*

(1) $JP_a^{\pm} = P_a^{\mp}$.

(2) $J\mathcal{H}_a^{0+1+2}(\Sigma; su(2)) = \mathcal{H}_a^{0+1+2}(\Sigma; su(2))$.

(3) $\mathcal{H}_a^{0+1+2}(\Sigma; su(2))$ *is a finite dimensional symplectic subspace of* $L^2(\Omega^{0+1+2}(\Sigma; su(2)))$.

(4) *If L is a Lagrangian in $\mathcal{H}_a^{0+1+2}(\Sigma; su(2))$, then $L \oplus P_a^{\pm}$ is Lagrangian in* $L^2(\Omega^{0+1+2}(\Sigma; su(2)))$.

2.8. Cauchy Data Spaces and Adiabatic Limits

Cauchy data spaces play an important role in this dissertation because of their relation to spectral flow. Liviu Nicolaescu analyzed this relationship in his thesis [**N1**]. Before we can state his theorems we need to learn about Cauchy data spaces and adiabatic limits.

We will state all results in this section in terms of the twisted odd-signature operator, but they apply to other Dirac type operators as well.

Let X and Y be oriented 3-manifolds with oriented boundary $\Sigma = \partial X = -\partial Y$ using the outward normal first convention. We introduce the notation

$$X^R = X \cup ([0, R] \times \Sigma) \quad \text{and} \quad X^\infty = X \cup ([0, \infty) \times \Sigma),$$
$$Y^R = Y \cup ([-R, 0] \times \Sigma) \quad \text{and} \quad Y^\infty = X \cup ((-\infty, 0] \times \Sigma)$$

for $R \geqslant 0$. Let A be a connection on X, which is in cylindrical form in a collar of Σ. Denote the restriction to Σ by a. As before denote by P_a^+ the L^2-span of the positive eigenspace and P_a^- the L^2-span of the negative eigenspace of S_a. There is a one-to-one correspondence given by restricting X^∞ to X between L^2 solutions to $D_A(\sigma, \tau) = 0$ on X^∞ and the solutions to $D_A(\sigma, \tau) = 0$ on X whose restriction to Σ lie in P_a^-.

Definition 2.8.1.

(1) The *Cauchy data space* of D_A is

$$\Lambda_{X,A} := \Lambda_X(D_A) := \overline{r(\mathrm{Ker} D_A)}^{L^2},$$

where $r : \Omega^{0+1}(X; su(2)) \to \Omega^{0+1+2}(\Sigma; su(2))$ is the restriction map from Definition 2.5.1 and $D_A : \Omega^{0+1}(X; su(2)) \to \Omega^{0+1}(X; su(2))$ is the odd signature operator twisted by A.

(2) We call

$$\mathscr{L}_{X,A} := \mathrm{proj}_{\mathrm{Ker} S_a}(\Lambda_{X,A} \cap (P^- \cup \mathrm{Ker} S_a))$$

the *scattering Lagrangian* or the *limiting values of extended L^2 solutions*,

See [**BB**] for more information on Cauchy data spaces, particularly Definition 2.20 in [**BB**]. Note that we can extend D_A to X^R and that $\Lambda^R_{X,A} := \Lambda_{X^R}(D_A)$ is a continuous family of Lagrangian subspaces by Lemma 3.2 of [**DanK**]. Denote $\Lambda^\infty_{X,A} := \lim_{R\to\infty} \Lambda^R_{X,A}$.

Lemma 8.6 in [**KL**] and Proposition 4.9 in [**APS**] describe the scattering Lagrangian at a flat connection in several useful ways, one of which is the following.

Lemma 2.8.2. *If A is a flat connection on X, then $\mathscr{L}_{X,A}$ is isomorphic to the image of $H^*(X; su(2)_{\mathrm{hol}(A)}) \to H^*(\partial X; su(2)_{\mathrm{hol}(A)})$.*

Here $H^*(X; su(2)_{\mathrm{hol}(A)})$ is the cohomology with values in $su(2)$ twisted by $\mathrm{hol}(A)$. We will write $H^*(X; su(2)_A)$ instead of $H^*(X; su(2)_{\mathrm{hol}(A)})$. Recall, that we have the orthogonal orthogonal decomposition

$$L^2(\Omega^{0+1+2}(\Sigma; su(2))) = P_a^- \oplus \mathrm{Ker} S_a \oplus P_a^+ = P^-_{a,\nu} \oplus E^-_{a,\nu} \oplus \mathrm{Ker} S_a \oplus E^+_{a,\nu} \oplus P^+_{a,\nu}$$

Kirk and Lesch [**KL**] give a more detailed decomposition in the spirit of the Hodge decomposition (dropping the index a everywhere including d_a)

$$L^2(\Omega^{0+1+2}(\Sigma; su(2))) = P_\nu^- \oplus d(E_\nu^+) \oplus d^*(E_\nu^+) \oplus \mathrm{Ker} S \oplus d(E_\nu^-) \oplus d^*(E_\nu^-) \oplus P_\nu^+,$$

which can be conveniently written as an orthogonal sum of symplectic spaces

(2.8.2)

$$L^2(\Omega^{0+1+2}(\Sigma; su(2))) = (P_\nu^- \oplus (P_\nu^+)) \oplus (d(E_\nu^+) \oplus d^*(E_\nu^-)) \oplus (d^*(E_\nu^+) \oplus d(E_\nu^-)) \oplus \mathrm{Ker} S.$$

Cauchy data spaces are complicated and we wish to relate them to simpler ones. It is particularly nice when D_A has non-resonance level 0.

DEFINITION 2.8.3. D_A has *non-resonance level* $\nu \geqslant 0$, if $P_{a,\nu}^- \cap \Lambda_{X,A} = 0$

Nicolaescu's adiabatic limit theorem describes the adiabatic limit of D_A.

THEOREM 2.8.4 (Corollary 4.11, [**N1**]). *Let A be flat. If D_A has non-resonance level 0, then*

$$\Lambda_{X,A}^\infty = P_a^+ \oplus \mathscr{L}_{X,A}$$

This adiabatic limit theorem can be extended as follows.

THEOREM 2.8.5 (Theorem 8.5, [**KL**]). *Let A be flat. There exists a subspace*

$$W_a \subset d(E_{a,\nu}^+) \subset P_{a,0}^-$$

isomorphic to

$$\mathrm{Im}\left(H^{0+1}(X, \Sigma; su(2)_A) \to H^{0+1}(X; su(2)_A)\right)$$

so that if $W_a^\perp$ denotes the orthogonal complement of W_a in $d(E_{a,\nu}^+)$, then with respect to the decomposition (2.8.2) into symplectic subspaces, the adiabatic limit of the Cauchy data spaces decomposes as a direct sum of Lagrangian subspaces:

$$\Lambda_{X,A}^\infty = P_{a,\nu}^+ \oplus (W_a \oplus J(W_a^\perp)) \oplus d(E_{a,\nu}^-) \oplus \mathscr{L}_A$$

where $\mathscr{L}_A \subset \mathrm{Ker} S_a \cong H^*(\Sigma; su(2)_A)$ denotes the scattering Lagrangian on X.

If A is irreducible, then we have $\mathrm{Im}\,(H^0(X,\Sigma; su(2)_A) \to H^0(X; su(2)_A)) = 0$ (Proposition 4.3 in [**KK2**]). The following proposition will help with our computation.

Proposition 2.8.6 (Proposition 2.10, [**BHKK**]). *Suppose that A is a flat connection on a 3-manifold X with boundary Σ. Let a be the restriction of A to Σ. If $Q \subset \mathrm{Ker} S_a$ is any subspace (not necessarily Lagrangian), then there is a short exact sequence*

$$0 \to \mathrm{Im}(H^1(X,\Sigma;su(2)_A) \to H^1(X;su(2)_A) \to \mathrm{Ker} D_A(P^+ \oplus Q) \xrightarrow{p} \mathscr{L}_{X,A} \cap Q \to 0,$$

where $\mathrm{Ker} D_A(P_a^+ \oplus Q)$ consists of solutions to $D_A(\sigma,\tau)=0$ whose restriction to the boundary lie in $P_a^+ \oplus Q$.

In particular Proposition 2.8.6 implies:

Corollary 2.8.7.

(1) *By letting $Q = 0$, we get the isomorphism given in* [**APS**]

$$\Lambda_{X,A} \cap P_a^+ \cong \mathrm{Im}(H^1(X,\Sigma;su(2)_{\mathrm{hol}(A)}) \to H^1(X;su(2)_{\mathrm{hol}(A)})).$$

Thus 0 non-resonance level is equivalent to

$$\mathrm{Im}(H^1(X,\Sigma;su(2)_{\mathrm{hol}(A)}) \to H^1(X;su(2)_{\mathrm{hol}(A)})) = 0.$$

(2) *Assuming 0 non-resonance level, we get the isomorphism*

$$\Lambda_{X,A} \cap (P_a^+ \oplus Q) \cong \mathscr{L}_X \cap Q.$$

The following fact about stretching collars will also be very useful.

Lemma 2.8.8 (Lemma 8.10, [**KL**]). *Let A be a connection on $M = X \cup_\Sigma Y$, and let a be its restriction to Σ. Let $V \subset \mathrm{Ker} S_a$ be a Lagrangian subspace.*

(1) *The dimension of the intersection $\Lambda^R_{X,A} \cap \Lambda^R_{Y,A}$ is independent of $R \in [0,\infty]$.*
(2) *The dimension of the intersection $\Lambda^R_{X,A} \cap (P_a^+ \oplus V)$ is independent of $R \in [0,\infty]$.*
(3) *The dimension of the intersection $(P_a^- \oplus V) \cap \Lambda^R_{Y,A}$ is independent of $R \in [0,\infty]$.*

2.9. Boundary Conditions

Let X be a 3-manifold with boundary Σ. We will review boundary conditions in the context of the odd signature operator D_A twisted by $A \in \mathcal{A}_X$ and of the tangential operator S_a twisted by $a \in \mathcal{A}_\Sigma$.

Let $\mathcal{B} \subset L^2(\Omega^{0+1+2}(\Sigma; su(2)))$ be a closed subspace. A solution $\varphi \in L^2_1(X)$ to $D_A(\varphi) = 0$ satisfies the *boundary condition* $\mathcal{B}$, if $r(\varphi) \in \mathcal{B} \cap L^2_{\frac{1}{2}}$.

A Lagrangian subspace $\mathcal{P} \subset L^2(\Omega^{0+1+2}(\Sigma; su(2)))$ is called an *Atiyah-Patodi-Singer (APS) boundary condition*, if $\mathcal{P}$ contains all eigenvectors of the tangential operator S_a with sufficiently large eigenvector.

The APS boundary conditions are non-local (or global) boundary conditions. A local boundary condition is a boundary condition $\mathcal{B} \subset L^2(\Omega^{0+1+2}(\Sigma; su(2)))$ which for any open cover U_i of Σ can be written as the intersection

$$\bigcap_i \{\varphi \subset L^2(\Omega^{0+1+2}(\Sigma; su(2))) \mid \varphi|_{U_i} \in \mathcal{B}_{U_i}\},$$

where $\mathcal{B}_{U_i} \subset L^2(\Omega^{0+1+2}(U_i; su(2)))$.

2.10. Maslov Index

Let H be a symplectic Hilbert space with compatible almost complex structure J. A pair of Lagrangians (L, M) in H is called *Fredholm* if $L + M$ is closed and both $\dim(L \cap M)$ and $\operatorname{codim}(L + M)$ are finite. Consider a continuous path (L_t, M_t) of Fredholm pairs of Lagrangians in H. Continuity is measured in the gap topology. If L_t and M_t are transverse at the end points, that is, intersect trivially, then the *Maslov index* $\operatorname{Mas}(L_t, M_t)$ is roughly defined to be a count of how many times L_t and M_t intersect with sign and multiplicity, that is, counting the dimension of the intersection. For a careful definition see [**CLM**], [**N1**] or [**Da1**].

Nicolaescu's main theorem in [**N1**] applied in our context states

$$\mathrm{SF}(D_{A_t}) = \mathrm{Mas}(\Lambda_X(D_{A_t}), \Lambda_Y(D_{A_t})),$$

where $\Lambda_X(D_{A_t})$ and $\Lambda_S(D_{A_t})$ are the Cauchy data spaces associated to the restrictions of D_{A_t} to X and S, which are Lagrangian subspaces of $L^2(\Omega^{0+1+2}(X; su(2)))$. We want to use the homotopy invariance and additivity of the Maslov index facts as well as facts about the adiabatic limits to express spectral flow in terms of a sum of Maslov indices which will in turn correspond to spectral flow on X and Y with certain boundary conditions.

Note that Nicolaescu assumes that the Cauchy data spaces are transverse at the end points. In [**Da2**] this result has been extended to the situation when the Dirac operators are not invertible at the endpoints. Also see Theorem 7.6 in [**KL**] for a proof of the same result.

In this case a compatible convention must be chosen for computing Maslov index.

DEFINITION 2.10.1. Given a continuous 1-parameter family of Fredholm pairs of Lagrangians (L_t, M_t), $t \in [0,1]$, choose $\varepsilon > 0$ small enough so that

(1) $e^{sJ}L_i$ is transverse to M_i for $i = 0, 1$ and $0 < s \leqslant \varepsilon$, and

(2) $(e^{sJ}L_t, M_t)$ is a Fredholm pair for all $t \in [0,1]$ and all $0 \leqslant s \leqslant \varepsilon$.

Then define the *Maslov index* of the pair (L_t, M_t) to be the Maslov index of $(e^{\varepsilon L}L_t, M_t)$.

Then the precise statement of the extended version of Nicolaescu's splitting theorem in the context of the odd-signature operator and $SU(2)$ connections is the following.

THEOREM 2.10.2 (Theorem 4.3,[**Da2**]). *Suppose M is a closed 3-manifold decomposed along a surface Σ into two pieces X and Y, with Σ oriented so that $\Sigma = \partial X = -\partial Y$. Suppose A_t is a continuous path of $SU(2)$ connections on M in cylindrical* *form in a collar of* Σ. *Then* $(\Lambda_X(D_{A_t})), \Lambda_Y(D_{A_t}))$ *is a Fredholm pair of Lagrangians and*

$$\mathrm{SF}(D_{A_t}) = \mathrm{Mas}(\Lambda_X(D_{A_t}), \Lambda_Y(D_{A_t})).$$

We also have a relative version of this theorem (see [**Da1**] and [**N1**]) which relates spectral flow on a manifold with boundary with APS boundary conditions to some Maslov index. It is implied by the results in [**Da2**].

THEOREM 2.10.3. *Suppose* X *is a* 3*-manifold with boundary* Σ. *If* A_t *is a path of connections on* X *in cylindrical form near* Σ *and* $\mathcal{P}_t$ *is a continuous family of self-adjoint APS boundary conditions, then the spectral flow* $\mathrm{SF}(D_{A_t}|X; \mathcal{P}_t)$ *is well defined and* $\mathrm{SF}(D_{A_t}|X; \mathcal{P}_t) = \mathrm{Mas}(\Lambda_X(D_{A_t}), \mathcal{P}_t)$.

Given three Lagrangians L_1, L_2 and L_3 in H, we may define an integer $\tau_H(L_1, L_2, L_3)$ as the signature of the symmetric quadratic form

$$\begin{aligned} Q : L_1 \oplus L_2 \oplus L_3 &\rightarrow \mathbf{R} \\ (x_1, x_2, x_3) &\rightarrow \langle x_1, x_2\rangle_H + \langle x_2, x_3\rangle_H + \langle x_3, x_1\rangle_H \end{aligned}$$

This is the maslov triple index usually considered in the literature. See [**CLM**].

We will use a different Maslov triple index as defined in [**KL**], which is up to normalization the same as Bunke's Maslov triple index in [**Bu**]. (For an alternative on the subject of the η-invariant see [**Wo1**] and [**Wo2**]). It is convenient for our purpose to define it using the main characterization of this Maslov triple index. The following definition summarizes formula (6.24) and (6.21) in [**KL**].

DEFINITION 2.10.4. Let P_t, Q_t and R_t, $t \in [0, 1]$, be paths of Lagrangians of H so that they are pairwise Fredholm. Then the (twisted) Maslov triple index τ_μ is determined by

$$\tau_\mu(P_0, Q_0, R_0) - \tau_\mu(P_1, Q_1, R_1) = \mathrm{Mas}(P, Q) + \mathrm{Mas}(Q, R) - \mathrm{Mas}(P, R)$$

τ_μ and τ_H share some properties. By Proposition 6.11 in [**KL**] we have the useful property $\tau_\mu(P,P,Q) = \tau_\mu(Q,P,P) = 0$. Furthermore τ_μ is additive under direct sums (symplectic additivity).

2.11. Twisted Cohomology

Twisted cohomology plays a crucial role in the computation of the scattering Lagrangian. Computing the scattering Lagrangian is difficult in general because it depends on the metric on X. However, in the case of the odd signature operator twisted by a flat connection A, the De Rham and Hodge theorems provide an identification of the scattering Lagrangian with the image of the twisted cohomology of X in the cohomology of ∂X as in Lemma 2.8.2.

Let X be a space and $\rho : \pi \to \mathrm{Aut}(V)$ be a representation of $\pi = \pi_1 X$ on a vector space V. Let $H^*(X; V_\rho)$ be the *cohomology of X twisted by ρ*. See for example page 98 in [**DavK**] for a definition.

If $X = K(\pi, 1)$ is an Eilenberg-MacLane space, then $H^*(X; V_\rho)$ is isomorphic to the group cohomology $H^*(\pi, V_\rho)$. We will review some facts about group cohomology. For detailed information see [**E**] and [**Brown**].

One chain complex computing $H^*(\pi, V_\rho)$ is the bar resolution. This complex consists of $C^0(\pi, V_\rho) = V$ and $C^n(\pi, V_\rho) = \mathrm{Maps}(\pi^n, V)$. The coboundary maps are defined by

$$\begin{aligned} \delta^0(v)(z_1) &= z_1 \cdot v - v = (z_1 - 1) \cdot v, \text{ and} \\ \delta^n f(z_1, \ldots, z_{n+1}) &= z_1 \cdot f(z_2, \ldots, z_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(z_1, \ldots, z_i z_{i+1}, \ldots, z_{n+1}) \\ &\quad + (-1)^{n+1} f(z_1, \ldots, z_n) \end{aligned}$$

for $v \in V$, $z_i \in \pi$ and $f \in \mathrm{Maps}(\pi, su(2))$, where $(z + z') \cdot v$ means $\rho(z)(v) + \rho(z')(v)$.

We will mostly be interested in computing the zeroth and first cohomology. The relevant coboundary maps are

$$\begin{aligned}\delta^0(v)(z) &= z\cdot v - v = (z-1)\cdot v,\\ \delta^1(f)(z,z') &= f(z) + z\cdot f(z') - f(zz')\end{aligned}$$

for $v \in V$, $z, z' \in \pi$ and $f \in \mathrm{Maps}(\pi, V)$.

DEFINITION 2.11.1. Let ρ be a representation. A *(Fox) derivation with respect to* ρ is a function $f : \pi \to V$ satisying

$$f(zz') = f(z) + z\cdot f(z') \text{ for all } z, z' \in \pi$$

A derivation has the following nice properties, which follow immediately from the definition, but help in computations:

- $f(1) = 0$
- $f(z^{-1}) = -z^{-1}\cdot f(z)$.

If F is the free group generated by $A = \{z_1, \ldots, z_n\}$ and $\rho : F \to \mathrm{Aut}(V)$ a representation, we have the following.

LEMMA 2.11.2. *Any function $f : A \to V$ uniquely determines a derivation on F, which we will call f as well.*

This is not surprising. Its not very illuminating proof may be skipped. The trouble is that a priori it is not clear whether the extension of f to F is well-defined.

PROOF. Observe that we can extend f to a function on $A \cup \{1\} \cup A^{-1}$ by the above properties, where A^{-1} denotes the inverses of all elements in A. For $x \in A$ and $a \in \mathbf{Z}$ a *syllable in A* is given by x^a, where we write $x^a = \underbrace{x\cdots x}_{a}$ and $x^{-a} = \underbrace{x^{-1}\cdots x^{-1}}_{a}$, for $a > 0$ as well as $x^0 = 1$. A *word in A* is a sequence of syllables in A. Then define inductively $f(xy) := f(x) + x\cdot f(y)$, where x and y are a words in A. Furthermore we let $f(x) = 0$ if x is the empty word. First we need to check that f is well-defined by verifying that f preserves associativity. For words v, w, x, y, z in A, we have

$$\begin{aligned} f(vx(yz)w) &= f(v) + v \cdot f(x) + (vx) \cdot f(y) + (vxy) \cdot f(z) + (vxyz) \cdot f(w) \\ &= f(v) + v \cdot (f(xy) + (xy) \cdot f(z)) + (vxyz) \cdot f(w) \\ &= f(v) + f((xy)z) + (vxyz) \cdot f(w) \\ &= f(v) + v \cdot (f(xy) + (xy) \cdot f(z)) + (vxyz) \cdot f(w) \\ &= f(v) + f((xy)z) + (vxyz) \cdot f(w). \end{aligned}$$

For f to be well-defined when considered as a function on the group π, we need to check that f takes the same value on words in A that represent the same element in π. Two words represent the same element in π if they are related by elementary expansions and contractions of the form $xy^0z \sim xz$ and $xy^ay^bz \sim xy^{a+b}z$ for words x, y, z in A and $a, b \in \mathbf{Z}$. We have

$$f(xy^0z) = f(x) + x \cdot f(y^0z) = f(x) + x \cdot f(y^0) + (xy^0) \cdot f(z) = f(x) + x \cdot f(z) = f(xz)$$

and

$$\begin{aligned} f(xy^ay^bz) &= f(x) + x \cdot f(y^a) + (xy^a) \cdot f(y^b) + (xy^ay^b)f(z) \\ &= f(x) + x \cdot (f(y^a) + y^a \cdot f(y^b)) + (xy^ay^b)f(z) \\ &= f(x) + x \cdot f(y^{a+b}) + (xy^{a+b}) \cdot f(z) = f(xy^{a+b}z). \end{aligned}$$

Thus $f : \pi \to Aut(V)$ is well-defined. □

We can extend this to arbitrary finitely generated groups. Let

$$\pi = \langle z_1, \ldots, z_n \mid r_1, \ldots r_m \rangle$$

be the group with generators $S = \{z_1, \ldots, z_n\}$ and relators $R = \{r_1, \ldots, r_m\}$. Let F be the free group generated by S and let $p : F \to \pi$ be the projection to the quotient $\pi = F/R$. Consider a representation $\rho : F \to \mathrm{Aut}(V)$ which descends to a representation $\bar{\rho} : \pi \to \mathrm{Aut}(V)$, that is, $\bar{\rho} \circ p = \rho$. Denote $\bar{\rho}(z)(v)$ by $z \bullet v$. The choice of notation is supposed to remind us that the representation plays a crucial role in derivations, even though their notation is suppressed in the definition of a derivation. Observe that in this notation, $x \cdot v = \bar{x} \bullet v$, if $p(x) = \bar{x}$. Note also that for all $x \in F$ with $p(x) = 1$ get $\rho(x) = \mathrm{Id}$, that is, $x \cdot v = v$ for all $v \in V$.

LEMMA 2.11.3. *Let $f : F \to V$ be a derivation with respect to ρ, and assume that ρ factors through π. Then f descends to a derivation $\bar{f} : \pi \to V$, that is, $\bar{f} \circ p = f$, if and only if $f(r) = 0$ for $r \in R$.*

PROOF. Assume that $f(r) = 0$ for $r \in R$. Then we have $f(r^{-1}) = -r^{-1}f(r) = 0$ as well, and we get for $x, y \in F$

$$f(xr^{\pm 1}y) = f(x) + x \cdot f(r^{\pm 1}) + (xr^{\pm 1}) \cdot f(y) = f(x) + x \cdot f(y) = f(xy).$$

Thus we may define $\bar{f}(\bar{x}) := f(x)$ if $p(x) = \bar{x}$. Furthermore, if $p(x) = \bar{x}$ and $p(y) = \bar{y}$, then

$$\bar{f}(\bar{x}\bar{y}) = f(xy) = f(x) + x \cdot f(y) = \bar{f}(\bar{x}) + x \cdot \bar{f}(\bar{y}) = \bar{f}(\bar{x}) + \bar{x} \bullet \bar{f}(\bar{y}).$$

Thus f descends to a derivation $\bar{f} : \pi \to V$.

On the other hand, if f is a derivation $F \to V$ which descends to π, then in particular $f(r) = f(1) = 0$. ☐

Lemmas 2.11.2 and 2.11.3 provide us with an easy way to keep track of derivations. This will be used in Chapter 6 for group cohomology computations. Lemmas 2.11.2 and 2.11.3 imply the following.

COROLLARY 2.11.4. *Let $\rho : \pi \to V$ be a representation. Then a derivation on π is determined by its values on its generators.*

Since $0 = \delta^1(f)(z, z') = f(z) + z \cdot f(z') - f(zz')$ for all $z, z' \in \pi$ if and only if f is a cocycle, we get

LEMMA 2.11.5. *The 1-cocycles of the bar-resolution are precisely the derivations in Maps*(π, V). □

EXAMPLE 2.11.6. Let M be a torus bundle over S^1. Notice that the universal cover of M is $\mathbf{R}^3$. $\mathbf{R}^3$ is contractible, thus $\pi_i M = \pi_i \mathbf{R}^3$ is trivial for $i \geqslant 2$. It follows that $M = K(\pi_1 M, 1)$, and so we can use the bar resolution to compute its twisted cohomology.

EXAMPLE 2.11.7. More generally, let X be any compact 3 manifold with torsion-free fundamental group. Then it is shown in [**He**] that X is a $K(\pi_1(X), 1)$.

The De Rham map provides the isomorphism between De Rham cohomology of the twisted De Rham complex d_A, where A is a flat $SU(2)$ connection on M and (singular) cohomology of M twisted by $\mathrm{hol}(A)$ $H^*(M; su(2)_{\mathrm{hol}(A)})$.

DEFINITION 2.11.8. Let $\pi : \tilde{M} \to M$ be the universal cover of M. In our context the De Rham map dR is given by the composition:

$$dR : \Omega^i(\mathrm{ad}P_{\mathrm{hol}(A)}) \xrightarrow{\pi^*} \Omega^i(\pi^*\mathrm{ad}P_{\mathrm{hol}(A)}) \overset{\mathrm{hol}(A)}{\cong} \Omega^i(\tilde{M}) \otimes su(2) \xrightarrow{\Phi} \mathrm{Hom}_{\mathbf{Z}\pi}(S_i(\tilde{M}), su(2)),$$

where $\Phi(\sigma)(c) = \int_c \sigma$ denotes integrating an $su(2)$ valued i-form over a singular chain in $S_i(\tilde{M})$.

Note that we need to fix a trivialization of the fiber at a fixed lift of the base point to the universal cover, before we can trivialize $\Omega^i(\pi^*\mathrm{ad}P_{\mathrm{hol}(A)})$ using the holonomy of A.

CHAPTER 3

Atiyah-Patodi-Singer Boundary Conditions

Our distant goal is to relate spectral flow on a closed 3-manifold to spectral flow on pieces of a certain splitting. In order to be able to even talk about spectral flow on a manifold with boundary, we need to impose boundary conditions, which will ultimately stem from the boundary conditions used in Nicolaescu's theorem 2.10.3. In particular the kernel of the path of differential operators will be infinite dimensional if we do not restrict the domain, that is, impose boundary conditions.

Chapter 18 in [**BW**] is dedicated to the study of *(global) elliptic boundary conditions*. We are concerned with particular global elliptic boundary conditions called *APS boundary conditions* (see Section 2.9). In the special case of the twisted odd-signature operator D_A twisted by $A \in \mathcal{A}_Y$ on a 3-manifold Y with torus boundary T, the APS boundary conditions ensure that D_A is self-adjoint and elliptic.

We define a specific family of APS boundary conditions which is large enough to be useful and small enough to have a nice parametrization. We first choose a family of connections $a_{\alpha,\beta}$ on the torus T parametrized by $(\alpha, \beta) \in \mathbf{R}^2$ which covers all gauge equivalence classes of flat connections on T. Then we introduce a space $\widetilde{\mathbf{R}}^2$ (see Figure 3.3.1) with a continuous smooth surjection $\pi : \widetilde{\mathbf{R}}^2 \to \mathbf{R}^2$, where $\pi^{-1}((\frac{1}{2}\mathbf{Z})^2)$ is a collection of circles and π is a homeomorphism on $\pi^{-1}(\mathbf{R}^2 - (\frac{1}{2}\mathbf{Z})^2)$, and we construct a nice family $\mathcal{P}$ of boundary conditions parametrized by a $\alpha \in \widetilde{\mathbf{R}}^2$. In particular we show that given a family of connections A_φ on a manifold with boundary T parametrized by $\varphi \in U \subset \widetilde{\mathbf{R}}^2$, the odd signature operator D_{A_φ} with boundary conditions $\mathcal{P}_\varphi$ is self-adjoint, elliptic and varies continuously in $\varphi \in U$. This provides a convenient framework for working with the odd-signature on a manifold with torus boundary.

The results will be applied to the splitting $M = X \cup_T S$ in Chapter 4, where S is the solid torus. In Chapter 5 we will explicitely compute spectral flow on S with these boundary conditions.

3.1. Connections on the Torus

All of our analysis requires the $SU(2)$ connections on Y to be flat on the torus, because then the dimension of the kernel of the tangential operator is independent of the metric on T and the kernel is isomorphic to the twisted De Rham cohomology of T. The gauge-equivalence classes of flat $SU(2)$-connections on T are in one-to-one correspondence with the conjugacy classes of $SU(2)$-representations of the fundamental group $\pi_1 T$ by Proposition 2.4.5. Furthermore it is induced by the holonomy map, which we will use to specify a convenient family of flat $SU(2)$-connections.

Let χ_T be the variety of $SU(2)$-conjugacy classes of representations of the fundamental group $\pi_1 T$ of the torus $T = S^1 \times S^1$

$$\chi_T := \mathrm{Hom}(\pi_1 T, SU(2))/\mathrm{conj}.$$

Denote the space of $SU(2)$-connections on the torus T by $\mathcal{A}_T$. Pick a trivialization of the principal $SU(2)$-bundle $P_Y = Y \times SU(2)$ over Y, which trivializes the principal $SU(2)$-bundle over T by restricting P_Y to T. Then we can identify the affine space $\mathcal{A}_T$ with $\Omega^1(T; su(2)) := \Omega^1(T) \otimes su(2)$ in the usual way, where the product (or trivial) connection θ corresponds to $0 \in \Omega^1(T; su(2))$. Identify $SU(2)$ and $su(2)$ with the unitary quaternions and imaginary quaternions respectively as in Section 2.3.

Recall that with our identification of $\mathcal{A}_T$ the curvature can be expressed as $F_a = da + a \wedge a$. Let $\mathcal{F}_T = \{a \in \mathcal{A}_T \mid F_a = 0\}$ be the space of flat $SU(2)$-connections on T.

Let $\mathcal{G}_T = C^\infty(T, SU(2))$ be the group of gauge transformations. Then the action of $g \in \mathcal{G}_T$ on $A \in \mathcal{A}$ is given by

$$g \cdot A = g^{-1}Ag + g^{-1}dg.$$

By Proposition 2.4.5 the holonomy map gives a homeomorphism from $\mathscr{M}_T := \mathcal{F}_T/\mathcal{G}_T$ to $\chi(T)$. Let us analyze this map. Let $A \in \mathcal{F}_T$. Let $\gamma : I \to T = T \times 0 \subset P$ be an element of $\pi_1(T)$, that is, $\gamma(0) = \gamma(1)$, and think of γ as a loop in the zero section of P. Consider $g : I \to SU(2)$ with $g(0) = 1$. Then by the rules in Section 2.4 we have

$$\begin{aligned} A(\dot{\gamma_t g_t}) &= A(\dot{\gamma}_t g_t) + A(\gamma_t \dot{g}_t) \\ &= A((R_{g_t})_* \dot{\gamma}_t) + A((i_{\gamma_t g_t})_* g_t^{-1}\dot{g}_t) \\ &= R^*_{g_t}(A)(\dot{\gamma}_t) + A((g_t^{-1}\dot{g}_t)^\#) \\ &= \mathrm{ad}(g_t^{-1})A(\dot{\gamma}_t) + g_t^{-1}\dot{g}_t. \end{aligned}$$

Let g be the unique solution $A(\dot{\gamma_t g_t}) = 0$ and thus to the ordinary differential equation

$$\begin{aligned} -\dot{g}(t)g^{-1}(t) &= A(\dot{\gamma}(t)) \\ g(0) &= \mathrm{Id}. \end{aligned}$$

Then $\mathrm{hol}_\gamma(A) = g(1) \in SU(2)$. If $A = -i\alpha\, dm - i\beta\, dl$ with $(\alpha, \beta) \in \mathbf{R}^2$, then $\mathrm{hol}_\lambda(A) = \rho_{(\alpha,\beta)}$ is given in quaternionic notation (see Section 2.3) by

$$\begin{aligned} \rho_{(\alpha,\beta)} : \pi_1(T) &\to SU(2) \\ \mu &\mapsto e^{2\pi i\alpha} \\ \lambda &\mapsto e^{2\pi i\beta}. \end{aligned} \tag{3.1.3}$$

Notice, that $\mathbf{R}^2 \to \chi_T, (\alpha, \beta) \to \rho_{(\alpha,\beta)}$ is a branched cover of χ_T, with branch points the half integer lattice, which map to central representations, and covering transformations $(\alpha, \beta) \to (\pm\alpha + m, \pm\beta + n)$, $(m, n) \in \mathbf{Z}^2$. Each (α, β) then also corresponds to an $SU(2)$-connection $-i\alpha\, dm - i\beta\, dl$ on T. We summarize:

Lemma 3.1.1. *We have a smooth family* $\{-i\alpha\, dm - i\beta\, dl\}$ *of flat connections with holonomy* $\rho_{(\alpha,\beta)}$ *as in (3.1.3) parametrized by* $\mathbf{R}^2$*, which projects onto* $\mathscr{M}_T = \chi_T$*. The central representations correspond to the half integer lattice.* □

The Hodge star operator $*$ depends on the orientation as well as the metric used for T. Let $T = S^1 \times S^1$ be parametrized by $(m, l) \mapsto (e^{im}, e^{il})$, where $m, l \in [0, 2\pi)$. Choose the product metric on T, that is, $\{\frac{\partial}{\partial m}, \frac{\partial}{\partial l}\}$ is an orthonormal basis. Choose the orientation determined by $dm \wedge dl \in \Omega^2(T)$.

Definition 3.1.2. Let $a_{\alpha,\beta} := -i\alpha\, dm - i\beta\, dl$. In view of the above Lemma we write $P^{\pm}_{\alpha,\beta} := P^{\pm}_{a_{\alpha,\beta}}$, $P^{\pm}_{\mathbf{C}j,\alpha,\beta} := P^{\pm}_{\mathbf{C}j,a_{\alpha,\beta}}$, $\mathcal{H}^{0+1+2}_{\alpha,\beta}(T; su(2)) := \mathcal{H}^{0+1+2}_{a_{\alpha,\beta}}(T; su(2))$, $\Delta_{\alpha,\beta} := \Delta_{a_{\alpha,\beta}}$, $S_{\alpha,\beta} := S_{a_{\alpha,\beta}}$, $d_{\alpha,\beta} = d_{a_{\alpha,\beta}}$.

3.2. Harmonic Forms on the Torus

Proposition 3.2.1.

(1) $\mathcal{H}^0_{\alpha,\beta}(T; su(2))$ *is isomorphic to*
- $\mathbf{R}$ *and generated by* i *for* $(\alpha, \beta) \notin (\frac{1}{2}\mathbf{Z})^2$*, and*
- $\mathbf{R}^3$ *and generated by* i, $e^{i(2\alpha m + 2\beta l)}j$ *and* $e^{i(2\alpha m+2\beta l)}k$ *for* $(\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2$.

(2) $\mathcal{H}^1_{\alpha,\beta}(T; su(2))$ *is isomorphic to*
- $\mathbf{R}^2$ *and generated by* $i\, dm$ *and* $i\, dl$ *for* $(\alpha, \beta) \notin (\frac{1}{2}\mathbf{Z})^2$*, and*
- $\mathbf{R}^6$ *and generated by* $i\, dm$, $i\, dl$, $e^{i(2\alpha m+2\beta l)}j\, dm$, $e^{i(2\alpha m+2\beta l)}k\, dm$, $e^{i(2\alpha m+2\beta l)}j\, dl$ *and* $e^{i(2\alpha m+2\beta l)}k\, dl$ *for* $(\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2$.

(3) $\mathcal{H}^2_{\alpha,\beta}(T; su(2)) \cong \mathcal{H}^0_{\alpha,\beta}(T; su(2))$ *via the Hodge star.*

Proof. By Lemma 2.5.5 the twisted Laplacian preserves the splitting

$$\mathcal{H}^{0+1+2}_{\alpha,\beta}(T; su(2)) = \mathcal{H}^{0+1+2}_{\alpha,\beta}(T; \mathbf{R}i) \oplus \mathcal{H}^{0+1+2}_{\alpha,\beta}(T; \mathbf{C}j).$$

We recall that $\mathcal{H}^*_{(\alpha,\beta)}(T; su(2)) = \mathrm{Ker}(d_{\alpha,\beta}) \cap \mathrm{Ker}(d^*_{\alpha,\beta})$. For the 0-forms we have $d^*_{\alpha,\beta} = 0$.

First consider $\varphi \in \mathcal{H}^0_{\alpha,\beta}(T;\mathbf{R}i)$. Then $[a_{\alpha,\beta},\varphi]=0$, that is, $d\varphi = d\varphi + [a_{\alpha,\beta},\varphi] = 0$. Thus $\varphi \in \mathbf{R}i$ is constant. Suppose $\varphi j \in \mathcal{H}^0_{\alpha,\beta}(T;\mathbf{C}j)$. Then

$$d(\varphi j) = -[a_{\alpha,\beta},\varphi j] = -2a_{\alpha,\beta}\varphi j = i(2\alpha\, dm + 2\beta\, dl)j$$

yields $\varphi j = Ce^{i(2\alpha m+2\beta l)}j$, $C \in \mathbf{C}$, which is a function on T for $C \neq 0$ if and only if $\alpha,\beta \in \frac{1}{2}\mathbf{Z}$.

A quick computation shows for all $su(2)$-valued functions φ_m on T that $\Delta_a(\varphi_m\, dm) = \tilde{\varphi}_m\, dm$, where $\tilde{\varphi}_m$ also denotes some $su(2)$-valued function on T. The same holds for dl. (Actually this fact holds in general.) Therefore any eigenfunction is a sum of eigenfunctions of the form $\varphi_m\, dm$ and $\varphi_l\, dl$.

Thus let $\varphi = \varphi_m dm \in \mathcal{H}^1_{\alpha,\beta}(T;\mathbf{R}i)$. Then again $[a_{\alpha,\beta},\varphi]=0$. Thus $d\varphi = 0$ and $d * \varphi = 0$, that is, $\varphi_m \in \mathbf{R}i$ is constant. The same argument works for dl.

Now consider $\varphi = \varphi_m j\, dm \in \mathcal{H}^1_{\alpha,\beta}(T;\mathbf{C}j)$. Then by $d_{\alpha,\beta}\varphi j = 0$ we must have

$$\frac{\partial \varphi_m}{\partial l} j\, dl \wedge dm = d\varphi j = -[a_{\alpha,\beta},\varphi j] = 2\beta i \varphi_m j\, dl \wedge dm$$

and similarly by $d^*_{\alpha,\beta}\varphi j = 0$

$$\frac{\partial \varphi_m}{\partial m} j\, dm \wedge dl = -\frac{\partial \varphi_m}{\partial m} j\, dl \wedge dm = d * \varphi j = -[a_{\alpha,\beta}, *\varphi j] = 2\alpha i \varphi_m j\, dm \wedge dl.$$

Thus like before $\varphi j = Ce^{i(2\alpha m+2\beta l)}j\, dm$ which is a function on T if and only if $\alpha,\beta \in \frac{1}{2}\mathbf{Z}$. The same holds for dl. □

3.3. Boundary Conditions

We want a path of self-adjoint operators which is continuous in the graph topology. Boundary conditions are necessary to make the odd signature on Y (unbounded) self-adjoint. In order for the odd signature operator D_{A_t} to vary continuously and be self-adjoint, we need a path of boundary conditions which is continuous in the gap topology. See [**BLP**] for details.

It would be particularly nice to have a continuous family of Lagrangians in $L^2(\Omega^{0+1+2}(T; su(2)))$ parametrized by the family $a_{\alpha,\beta}$ of connections on the torus T from Definition 3.1.2, and thus by $\mathbf{R}^2$, because any path of connections on Y is gauge-equivalent to a path which restricts to a path in the family $a_{\alpha,\beta}$. We would like to pick a continuous family of Lagrangians in $\mathcal{H}^{0+1+2}_{(\alpha,\beta)}(T; su(2))$ and extend it to a family of APS boundary conditions in $L^2(\Omega^{0+1+2}(T; su(2)))$ by P^+. Unfortunately we run into problems when passing through $(\frac{1}{2}\mathbf{Z})^2$, because the dimension of the kernel of the tangential operator jumps up. This means that eigenvectors (with non-zero eigenvalues) from $P^\pm$ have eigenvalue 0 when limiting to $(\frac{1}{2}\mathbf{Z})^2$. Even worse, for a smooth path ρ_t, $t \in [-1, 1]$, through $\rho_0 \in (\frac{1}{2}\mathbf{Z})^2$ we have

$$\lim_{t \to 0^+} P^+_{\rho_t} \neq \lim_{t \to 0^-} P^+_{\rho_t}.$$

In fact the $\mathcal{H}^{0+1+2}_{\alpha,\beta}(T; \mathbf{C}j)$ part of the limits turn out to be orthogonal to each other. Thus we will introduce the following space to parametrize the Lagrangians. We will see later, why this is exactly what we need. The advantage of this space is that we can easily homotop paths of connections together with boundary conditions, thus getting continuous paths of self-adjoint operators.

Definition 3.3.1. Let $\tilde{\mathbf{R}}^2 := \mathbf{R}^2 \times S^1/\sim$, where $(\alpha, \beta, \theta) \sim (\alpha, \beta, 1)$ if $(\alpha, \beta) \notin (\frac{1}{2}\mathbf{Z})^2$. We will simply write $(\alpha, \beta, \theta) \in \tilde{\mathbf{R}}^2$. Alternatively it is convenient to think of elements in $\tilde{\mathbf{R}}^2$ as being of the form

$$\begin{cases} (\alpha, \beta, \theta) & \text{if } (\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2 \text{ and } \theta \in S^1, \\ (\alpha, \beta) & \text{if } (\alpha, \beta) \notin (\frac{1}{2}\mathbf{Z})^2. \end{cases}$$

Denote by $\pi : \tilde{\mathbf{R}}^2 \to \mathbf{R}^2$ the projection $(\alpha, \beta, \theta) \to (\alpha, \beta)$. We are also going to define a bijection h between $\tilde{\mathbf{R}}^2$ and the space $\dot{\mathbf{R}}^2$ shown in Figure 3.3.1, namely $\mathbf{R}^2$ with open disks of radius $\frac{1}{8}$ removed around all half integer lattice points with the induced topology.

We will describe what this bijection looks like around the origin. At all the other half integer lattice points we get a similar bijection via translation. Away from disks of radius $\frac{1}{4}$ around each half integer lattice point the bijection is the identity map. Identify $\mathbf{R}^2$ with $\mathbf{C}$ in the usual way. Let $D \subset \mathbf{C}$ be the disk of radius $\frac{1}{4}$, $\tilde{D} := D \times S^1/\sim$ and $A \subset \mathbf{C}$ the disk of radius $\frac{1}{4}$ with an open disk of radius $\frac{1}{8}$ around the origin removed, that is, an annulus. Let $\eta : \mathbf{R} \to [0,1]$ be a smooth (cut-off) function with

$$\eta(t) = \begin{cases} 0 & t \leqslant \frac{1}{8} \\ \text{a homeomorphism onto } [0,1] & \frac{1}{8} \leqslant t \leqslant \frac{1}{4} \\ 1 & t \geqslant \frac{1}{4} \end{cases}$$

Then the bijection $h : A \to \tilde{D}$ is given by $h(z) := (\eta(|z|) \cdot z, \frac{z}{|z|}) \in \tilde{D}$.

We give $\widetilde{\mathbf{R}}^2$ the topology that makes the bijection $h : \widetilde{\mathbf{R}}^2 \to \dot{\mathbf{R}}^2$ into a homeomorphism. Notice that h is a diffeomorphism away from $(\frac{1}{2}\mathbf{Z})^2$.

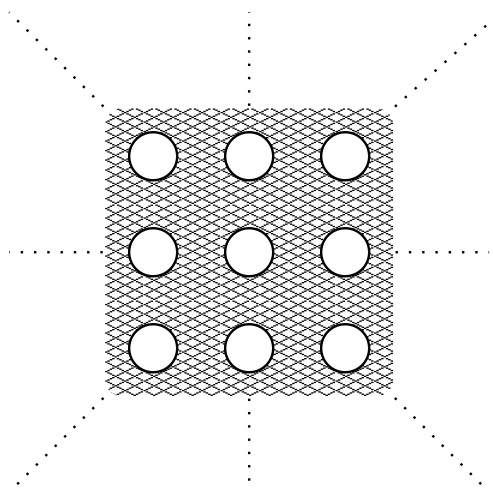

FIGURE 3.3.1. $\dot{\mathbf{R}}^2 \approx \widetilde{\mathbf{R}}^2$

THEOREM 3.3.2. *Let $(\alpha, \beta) \in \mathbf{R}^2$ and $\theta \in S^1 \subset \mathbf{C}$. Then*

(1) $P^{\pm}_{\alpha,\beta} : \mathbf{R}^2 - (\frac{1}{2}\mathbf{Z})^2 \to$ *closed subspaces of* $L^2(\Omega^{0+1+2}(T; su(2)))$ *is continuous.*

(2) *Moreover* $\lim_{t\to 0^+} P^{\pm}_{(\alpha,\beta)\pm t(\mathrm{Re}\,\theta,\mathrm{Im}\,\theta)}$ *exists and we can define*

$$K^{\pm}_{(\alpha,\beta,\theta)} := \lim_{t\to 0^+} P^{\pm}_{(\alpha,\beta)\pm t(\mathrm{Re}\,\theta,\mathrm{Im}\,\theta)}/P^{\pm}_{(\alpha,\beta)} \subset \mathcal{H}^{0+1+2}_{(\alpha,\beta)}(T;\mathbf{C}j).$$

Note that $K^{\pm}_{(\alpha,\beta,\theta)} = 0$ *for* $(\alpha,\beta)\notin(\frac{1}{2}\mathbf{Z})^2$.

(3) $P^{\pm}_{\alpha,\beta}\oplus K^{\pm}_{(\alpha,\beta,\theta)} : \widetilde{\mathbf{R}}^2 \to$ *closed subspaces of* $L^2(\Omega^{0+1+2}(T;su(2)))$ *is continuous.*

(4) *If* $(\alpha,\beta)=\frac{1}{2}(r,s)\in(\frac{1}{2}\mathbf{Z})^2$, *then* $K^{\pm}_{(\alpha,\beta,\theta)} = \langle\{\psi^{\pm}_1 j, \psi^{\pm}_2 j, \psi^{\pm}_1 k, \psi^{\pm}_2 k\}\rangle$ *where*

$$\psi^{\pm}_1 = e^{i(rm+sl)}(1 \mp (i\mathrm{Im}\theta\, dm - i\mathrm{Re}\theta\, dl))$$

and

$$\psi^{\pm}_2 = e^{i(rm+sl)}(dm\wedge dl \pm (i\mathrm{Re}\theta\, dm + i\mathrm{Im}\theta\, dl)).$$

Notice from the explicit description of $K^{\pm}$ that $K^{\pm}_{(\alpha,\beta,\theta)} = K^{\mp}_{(\alpha,\beta,-\theta)}$. Before we prove this theorem, an eigenspace decomposition will be useful to compute $K^{\pm}_{(\alpha,\beta,\theta)}$ explicitly and study the behaviour of the family $P^{\pm}_{(\alpha,\beta)}\oplus K^{\pm}_{(\alpha,\beta,\theta)}$ around the half integer lattice. It is a lengthy but straight forward computation.

Proposition 3.3.3. *Fix* $a = a_{\alpha,\beta}$. *We have the following decomposition of*

$$L^2(\Omega^{0+1+2}(T;su(2))) = L^2(\Omega^{0+1+2}(T;\mathbf{R}i)) \oplus L^2(\Omega^{0+1+2}(T;\mathbf{C}j))$$

into eigenspaces of Δ_a:

$$L^2(\Omega^{0+1+2}(T;\mathbf{R}i)) = \overline{\bigoplus_{(r,s)\in(\frac{1}{2}\mathbf{Z})^2} \sin(rm+sl)\mathcal{H}^{0+1+2}(T;\mathbf{R}i)\oplus\cos(rm+sl)\mathcal{H}^{0+1+2}(T;\mathbf{R}i)}^{L^2}$$

$$L^2(\Omega^{0+1+2}(T;\mathbf{C}j)) = \overline{\bigoplus_{(r,s)\in(\frac{1}{2}\mathbf{Z})^2} e^{i(rm+sl)}\mathcal{H}^{0+1+2}(T;\mathbf{C}j)}^{L^2}$$

where $\mathcal{H}^{0+1+2}(T;\mathbf{R}i)$ *and* $\mathcal{H}^{0+1+2}(T;\mathbf{C}j)$ *are the harmonic forms (of the Laplacian* Δ*) at the trivial connection,* m *and* l *are coordinates on the torus and*

(1) *the forms in* $\sin(rm+sl)\mathcal{H}^{0+1+2}(T;\mathbf{R}i) \oplus \cos(rm+sl)\mathcal{H}^{0+1+2}(T;\mathbf{R}i)$ *are eigenvectors of* $\Delta_{a_{\alpha,\beta}}$ *with eigenvalue* r^2+s^2*, and*

(2) *the forms in* $e^{i(rm+sl)}\mathcal{H}^{0+1+2}(T;\mathbf{C}j)$ *are eigenvectors of* $\Delta_{a_{\alpha,\beta}}$ *with eigenvalue* $(r-2\alpha)^2+(s-2\beta)^2$.

Proof. We calculate

$$\begin{aligned}
-*d_a*d_a e^{i(rm+sl)}j &= -*d_a*\big(e^{i(rm+sl)}(ir\,dm+is\,dl)j+\big[-\alpha i\,dm-\beta i\,dl, e^{i(rm+sl)}j\big]\big)\\
&= -*d_a*\big(e^{i(rm+sl)}(ir\,dm+is\,dl)j+e^{i(rm+sl)}(-2\alpha i\,dm-2\beta i\,dl)j\big)\\
&= -*d_a*\big(e^{i(rm+sl)}[i(r-2\alpha)\,dm+i(s-2\beta)\,dl]j\big)\\
&= -*d_a\left(e^{i(rm+sl)}[i(r-2\alpha)\,dl-i(s-2\beta)\,dm]j\right)\\
&= -*\big(e^{i(rm+sl)}(ir\,dm+is\,dl)\wedge[i(r-2\alpha)\,dl-i(s-2\beta)\,dm]j\\
&\quad +\big[-\alpha i\,dm-\beta i\,dl, e^{i(rm+sl)}[i(r-2\alpha)\,dl-i(s-2\beta)\,dm]j\big]\big)\\
&= -*\big(e^{i(rm+sl)}[-r(r-2\alpha)\,dm\wedge dl+s(s-2\beta)\,dl\wedge dm]j\\
&\quad + e^{i(rm+sl)}[-2\alpha(r-2\alpha)\,dm\wedge dl+2\beta(s-2\beta)\,dl\wedge dm]j\big)\\
&= -*\big(e^{i(rm+sl)}[-(r-2\alpha)^2\,dm\wedge dl-(s-2\beta)^2\,dm\wedge dl]j\big)\\
&= [(r-2\alpha)^2+(s-2\beta)^2]e^{i(rm+sl)}j
\end{aligned}$$

Thus $e^{i(rm+sl)}j$ is an eigenvector with eigenvalue $(r-2\alpha)^2+(s-2\beta)^2$. $e^{i(rm+sl)}j\,dm$ is also eigenvector with eigenvalue $(r-2\alpha)^2+(s-2\beta)^2$, because

$$\begin{aligned}
&-*d_a*d_a e^{i(rm+sl)}j\,dm \\
&\quad= -*d_a*\big(e^{i(rm+sl)}(is\,dl\wedge dm)j+\big[-\alpha i\,dm-\beta i\,dl, e^{i(rm+sl)}j\,dm\big]\big) \\
&\quad= -*d_a*\big(e^{i(rm+sl)}(is\,dl\wedge dm)j+e^{i(rm+sl)}(-2\beta i\,dl\wedge dm)j\big) \\
&\quad= -*d_a*\big(e^{i(rm+sl)}i(s-2\beta)\,dl\wedge dmj\big) \\
&\quad= *d_a\big(e^{i(rm+sl)}i(s-2\beta)j\big) \\
&\quad= *\big(-e^{i(rm+sl)}(s-2\beta)(r\,dm+s\,dl)j+\big[-\alpha i\,dm-\beta i\,dl, e^{i(rm+sl)}i(s-2\beta)j\big]\big) \\
&\quad= -*\big(e^{i(rm+sl)}(s-2\beta)(r\,dm+s\,dl-2\alpha\,dm-2\beta\,dl)j\big) \\
&\quad= -e^{i(rm+sl)}[(r-2\alpha)(s-2\beta)\,dl-(s-2\beta)^2dm]j
\end{aligned}$$

and

$$\begin{aligned}
&-d_a*d_a*e^{i(rm+sl)}j\,dm \\
&\quad= -d_a*d_a e^{i(rm+sl)}j\,dl \\
&\quad= -d_a*\big(e^{i(rm+sl)}(ir\,dm\wedge dl)j+\big[-\alpha i\,dm-\beta i\,dl, e^{i(rm+sl)}j\,dl\big]\big) \\
&\quad= -d_a*\big(e^{i(rm+sl)}(ir\,dm\wedge dl)j+e^{i(rm+sl)}(-2\alpha i\,dm\wedge dl)j\big) \\
&\quad= -d_a*\big(e^{i(rm+sl)}i(r-2\alpha)\,dm\wedge dlj\big) \\
&\quad= -d_a\big(e^{i(rm+sl)}i(r-2\alpha)j\big) \\
&\quad= e^{i(rm+sl)}(r-2\alpha)(r\,dm+s\,dl)j+\big[-\alpha i\,dm-\beta i\,dl, e^{i(rm+sl)}i(r-2\alpha)j\big] \\
&\quad= e^{i(rm+sl)}(r-2\alpha)(r\,dm+s\,dl-2\alpha\,dm-2\beta\,dl)j \\
&\quad= -e^{i(rm+sl)}[-(r-2\alpha)(s-2\beta)\,dl-(r-2\alpha)^2dm]j.
\end{aligned}$$

Through multiplication by i or application of J we get 6 more (linearly independent) eigenvectors with eigenvalue $(r-2\alpha)^2+(s-2\beta)^2$ from $e^{i(rm+sl)}j$ and $e^{i(rm+sl)}j\,dm$.

Similarly $\sin(rm+sl)i$, $\cos(rm+sl)i$ are eigenvectors with eigenvalue r^2+s^2. Multiplication by i or application of J gives 6 more eigenvectors. Four of these eight are 0 if $(r,s)=(0,0)$, otherwise they are linearly independent.

It is well known that $\{e^{i(rm+sl)}\}_{r,s}$ and $\{\sin(rm+sl),\cos(rm+sl)\}_{r,s}$ is an eigenbasis of $L^2(\mathbf{C})$ and $L^2(\mathbf{R})$ for Δ respectively. Furthermore the twisted Laplacian preserves the form-part, for example $\Delta_a(\varphi\, dm) = \tilde{\varphi}\, dm$. Thus the above vectors form an eigenbasis for Δ_a. □

Proof of Theorem 3.3.2. For the continuity of $P^{\pm}_{\alpha,\beta}$ away from $(\frac{1}{2}\mathbf{Z})^2$ see [**KK3**]. We could use analytic perturbation theory as in [**K**] to show that for a continuous path of eigenvalues we have a continuous path of eigenvectors and deduce the existence of the limit. But we have to explicitly compute $K^{\pm}_{(\alpha,\beta,\theta)}$ for $(\alpha,\beta)=\frac{1}{2}(r,s)\in(\frac{1}{2}\mathbf{Z})^2$ using Proposition 3.3.3 anyway to show the continuouity of $P^{\pm}_{\alpha,\beta}\oplus K^{\pm}_{(\alpha,\beta,\theta)}$. This will imply the existence of the limit when we approach half integer lattice points.

By Lemma 3.3.3, $\varphi\in e^{i(rm+sl)}\mathcal{H}^{0+1+2}(T;\mathbf{C}j)$ is an eigenvector of $\Delta_{(\alpha_t,\beta_t)}=\Delta_{a_{(\alpha_t,\beta_t)}}$ for $(\alpha_t,\beta_t)=\frac{1}{2}(r,s)+\frac{1}{2}t(\mathrm{Re}\theta,\mathrm{Im}\theta)$ with eigenvalue $t^2=(r-2\alpha_t)^2+(s-2\beta_t)^2$. For $\varphi=e^{i(rm+sl)}j$ we can apply Lemma 2.6.2 to see that

$$\varphi+\frac{1}{t}S_{(\alpha_t,\beta_t)}\varphi = e^{i(rm+sl)}\left(1-\frac{1}{t}i(r-2\alpha_t)\,dl+\frac{1}{t}i(s-2\beta_t)\,dm\right)j$$

is a t-eigenvector of $S_{(\alpha_t,\beta_t)}$. This yields

$$\lim_{t\to 0}(\varphi\pm\frac{1}{t}S_{(\alpha_t,\beta_t)}\varphi)=e^{i(rm+sl)}(1\mp(i\mathrm{Im}\theta\,dm-i\mathrm{Re}\theta\,dl)).$$

For $\varphi=e^{i(rm+sl)}j\,dm\wedge dl$ we get

$$S_{(\alpha_t,\beta_t)}\varphi=e^{i(rm+sl)}(i(r-2\alpha_t)\,dm+i(s-2\beta_t)\,dl)j.$$

Thus we similarly get

$$\lim_{t\to 0}(\varphi\mp\frac{1}{t}S_{(\alpha_t,\beta_t)}\varphi)=e^{i(rm+sl)}(dm\wedge dl\pm(i\mathrm{Re}\theta\,dm+i\mathrm{Im}\theta\,dl))j.$$

By doing this computation for $\varphi = e^{i(rm+sl)}k$ and $\varphi = e^{i(rm+sl)}k\,dm \wedge dl$ as well, we get 8 linearly independent eigenvectors, which by definition lie in either $K^+_{(\alpha,\beta,\theta)}$ or $K^-_{(\alpha,\beta,\theta)}$. By Lemma 2.6.2 the kernels of the Laplacian and the tangential operators are the same, which in particular implies that the $\mathbf{C}j$ part of $\mathrm{Ker}S_{(\alpha,\beta,\theta)}$ is 8-dimensional. Thus the L^2-span of the above eigenvectors make up $K^+_{(\alpha,\beta,\theta)}$ or $K^-_{(\alpha,\beta,\theta)}$. This completes the computation of $K^\pm_{(\alpha,\beta,\theta)}$.

Now we are ready to prove that $P^+_{\alpha,\beta} \oplus K^+_{\alpha,\beta,\theta}$ is a continuous family parametrized by $(\tilde{\mathbf{R}})^2$. Away from $(\frac{1}{2}\mathbf{Z})^2$ it is clear that the family $P^+_{\alpha,\beta} \oplus K^+_{\alpha,\beta,\theta}$ is continuous, because $K^+_{\alpha,\beta,\theta} = 0$ and $P^+_{\alpha,\beta}$ changes continuously as long as the dimension of $\mathrm{Ker}S_{\alpha,\beta}$ does not change. To show continuity at half integer lattice points, it suffices to show that for any continuous path $\tilde{\rho}_t = (\tilde{\alpha}_t, \tilde{\beta}_t, \theta_t)$ in $\tilde{\mathbf{R}}^2$ that limits to $\tilde{\rho}(0) = (0, 0, \theta_0)$ we have

$$\lim_{t\to 0} P^+_{\tilde{\alpha}_t,\tilde{\beta}_t} \oplus K^+_{\tilde{\rho}_t} = P^+_{0,0} \oplus K^+_{0,0,\theta}.$$

For all other half integer lattice points the argument is the same, only the notation would be cumbersome. By definition a path in $\tilde{\mathbf{R}}^2$ is continuous in a ball of radius $\frac{1}{8}$ around $(0, 0)$, if $h \circ \tilde{\rho}_t = \rho_t = (\alpha_t, \beta_t)$ is continuous in $\mathbf{R}^2$, where $h : \tilde{\mathbf{R}}^2 \to \dot{\mathbf{R}}^2 \subset \mathbf{R}^2$ is the homeomorphism given in 3.3.1. We consider two cases:

(1) If $|\rho_t| = \frac{1}{8}$ for t small, then $\tilde{\rho}_t = (0, 0, \theta_t)$ and θ_t continuous for small t. Elementary triangle equality arguments applied to $P^+_{(0,0)+s(\mathrm{Re}\theta,\mathrm{Im}\theta)}$, s small, show that $K^+_{0,0,\theta_t}$ is continuous.

(2) Consider $|\rho_t| \neq \frac{1}{8}$ for $t > 0$ small. We have $(\tilde{\alpha}_t, \tilde{\beta}_t) = (\alpha_t, \beta_t)\eta(\sqrt{\alpha_t^2 + \beta_t^2})$ and $\theta_0 = 8(\alpha_0 + i\beta_0) \in S^1$. We need to check that $K^+_{0,0,\theta} = \lim_{t\to 0}(P^+_{\tilde{\alpha}_t,\tilde{\beta}_t}/P^+_{0,0})$. We can apply Lemma 2.6.2 to see that for the $((2\tilde{\alpha}_t)^2 + (2\tilde{\beta}_t)^2)$-eigenvector

$\varphi = j$ of $\Delta_{\tilde{\alpha}_t,\tilde{\beta}_t}$ we get a $2\sqrt{\tilde{\alpha}_t^2+\tilde{\beta}_t^2}$-eigenvector of $S_{\tilde{\alpha}_t,\tilde{\beta}_t}$:

$$\begin{aligned}
&\varphi + \frac{1}{2\sqrt{\tilde{\alpha}_t^2+\tilde{\beta}_t^2}} S_{\tilde{\alpha}_t,\tilde{\beta}_t}\varphi \\
&= \quad \varphi + \frac{2i\tilde{\alpha}_t\,dl - 2i\tilde{\beta}_t\,dm}{2\sqrt{\tilde{\alpha}_t^2+\tilde{\beta}_t^2}} \\
&= \quad j + \frac{i\alpha_t\eta(\sqrt{\alpha_t^2+\beta_t^2})dl - i\beta_t\eta(\sqrt{\alpha_t^2+\beta_t^2})dm}{\sqrt{\alpha_t^2\eta(\sqrt{\alpha_t^2+\beta_t^2})^2+\beta_t^2\eta(\sqrt{\alpha_t^2+\beta_t^2})^2}} j \\
&= \quad j + \frac{i\alpha_t dl - i\beta_t dm}{\sqrt{\alpha_t^2+\beta_t^2}} j \\
&\xrightarrow{t\to 0} \quad j + 8(i\alpha_0 dl - i\beta_0 dm)j = j - (i\mathrm{Im}\theta\,dm - i\mathrm{Re}\theta\,dl)j
\end{aligned}$$

Similarly we get for the other $2\sqrt{\tilde{\alpha}_t^2+\tilde{\beta}_t^2}$-eigenvectors of $S_{\tilde{\alpha}_t,\tilde{\beta}_t}$, that they limit to the other basis elements of $K^+_{0,0,\theta}$ as given in the statement of Theorem 3.3.2.

□

DEFINITION 3.3.4. Given a continuous family of Lagrangians L of $\mathcal{H}^{0+1+2}(T;\mathbf{R}i)$ parametrized by a subset $U \subset \tilde{\mathbf{R}}^2$, define – in view of the splitting in Lemma 2.5.5 – a family $\mathcal{P}^{\pm}(L)$ of subspaces of $L^2(\Omega^{0+1+2}(T;su(2)))$ parametrized by U as follows

$$\mathcal{P}^{\pm}_{(\alpha,\beta,\theta)}(L) := (P^{\pm}_{\mathbf{R}i} \oplus L_{(\alpha,\beta,\theta)}) \oplus (P^{\pm}_{\mathbf{C}j,(\alpha,\beta)} \oplus K^{\pm}_{(\alpha,\beta,\theta)})$$

For our application, the family of Lagrangians $L_{\alpha,\beta,\theta}$ will be independent of (α,β,θ). Notice that $K_{(\alpha,\beta,\theta)}$ vanishes away from $(\frac{1}{2}\mathbf{Z})^2$, while we have "blown up" the points of $(\frac{1}{2}\mathbf{Z})^2$ and removed the singularities that paths through $(\frac{1}{2}\mathbf{Z})^2$ encounter. Here is a corollary of Proposition 3.3.2.

COROLLARY 3.3.5. *$\mathcal{P}^{\pm}(L)$ is a continuous family of Lagrangians parametrized by* $\tilde{\mathbf{R}}^2$. □

The family $\mathcal{P}^{\pm}(L)$ and Corollary 3.3.5 will be used in the following two ways in the following chapters for a splitting of a closed manifold $M = S \cup_T X$ into a solid torus S and its complement X.

(1) On S we want to study the spectral flow of the path of odd signature operators twisted by a path of connections in a family $A_{\alpha,\beta}$ given in Definition 5.2.1. After possibly reparametrizing this path so that it lifts to $\widetilde{\mathbf{R}}^2$, we can find continuous boundary conditions of the form $\mathcal{P}^{\pm}(L)$. Then by Corollary 3.3.5 a path in $\widetilde{\mathbf{R}}^2$ gives a path of continuous connections together with a continuous path of boundary conditions. Thus we get a continuous family of self-adjoint, elliptic odd signature operators on S parametrized by $\widetilde{\mathbf{R}}^2$.

(2) Given a path of connections on X which restricts to a path $a_{\rho(t)}$ of connections on T in the family $a_{\alpha,\beta}$ from Definition 3.1.2, we can reparametrize ρ_t so that $\rho(t)$ lifts to a path $\tilde{\rho}(t)$ in $\widetilde{\mathbf{R}}^2$. This will give us a path of self-adjoint, elliptic odd signature operators on X.

It is a priori not clear what a (continuous) path in $\widetilde{\mathbf{R}}^2$ looks like near $(\frac{1}{2}\mathbf{Z})^2$ when viewed as a path in $\mathbf{R}^2$ with the usual topology. In order to apply the tools provided in Chapter 4, one usually starts out with a path in $\mathbf{R}^2$ and needs to know, that this path will lift to a path in $\widetilde{\mathbf{R}}^2$. Thus it would be useful to know what continuous paths in $\widetilde{\mathbf{R}}^2$ look like. However, the following chapters will only deal with linear paths, which are continuous by Definition 3.3.2.

So even though we will never use them in this thesis, here are some examples of (continuous) paths in $\widetilde{\mathbf{R}}^2$.

LEMMA 3.3.6. *Suppose $(\alpha_t, \beta_t, \theta_t)$ is a (continuous) path in $\widetilde{\mathbf{R}}^2$ with $(\alpha_t, \beta_t) \notin (\frac{1}{2}\mathbf{Z})^2$ for $t \neq 0$ and $(\alpha_0, \beta_0) \in (\frac{1}{2}\mathbf{Z})^2$. Then $(\alpha_t, \beta_t, \theta_t)$ is continuous if one of the following conditions is satisfied:*

(1) $(\dot{\alpha}(0), \dot{\beta}(0)) \neq 0$ *exists and* $\frac{\dot{\alpha}(0)+i\dot{\beta}(0)}{\sqrt{\dot{\alpha}(0)^2+\dot{\beta}(0)^2}} = \theta$

(2) $(\dot{\alpha}(t), \dot{\beta}(t))$ *exists for small* t *and* $\lim_{t\to 0} \frac{\dot{\alpha}(t)+i\dot{\beta}(t)}{\sqrt{\dot{\alpha}(t)^2+\dot{\beta}(t)^2}} = \theta$

□

CHAPTER 4

Spectral Flow on a Closed 3-Manifold

We develop a splitting formula for spectral flow, which expresses spectral flow of the odd signature operator between flat connections on a closed three manifold in terms of spectral flow on the solid torus and its complement with the APS boundary conditions from Chapter 3. Even though Nicolaescu provides us with an elegant expression of spectral flow in terms of the Maslov index of the respective Cauchy data spaces, it is not of immediate use for actually computing spectral flow, since the Cauchy data spaces themselves are complicated objects. The purpose of the splitting formula in Theorem 4.5.5 however is to make computations of spectral flow easier by shifting the problem to two more tractable ones. The main application to keep in mind is the computation of spectral flow of the twisted odd signature operator between flat connections, whenever it is possible to find a path between them which is flat in the complement of a solid torus.

Let $M = X \cup_T S$ be a closed 3-manifold with S being the solid torus and T the torus. Let A_t be a path of $SU(2)$-connections on M with the following properties:

(1) A_t is in cylindrical form and flat in a collar of T.

(2) A_t restricts to the path $a_{\rho(t)}$ (see Definition 3.1.2) on T for some path $\tilde{\rho}$ in $\widetilde{\mathbf{R}}^2$ with $\pi \circ \tilde{\rho} = \rho$, where $\pi : \widetilde{\mathbf{R}}^2 \to \mathbf{R}^2$ is the projection onto the $\mathbf{R}^2$-factor.

(3) A_0 and A_1 are flat on M.

The goal is to compute $\mathrm{SF}(D_{A_t})$ in terms of spectral flow on X and S.

Spectral flow does not depend on the path of connections since the space of all connections is contractible. Thus write $\mathrm{SF}(A_0, A_1) := \mathrm{SF}(D_{A_t})$. Furthermore note

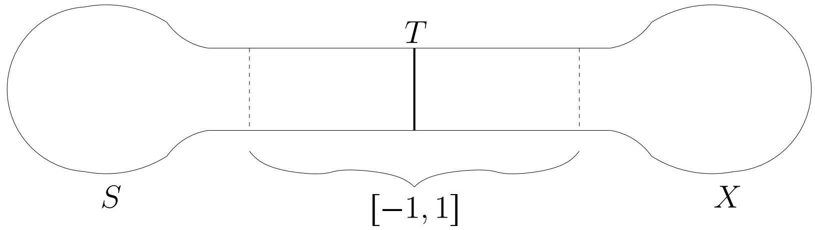

FIGURE 4.0.1. The collar around T

that since we have control over how spectral flow changes if we change endpoints by a gauge transformation (see Lemma 5.3.10), the splitting formula in Theorem 4.5.5 can be applied to any path of connections A'_t with flat endpoints. Since we have an $\mathbf{R}^2$ branched cover $a_{\alpha,\beta}$ of our moduli space of flat connections on the torus, we can find a gauge transformations g_ε so that $g_\varepsilon \cdot A'_\varepsilon|_T = a_{\alpha_\varepsilon,\beta_\varepsilon}$.

The main ingredients that we need for developing a splitting formula are the following.

- Nicolaescu's splitting theorem in [**N1**] and its extension by Daniel in [**Da2**], Theorem 2.10.2, which expresses spectral flow as the Maslov index of paths

$$\mathrm{SF}(D_{A_t}) = \mathrm{Mas}(\Lambda_{X,t}, \Lambda_{S,t}),$$

  where $\Lambda_{X,t} := \Lambda_X(D_{A_t})$ and $\Lambda_{S,t} := \Lambda_S(D_{A_t})$ are the respective Cauchy data spaces.
- The relative version of Nicolaescu's main theorem, Theorem 2.10.3 for manifolds with boundary.
- Nicolaescu's adiabatic limit theorem, Theorem 2.8.4, as well as Kirk and Lesch's generalization, Theorem 2.8.5.
- Kirk and Lesch's result, Lemma 2.8.8. It gives us control over the intersection of Cauchy data spaces and boundary conditions behaves with respect to stretching the collar. Specifically the Maslov indices in these situations vanish.

Before we can start with the discussion of the spectral flow on M, we need to analyze the scattering Lagrangian of D_A on S, because it plays a central role in the splitting theorem. This is an indication of why our splitting theorem only works for splittings along the torus T.

4.1. Connections on the Solid Torus

First we need a few conventions, that are necessary to avoid confusion with signs and orientations.

Denote the space of $SU(2)$-connections on S by $\mathcal{A}_S$ and the space of flat $SU(2)$-connections on S by $\mathcal{F}_S$.

Using the chosen trivialization of the principal $SU(2)$-bundle $P_S = S \times SU(2)$, identify $\mathcal{A}_S$ with

$$\Omega^1(S; su(2)) := \Omega^1(S) \otimes su(2)$$

as before.

Let $S = D^2 \times S^1$ be parametrized by polar coordinates (ne^{im}, e^{il}), where $n \in [0,1]$, $m, l \in [0, 2\pi)$. This way we can consider $N_\varepsilon(T) = [1, 1-\varepsilon] \times S^1 \times S^1$ as a collar of T in S for $\varepsilon < 1$. Fix some small ε. Choose a metric on S, so that its restriction to $N_\varepsilon(T)$ is simply the product metric, that is, $\{\frac{\partial}{\partial n}, \frac{\partial}{\partial m}, \frac{\partial}{\partial l}\}$ is an orthonormal basis on $N_\varepsilon(T)$ with the outward normal first convention. Choose the orientation determined by some nowhere zero 2-form $\alpha \in \Omega^2(T)$ and $\beta \in \Omega^3(S)$, such that $\alpha(\frac{\partial}{\partial m}, \frac{\partial}{\partial l}) > 0$ and $\beta_p(\frac{\partial}{\partial n}, \frac{\partial}{\partial m}, \frac{\partial}{\partial l}) > 0$ for $p \in T$. Notice that the coordinates n, m and l correspond to the outer unit normal $\nu = \frac{\partial}{\partial n}$, the meridian μ and longitude λ on T respectively (via the exponential map), as shown in Figure 4.1.2.

It is always convenient to split paths of connections into smaller segments that we may deal with separately. All of our computational tools for spectral flow depend on us beginning and ending at flat connections, because only then does the dimension of

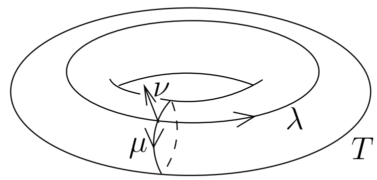

FIGURE 4.1.2. The torus

the kernels of the odd signature operator and the tangential operator not depend on the metric. Thus we require the connections on S to be flat whenever its restriction to the torus can be extended to a flat connection. The following determines when a flat connection on T cannot be extended to a flat connection on all of S.

LEMMA 4.1.1. *If an $SU(2)$-connection $a_{\alpha,\beta} = -i\alpha\, dm - i\beta\, dl$ on T extends to a flat $SU(2)$-connection on S then $\alpha \in \mathbf{Z}$.*

PROOF. Suppose A is a flat $SU(2)$-connection on S with $A|_T = a_{\alpha,\beta}$. Then $\text{hol}(A)$ is a $SU(2)$-representation of $\pi_1(S) = \langle \pi_1(T) | \mu = 1 \rangle$, such that its restriction $\text{hol}(A)|_T = \text{hol}(a_{\alpha,\beta})$ factors through $\pi_1(S)$. Thus we have $\text{hol}(a_{\alpha,\beta})(\mu) = 1$, that is, $\alpha \in \mathbf{Z}$. □

4.2. The Scattering Lagrangian of D_A on S

In view of Lemma 4.1.1 let A be a flat connection on S which restricts to $a_{\alpha,\beta}$, $\alpha \in \mathbf{Z}$. In this section we will compute the scattering Lagrangian of D_A on S.

One observes that $p : \pi(T) \to \pi(S)$ is a surjection and thus $\text{hol}(A) \circ p = \text{hol}(a_{\alpha,\beta})$. Thus the holonomy of A is determined by $a_{\alpha,\beta}$. Furthermore, the De Rham map only depends on $\text{hol}(A)$ and the corresponding group cohomology computation only depends on $\text{hol}(A)$ and $\text{hol}(a_{\alpha,\beta})$. Thus notice that the scattering Lagrangian does not depend on the connection A but only on its restriction $a_{\alpha,\beta}$.

LEMMA 4.2.1.

- *If* $\beta \in \mathbf{R} - \frac{1}{2}\mathbf{Z}$, *then* $\mathscr{L}_{S,A} = \mathbf{R}i \oplus \mathbf{R}i\, dl$
- *If* $\beta \in \frac{1}{2}\mathbf{Z}$, *then* $\mathscr{L}_{S,A} = \widehat{\mathscr{L}}_{S,A} \oplus \breve{\mathscr{L}}_{S,A}$, *where*

$$\begin{aligned}\widehat{\mathscr{L}}_{S,A} &= \mathbf{R}i \oplus \mathbf{R}i\, dl \\ \breve{\mathscr{L}}_{S,A} &= e^{i(2\alpha m + 2\beta l)}(\mathbf{C}j \oplus \mathbf{C}j\, dl)\end{aligned}$$

SKETCH OF PROOF. This can be verified directly by applying D_A for $A = -i\beta\, dl$ and observing that the scattering Lagrangian depends only on the restriction $a_{\alpha,\beta}$ of A to T. Then we use a gauge transformation g with $g|_T = e^{i\alpha m}$ on the boundary to compute $\mathscr{L}_{S,g\cdot A} = \mathrm{ad}_g \mathscr{L}_{S,A}$. □

4.3. Splitting the Spectral Flow – the Simple Case

We will first describe a proof of the splitting theorem for the case when

(1) D_{A_0} and D_{A_1} restricted to X as well as S have non-resonance level 0 as defined in 2.8.3, and

(2) $\rho(t) \notin (\frac{1}{2}\mathbf{Z})^2$ for all $t \in [0,1]$, in which case $K^{\pm}_{(\rho(t),\theta)} = 0$.

Once the principle of the proof is understood with these two assumptions, then the technical difficulties arising in the general case will hopefully not obscure the argument.

The first assumption allows us to apply Nicolaescu's adiabatic limit theorem. Fix the path $R_t := \frac{t}{1-t}$ which corresponds to the parameter that stretches the collar from 0 to ∞. Stretching the collar of S and X yields the adiabatic limit $\Lambda^{R_1}_{S,\varepsilon} = P^-_{S,\varepsilon} \oplus \mathscr{L}_{S,\varepsilon}$ and $\Lambda^{R_1}_{X,\varepsilon} = P^+_{X,\varepsilon} \oplus \mathscr{L}_{X,\varepsilon}$ when $\varepsilon = 0$ and $\varepsilon = 1$, because A_0 and A_1 are flat, where $\mathscr{L}_{S,\varepsilon}$ and $\mathscr{L}_{X,\varepsilon}$ are the scattering Lagrangians of $D_A|_S$ and $D_A|_X$. The second assumption ensures that $K^{\pm}_{(\rho(t),\theta)} = 0$ for all t. In particular the space $P^-_{S,t} \oplus \widehat{\mathscr{L}}_{S,t}$ varies continuously in t, where $\widehat{\mathscr{L}}_{S,t} = \mathbf{R}i \oplus \mathbf{R}i\, dl \subset \mathcal{H}^{0+1+2}_{\alpha,\beta}(T; su(2))$ as in Definition

5.3.8. The second assumption also implies that $\mathscr{L}_{S,t} = \widehat{\mathscr{L}_{S,t}}$ for all t as we will see later in this chapter.

The idea of the proof is the following. First we relate the spectral flow to the Maslov index of the Cauchy data spaces using Nicolaescu's splitting theorem. Next we use homotopy invariance of the Maslov index and Nicolaescu's adiabatic limit theorem to find homotopic paths of Lagrangians, parts of which correspond to a Maslov index of Cauchy data space with APS boundary conditions. Now we can use additivity of pairs of paths of Lagrangians and apply the relative version of Nicolaescu's splitting theorem to two of the parts. The remaining summands will either vanish or simplify a lot.

Figure 4.3.3 may be useful for some to keep in mind the simple structure of the argument. Once the meaning of the picture is clear, one can easily reconstruct the argument from it. The diagram indicates how we homotop and split up our path of Fredholm pairs of Lagrangians using homotopy invariance and additivity of the Maslov index. More precisely the picture indicates a sequence of paths of Fredholm pairs of Lagrangians $(\mathscr{M}_i(t), \mathscr{N}_i(t))$, $i = 1, \ldots, 9$, such that the composition $(\mathscr{M}_9, \mathscr{N}_9) * \cdots * (\mathscr{M}_1, \mathscr{N}_1)(t)$ is homotopic to $(\Lambda_{S,t}, \Lambda_{X,t})$. Then the additivity of the Maslov index yields $\mathrm{Mas}(\Lambda_{S,t}, \Lambda_{X,t}) = \sum_{i=1}^{9} \mathrm{Mas}(\mathscr{M}_i(t), \mathscr{N}_i(t))$.

Moreover Figure 4.3.3 encodes, which paths correspond to stretching, the path of Cauchy data spaces, and the path of boundary conditions, and which ones match up, so that we can apply the Maslov index to them.

- The vertical arrows corresponds to stretching the collar at a flat connection. They denote $\Lambda_{S,0}^{R_t}$ and $\Lambda_{X,1}^{R_{1-t}}$ respectively.
- The lower horizontal arrows are the paths of Cauchy data spaces $\Lambda_{S,t}$ and $\Lambda_{X,t}$.

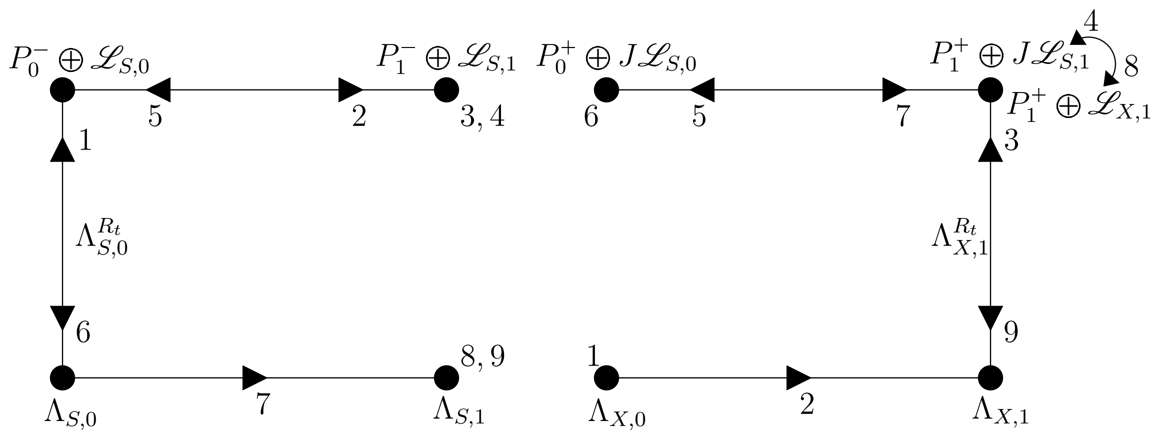

FIGURE 4.3.3. The path segments $\mathscr{M}_i(t)$ (left) and $\mathscr{N}_i(t)$ (right)

- The upper horizontal arrows are the paths of boundary conditions $P_t^- \oplus \mathscr{L}_{S,0}$ and $P_t^+ \oplus J\mathscr{L}_{S,t}$.
- Horizontal and vertical paths on the left match up with horizontal and vertical paths on the right respectively.

The following is a description of each of the segments $(\mathscr{M}_i, \mathscr{N}_i)$, $i = 1, \ldots, 9$.

(1) We stretch the collar of S to infinity, so $\mathscr{M}_1 = \Lambda_{S,0}^{R_t}$. $\mathscr{N}_1$ is the constant path $\Lambda_{X,0}$.

(2) The path $\mathscr{M}_2$ follows the boundary conditions $P_t^- \oplus \mathscr{L}_{S,t}$, while $\mathscr{N}_2$ is the path of Cauchy data spaces of $D_{A_t}|_X$.

(3) $\mathscr{M}_3$ is the constant path $P_1^- \oplus \mathscr{L}_{S,1}$. We stretch the collar of X to infinity, so $\mathscr{N}_3 = \Lambda_{X,0}^{R_t}$.

(4) $\mathscr{M}_4$ is again the constant path $P_1^- \oplus \mathscr{L}_{S,1}$. $\mathscr{N}_3$ is an arbitrary path from $P_1^+ \oplus \mathscr{L}_{X,1}$ to $P_1^+ \oplus J\mathscr{L}_{S,1}$. We might as well choose a path which moves only within $\mathcal{H}_1^{0+1+2}(T; su(2))$, meaning $\mathscr{N}_3 = \mathscr{N}_3' \oplus \mathscr{N}_3''$, where $\mathscr{N}_3'$ is the constant path and $\mathscr{N}_3''$ is a path in $\mathcal{H}_1^{0+1+2}(T; su(2))$.

(5) $\mathscr{M}_5$ is $\mathscr{M}_2$ backwards. $\mathscr{N}_5$ corresponds to the path $P_{1-t}^+ \oplus J\mathscr{L}_{S,1-t}$.

(6) For $\mathscr{M}_6$ we unstretch the collar of S, so $\mathscr{M}_6 = \Lambda_{S,0}^{R_{1-t}}$. $\mathscr{N}_6$ is the constant path at $P_0^+ \oplus J\mathscr{L}_{S,0}$.

(7) $\mathscr{M}_7$ is the path of Cauchy Data spaces of $D_{A_t}|_S$. $\mathscr{N}_7$ is $\mathscr{N}_5$ backwards.

(8) $\mathscr{M}_8$ is the constant path at the Cauchy Data space of $D_{A_1}|_S$. $\mathscr{N}_8$ is $\mathscr{N}_4$ backwards.

(9) $\mathscr{M}_9$ is again constant at $\Lambda_{S,1}$. $\mathscr{N}_9$ unstretches the collar of X, so $\mathscr{N}_9 = \Lambda_{X,0}^{R_{1-t}}$.

In Figure 4.3.3 all the horizontal paths are parametrized by the connection a_t, while all vertical paths are parametrized by the length R_t of the collar of S or T. By meditating on the picture one might understand the simplicity of the approach for splitting the spectral flow. It is apparent from the picture, that by "retracting" everything to the lower horizontal line, the left path is homotopic to $\Lambda_{S,t}$ and the right path is homotopic to $\Lambda_{X,t}$. In fact the composition of $\mathscr{M}_i$, $i = 1, \ldots, 6$ and $\mathscr{N}_i$, $i = 3, \ldots, 9$ are homotopic to the constant path at $\Lambda_{S,0}$ and $\Lambda_{X,1}$ respectively, because it is the composition of a path and its reverse. Furthermore one observes, that we are not following the two lower horizontal lines corresponding to the path of pairs of Cauchy data spaces simultaneously, the Maslov of which we are interested in. This will enable us to get a splitting formula.

Let us discuss the Maslov indices of the paths of pairs of Lagrangians in Figure 4.3.3 a little bit. Later we will discuss all the paths in detail. Observe, that by construction the Maslov index of the 5th path will be 0. By stretching the collar of S we can reduce the Maslov index of the sum of paths 4 and 8 to the Maslov index of paths only involving stretching the collar (and not the rotation between $\mathscr{L}_{X,1}$ and $J\mathscr{L}_{S,1}$). The Maslov index of the path 2 and 7 are the spectral flow on X and S with the APS boundary conditions given by the paths 2 and 7 on the left and right respectively. All the remaining paths only involve stretching the collar, which turn out to have vanishing Maslov indices. Thus in this case we would get the splitting formula $\mathrm{SF}(D_{A_t}) = \mathrm{SF}_S(D_{A_t}; P_t^+ \oplus J\mathscr{L}_{S,t}) + \mathrm{SF}_X(D_{A_t}; P_t^- \oplus \mathscr{L}_{S,t})$.

It is a good idea to at least once follow the paths $\mathscr{M}_i$ and $\mathscr{N}_i$ in Figure 4.3.3 with the left and right hand simultaneously. Imagine the meaning of each segment and the consequence toward the Maslov index.

4.4. Splitting the Spectral Flow – the General Case

In the general case we have to fill gaps in the paths of subspaces and match up the ends in Figure 4.3.3:

- If we pass through representations, which are central on T, the paths of APS boundary conditions (the upper horizontal lines) are not continuous. We have to fill in the gaps.
- Stretching the collar on X will in general give you a different limit than the one given in Figure 4.3.3. We will have to match that up the ends of stretching the collar with the path of APS boundary conditions.

Unfortunately this is a bit technical and hard to understand unless one keeps Figure 4.3.3 in mind.

Since $K^{\pm}_{\alpha,\beta,\theta} \neq K^{\pm}_{\alpha,\beta,-\theta}$ for $(\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2$, the upper horizontal path in Figure 4.3.3 is not continuous if ρ passes through $(\frac{1}{2}\mathbf{Z})^2$. Thus, instead of $P_t^+ \oplus \mathscr{L}_{S,t}$ we will use the APS boundary conditions $\mathcal{P}^{\pm}_{\tilde{\rho}(t)}$ with the notation introduced in Definition 3.3.4 and Lemma 4.2.1, which were shown to be continuous for a continuous lift $\tilde{\rho}$ of the path ρ from $\mathbf{R}^2$ to $\widetilde{\mathbf{R}}^2$.

If $\rho(0)$ or $\rho(1)$ are in the half integer lattice, then the scattering Lagrangians $\mathscr{L}_{S,\varepsilon}$, $\varepsilon = 0, 1$ will not be equal to $\widehat{\mathscr{L}_{S,\varepsilon}}$. Also the adiabatic limit on X will generally look different from $P_1^+ \oplus \mathscr{L}_{X,1}$ (see Theorem 2.8.5). We need to match up the adiabatic limit with our preferred APS boundary conditions $\mathcal{P}^{+}_{\tilde{\rho}(1)}(J\widehat{\mathscr{L}_S})$. This is accomplished by introducing the paths $P^+_{1,\nu} \oplus L_{W,t} \oplus d(E^-_{1,\nu}) \oplus L_{X,t}$ for X and $P^-_0 \oplus L_{S,t}$ for S as follows:

Definition 4.4.1.

(1) Fix a path of Lagrangians $L_{X,t}$ in $H^{0+1+2}(T; su(2))$ from the scattering Lagrangian $\mathscr{L}_{X,1}$ to $J\widehat{\mathscr{L}_{S,1}} \oplus K^+_{\rho_1,i}$.
(2) Fix the path of Lagrangians $L_{W,t} = e^{J\pi t} W_{a_1} \oplus J(W^\perp_{a_1})$ in $d(E^+_{a_1,\nu}) \oplus d^*(E^-_{a_1,\nu})$ from $W_{a_1} \oplus J(W^\perp_{a_1})$ to $d^*(E^-_{a_1,\nu}) = J(W_\alpha) \oplus J(W^\perp_a) \subset P^+_1$.
(3) Fix a path of Lagrangians $L_{S,\varepsilon,t} := \widehat{\mathscr{L}_{S,\varepsilon}} \oplus \check{L}_{S,\varepsilon,t}$, $\varepsilon = 0, 1$, in $H^{0+1+2}(T; su(2))$, where $\check{L}_{S,\varepsilon,t}$ is an arbitrary path from $\check{\mathscr{L}}_{S,\varepsilon}$ to $K^-_{\rho_\varepsilon,i}$. As in Lemma 4.2.1, $\check{\mathscr{L}}_{S,\varepsilon}$ denotes the $\mathbf{C}j$ part of $\mathscr{L}_{S,\varepsilon}$. Set $L_{S,t} := L_{S,0,t}$.

In the end the choice of paths in definition 4.4.1 does not matter, because either way the composition of paths in Figure 4.3.3 is such that it is homotopic to $\Lambda_{S,t}$ and $\Lambda_{X,t}$ respectively, and because the Maslov index is a homotopy invariant of paths of Fredholm pairs of Lagrangians. We chose paths that were as simple as possible.

4.5. A Splitting Formula for Spectral Flow

With the homotopy invariance of the Maslov index we can try to find a formula for the spectral flow on M in terms of spectral flow on S and X by computing the Maslov indices of the paths in Figure 4.3.3 separately, as we have indicated before. A slightly more complicated but more general description of the paths that does not make any assumptions on the path of connections are given in the Table 4.5.4. Notice that the indices of the paths have changed slightly. The ends of the paths $\mathscr{M}_i$ and $\mathscr{N}_i$ are listed in the two middle columns. To the left and right of them, adjacent two both ends (in the middle columns), are the paths themselves with their indices in the first column. There are eleven paths altogether. Roughly, paths 1, 4, 8 and 11

i	paths $\mathscr{M}_i(t)$	Endpoints of $\mathscr{M}_i$	and $\mathscr{N}_i$	paths $\mathscr{N}_i(t)$
		$\Lambda_{S,0}$	$\Lambda_{X,0}$	
1	$\Lambda_{S,0}^{R_t}$			$\Lambda_{X,0}^{R_t}$
		$P_0^- \oplus \mathscr{L}_{S,0}$	$\Lambda_{X,0}^{\infty}$	
2	$P_0^- \oplus L_{S,t}$			$\Lambda_{X,0}^{R_{1-t}}$
		$\mathcal{P}^-_{\tilde\rho(0)}(\widehat{\mathscr{L}_S})$	$\Lambda_{X,0}$	
3	$\mathcal{P}^-_{\tilde\rho(t)}(\widehat{\mathscr{L}_S})$			$\Lambda_{X,t}$
		$\mathcal{P}^-_{\tilde\rho(1)}(\widehat{\mathscr{L}_S})$	$\Lambda_{X,1}$	
4	constant			$\Lambda_{X,1}^{R_t}$
		$\mathcal{P}^-_{\tilde\rho(1)}(\widehat{\mathscr{L}_S})$	$\Lambda_{X,1}^{\infty}$	
5	constant			$P^+_{a_1,\nu} \oplus L_{W,t} \oplus d(E^-_{a_1,\nu}) \oplus L_{X,t}$
		$\mathcal{P}^-_{\tilde\rho(1)}(\widehat{\mathscr{L}_S})$	$\mathcal{P}^+_{\tilde\rho(1)}(J\widehat{\mathscr{L}_S})$	
6	$\mathcal{P}^-_{\tilde\rho(1-t)}(\widehat{\mathscr{L}_S})$			$\mathcal{P}^+_{\tilde\rho(1-t)}(J\widehat{\mathscr{L}_S})$
		$\mathcal{P}^-_{\tilde\rho(0)}(\widehat{\mathscr{L}_S})$	$\mathcal{P}^+_{\tilde\rho(0)}(J\widehat{\mathscr{L}_S})$	
7	$P_0^- \oplus L_{S,(1-t)}$			constant
		$P_0^- \oplus \mathscr{L}_{S,0}$	$\mathcal{P}^+_{\tilde\rho(0)}(J\widehat{\mathscr{L}_S})$	
8	$\Lambda_{S,0}^{R_{1-t}}$			constant
		$\Lambda_{S,0}$	$\mathcal{P}^+_{\tilde\rho(0)}(J\widehat{\mathscr{L}_S})$	
9	$\Lambda_{S,t}$			$\mathcal{P}^+_{\tilde\rho(t)}(J\widehat{\mathscr{L}_S})$
		$\Lambda_{S,1}$	$\mathcal{P}^+_{\tilde\rho(1)}(J\widehat{\mathscr{L}_S})$	
10	$\Lambda_{S,1}^{R_t}$			$P^+_{a_1,\nu} \oplus L_{W,1-t} \oplus d(E^-_{a_1,\nu}) \oplus L_{X,1-t}$
		$\Lambda_{S,1}^{\infty}$	$\Lambda_{X,1}^{\infty}$	
11	$\Lambda_{S,1}^{R_{1-t}}$			$\Lambda_{X,1}^{R_{1-t}}$
		$\Lambda_{S,1}$	$\Lambda_{X,1}$	

FIGURE 4.5.4. The paths homotopic to $\Lambda_{S,t}$ and $\Lambda_{X,t}$ broken up into pieces

correspond to the vertical paths in Figure 4.3.3. Paths 3, 6 and 9 correspond to the horizontal paths. The rest, namely paths 2, 5, 7 and 10 match up endpoints of the vertical and horizontal paths. In paths 1 and 2 on X, as well as 10 and 11 on S we follow stretching the collar with "unstretching" it, because this way paths 1 and 2 are readily computable. By unstretching we mean the reverse of the path corresponding to stretching the collar.

By Lemma 8.10 of [**KL**] $\mathrm{Mas}(\mathscr{M}_i, \mathscr{N}_i) = 0$ for $i = 1, 4, 8, 11$. Furthermore, we have $\mathrm{Mas}(\mathscr{M}_9, \mathscr{N}_9) = \mathrm{SF}_S(A_t; \mathcal{P}^+_{\tilde\rho(t)}(J\widehat{\mathscr{L}_S}))$. Since $\mathcal{P}^-(\widehat{\mathscr{L}_{S,t}}) = (\mathcal{P}^+_{\tilde\rho(t)}(J\widehat{\mathscr{L}_S}))^\perp$ for all t, the Maslov index for the sixth path $\mathrm{Mas}(\mathscr{M}_6, \mathscr{N}_6)$ vanishes. The piece $\mathrm{Mas}(\mathscr{M}_3, \mathscr{N}_3) =$

$\mathrm{SF}_X(A_t; \mathcal{P}^-(\widehat{\mathscr{L}_S}))$ of paths of flat connections has been studied well, but a detailed computation will depend on X. We will do this explicitly for the case of torus bundles over S^1 in Chapter 6. The following lemma deals with the rest of the paths, namely paths 2, 5, 7 and 10.

LEMMA 4.5.1.

(1) $\mathrm{Mas}(\mathscr{M}_2, \mathscr{N}_2) = \mathrm{Mas}(L_{S,0,t}, \mathscr{L}_{X,0})$
(2) $\mathrm{Mas}(\mathscr{M}_7, \mathscr{N}_7) = -\mathrm{Mas}(L_{S,0,t}, K^+_{\rho_0,i} \oplus J\widehat{\mathscr{L}_{S,0}})$
(3) $\mathrm{Mas}(\mathscr{M}_5, \mathscr{N}_5) + \mathrm{Mas}(\mathscr{M}_{10}, \mathscr{N}_{10}) = \mathrm{Mas}(L_{S,1,t}, K^+_{\rho_1,i} \oplus J\widehat{\mathscr{L}_{S,1}}) - \mathrm{Mas}(L_{S,1,t}, \mathscr{L}_{X,1})$

PROOF.

(1) Notice that we may homotop $\mathscr{M}_2$ to the composition of the two paths

$$\begin{aligned}\mathscr{M}_{2a} &= P_0^- \oplus L_{S,0,t}\\ \mathscr{M}_{2b} &= P_0^- \oplus K^+_{\rho_0,i} \oplus \widehat{\mathscr{L}_0}.\end{aligned}$$

and $\mathscr{N}_2$ to the composition of the two paths

$$\begin{aligned}\mathscr{N}_{2a} &= P^+_{0,\nu} \oplus (W_0 \oplus J(W_0^\perp)) \oplus d^*(E^-_{0,\nu}) \oplus \mathscr{L}_{X,0}\\ \mathscr{N}_{2b} &= \Lambda^{R_{1-t}}_{X,0}.\end{aligned}$$

By Lemma 8.10 of [**KL**] $\mathrm{Mas}(\mathscr{M}_{2b}, \mathscr{N}_{2b}) = 0$. Then by our choice of $L_{S,0,t}$ we get

$$\mathrm{Mas}(\mathscr{M}_2, \mathscr{N}_2) = \mathrm{Mas}(\mathscr{M}_{2a}, \mathscr{N}_{2a}) = \mathrm{Mas}(L_{S,0,t}, \mathscr{L}_{X,0}).$$

(2) We can directly check:

$$\mathrm{Mas}(\mathscr{M}_7, \mathscr{N}_7) = \mathrm{Mas}(L_{S,0,(1-t)}, K^+_{\rho_0,i} \oplus J\widehat{\mathscr{L}_{S,0}})) = -\mathrm{Mas}(\check{L}_{S,0,t}, K^+_{\rho_0,i}).$$

(3) The path $\mathscr{M}_{10}$ can be homotoped to the composition of the two paths

$$\begin{aligned}\mathscr{M}_{10a} &= \Lambda^t_{S,1}\\ \mathscr{M}_{10b} &= \Lambda^\infty_{S,1} = P_0^- \oplus \mathscr{L}_{S,1}\end{aligned}$$

and $\mathscr{N}_{10}$ to the composition of the two paths

$$\begin{aligned}\mathscr{N}_{10a} &= \mathcal{P}^+_{\tilde{\rho}(1)}(J\widehat{\mathscr{L}_S})\\ \mathscr{N}_{10b} &= P^+_\nu \oplus L_{W,t}\end{aligned}$$

Again by Lemma 8.10 of [**KL**] we get $\operatorname{Mas}(\mathscr{M}_{10a}, \mathscr{N}_{10a}) = 0$. Thus

$$\begin{aligned}\operatorname{Mas}(\mathscr{M}_{10}, \mathscr{N}_{10}) &= \operatorname{Mas}(P_1^- \oplus \mathscr{L}_{S,1}, P^+_{1,\nu} \oplus L_{W,1-t} \oplus d(E^-_{1,\nu}) \oplus L_{X,1-t})\\ &= \operatorname{Mas}(E_1^- \oplus \mathscr{L}_{S,1}, L_{W,1-t} \oplus d(E^-_{1,\nu}) \oplus L_{X,1-t})\\ &= \operatorname{Mas}(E_1^-, L_{W,1-t} \oplus d(E^-_{1,\nu})) + \operatorname{Mas}(\mathscr{L}_{S,1}, L_{X,1-t})\end{aligned}$$

while

$$\begin{aligned}\operatorname{Mas}(\mathscr{M}_5, \mathscr{N}_5) &= \operatorname{Mas}(P_1^- \oplus K^-_{1,i} \oplus \widehat{\mathscr{L}}_{S,1}, P^+_{1,\nu} \oplus L_{W,t} \oplus d(E^-_{1,\nu}) \oplus L_{X,t})\\ &= \operatorname{Mas}(E_1^-, L_{W,t} \oplus d(E^-_{1,\nu})) + \operatorname{Mas}(K^-_{1,i} \oplus \widehat{\mathscr{L}}_{S,1}, L_{X,t})\end{aligned}$$

Thus

$$\operatorname{Mas}(\mathscr{M}_5, \mathscr{N}_5) + \operatorname{Mas}(\mathscr{M}_{10}, \mathscr{N}_{10}) = \operatorname{Mas}(\mathscr{L}_{S,1}, L_{X,1-t}) + \operatorname{Mas}(K^-_{1,i} \oplus \widehat{\mathscr{L}}_{S,1}, L_{X,t})$$

Notice that since $L_{S,1,t} \circ L_{S,1,1-t}$ and $L_{X,t} \circ L_{X,1-t}$ are both homotopic to constant paths, we can compute

$$\begin{aligned}
&\mathrm{Mas}(\mathscr{M}_5, \mathscr{N}_5) + \mathrm{Mas}(\mathscr{M}_{10}, \mathscr{N}_{10}) \\
&= \mathrm{Mas}(\mathscr{L}_{S,1}, L_{X,1-t}) + \mathrm{Mas}(K^-_{1,i} \oplus \widehat{\mathscr{L}_{S,1}}, L_{X,t}) \\
&= -\mathrm{Mas}(L_{S,1,t}, \mathscr{L}_{X,1}) - \mathrm{Mas}(L_{S,1,1-t}, J\widehat{\mathscr{L}_{S,1}} \oplus K^+_{\rho_1,i}) \\
&= \mathrm{Mas}(\check{L}_{S,1,t}, K^+_{\rho_1,i}) - \mathrm{Mas}(L_{S,1,t}, \mathscr{L}_{X,1})
\end{aligned}$$

□

Lemma 4.5.1, together with the definition and properties of the twisted Maslov triple index in [**KL**] yield the following.

COROLLARY 4.5.2. *The sums*

$$\begin{aligned}
&\mathrm{Mas}(\mathscr{M}_2, \mathscr{M}_2) + \mathrm{Mas}(\mathscr{M}_7, \mathscr{N}_7) \\
&= \tau_\mu(\mathscr{L}_{S,0}, \mathscr{L}_{X,0}, K^+_{\rho_0,i} \oplus J\widehat{\mathscr{L}_{S,0}}) - \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}_{S,0}}, \mathscr{L}_{X,0}, K^+_{\rho_0,i} \oplus J\widehat{\mathscr{L}_{S,0}})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Mas}(\mathscr{M}_5, \mathscr{M}_5) + \mathrm{Mas}(\mathscr{M}_{10}, \mathscr{N}_{10}) \\
&= -\tau_\mu(\mathscr{L}_{S,1}, \mathscr{L}_{X,1}, K^+_{\rho_1,i} \oplus J\widehat{\mathscr{L}_{S,1}}) + \tau_\mu(K^-_{\rho_1,i} \oplus \widehat{\mathscr{L}_{S,1}}, \mathscr{L}_{X,1}, K^+_{\rho_1,i} \oplus J\widehat{\mathscr{L}_{S,1}})
\end{aligned}$$

are sums of Maslov triple indices and are thus, as expected, independent of the chosen path $L_{S,\varepsilon,t}$, $\varepsilon = 0, 1$.

LEMMA 4.5.3. $\tau_\mu(L_1, L_2, L_3) - \tau_\mu(L_1, L_2, L_4) = \tau_\mu(L_2, L_3, L_4) - \tau_\mu(L_1, L_3, L_4)$

PROOF. Fix a path of Lagrangians L_{34} from L_3 to L_4 and L_{12} from L_1 to L_2. Let L_{43} be the inverse path of L_{34} and L_{21} the inverse path of L_{12}. Notice that then

$$\mathrm{Mas}(L_2, L_{34}) + \mathrm{Mas}(L_{21}, L_4) + \mathrm{Mas}(L_1, L_{43}) + \mathrm{Mas}(L_{12}, L_3) = 0.$$

This yields

$$\begin{aligned}
&\tau_\mu(L_1, L_2, L_3) - \tau_\mu(L_1, L_2, L_4) \\
&= \operatorname{Mas}(L_2, L_{34}) - \operatorname{Mas}(L_1, L_{34}) = \operatorname{Mas}(L_2, L_{34}) + \operatorname{Mas}(L_1, L_{43}) \\
&= -\operatorname{Mas}(L_{21}, L_4) - \operatorname{Mas}(L_{12}, L_3) = \operatorname{Mas}(L_{12}, L_4) - \operatorname{Mas}(L_{12}, L_3) \\
&= \tau_\mu(L_1, L_4, L_3) - \tau_\mu(L_2, L_4, L_3) = -\tau_\mu(L_1, L_3, L_4) + \tau_\mu(L_2, L_3, L_4).
\end{aligned}$$

□

In order to simplify our formula even more, we compute the following

COROLLARY 4.5.4.

$$\operatorname{Mas}(\mathscr{M}_2, \mathscr{M}_2) + \operatorname{Mas}(\mathscr{M}_7, \mathscr{N}_7) = \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}_{S,0}}, \mathscr{L}_{S,0}, \mathscr{L}_{X,0}) - \tau_\mu(K^-_{\rho_0,i}, \widecheck{\mathscr{L}_{S,0}}, K^+_{\rho_0,i})$$

and

$$\operatorname{Mas}(\mathscr{M}_5, \mathscr{M}_5) + \operatorname{Mas}(\mathscr{M}_{10}, \mathscr{N}_{10}) = -\tau_\mu(K^-_{\rho_1,i} \oplus \widehat{\mathscr{L}_{S,1}}, \mathscr{L}_{S,1}, \mathscr{L}_{X,1}) + \tau_\mu(K^-_{\rho_1,i}, \widecheck{\mathscr{L}_{S,1}}, K^+_{\rho_1,i})$$

PROOF. Lemma 4.5.3 together with linearity under the direct sum of triples and proposition 6.11 of [**KL**] shows that:

$$\begin{aligned}
&\operatorname{Mas}(\mathscr{M}_2, \mathscr{M}_2) + \operatorname{Mas}(\mathscr{M}_7, \mathscr{N}_7) \\
&= \tau_\mu(\mathscr{L}_{S,0}, \mathscr{L}_{X,0}, K^+_{\rho_0,i} \oplus J\widehat{\mathscr{L}_{S,0}}) - \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}_{S,0}}, \mathscr{L}_{X,0}, K^+_{\rho_0,i} \oplus J\widehat{\mathscr{L}_{S,0}}) \\
&= \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}_{S,0}}, \mathscr{L}_{S,0}, \mathscr{L}_{X,0}) - \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}_{S,0}}, \mathscr{L}_{S,0}, K^+_{\rho_0,i} \oplus J\widehat{\mathscr{L}_{S,0}}) \\
&= \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}_{S,0}}, \mathscr{L}_{S,0}, \mathscr{L}_{X,0}) - \tau_\mu(K^-_{\rho_0,i}, \widecheck{\mathscr{L}_{S,0}}, K^+_{\rho_0,i}) - \tau_\mu(\widehat{\mathscr{L}_{S,0}}, \widehat{\mathscr{L}_{S,0}}, J\widehat{\mathscr{L}_{S,0}}) \\
&= \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}_{S,0}}, \mathscr{L}_{S,0}, \mathscr{L}_{X,0}) - \tau_\mu(K^-_{\rho_0,i}, \widecheck{\mathscr{L}_{S,0}}, K^+_{\rho_0,i})
\end{aligned}$$

The other sum is simplified analogously. □

Notice that $\tau_\mu(K^-_{\rho_0,i}, \breve{\mathscr{L}}_{S,0}, K^+_{\rho_0,i}) = \tau_\mu(K^-_{\rho_1,i}, \breve{\mathscr{L}}_{S,1}, K^+_{\rho_1,i})$, since we have an explicit description of all the ingredients given by Theorem 3.3.2 and Lemma 4.2.1. Thus we have proven the following.

THEOREM 4.5.5. *Let $M = X \cup_T S$ be a closed 3-manifold with S being the solid torus and T the torus. The orientations are as in Section 4.1. Let A_t be a path of $SU(2)$-connections on M with the following properties:*

(1) *A_t is in cylindrical form and flat in a collar of T.*

(2) *A_t restricts to the path $a_{\rho(t)}$ (see Lemma 3.1.2) on T for some path $\tilde\rho$ in $\tilde{\mathbf{R}}^2$ with $\pi \circ \tilde\rho = \rho$, where $\pi : \tilde{\mathbf{R}}^2 \to \mathbf{R}^2$ is the projection onto the $\mathbf{R}^2$-factor.*

(3) *A_0 and A_1 are flat on M.*

Then we have the splitting formula:

$$\begin{aligned}\mathrm{SF}(D_{A_t}) \;&=\; \mathrm{SF}_S(A_t; \mathcal{P}^+_{\tilde\rho(t)}(J\widehat{\mathscr{L}_S}))) + \mathrm{SF}_X(A_t|_X; \mathcal{P}^-(\widehat{\mathscr{L}_S})_{\tilde\rho})) \\ &\quad + \tau_\mu(K^-_{\rho_0,i} \oplus \widehat{\mathscr{L}}_{S,0}, \mathscr{L}_{S,0}, \mathscr{L}_{X,0}) - \tau_\mu(K^-_{\rho_1,i} \oplus \widehat{\mathscr{L}}_{S,1}, \mathscr{L}_{S,1}, \mathscr{L}_{X,1}).\end{aligned}$$

A way to compute $\mathrm{SF}_S(A_t; \mathcal{P}^+_{\tilde\rho(t)}(J\widehat{\mathscr{L}_S}))$ will be given in Theorem 5.3.9. We will compute $\mathrm{SF}_X(A_t|_X; \mathcal{P}^-(\widehat{\mathscr{L}_S}))$ for our main application in Chapter 6, where M is a torus-bundle over S^1 and A_t is flat on X.

COROLLARY 4.5.6. *We make the same assumptions as in Theorem 4.5.5. Then, if $\rho_\varepsilon \notin (\frac{1}{2}\mathbf{Z})^2$ for $\varepsilon = 0, 1$ we get:*

$$\mathrm{SF}(D_{A_t}) = \mathrm{SF}_S(A_t; \mathcal{P}^+_{\tilde\rho(t)}(J\widehat{\mathscr{L}_S})) + \mathrm{SF}_X(A_t; \mathcal{P}^-_{\tilde\rho(t)}(\widehat{\mathscr{L}_S})).$$

PROOF. We have $K^\pm_{\rho_\varepsilon,i} = 0$, thus

$$\tau_\mu(K^-_{\rho_\varepsilon,i} \oplus \widehat{\mathscr{L}}_{S,\varepsilon}, \mathscr{L}_{S,\varepsilon}, \mathscr{L}_{X,\varepsilon}) = \tau_\mu(\mathscr{L}_{S,\varepsilon}, \mathscr{L}_{S,\varepsilon}, \mathscr{L}_{X,\varepsilon}) = 0$$

. ∎

We make a final observation, which is related and will be useful later.

LEMMA 4.5.7. *Let A_t be a path of connections and g a gauge transformation, then*

$$\mathrm{SF}_X(A_t; \mathcal{P}_t) = \mathrm{SF}_X(g \cdot A_t; \mathrm{ad}_g \mathcal{P}_t).$$

In particular if $A_t|_T = a_{\tilde{\rho}(t)}$ and $g \cdot A_t|_T = a_{\tilde{\rho}'(t)}$, then

$$\mathrm{SF}_X(A_t; \mathcal{P}^-_{\tilde{\rho}(t)}(\widehat{\mathscr{L}_S})) = \mathrm{SF}_X(g \cdot A_t; \mathcal{P}^-_{\tilde{\rho}'(t)}(\widehat{\mathscr{L}_S})). \quad \square$$

CHAPTER 5

Spectral Flow on the Solid Torus

In our main application discussed in Chapter 6 we are interested in computing spectral flow of the twisted odd signature operator along a path of flat $SU(2)$ connections on torus bundles M over S^1. The main tool is our splitting formula for spectral flow from Theorem 4.5.5, by which we ultimately need to compute spectral flow on a solid torus $S \subset M$ and on its complement $M - S$ with certain APS boundary conditions.

While the spectral flow computation on $M - S$ certainly depends on M itself, we can discuss spectral flow on S separately, as it might be interesting and applicable in other settings.

However, we shall keep in mind that S is a submanifold of some closed manifold M and that we need our spectral flow computation for a splitting formula like in Theorem 4.5.5. Specifically, by exploiting the way spectral flow on M behaves under gauge transformations, we can restrict our spectral flow computation on the solid torus to the fixed family of connections given in Definition 5.2.1 and are still able to compute spectral flow on M using our splitting formula.

Let us take a closer look at this. The space of connections on M is contractible. Thus spectral flow depends only on the endpoints A_0 and A_1, not on the path connecting them, and changes by a multiple of 8 if one changes A_0 or A_1 by gauge transformations. One can compute this change using the Atiyah-Patodi-Singer index theorem [**APS**]. Therefore:

(1) Fix a family of flat connections $a_{\alpha,\beta}$, $\alpha, \beta \in \mathbf{R}$, on the torus T as in Definition 3.1.2, which covers the gauge equivalence classes of flat connections on T.

(2) Extend $a_{\alpha,\beta}$ to a family of connections $A_{\alpha,\beta}$ on S which includes all flat connections up to gauge equivalence, in particular the the family constructed in Definition 5.2.1 suffices.

Then we may find gauge-equivalent connections $A_0' \sim A_0$ and $A_1' \sim A_1$ on M that restrict to some $A_{\alpha,\beta}$ on S. This can be seen by recalling that any map from a 2-dimensional submanifold of M to $SU(2)$ can be extended to all of M, because $SU(2)$ is 2-connected. In this case the submanifold is $T = \partial S$. Furthermore we can extend the family $A_{\alpha,\beta}$ on S to a family of connections on M parametrized by $\mathbf{R}^2$, which includes A_0' and A_1'. We will give an explicit family of extensions to S in Definition 5.2.1, but we can also get such a family for M by obstruction theory, because $\mathbf{R}^2 \times \Omega^*(M; su(2))$ is contractible. We conclude that we only need to compute spectral flow on S for the family of connections $A_{\alpha,\beta}$ on S for it to be useful for splitting formulas.

5.1. Objective

Let $S = D^2 \times S^1$ be the solid torus and $T = \partial S$ its boundary. Let $A_{\alpha,\beta}$ be the family of connections on S as described in Definition 5.2.1 and let $A_t = A_{\alpha_t,\beta_t}$ be a path of connections on the solid torus S, so that A_0 and A_1 are flat connections. Boundary conditions are necessary to make the odd signature D_{A_t} on S (unbounded) self-adjoint. The Atiyah-Patodi-Singer (APS) boundary conditions discussed in Chapter 3 (after possibly reparametrizing A_t) are suitable for our purpose. The goal is to compute the spectral flow $\mathrm{SF}(D_{A_t}; \mathcal{P}_t)$ of D_{A_t} on S, which depends on the chosen APS boundary conditions $\mathcal{P}_t$.

5.2. A Smooth Family of Connections on S Parametrized by $\mathbf{R}^2$

We would like to extend the family of flat connections on T given in Lemma 3.1.2 to a family $A_{(\alpha,\beta)}$ (smoothly parametrized by $\mathbf{R}^2$) of connections on S, so that

(1) $A_{(\alpha,\beta)}$ is flat for $\alpha \in \mathbf{Z}$ (by Lemma 4.1.1 there is no flat extension when $\alpha \notin \mathbf{Z}$),

(2) the restriction of $A_{(\alpha,\beta)}$ to a collar $N_\varepsilon(T)$ of T in S is $a_{(\alpha,\beta)} := -i\alpha\, dm - i\beta\, dl \in \mathcal{A}_{N_\varepsilon(T)}$.

Notice that $i\beta\, dl$ makes sense globally as a (flat) connection on S for all $\beta \in \mathbf{R}$, whereas $i\alpha\, dm$ does not for $\alpha \neq 0$. Thus we can try to construct the family, so that

(3) $A_{(\alpha,\beta)}$ is equal to $-i\beta\, dl$ for $\alpha = 0$ and $\beta \in \mathbf{R}$.

Since $\pi_2 SU(2) = 0$, we can find a gauge transformation $\mathfrak{a}$ on S, such that

$$\begin{array}{rccc} \mathfrak{a}|_{N_\varepsilon(T)}: & N_\varepsilon(T) & \to & SU(2) \\ & (n, e^{im}, e^{il}) & \mapsto & e^{im}. \end{array}$$

The gauge transformation $\mathfrak{a}$ can easily be constructed factoring through D^2. It will be convenient to fix $\mathfrak{a}(ne^{im}, e^{il}) = q(n)e^{im} + \sqrt{1-(q(n))^2}j$ as in Definition 5.3.12. Since

$$\begin{aligned} \text{(5.2.4)} \qquad \mathfrak{a}|_{N_\varepsilon(T)} \cdot a_{(\alpha,\beta)} &= \mathfrak{a} a_{(\alpha,\beta)} \mathfrak{a}^{-1} + \mathfrak{a} d(\mathfrak{a}^{-1}) = a_{(\alpha,\beta)} + e^{im} \tfrac{\partial}{\partial m} e^{-im} dm \\ &= a_{(\alpha,\beta)} - idm = a_{(\alpha+1,\beta)}, \end{aligned}$$

we get a smooth family of connections $A_{(\alpha,\beta)} := -\mathfrak{a}^\alpha \cdot i\beta\, dl$ on S, $(\alpha,\beta) \in \mathbf{Z} \times \mathbf{R}$, with the desired property

$$A_{(\alpha,\beta)}|_{N_\varepsilon(T)} = \mathfrak{a}^\alpha|_{N_\varepsilon(T)} \cdot a_{(0,\beta)} = a_{(\alpha,\beta)}.$$

This can be easily extended to a smooth family of connections parametrized by $\mathbf{R}^2$ with $A_{(\alpha,\beta)}|_{N(T)} = a_{(\alpha,\beta)}$. One possible way is the following:

DEFINITION 5.2.1. Let $\eta : S \to [0,1]$ be a smooth function with $\eta(x) = 1$ for $x \in N_\varepsilon(T)$ and $\eta(x) = 0$ for $x \in S - N_{\frac{1}{2}}(T)$, which may as well factor through D^2. Let $\{\tau_r : \mathbf{R} \to \mathbf{R}\}_{r \in \mathbf{Z}}$ be a partition of unity subordinate to $\{(r-1, r+1)\}_{r \in \mathbf{Z}}$ (see Figure 5.2.1). For $(\alpha, \beta) \in \mathbf{R}^2$ define the family

$$A_{(\alpha,\beta)} := \sum_{r \in \mathbf{Z}} \tau_r(\alpha)\, \mathfrak{a}^r \cdot (\eta\, i(r - \alpha)\, dm - i\beta\, dl).$$

We need to check that $A_{(\alpha,\beta)}$ has the desired properties.

LEMMA 5.2.2. *The family $A_{(\alpha,\beta)}$ defined in 5.2.1 has the following properties:*

(1) *$A_{(\alpha,\beta)}$ is flat for $\alpha \in \mathbf{Z}$ (by Lemma 4.1.1 there is no flat extension when $\alpha \notin \mathbf{Z}$),*
(2) *the restriction of $A_{(\alpha,\beta)}$ to a collar $N_\varepsilon(T)$ of T in S is $a_{(\alpha,\beta)} = -i\alpha\, dm - i\beta\, dl \in \mathcal{A}_{N_\varepsilon(T)}$.*
(3) *$A_{(\alpha,\beta)}$ is equal to $-i\beta\, dl$ for $\alpha = 0$ and $\beta \in \mathbf{R}$.*
(4) *$A_{(\alpha,\beta)} = \mathfrak{a} \cdot A_{(\alpha+1,\beta)}$*

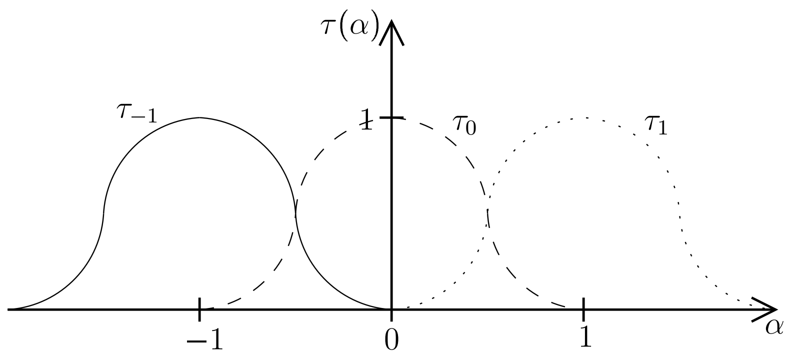

FIGURE 5.2.1. The partition of unity

PROOF. First notice, that $\eta\, i(r-\alpha)\, dm - i\beta\, dl$ is an $SU(2)$-connection (in normal form) for $(\alpha, \beta) \in \mathbf{R}^2$ and $r \in \mathbf{Z}$ on S. Thus $A_{(\alpha,\beta)}$ is a smooth family of $SU(2)$-connections (in normal form) on S. It is easy to check, that this family has the desired properties:

(1) Let $\alpha \in \mathbf{Z}$. Then we have $\tau_r(\alpha) = 0$ for $r \neq \alpha$. Thus $A_{(\alpha,\beta)} = -\mathfrak{a}^{-\alpha} \cdot i\beta\, dl$, which is flat.

(2) Since $\eta(x) = 1$ for $x \in N_\varepsilon(T)$, we get

$$A_{(\alpha,\beta)}|_{N_\varepsilon(T)} = \sum_{r\in\mathbf{Z}} \tau_r(\alpha)\, \mathfrak{a}^r \cdot (i(r-\alpha)\, dm - i\beta\, dl) = \sum_{r\in\mathbf{Z}} \tau_r(\alpha)\, \mathfrak{a}^r \cdot a_{\alpha-r,\beta}.$$

Formula (5.2.4) for the action of a gauge transformation then implies

$$A_{(\alpha,\beta)}|_{N_\varepsilon(T)} = \sum_{r\in\mathbf{Z}} \tau_r(\alpha)\, a_{(\alpha,\beta)} = a_{(\alpha,\beta)}.$$

The last two points are clear from the definition. □

5.3. Computation of Spectral Flow on the Solid Torus

In this section all connections and odd signature operators are considered on the solid torus only. Everything that follows depends on a continuous family of Lagrangians $L_{\alpha,\beta,\theta}$ in $\mathcal{H}^{0+1+2}(T;\mathbf{R}i)$ parametrized by $\widetilde{\mathbf{R}}^2$. Recall the family $A_{\alpha,\beta}$ of connections from Definition 5.2.1 and the scattering Lagrangian $\mathscr{L}_{S,A}$ from Definition 2.8.1 computed explicitly in Lemma 4.2.1.

To avoid technical difficulties we are going to assume that $L_{\alpha,\beta,\theta}$ is a family of Lagrangians in $\mathcal{H}^{0+1+2}(T;\mathbf{R}i)$, which is transverse to $\widehat{\mathscr{L}_{S,A_{(\alpha,\beta)}}}$ for all $(\alpha,\beta,\theta) \in \mathbf{Z} \times \mathbf{R} \times \{\pm i\} \subset \widetilde{\mathbf{R}}^2$. Later we will fix a specific family of Lagrangians $L_{\alpha,\beta,\theta}$ in view of the splitting formula in Chapter 4.

DEFINITION 5.3.1. Define $\mathrm{SF}(\tilde\rho) := \mathrm{SF}(A_{\pi\circ\tilde\rho(t)}, \mathcal{P}^+_{\tilde\rho(t)}(L))$ as a function of paths $\tilde\rho$ in $\widetilde{\mathbf{R}}^2$, where $\pi : \widetilde{\mathbf{R}}^2 \to \mathbf{R}^2$ is the projection mentioned in Definition 3.3.1.

First we will compute spectral flow on S along a path of flat connections. By comparing Theorem 3.3.2 and Lemma 4.2.1 we get the following.

LEMMA 5.3.2. *Let* $(\alpha, \beta) \in \mathbf{Z} \times \frac{1}{2}\mathbf{Z}$. *We have*

$$\dim(\mathscr{L}_{\alpha,\beta} \cap K^+_{\alpha,\beta,\theta}) = \begin{cases} 2 & \text{if } \theta = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

The next lemma could be obtained by comparing Lemma 5.3.2 to Figure 1 on page 531 in [**KK2**]. However we can directly compute the Maslov index. This works just like the proof of Lemma 6.4.2.

LEMMA 5.3.3. *Let* $(\alpha, \beta) \in \mathbf{Z} \times \frac{1}{2}\mathbf{Z}$. *Then for small enough* $\varepsilon > 0$ *and varying* $t \in (-\varepsilon, \varepsilon)$ *we have*

$$\mathrm{Mas}(\mathscr{L}_{\alpha,\beta}, K^+_{\alpha,\beta,\theta e^{\pi i t}}) = \begin{cases} 2 & \text{if } \theta = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

LEMMA 5.3.4. SF *is additive under compositions of paths in* $\tilde{\mathbf{R}}^2$. □

LEMMA 5.3.5. SF *is a homotopy invariant rel endpoints.* □

LEMMA 5.3.6. $\mathrm{SF}(\tilde{\rho}) = 0$ *if* $\tilde{\rho}$ *lies entirely in* $\mathbf{Z} \times \mathbf{R} \times \{\pm i\} \subset \tilde{\mathbf{R}}^2$. *Note, that* $\mathbf{Z} \times \mathbf{R} \times \{\pm i\}$ *corresponds to the thickened vertical lines in Figure 5.3.2.*

PROOF. Since $\tilde{\rho}([0,1]) \subset \mathbf{Z} \times \mathbf{R} \times \{\pm i\}$, $A_{\tilde{\rho}}$ is a path of flat connections on S. Thus we can use cohomology to compute the kernel of $D_{A_{\alpha,\beta}}$ on S with boundary conditions $\mathcal{P}^+_{\alpha,\beta}(L)$. We have $\mathrm{Ker}(H^1(S; su(2)_{\mathrm{hol}(A_{\alpha,\beta})}) \to H^1(T; su(2)_{\mathrm{hol}(A_{\alpha,\beta})})) = 0$.

Then Proposition 2.8.6 and $L_{\alpha,\beta} \cap \mathscr{L}_{S,A_{\alpha,\beta}} = 0$ imply that $D_{A_{\alpha,\beta}}$ (with boundary conditions $\mathcal{P}^+_{\alpha,\beta}(L)$) has no kernel. Furthermore $K^+_{\alpha,\beta,\pm i} \cap \breve{\mathscr{L}}_{S,A_{\alpha,\beta}} = 0$ when $(\alpha,\beta) \in \mathbf{Z} \times \{\frac{1}{2}\}$. Thus $\mathrm{SF}(\tilde{\rho}) = 0$. □

LEMMA 5.3.7. *There is a cohomology class $z \in H^1(\tilde{\mathbf{R}}^2, \mathbf{Z} \times \mathbf{R} \times \{\pm i\})$, such that for a path $\tilde{\rho}$ in $\tilde{\mathbf{R}}^2$ that starts and ends in $\mathbf{Z} \times \mathbf{R} \times \{\pm i\} \subset \tilde{\mathbf{R}}^2$ the spectral flow equals $z(u)$, where $u := [\tilde{\rho}([0,1])] \in H_1(\tilde{\mathbf{R}}^2, \mathbf{Z} \times \mathbf{R} \times \{\pm i\})$.*

PROOF. By defining $\zeta(\sum_i a_i\sigma_i) := \sum_i a_i \mathrm{SF}(\sigma_i)$ for singular 1-simplices σ_i and integers a_i, we get a map $\zeta : S_1(\tilde{\mathbf{R}}^2) \to \mathbf{Z}$, i.e. $\zeta \in S^1(\tilde{\mathbf{R}}^2)$. By Lemma 5.3.6 the spectral flow descends to a map $\bar{\zeta} : S_1(\tilde{\mathbf{R}}^2)/S_1(Z \times \mathbf{R} \times \{\pm i\}) \to \mathbf{Z}$. Thus $\bar{\zeta} \in S^1(\tilde{\mathbf{R}}^2, Z \times \mathbf{R} \times \{\pm i\})$.

If ω is a 2-simplex in $\tilde{\mathbf{R}}^2$, then $\partial\omega = \omega \circ f_0^2 - \omega \circ f_1^2 + \omega \circ f_2^2$, where f_0^2 are the face maps. Then as paths, $(\omega \circ f_0^2) * (\omega \circ f_2^2)$ and $\omega \circ f_1^2$ are homotopic in $\tilde{\mathbf{R}}^2$. Thus by Lemmas 5.3.4 and 5.3.5 we have

$$\zeta(\partial\omega) = \zeta((\omega \circ f_0^2) * (\omega \circ f_2^2)) - \zeta(\omega \circ f_1^2) = 0.$$

Now let $\sum_j b_j\omega_j \in S_2(\tilde{\mathbf{R}}^2, Z \times \mathbf{R} \times \{\pm i\})$ with integers b_j and 2-simplices ω_j. Let $\partial\omega_j = \sum_i a_{ij}\sigma_{ij}$. Then

$$\mathfrak{d}\zeta(\sum_j b_j\omega_j) = \zeta(\partial \sum_j b_j\omega_j) = \sum_{i,j} a_{ij}\zeta(\partial\omega_{ij}) = 0.$$

Since there is an injection $i : S^*(\tilde{\mathbf{R}}^2, Z \times \mathbf{R} \times \{\pm i\}) \to S^*(\tilde{\mathbf{R}}^2)$ which is a chain map, we have $i \circ \mathfrak{d}\bar{\zeta} = \mathfrak{d} \circ i\zeta = 0$. Thus $\bar{\zeta}$ is a cocycle. We found the desired cohomology class $z = [\bar{\zeta}]$. □

Equivalently we may say (by Poincaré duality) that there exists an infinite homology class in $H_1(\tilde{\mathbf{R}}^2 - (\mathbf{Z} \times \mathbf{R} \times \{\pm i\}))$, such that the intersection number with (the homology class representing) the path is equal to the spectral flow.

Let us study the spectral flow on the solid torus for a specific family of Lagrangians $L_{\alpha,\beta,\theta}$ in $\mathcal{H}^{0+1+2}(T;\mathbf{R}i)$. It will roughly be the $\mathbf{R}i$ part of orthogonal complement of the the scattering Lagrangian on S. With Lemma 4.2.1 in mind we are going to make the following definition.

DEFINITION 5.3.8. For $(\alpha,\beta) \in \mathbf{R}^2$ define $\widehat{\mathscr{L}}_{S,A_{(\alpha,\beta)}} := \mathbf{R}i \oplus \mathbf{R}i\,dl$. Note that $\widehat{\mathscr{L}}_{S,A_{(\alpha,\beta)}} = \mathcal{H}^0(T;\mathbf{R}i) \oplus \mathbf{R}i\,dl \subset \mathcal{H}^{0+1+2}(T;\mathbf{R}i)$.

Thus, fix $L_{\alpha,\beta,\theta} := J\widehat{\mathscr{L}}_{S,A_{(\alpha,\beta)}} = \mathbf{R}i\,dm \oplus \mathbf{R}i\,dm \wedge dl = \mathbf{R}i\,dm \oplus \mathcal{H}^2(T;\mathbf{R}i)$. Our goal is to compute an (infinite) cycle as shown in Figure 5.3.2.

With Lemma 5.3.7 in place, we could refer to [**KK2**] for the mod 8 computation of spectral flow. This would be sufficient for the main application in Chapter 6. However, we will extend the results in [**KK2**] to compute spectral flow mod 0. This enables us to compute mod 0 spectral flow using the splitting formula for spectral flow developed in Chapter 4. In any case, it will be interesting to see how this piece of the puzzle fits together with the rest. Coincidentally we are going to use the splitting formula for the computation of spectral flow on S, particularly the fact that the splitting formula does not depend on the lift of ρ to $\widetilde{\mathbf{R}}^2$.

THEOREM 5.3.9. *If $\tilde{\rho}$ starts and ends in $\mathbf{Z} \times \mathbf{R} \times \{\pm i\}$, then* $\mathrm{SF}(\tilde{\rho})$ *is given by the intersection number $\tilde{\rho} \cdot z$ where z is the cycle shown in Figure 5.3.2.*

An example of how to compute the intersection number and thus the spectral flow is shown in Figure 5.3.3

The lengthy proof of Theorem 5.3.9 is deferred to the next section. However, let me give a short outline of the proof. Consider a path of connections A_t between flat connections on a certain closed manifold $M = S \cup_T X$, which restricts to a path $A_{\rho(t)}$ in our family of connections on S defined in 5.2.1. An example such a path $\rho(t)$ in

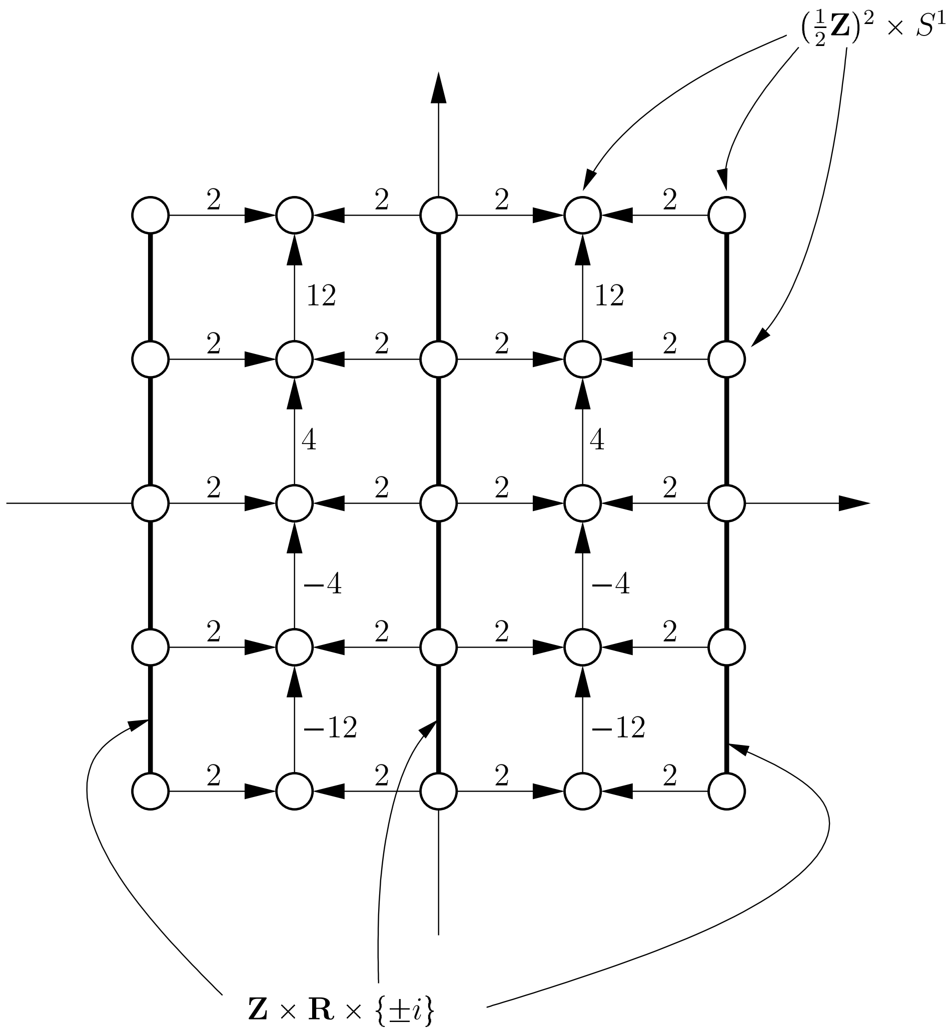

FIGURE 5.3.2. Cycle in $\widetilde{\mathbf{R}}^2$

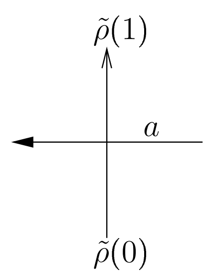

FIGURE 5.3.3. An example with intersection number a

$\widetilde{\mathbf{R}}^2$ is shown in Figure 5.4.5. Now consider a particular gauge transformation g on M. Suppose $(g \cdot A_t)|T = A_{\rho'(t)}|T$. Then we are going to compare the spectral flow of $g \cdot A_t$ with the spectral flow of B_t given by

$$B_t = \begin{cases} g \cdot A_t & \text{when restricted to X} \\ A_{\rho'(t)} & \text{when restricted to S,} \end{cases}$$

using some properties of a particular family of gauge transformations, which are derived in [**BHKK**], as well as the following well-known fact:

LEMMA 5.3.10. *Suppose M is a closed 3-manifold and $g : M \to SU(2)$ is a gauge transformation. If A_t is any path of $SU(2)$ connections on M from A_0 to $A_1 = g \cdot A_0$, then*

$$\mathrm{SF}(D_{A_t}) = 8 \deg(g)$$

PROOF. This is an application of the Atiyah-Patodi-Singer index theorem. □

By repeating the above process a few times and applying the splitting formula for spectral flow in Theorem 4.5.5 for different paths ρ we will learn how the chains relate to each other, because the terms of the spectral flow on X cancel. This yields the (infinite) cycle with three unknowns a,b and c. The coefficients in Figure 5.3.2 can then be computed using some lens space examples and by applying the splitting formula for spectral flow in Theorem 4.5.5.

Before we embark on a proof of Theorem 5.3.9 we adapt some of the definitions and results in [**BHKK**] to our needs.

Consider the following group of gauge transformations:

$$\mathcal{G}_{\mathrm{nf}} = \{\text{smooth maps } g : S \to SU(2) \mid g|_{N_\varepsilon(T)}(ne^{im}, e^{il}) = e^{i\alpha m + i\beta l} \text{ for some } \alpha, \beta \in \mathbf{Z}\}$$

There is a well-defined degree for maps $(X,T) \to (S^3,S^1)$ with $\partial X = T$ by observing that $H_3(S^3,S^1;\mathbf{Z}) = \mathbf{Z}$ by the long exact sequence of the pair (S^3,S^1) and $H_3(X,T;\mathbf{Z}) = \mathbf{Z}$. In particular we have a well-defined degree for elements of $\mathcal{G}_{\text{nf}}$.

Lemma 5.3.11 (Lemma 4.1, [**BHKK**]). *Let $g, g' \in \mathcal{G}_{nf}$. Then g and g' are homotopic if and only if $g|_T = g'|_T$ and* $\deg(g) = \deg(h)$.

Definition 5.3.12. Let $q : [0,1] \to [0,1]$ be a smooth non-decreasing cutoff function with $q(n) = 0$ for n near 0 and $q(n) = 1$ for $n \in [1-\varepsilon, 1]$, where ε is such that the family $A_{\alpha,\beta}$ is in cylindrical form in the collar $N_\varepsilon(T)$ of T. Then consider the gauge transformations on $S = D^2 \times S^1 \to SU(2)$:

(1) $\mathfrak{a}(ne^{im}, e^{il}) = q(n)e^{im} + \sqrt{1-(q(n))^2}j$,
(2) $\mathfrak{b}(ne^{im}, e^{il}) = e^{il}$,
(3) $\mathfrak{c}(ne^{im}, e^{il})$ is a gauge transformation of degree 1 with $\mathfrak{c}|_{N_\varepsilon(T)} \equiv 1$.

The exact description of these gauge transformations is not relevant for us. However, we will need to exploit some of their properties. We have the following up to homotopy.

Lemma 5.3.13 (Lemma 4.3, [**BHKK**]). $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{c}^{-2}$.

We also have

Theorem 5.3.14 (Theorem 4.4, [**BHKK**]). $\deg(\mathfrak{a}^a\mathfrak{b}^b\mathfrak{c}^c) = c - ab$.

The following lemma will be used frequently

Lemma 5.3.15. *If $g : (X,T) \to (S^3,S^1)$ misses a point $p \notin S^1$, then* $\deg(g) = 0$.

Proof. First note that by the long exact sequence of a pair $H_3(S^3,S^1) \cong H_3(S^3)$ and $H_3(S^3,S^3-p) \cong H_3(S^3)$. Thus $H_3(S^3,S^1) \cong H_3(S^3,S^3-p)$. The long exact sequence of the triple $(S^3, S^3 - p, S^1)$, $p \notin S^1$, then implies, that

$$H_3(S^3 - p, S^1) \to H_3(S^3, S^1)$$

is the zero map. Thus, if a map $g : (X, T) \to (S^3, S^1)$ misses a point $p \notin S^1$, then $\deg(g) = 0$. $\square$

We want to consider certain closed manifolds to deduce information about the spectral flow on the solid torus. We will need to consider gauge transformations on M.

LEMMA 5.3.16. *If $\partial X = T$, any map $g : T \to SU(2)$ extends to X.*

PROOF. Since $\pi_1(SU(2)) = \pi_2(SU(2)) = 0$, there is no obstruction of extending g to X. $\square$

We are going to use one additional gauge transformation on S. We set $\mathfrak{d} \equiv j$. Notice, that even though $\mathfrak{d} \notin \mathcal{G}_{\text{nf}}$, we have $\mathfrak{d} g \mathfrak{d}^{-1} \in \mathcal{G}_{\text{nf}}$ for all $g \in \mathcal{G}_{\text{nf}}$. Furthermore the following observation is noteworthy

LEMMA 5.3.17. $\deg(\mathfrak{a}\mathfrak{d}\mathfrak{a}\mathfrak{d}^{-1}) = 0.$

PROOF. Note that $\mathfrak{a}$ is homotopic to $\mathfrak{a}'(ne^{im}, e^{il}) = ne^{im} + \sqrt{1-n^2}j$. Then for $(ne^{im}, e^{il}) \in S$ we have

$$\mathfrak{a}'\mathfrak{d}\mathfrak{a}'\mathfrak{d}^{-1}(ne^{im}, e^{il}) = 2n^2 - 1 + 2n\sqrt{1-n^2}e^{im}j \neq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$$

which implies $\deg(\mathfrak{a}\mathfrak{d}\mathfrak{a}\mathfrak{d}^{-1}) = \deg(\mathfrak{a}'\mathfrak{d}\mathfrak{a}'\mathfrak{d}^{-1}) = 0$ by Lemma 5.3.15. $\square$

We will make one final observation. Let $M = S \cup_T X$.

LEMMA 5.3.18. *If $g : M \to S^3$ with $g(T) \subset S^1$, then* $\deg(g) = \deg(g|_S) + \deg(g|_X)$.

Proof. By the relative Mayer-Vietoris sequence we get the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_3(S,T)\oplus H_3(X,T) & \xrightarrow{\Phi_*} & H_3(M,T) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (g|_S)_*\oplus(g|_X)_*} & & \downarrow{\scriptstyle g_*} & & \\ & & H_3(S^3,S^1)\oplus H_3(S^3,S^1) & \xrightarrow{\Psi_*} & H_3(S^3,S^1) & \longrightarrow & 0 \end{array},$$

where on the chain level $\Phi(a,b)=a+b$ and $\Phi(a,b)=a+b$.

The long exact sequences of the pairs (M,T) and (S^3,S^1) yield a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_3(M) & \xrightarrow{i_*} & H_3(M,T) & & \\ & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} & & \\ 0 & \longrightarrow & H_3(S^3) & \xrightarrow{j_*} & H_3(S^3,S^1) & \longrightarrow & 0 \end{array}.$$

We denote the fundamental class of $H_3(S^3)$ by $[S^3]$. We know, that for the fundamental homology class $[M]$ we have $g_*([M])=\deg(g)[S^3]$. Then the fundamental class of $H_3(S^3,S^1)$ is given by $j_*([S^3])$. The fundamental classes $[S]$ and $[X]$ of $H_3(S,T)$ and $H_3(X,T)$ can be written as $([S],[X])=(\Phi_*)^{-1}\circ i_*([M])$ and satisfy $g_*([S])=\deg(g|_S)j_*([S^3])$ and $g_*([X])=\deg(g|_X)j_*([S^3])$. The commutativity of the above diagrams shows

$$\begin{aligned} \deg(g)[S^3] &= g_*([M]) = (j_*)^{-1}\circ\Psi_*\circ((g|_S)_*\oplus(g|_X)_*)\circ(\Phi_*)^{-1}\circ i_*([M]) \\ &= (j_*)^{-1}\circ\Psi_*(\deg(g|_S)j_*([S^3]),\deg(g|_X)j_*([S^3])) \\ &= (j_*)^{-1}((\deg(g|_S)+\deg(g|_X))j_*([S^3])) = (\deg(g|_S)+\deg(g|_X))[S^3]. \end{aligned}$$

□

5.4. Proof of Theorem 5.3.9

For simplicity consider the lens space $M = S \cup_h S'$, where $h : \partial S = T \to \partial S'$, $(e^{im}, e^{il}) \mapsto (e^{i3m+i4l}, e^{im+il})$. Then we have a family flat connections on M which restricts to a noncentral connections on T. Figure 5.4.4 shows which flat connections of the family of connections $a_{\alpha,\beta}$ on T extend to flat connections on S and S'. The intersection of the lines corresponding to these flat connections on T correspond to flat connections on M.

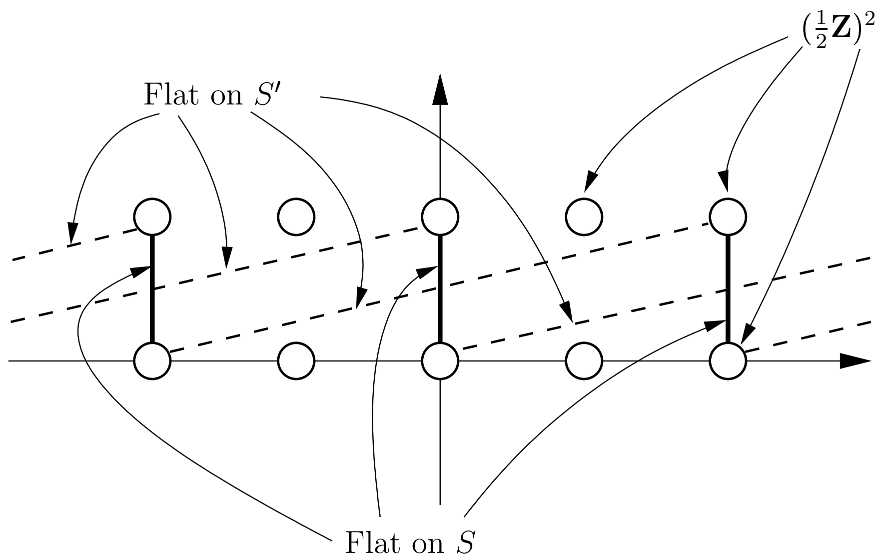

FIGURE 5.4.4. Flat connections S and S'

By Lemma 5.3.16 we may extend $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ from Definition 5.3.12 to all of M. Denote the extensions by $\bar{\mathfrak{a}}$, $\bar{\mathfrak{b}}$, $\bar{\mathfrak{c}}$ and $\bar{\mathfrak{d}}$ respectively. (Note, that we can simply choose $\bar{\mathfrak{d}} = j$.) Let $\rho(t)$ be a (straight line) path in $\tilde{\mathbf{R}}^2$ which starts at $(0, \frac{1}{4}) \in \tilde{\mathbf{R}}^2$ and ends at $(1, \frac{1}{4}) \in \tilde{\mathbf{R}}^2$ as shown in Figure 5.4.5.

Now let A_t be a fixed but arbitrary path of connections on M, which is flat at the endpoints and equals $A_{\rho(t)}$ when restricted to S as in Definition 5.2.1. Consider the

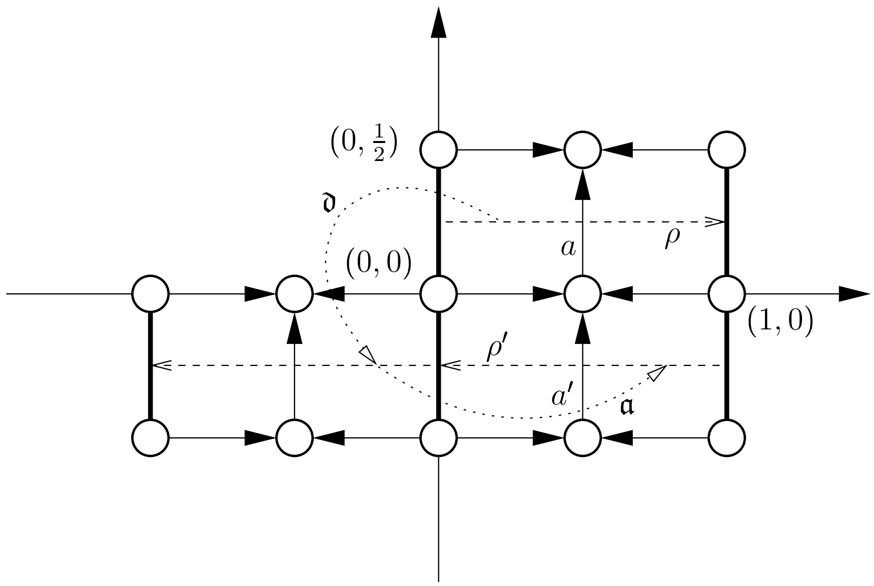

FIGURE 5.4.5. Computing $a' = -a$

gauge transformation $g = \bar{\mathfrak{a}} \cdot \bar{\mathfrak{d}}$. Define a path of connections on M by

$$B_t = \begin{cases} g \cdot A_t & \text{when restricted to X} \\ A_{\rho'(t)} & \text{when restricted to S,} \end{cases}$$

where $\rho'(t) = -\rho(t) + (1,0)$. Notice that $A_{\rho'(t)}|_T = g \cdot A_{\rho(t)}|_T$.

Let us compare $\mathrm{SF}(A_t)$ with $\mathrm{SF}(B_t)$ by utilizing Theorem 5.3.10. By Definition 5.2.1 we have

$$B_0|_S = \mathfrak{a}(-A_0)|_S = \mathfrak{a}\mathfrak{d}A_0|_S$$

and

$$B_1|_S = \mathfrak{d}A_0|_S = \mathfrak{d}\mathfrak{a}^{-1}A_1|_S = \mathfrak{d}\mathfrak{a}^{-1}\mathfrak{d}^{-1}\mathfrak{a}^{-1}(\mathfrak{a}\mathfrak{d}A_1)|_S.$$

Notice that $B_1|_X = \bar{\mathfrak{a}}\bar{\mathfrak{d}}A_1|_X$. Consider the gauge transformation

$$g'(x) = \begin{cases} 1 & \text{if } x \in X \\ \mathfrak{a}\mathfrak{d}\mathfrak{a}\mathfrak{d}^{-1}(x) & \text{if } x \in S. \end{cases}$$

By exploiting Lemmas 5.3.5, 5.3.4, 5.3.10, 5.3.18 and 5.3.17 (in this order) we get

$$\begin{aligned} \mathrm{SF}(A_t) &= \mathrm{SF}(g \cdot A_t) = \mathrm{SF}(\bar{\mathfrak{a}}\bar{\mathfrak{d}}A_0, \bar{\mathfrak{a}}\bar{\mathfrak{d}}A_1) = \mathrm{SF}(B_0, B_1) + \mathrm{SF}(B_1, \bar{\mathfrak{a}}\bar{\mathfrak{d}}A_1) \\ &= \mathrm{SF}(B_0, B_1) + 8\deg(g') = \mathrm{SF}(B_0, B_1) + 8\deg(\mathfrak{a}\mathfrak{d}\mathfrak{a}\mathfrak{d}^{-1}) \\ &= \mathrm{SF}(B_t) + 8\deg(\mathfrak{a}\mathfrak{d}\mathfrak{a}\mathfrak{d}^{-1}) = \mathrm{SF}(B_t) = -\mathrm{SF}(B_{1-t}). \end{aligned}$$

After applying the splitting formula in Corollary 4.5.6 and Lemma 4.5.7, we see that the spectral flow terms for X vanish, so that $a' = -a$.

Now let $k, l \in \mathbf{Z}$ and consider straight line path $\rho(t)$ which starts at $(k, \frac{l}{2} + \frac{1}{4})$ and ends at $(k, \frac{l}{2} + \frac{5}{4})$, as shown in Figure 5.4.6.

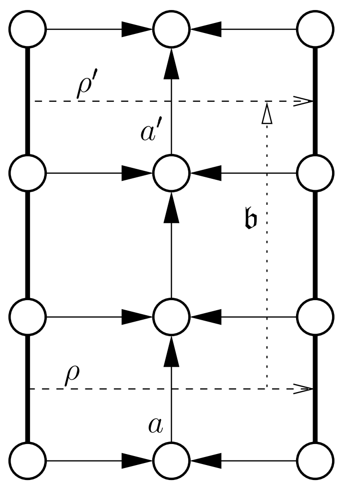

Figure 5.4.6. Computing $a' = 16 + a$

As before, let A_t be a fixed but arbitrary path of connections on M, which equals $A_{\rho(t)}$ when restricted to S as in definition 5.2.1. Consider the gauge transformation $g = \bar{\mathfrak{b}}$. Define a path of connections on M by

$$B_t = \begin{cases} g \cdot A_t & \text{when restricted to X} \\ A_{\rho'(t)} & \text{when restricted to S,} \end{cases}$$

where $\rho'(t) = \rho(t) + (0, 1)$. Notice that $A_{\rho'(t)}|_T = g \cdot A_{\rho(t)}|_T$.

We have

$$B_0|_S = \mathfrak{b}A_0|_S$$

and

$$B_1|_S = \mathfrak{a}\mathfrak{b}A_0|_S = \mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{b}^{-1}(\mathfrak{b}\mathfrak{a}A_0)|_S = \mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{b}^{-1}(\mathfrak{b}A_1)|_S.$$

Consider the gauge transformation

$$g'(x) = \begin{cases} 1 & \text{if } x \in X \\ (\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{b}^{-1})^{-1}(x) & \text{if } x \in S. \end{cases}$$

Then we get

$$\begin{aligned} \mathrm{SF}(A_t) &= \mathrm{SF}(g \cdot A_t) = \mathrm{SF}(\bar{\mathfrak{b}}A_0, \bar{\mathfrak{b}}A_1) = \mathrm{SF}(B_0, B_1) + \mathrm{SF}(B_1, \bar{\mathfrak{b}}A_1) \\ &= \mathrm{SF}(B_0, B_1) + \deg(g') = \mathrm{SF}(B_0, B_1) + \deg((\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1}\mathfrak{b}^{-1})^{-1}) \\ &= \mathrm{SF}(B_t) + \deg(\mathfrak{c}^2) = \mathrm{SF}(B_t) + 16. \end{aligned}$$

Application of the splitting formula in Corollary 4.5.6 then shows that $a' = a + 16$.

Similarly if for the same path ρ we consider $\rho'(t) = \rho(t) + (1,0)$, the gauge transformation $g = \bar{\mathfrak{a}}$, and we define B_t as before, then we have $B_t = g \cdot A_t$ and thus $\mathrm{SF}(A_t) = \mathrm{SF}(B_t)$.

A similar computation gives us the relationship between coefficients for the horizontal simplices.

Let $\rho(t)$ be a path going half around the origin as shown in Figure 5.4.7, e.g. $\rho(t) = \frac{1}{4}e^{\pi i(t-\frac{1}{2})}$. And let $\rho'(t)$ be $-\rho(t)$.

Let $g = \bar{\mathfrak{d}}$ and define B_t as above. Since $B_t = \bar{\mathfrak{d}} \cdot A_t$, we immediately get $\mathrm{SF}(A_t) = \mathrm{SF}(B_t)$. The splitting formula from Corollary 4.5.6 yields $b = b'$. Similarly, if $g = \bar{\mathfrak{a}}$ or $g = \bar{\mathfrak{b}}$ we have $B_t = g \cdot A_t$. Certainly, we get the same relationship for the coefficients c and c' in Figure 5.4.7.

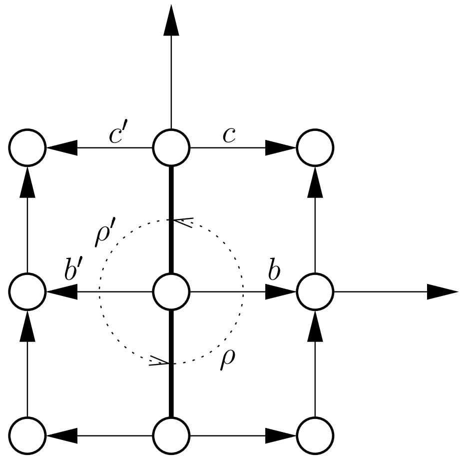

FIGURE 5.4.7. Computing $b' = b$

Thus we are left with determining the coefficients a, b and c in our cycle in Figure 5.4.8.

To accomplish this we will use our splitting formula from Theorem 4.5.5. The advantage our splitting formula has over Lemma 4.5 in [**BHKK**] or Corollary 1.25 in [**Bu**] is that we do not need to assume that $\mathrm{Ker} S_{a_t}$ is 0 or constant.

To find the values of b and c we refer to Lemma 5.3.3. For a we consider the 3-sphere M constructed by gluing two solid tori S and S' with the same orientation along the boundary using the orientation reversing homeomorphism on the boundary $(e^{im}, e^{il}) \mapsto (e^{il}, e^{im})$. For $X = M - S$ we get a similar statement as in Lemma 5.3.3, then we can once more employ the splitting formula in Corollary 4.5.6 for the paths

$$\rho(t) = \begin{cases} (0, 0, e^{\pi i(\frac{1}{2} - 4t)}) & \text{for } t \in [0, \frac{1}{8}] \\ (4t - \frac{1}{2}, 0, -1) & \text{for } t \in [\frac{1}{8}, \frac{1}{4}) \\ (\frac{1}{2}, 0, e^{4\pi i t}) & \text{for } t \in [\frac{1}{4}, \frac{3}{4}] \\ (4(1-t) - \frac{1}{2}, 0, -1) & \text{for } t \in (\frac{3}{4}, \frac{7}{8}] \\ (0, 0, e^{4\pi i(t - \frac{7}{8})}) & \text{for } t \in [\frac{7}{8}, 1] \end{cases}$$

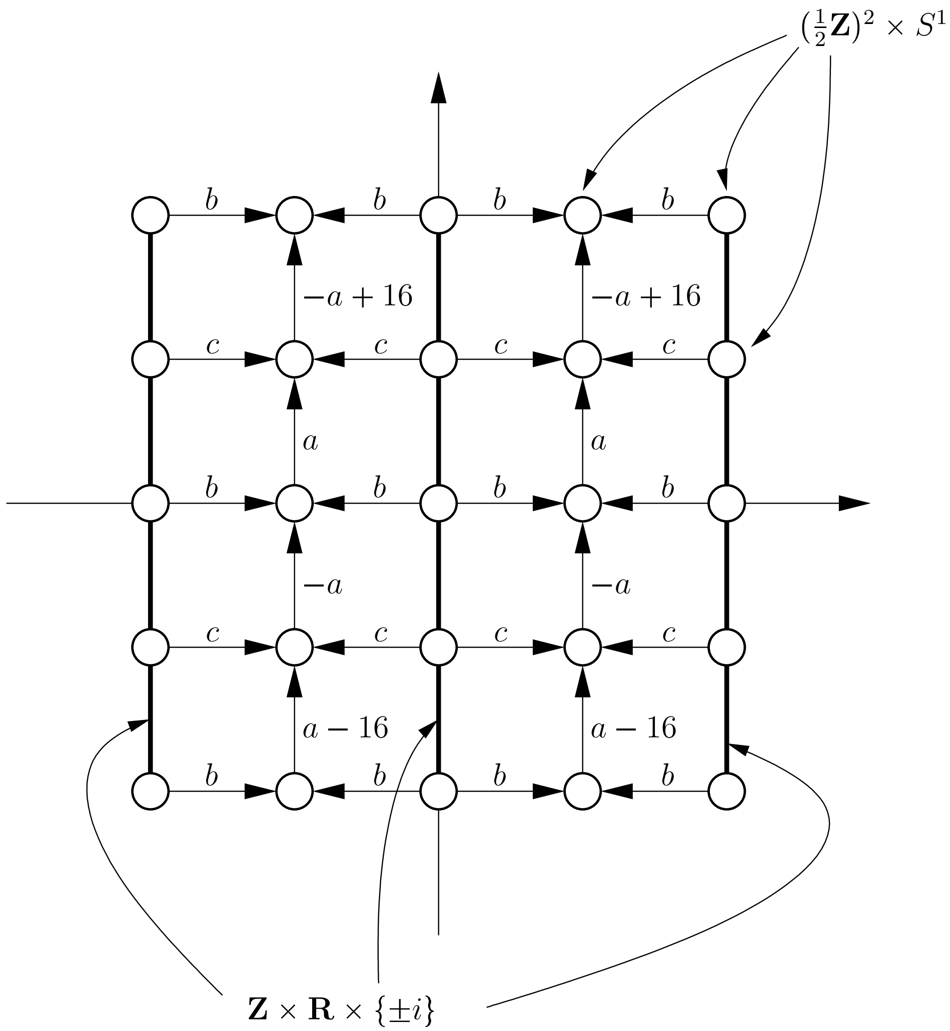

FIGURE 5.4.8. Cycle in $\widetilde{\mathbf{R}}^2$

and

$$\rho'(t) = \begin{cases} (0, 0, e^{\pi i(\frac{1}{2}-4t)}) & \text{for } t \in [0, \frac{1}{8}] \\ (4t - \frac{1}{2}, 0, -1) & \text{for } t \in [\frac{1}{8}, \frac{1}{4}) \\ (\frac{1}{2}, 0, -1) & \text{for } t \in [\frac{1}{4}, \frac{3}{4}] \\ (4(1-t) - \frac{1}{2}, 0, -1) & \text{for } t \in (\frac{3}{4}, \frac{7}{8}] \\ (0, 0, e^{4\pi i(t-\frac{7}{8})}) & \text{for } t \in [\frac{7}{8}, 1] \end{cases}$$

to see that

$$
\begin{aligned}
0 &= \mathrm{SF}(D_{A_{\rho(t)}}) - \mathrm{SF}(D_{A_{\rho'(t)}}) \\
&= \mathrm{SF}_S(D_{A_{(\frac{1}{2},0,e^{2\pi i t})}}; \mathcal{P}^+(J\widehat{\mathscr{L}_S})) + \mathrm{SF}_X(D_{A_{(\frac{1}{2},0,e^{2\pi i t})}}; \mathcal{P}^-(\widehat{\mathscr{L}_S})) \\
&= (2 - a + 2 - a) + (2 + 2).
\end{aligned}
$$

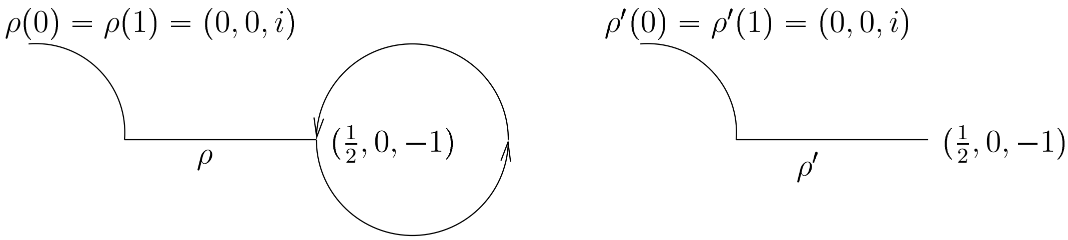

FIGURE 5.4.9. The paths ρ and ρ'

The image of the paths in $\tilde{\mathbf{R}}^2$ is shown in Figure 5.4.9. This shows that a has to equal 4. $\square$

CHAPTER 6

Spectral Flow on a Torus Bundle over the Circle

We are interested in computing spectral flow on torus bundles over S^1, because it addresses Lisa Jeffrey's Conjecture 5.8 in [**J**], the missing piece in her work on Witten's 3-manifold invariants [**J**].

Her conjecture, based on physical reasoning, implies that the spectral flow between irreducible flat $SU(2)$-connections on torus bundles over S^1 is 0 mod 4, which shall be confirmed in this chapter.

This chapter also serves as an example showing the spirit of applications for Theorem 4.5.5. Anywhere where spectral flow computations between flat connections are necessary, Theorem 4.5.5 or a suitable adaptation gives a possible approach.

The full meaning of Lisa Jeffrey's conjecture needs some interpretation when the trace of the monodromy matrix is zero as well as "when there are more than one wU fixing A" (see footnote to Conjecture 5.8 in [**J**]). Also there may be some other sign issues involved. Thus we are not able to analyze Lisa Jeffrey's conjecture in detail. However, it will be the subject of future research.

6.1. Irreducible $SU(2)$-Representations of $\pi_1 M$

By Proposition 2.4.5 flat $SU(2)$-connections on M and $SU(2)$-representations of the fundamental group of M go hand in hand. Group cohomology computations twisted by representations will be used to compute the scattering Lagrangian and other information about the kernel of the odd-signature operator and the tangential operator at flat connections. Therefore it is necessary to analyze the representations of the fundamental group. Along the way we will find a knot γ so that there exists a path of representations of $\pi_1(M-\gamma)$ connecting the two holonomies $\mathrm{hol}(A_0)$ and $\mathrm{hol}(A_1)$. The facts in this section are mostly due to Kirk and Klassen [**KK2**], though detailed (alternative) proofs have been added when it seemed appropriate.

Fix an orientation for T. Let $m : T \to T$ be an orientation preserving homeomorphism, $M = T \times I/(m(x),1) \sim (x,0)$ be its mapping torus and fix a base point $(*,0)$ in M. Consider the isomorphism $m_* : H_1(T;\mathbf{Z}) \to H_1(T;\mathbf{Z})$ induced by m on homology, and fix a meridian x and longitude y for T so that $H_1(T;\mathbf{Z}) \cong \pi_1 T = \langle x, y, |[x,y]\rangle \cong \mathbf{Z}^2$ and so that $dx \wedge dy$ is in the orientation class, when we consider $x, y \in H_1(T;\mathbf{Z}) \cong H^1(T;\mathbf{Z})$ (see Figure 6.1.1). After identifying $x = \binom{1}{0}$ and $y = \binom{0}{1}$ we can write m_* as an unimodular matrix. If $m_*(x) = ax + cy$ and $m_*(y) = bx + dy$, then

$$m_* = B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbf{Z}).$$

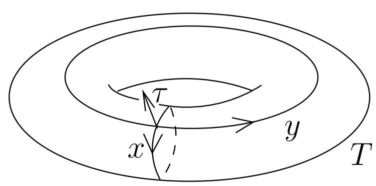

FIGURE 6.1.1. The torus

We will frequently make use of the following properties of B.

LEMMA 6.1.1.

(1) $|\mathrm{tr}B| \neq 2$ *if and only if* $\det(B \pm I) \neq 0$
(2) $|\mathrm{tr}B| \neq 2$ *implies* $c \neq 0$ *and* $b \neq 0$.

PROOF. Since $\det(B) = 1$ we have $\det(B+I) = (a+1)(d+1) - bc = ad - bc + 1 + a + d = 2 + \mathrm{tr}B$ and $\det(B-I) = 2 - \mathrm{tr}B$. Thus $|\mathrm{tr}B| \neq 2$ if and only if $\det(B \pm I) \neq 0$. Also if $c = 0$ or $b = 0$, then $1 = ad - bc = ad$, which implies $a = d = \pm 1$, that is, $|\mathrm{tr}B| = 2$. □

The fundamental group of M is an HNN extension of $\pi_1 T$, where τ is the loop from $(*, 0)$ to $(m(*), 1)$:

$$\pi_1 M = \{x, y, \tau | [x, y], \tau x \tau^{-1} = x^a y^c, \tau y \tau^{-1} = x^b y^d\}. \tag{6.1.5}$$

Recall that we have the identification

$$SU(2) \approx S^3 \approx \text{unit quaternions}$$

with the usual multiplication for unit quaternions.

Each irreducible representation can be conjugated to have a particularly nice description.

Lemma 6.1.2 (Proposition 5.5, [**KK1**]). *Let $\varphi = (\varphi_1, \varphi_2) \in \mathbf{R}^2$. The homomorphism*

$$\begin{array}{rcl} \rho_\varphi : \langle\{x, y, \tau\}\rangle & \to & SU(2) \\ \tau & \mapsto & j \\ x & \mapsto & e^{2\pi i \varphi_1} \\ y & \mapsto & e^{2\pi i \varphi_2} \end{array}$$

factors through $\pi_1 M$ if and only if $\varphi(B + I) \in \mathbf{Z}^2$.

Proof. ρ factors through $\pi_1 M$ if and only if it sends all relators to $1 \in SU(2)$. We have $\rho([x, y]) = 1$. For the other relations we get

$$\begin{array}{rrcl} & je^{2\pi i\varphi_1}(-j) = (e^{2\pi i\varphi_1})^a(e^{2\pi i\varphi_2})^c & \text{and} & je^{2\pi i\varphi_2}(-j) = (e^{2\pi i\varphi_1})^b(e^{2\pi i\varphi_2})^d \\ \Leftrightarrow & e^{-2\pi i\varphi_1} = e^{2\pi i\varphi_1 a + 2\pi i\varphi_2 c} & \text{and} & e^{-2\pi i\varphi_2} = e^{2\pi i\varphi_1 b + 2\pi i\varphi_2 d} \\ \Leftrightarrow & (a+1)\varphi_1 + c\varphi_2 \in \mathbf{Z} & \text{and} & b\varphi_1 + (d+1)\varphi_2 \in \mathbf{Z} \end{array}$$

that is, $(\varphi_1, \varphi_2)(B + I) \in \mathbf{Z}^2$. □

In other words we get:

Corollary 6.1.3. *The homomorphism* ρ_φ *is a* $SU(2)$*-representation of* $\pi_1 M$ *if and only if* $\varphi(B+I) \in \mathbf{Z}^2$*. An* $SU(2)$*-representation* ρ_φ *is abelian (or reducible) if and only if* $\varphi \in (\frac{1}{2}\mathbf{Z})^2$*.*

The following is useful when we analyze the space of nonabelian (or irreducible) representations modulo conjugation. (Note that this space is isomorphic to the space of flat irreducible connections modulo gauge transformations.)

Lemma 6.1.4. *Any irreducible* $SU(2)$*-representation of* $\pi_1 M$ *is conjugate to* ρ_φ *for some* φ*.*

Proof. Let ρ be an irreducible $SU(2)$-representation of $\pi_1 M$. It to maps x and y into a maximal torus, because they commute in $\pi_1 M$. Since all maximal tori are conjugate, we can assume that x and y map into $\mathbf{C}$, that is, $\rho(x) = e^{2\pi i\psi_1}$ and $\rho(y) = e^{2\pi i\psi_2}$.

Let $\rho(\tau) = \widehat{a}+\widehat{b}i+\widehat{c}j+\widehat{d}k$. Since ρ is nonabelian, we have $\rho(x) \neq \pm 1$ or $\rho(y) \neq \pm 1$ Thus assume without loss of generality that $\rho(x) \neq \pm 1$, that is $\varphi_1 \notin \frac{1}{2}\mathbf{Z}$. Then

$$\begin{aligned}\rho(x)^a\rho(y)^c = \rho(\tau)\rho(x)\rho(\tau)^{-1} &= \cos(2\pi\psi_1) + (\widehat{a}^2+\widehat{b}^2-\widehat{c}^2-\widehat{d}^2)\sin(2\pi\psi_1)i \\ &\quad + 2(\widehat{b}\widehat{c}+\widehat{a}\widehat{d})\sin(2\pi\psi_1)j + 2(\widehat{b}\widehat{d}-\widehat{a}\widehat{c})\sin(2\pi\psi_1)k\end{aligned}$$

is a complex number $\rho(x)\rho(y) \in \mathbf{C}$. Thus we have $\widehat{b}\widehat{c}+\widehat{a}\widehat{d} = 0$ and $\widehat{b}\widehat{d}-\widehat{a}\widehat{c} = 0$, which yields $\widehat{b}(\widehat{c}^2+\widehat{d}^2) = 0$. Since ρ is nonabelian, τ cannot get mapped into $\mathbf{C}$, that is, $\widehat{c}^2+\widehat{d}^2 \neq 0$, and thus we get $\widehat{b} = 0$. Similarly we get $\widehat{a} = 0$ from $\widehat{a}(\widehat{c}^2+\widehat{d}^2) = 0$.

Then we get for

$$C = \sqrt{\tfrac{1}{2}(1+\widehat{c})} - \operatorname{sign}(\widehat{d})\sqrt{\tfrac{1}{2}(1-\widehat{c})}k$$

that $C\rho C^{-1} = \rho_\varphi$, where $\varphi = (\psi_1, \psi_2)$. □

We will need to know which are the conjugacy classes of irreducible representations. The following criterion will help us find them.

LEMMA 6.1.5. *Two $SU(2)$-representations ρ_φ and ρ_ψ of $\pi_1 M$ are conjugate if and only if $\varphi = \pm\psi + \theta$ for some $\theta \in \mathbf{Z}^2$.*

PROOF. Let

$$C\rho_\varphi C^{-1} = \rho_\psi \tag{6.1.6}$$

with $C \in SU(2)$. In particular we have $CjC^{-1} = j$. The only quaternions that commute with j lie in the unique maximal torus containing j, so

$$C = \cos(2\pi\delta) + \sin(2\pi\delta)j \in \mathbf{R} \oplus \mathbf{R}j,$$

where $\delta \in \mathbf{R}$. Then computing (6.1.6) and comparing coefficients yields for $\varepsilon = 1, 2$

$$\cos(2\pi\varphi_\varepsilon) = \cos(2\pi\psi_\varepsilon) \tag{6.1.7}$$

$$\cos(4\pi\delta)\sin(2\pi\varphi_\varepsilon) = \sin(2\pi\psi_\varepsilon) \tag{6.1.8}$$

$$-\sin(4\pi\delta)\sin(2\pi\varphi_\varepsilon) = 0. \tag{6.1.9}$$

Line (6.1.7) implies that $\varphi_\varepsilon = \eta_\varepsilon\psi_\varepsilon + \theta_\varepsilon$ for some $\theta_\varepsilon \in \mathbf{Z}$ and $\eta_\varepsilon \in \{1, -1\}$ and also that $\varphi_\varepsilon \in \mathbf{Z}$ if and only if $\psi_\varepsilon \in \mathbf{Z}$. Thus if $\varphi_\varepsilon \in \mathbf{Z}$ for some ε, then we automatically get $\varphi = \pm\psi + \theta$.

Thus assume that $\varphi_\varepsilon, \psi_\varepsilon \notin \mathbf{Z}$ for $\varepsilon = 1, 2$. Then equation (6.1.9) implies $\delta \in \frac{1}{4}\mathbf{Z}$, that is, $\cos(4\pi\delta) = \pm 1$, which together with equation (6.1.8) yields $\eta_1 = \eta_2$. We get $\varphi = \pm\psi + \theta$.

On the other hand if φ and ψ were such that $\varphi = \pm\psi + \theta$, we can find a δ satisfying (6.1.8) and (6.1.9), that is, ρ_φ and ρ_ψ are conjugate. □

Lemma 6.1.5 allows us to restrict our attention to a small rectangle.

Corollary 6.1.6. *For a $SU(2)$ representation ρ_φ we can find a representation $\rho_{\varphi'}$ conjugate to ρ_φ, with $\varphi' \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \cup (\frac{1}{2}, 1) \times (0, \frac{1}{2})$.* □

Identify T with $T \times \{0\}$. Let $\gamma = px + qy$ be a simple closed curve in T and N_γ be a collar of γ in M. $M_\gamma = M - N_\gamma$ is homotopy equivalent to $T \times I/\sim$, where $(x, 0) \sim (m(x), 1)$ whenever $x \notin N_\gamma \cap T$. Later we will decompose the manifold M into the two pieces M_γ and N_γ, such that a given path of $SU(2)$-connections is flat on M_γ. Therefore let us study representation of $\pi_1(M_\gamma)$.

The fundamental group $\pi_1(M_\gamma)$ is again an HNN extension of $\pi_1(T)$:

$$\pi_1(M_\gamma) = \langle x, y, \tau | [x, y], \tau x^p y^q \tau^{-1} = x^{pa+qb} y^{pc+qd} \rangle. \tag{6.1.10}$$

Thus in analogy to the proof of Lemma 6.1.2, the homomorphism ρ_φ, $\varphi \in \mathbf{R}^2$, as in Lemma 6.1.2 factors through $\pi_1(M_\gamma)$ if and only if $\varphi(B + I)\binom{p}{q} = 0$. Furthermore, if $px + qy$ defines a curve, then such a pair $(p, q) \in \mathbf{Z}$ is relatively prime and thus unique up sign. This gives the following important result.

Lemma 6.1.7 (Corollary 7.2, [**KK2**]). *If ρ_φ and ρ_ψ are two $SU(2)$-representations of $\pi_1 M$, then $(1 - t)\varphi + t\psi$ defines a path of $SU(2)$-representations of $\pi_1(M_\gamma)$ as in Lemma 6.1.2, where $\gamma = px + qy$, (p, q) is a relatively prime pair satisfying*

$$(\varphi - \psi)(B + I)\begin{pmatrix} p \\ q \end{pmatrix} = 0. \tag{6.1.11}$$

Moreover, (6.1.11) uniquely determines γ (as an unoriented curve) up to isotopy in terms of φ and ψ if $\det(B + I) \neq 0$. □

By Corollary 6.1.6 we can find two representations $\rho_{\varphi'}$ and $\rho_{\psi'}$ conjugate to ρ_φ and ρ_ψ respectively such that $\varphi', \psi' \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \cup (\frac{1}{2}, 1) \times (0, \frac{1}{2})$. Particularly the straight line $(1 - t)\varphi' + t\psi'$ does not go through $(\frac{1}{2}\mathbf{Z})^2$. Given γ from Lemma 6.1.7, then $\rho_{(1-t)\varphi'+t\psi'}$ is a path of irreducible representations of $\pi_1(M_\gamma)$.

COROLLARY 6.1.8. *If ρ_φ and ρ_ψ are two irreducible representations, then we can find conjugate representations $\rho_{\varphi'}$ of ρ_φ and $\rho_{\psi'}$ of ρ_ψ so that $(1-t)\varphi' + t\psi'$ defines a path of irreducible $SU(2)$-representations of $\pi_1(M_\gamma)$.* □

It is important that the whole path of representations is irreducible, because then the non-resonance level of the odd-signature operator is 0 by Corollary 6.3.6 for all paths between irreducible representations that have not been covered by Kirk and Klassen [**KK2**], namely the paths that pass through representations which are central on the boundary. In these cases we can directly apply Nicolaescu's adiabatic limit theorem.

Even though M is not S^3 or $\mathbf{R}^3$, we still have the notion of meridian μ and longitude λ of γ. Let λ be the curve on T parallel to γ and μ the curve that bounds a disk in N_γ. Figure 6.1.2 explains convenient choices and orientations.

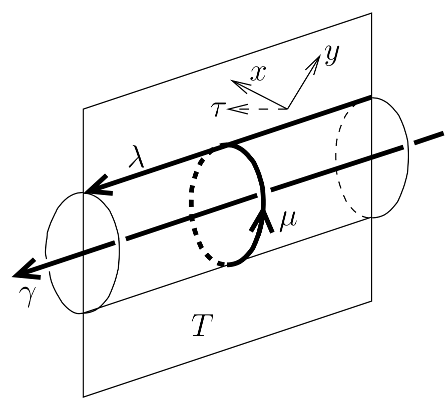

FIGURE 6.1.2. Longitude and meridian

After a basis change for $\pi_1 T$ change via

$$\widehat{B} = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in SL_2(\mathbf{Z})$$

where $ps - rq = 1$, we get

$$\pi_1 M = \langle x, y, \tau \mid [x, y], \tau x^p y^q \tau^{-1} = x^{pa+qb} y^{pc+qd}, \tau x^r y^s \tau^{-1} = x^{ra+sb} y^{rc+sd} \rangle.$$

By comparing this to (6.1.10) and since gluing in N_γ corresponds to killing the meridian, this suggests, that $\mu = x^{ra+sb} y^{rc+sd} \tau x^{-r} y^{-s} \tau^{-1}$ up to orientation. The following lemma gives a proof.

LEMMA 6.1.9. *The longitude λ and meridian μ are*

$$\lambda = x^p y^q$$

$$\mu = x^{ra+sb} y^{rc+sd} \tau x^{-r} y^{-s} \tau^{-1},$$

where r and s are any integers satisfying $ps - rq = 1$.

PROOF. Since λ is parallel to γ on T, we can choose $\lambda = x^p y^q$. Through a basis change on T we may assume that $\lambda = x$. A matrix corresponding to such basis change is

$$\widehat{B}^{-1} = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix} \in SL_2(\mathbf{Z}),$$

where $ps - rq = 1$. We know that even after this basis change x intersects y exactly once. Consider Figure 6.1.3. If we homotop the whole loop $\mu = m_*(y) \tau y^{-1} \tau^{-1}$ as well as $N_\gamma \cap T$ into the interior of $T \times I$ by pushing it upward and contracting τ a little, we get the meridian, because $(m_*(y), 1)$ gets identified with $(y, 0)$.

Reversing the basis change via

$$\widehat{B} = \begin{bmatrix} p & r \\ q & s \end{bmatrix} \in SL_2(\mathbf{Z})$$

yields $\mu = m_*(x^r y^s) \tau x^{-r} y^{-s} \tau^{-1} = x^{ra+sb} y^{rc+sd} \tau x^{-r} y^{-s} \tau^{-1}$. □

Thus we get the following.

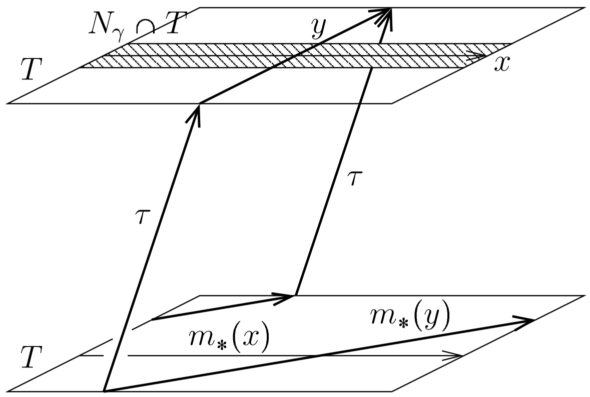

Figure 6.1.3. The representation of the meridian

Lemma 6.1.10. *For $\varphi = (\varphi_1, \varphi_2) \in \mathbf{R}^2$ we have*

$$\rho_\varphi(\lambda) = e^{2\pi i \alpha_\varphi}$$
$$\rho_\varphi(\mu) = e^{2\pi i \beta_\varphi},$$

where $\alpha_\varphi := \varphi\binom{p}{q}$ and $\beta_\varphi := \varphi(B+I)\binom{r}{s}$.

Proof. We calculate

$$\begin{aligned}\rho_\varphi(\lambda) &= e^{2\pi p\varphi_1} e^{2\pi q\varphi_2} = e^{2\pi\varphi\binom{p}{q}} = e^{2\pi i\alpha_\varphi}\\ \rho_\varphi(\mu) &= e^{2\pi i(ra+sb)\varphi_1} e^{2\pi i(rc+sd)\varphi_2} j e^{-2\pi i r\varphi_1} e^{-2\pi i s\varphi_2}(-j)\\ &= e^{2\pi i(r\varphi_1+s\varphi_2+(ra+sb)\varphi_1+(rc+sd)\varphi_2)}\\ &= e^{2\pi i\varphi(B+I)\binom{r}{s}} = e^{2\pi i\beta_\varphi}.\end{aligned}$$

□

We finish this section by a simple observation.

Corollary 6.1.11. *A representation ρ_φ of $\pi_1 M_\gamma$ restricts to a central representation on the boundary if and only if $\alpha_\varphi \in \frac{1}{2}\mathbf{Z}$ and $\beta_\varphi \in \frac{1}{2}\mathbf{Z}$.* □

6.2. An Example

Consider $B = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. By Lemmas 6.1.3 and 6.1.6, and since $\det(B+I) = 8$, conjugacy classes of $SU(2)$-representations of $\pi_1(M)$ are uniquely represented by all ρ_φ with $\varphi \in (\frac{1}{8}\mathbf{Z})^2 \cap ([0, \frac{1}{2}] \times [0, \frac{1}{2}] \cup (\frac{1}{2}, 1) \times (0, \frac{1}{2})$ for which $\varphi(B+I) \in \mathbf{Z}$. There are only two conjugacy classes of irreducible representations. They are represented by ρ_φ and ρ_ψ for $\varphi = (\frac{3}{4}, \frac{1}{4})$ and $\psi = (\frac{1}{4}, \frac{1}{4})$.

By Lemma 6.1.7 any $\rho_{t\varphi+(1-t)\psi}$ for $t \in [0,1]$ determines a representation of $\pi_1(M_\gamma)$, where $\gamma = px + qy$ is such that (p,q) is a relatively prime pair satisfying equation (6.1.11) in Lemma 6.1.7. In this case we have $\varphi(B+I) = (5,2)$ and $\psi(B+I) = (2,1)$. Thus we can choose $(p,q) = \pm(1,-3)$. A different choice of φ and ψ gives us a different (p,q). E.g. if we choose $\varphi' = \varphi$ and $\psi' = -\psi$, then $(p,q) = \pm(3,-7)$.

It is conjectured in [**KK2**], that one can always find some φ and ψ as in Lemma 6.1.6 so that the entire path $\rho_{t\varphi+(1-t)\psi}$ is noncentral, when restricted to ∂M_γ. If this had been true, then their work would have covered all cases of paths between irreducible flat connections. The following lemma shows that this is false and it is indeed necessary to consider paths that go through representations which are central on ∂M_γ.

Lemma 6.2.1. *Let* $B = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $\varphi = (\frac{3}{4}, \frac{1}{4})$ *and* $\psi = (\frac{1}{4}, \frac{1}{4})$. *Let* $\varphi' = \pm\varphi + \eta$ *and* $\psi' = \pm\psi + \theta$ *for* $\eta, \theta \in \mathbf{Z}^2$ *and* γ *a knot as in Lemma 6.1.7. Then both* $\rho_{\varphi'}$ *and* $\rho_{\psi'}$ *are central when restricted to* ∂M_γ.

Proof. We have $(\varphi-\psi)(B+I) = (3,1)$ and $(\varphi+\psi)(B+I) = (7,3)$. Furthermore for $\theta = (\theta_1, \theta_2) \in \mathbf{Z}^2$ we get $\theta(B+I) = (6\theta_1 + 2\theta_2, 2\theta_1 + 2\theta_2) = 2(3\theta_1 + \theta_2, \theta_1 + \theta_2)$. Similarly $\eta(B+I) = 2(3\eta_1 + \eta_2, \eta_1 + \eta_2)$. It follows that $(\varphi' - \psi')(B+I)$ is a pair of odd integers. Since (p,q) are required to be relatively prime, one of them must be odd. Since $(\varphi' - \psi')(B+I)\binom{p}{q} = 0$, both p and q must be odd. This implies that $\alpha_{\varphi'}$ and $\alpha_{\psi'}$ as in Lemma 6.1.10 are half integers. Since $\beta_{\varphi'}$ and $\beta_{\psi'}$ are also integers, $\rho_{\varphi'}|_{\partial M_\gamma}$ and $\rho_{\psi'}|_{\partial M_\gamma}$ are central. □

A similar example is given by $B = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$, $\varphi = (\frac{1}{4}, 0)$, $\psi = (\frac{1}{4}, \frac{1}{2})$.

In addition to the above examples there are also examples of paths with representations in the interior which are central on ∂M_γ. The examples

$$B = \begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix},\ \varphi = (\tfrac{1}{6}, \tfrac{1}{6}),\ \psi = (\tfrac{2}{3}, \tfrac{1}{6}) \text{ and } B = \begin{pmatrix} 9 & 4 \\ 2 & 1 \end{pmatrix},\ \varphi = (\tfrac{5}{6}, \tfrac{1}{3}),\ \psi = (\tfrac{1}{3}, \tfrac{1}{3})$$

are particularly interesting, because all conjugate choices of ρ_φ and ρ_ψ seem to still have representations in the interior which are central on the boundary, though no proof has been found.

6.3. Some Cohomology Computations

In order to compute spectral flow on the knot complement we will make use of the adiabatic limit theorem (see Theorem 2.8.5). Thus we will need to know more about the cohomology of M_γ and ∂M_γ, particularly about

$$\begin{aligned} &\operatorname{Im}(H^{0+1}(M_\gamma, \partial M_\gamma; su(2)_{\mathrm{hol}(A)}) \to H^{0+1}(M_\gamma; su(2)_{\mathrm{hol}(A)})) \\ = \ &\operatorname{Ker}(H^{0+1}(M_\gamma; su(2)_{\mathrm{hol}(A)}) \to H^{0+1}(\partial M_\gamma; su(2)_{\mathrm{hol}(A)})) \end{aligned}$$

and $\operatorname{Im}(H^{0+1}(M_\gamma; su(2)_{\mathrm{hol}(A)}) \to H^{0+1}(\partial M_\gamma; su(2)_{\mathrm{hol}(A)}))$.

We can compute $H^*(\pi_1 M_\gamma; su(2)_{\rho_\varphi})$ using the bar resolution as described in Section 2.11. By Poincaré duality we only need to find H^0 and H^1.

Let $\rho : \pi_1 M_\gamma \to SU(2)$ be a representation. By Corollary 2.11.4 a derivation on $\pi_1 M$ is determined by its values on its generators x, y and τ. Consider the set of 9 derivations $\{\widehat{\zeta}_{zv}\}$ on π_M determined for $v \in \{i, j, k\}$ and $z, z' \in \{x, y, \tau\}$ via

$$\widehat{\zeta}_{zv}(z') = \widehat{\zeta}^{\rho}_{zv}(z') := \begin{cases} v & \text{if } z = z' \\ 0 & \text{otherwise.} \end{cases}$$

The set of derivations on $\pi_1 M_\gamma$ is identified with the subset of those derivations on the free group generated by $\{x, y, \tau\}$ that take the relators to 0, as shown in Lemma 2.11.3.

Fix $\varphi \in \mathbf{R}^2$, such that $\varphi(B + I) \in \mathbf{Z}^2$. Then ρ_φ as in Lemma 6.1.2 is a representation of $\pi_1 M_\gamma$. We call its restriction to the boundary ρ_φ as well.

As in the proof of Lemma 6.1.9 we will henceforth assume after a basis change via $\widehat{B}^{-1}$ that $x = \gamma$ and $y = x^r y^s$. Then we get $\pi_1(M_\gamma) = \langle x, y, \tau | [x, y], \tau x \tau^{-1} = x^a y^c\rangle$ and $\mu = x^b y^d \tau y^{-1} \tau^{-1}$. We will denote a cohomology class by one of its representatives. Recall that $\rho_\varphi : \pi_1 M_\gamma \to SU(2)$ is reducible if and only if $\varphi \in (\frac{1}{2}\mathbf{Z})^2$.

Proposition 6.3.1. *Let $|\mathrm{tr} B| \neq 2$. Then we have*

$$H^0(M_\gamma; su(2)_{\rho_\varphi}) = \begin{cases} 0 & \text{if } \varphi \notin \left(\frac{1}{2}\mathbf{Z}\right)^2 \\ \mathbf{R}j & \text{if } \varphi \in \left(\frac{1}{2}\mathbf{Z}\right)^2 \end{cases}$$

$$H^1(M_\gamma; su(2)_{\rho_\varphi}) = \begin{cases} \langle\{(\widehat{\zeta}_{xi}, \widehat{\zeta}_{yi})\binom{-c}{a+1}\}\rangle \cong \mathbf{R} & \text{if } \rho_\varphi(\lambda) \neq \pm 1 \\ \langle\{(\widehat{\zeta}_{xi}, \widehat{\zeta}_{yi})\binom{-c}{a+1}, \widehat{\zeta}_{\tau j}, \widehat{\zeta}_{\tau k}\}\rangle \cong \mathbf{R}^3 & \text{if } \varphi \notin \left(\frac{1}{2}\mathbf{Z}\right)^2 \text{ and } \rho_\varphi(\lambda) = \pm 1. \\ \langle\{(\widehat{\zeta}_{xi}, \widehat{\zeta}_{yi})\binom{-c}{a+1}, (\widehat{\zeta}_{xj}, \widehat{\zeta}_{yj})\binom{-c}{a-1}, (\widehat{\zeta}_{xk}, \widehat{\zeta}_{yk})\binom{-c}{a+1}, \widehat{\zeta}_{\tau j}\}\rangle \cong \mathbf{R}^4 & \text{if } \varphi \in \left(\frac{1}{2}\mathbf{Z}\right)^2, \end{cases}$$

where $\widehat{\zeta}_{zv} = \widehat{\zeta}^{\rho_\varphi}_{zv}$.

PROOF. Let $\varphi \in \mathbf{R}^2$. Think of $su(2)$ as all imaginary quaternions $\mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k = \mathbf{R}i + \mathbf{C}j$. $SU(2)$ acts on $su(2)$ by the adjoint action, that is, by conjugation. In the case of ρ_φ we have to only consider conjugation by $e^{2\pi i\alpha}$ and j, where $\alpha \in \mathbf{R}$.

We can write conjugation by $u \in SU(2)$ as multiplication by a 3×3-matrix C_u on the left, where a column vector (v_i, v_j, v_k) corresponds to $v_i i + v_j j + v_k k \in su(2)$:

$$C_u \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} := u(v_i i + v_j j + v_k k)u^{-1}.$$

Then we get

$$C_{e^{2\pi i\alpha}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(4\pi\alpha) & -\sin(4\pi\alpha) \\ 0 & \sin(4\pi\alpha) & \cos(4\pi\alpha) \end{bmatrix}, \text{ and}$$

$$C_j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

By linearity we may denote conjugation followed by addition by one matrix in a similar way. By the definition of ρ_φ and of δ^0 we get

$$\delta^0(\cdot)(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(4\pi\varphi_1) - 1 & -\sin(4\pi\varphi_1) \\ 0 & \sin(4\pi\varphi_1) & \cos(4\pi\varphi_1) - 1 \end{bmatrix},$$

$$\delta^0(\cdot)(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(4\pi\varphi_2) - 1 & -\sin(4\pi\varphi_2) \\ 0 & \sin(4\pi\varphi_2) & \cos(4\pi\varphi_2) - 1 \end{bmatrix}, \text{ and}$$

$$\delta^0(\cdot)(\tau) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Thus $H^0(M_\gamma; su(2)_{\rho_\varphi}) = 0$ if ρ_φ is nonabelian (that is, $\varphi \notin (\frac{1}{2}\mathbf{Z})^2$) and $H^0(M_\gamma; su(2)_{\rho_\varphi}) = \mathbf{R}j \cong \mathbf{R}$ otherwise.

Let $[\zeta] \in H^1(M_\gamma; su(2)_{\rho_\varphi})$ with $\zeta(x) = X$, $\zeta(y) = Y$, and $\zeta(\tau) = T$, where $X, Y, T \in su(2)$ and $\varphi \in \mathbf{R}^2$.

Suppose $\varphi = (\varphi_1, \varphi_2) \notin \left(\frac{1}{2}\mathbf{Z}\right)^2$. Assume without loss of generality that $\varphi_1 \notin \frac{1}{2}\mathbf{Z}$. Then $\delta^0(\cdot)(x)$ maps $\mathbf{C}j$ onto $\mathbf{C}j$. That is, there is a unique $v \in \mathbf{C}j$, so that $\zeta(x) + \delta^0(v)(x) \in \mathbf{R}i$. Furthermore $\delta^0(\cdot)(\tau)$ maps $\mathbf{R}i$ onto $\mathbf{R}i$. That is, there is a unique $v' \in \mathbf{R}i$, so that $\zeta(\tau) + \delta(v')(\tau) \in \mathbf{C}j$. Note that $\delta^0(v')(x) = 0$ and $\delta^0(v)(\tau) \in \mathbf{R}k$. Thus we can change ζ by a unique coboundary, namely $\delta^0(v + v')$, so that $X \in \mathbf{R}i$ and $T \in \mathbf{C}j$. So let us assume $X \in \mathbf{R}i$ and $T \in \mathbf{C}j$.

A derivation ζ of $\pi_1(M_\gamma)$ is 0 on the relators of the group. Since $1 = [x, y] = xyx^{-1}y^{-1}$ in $\pi_1(M_\gamma)$ we get

$$0 = \zeta([x, y]) = X + x \cdot Y - xyx^{-1} \cdot X - xyx^{-1}y^{-1} \cdot Y = (1 - y) \cdot X + (x - 1) \cdot Y.$$

Then $X \in \mathbf{R}i$ and $\rho_\varphi(y) \in \mathbf{C}$ imply $(1 - y) \cdot X = 0$, which gives $(1 - x) \cdot Y = 0$. Thus we also get $Y \in \mathbf{R}i$ because $\rho_\varphi(x)$ is not a real number.

Since $X \in \mathbf{R}i$ and $Y \in \mathbf{R}i$ we have $y \cdot X = x \cdot X = X$, $\tau \cdot X = -X$ and $y \cdot Y = x \cdot Y = Y$. Then

$$\begin{aligned} 0 &= \zeta(x^a y^c \tau x^{-1} \tau^{-1}) \\ &= \sum_{k=0}^{a-1} x^k \cdot X + x^a \sum_{k=0}^{c-1} y^k \cdot Y + x^a y^c \cdot T - x^a y^c \tau x^{-1} \cdot X - x^a y^c \tau x^{-1} \tau^{-1} \cdot T \\ &= (a+1)X + cY + (x^a y^c - 1) \cdot T \end{aligned}$$

Comparing the $\mathbf{R}i$ coefficients gives $(X,Y)\binom{a+1}{c} = 0$. This equation defines a 1 real dimensional subspace of $(i\mathbf{R})^2$, namely $\langle\{\binom{-c}{a+1}\}\rangle$, since $c \neq 0$ by Lemma 6.1.1.

Also observe that the term $x^a y^c \cdot T$ can be written as

$$\begin{aligned} x^a y^c \cdot T &= \tau x \tau^{-1} \cdot T = j e^{2\pi i \varphi_1} j^{-1} T j e^{-2\pi i \varphi_1} j^{-1} \\ &= e^{-2\pi i \varphi_1} T e^{2\pi i \varphi_1} = x^{-1} \cdot T = \lambda^{-1} \cdot T \end{aligned}$$

Comparing $\mathbf{R}j$ and $\mathbf{R}k$ coefficients in

$$0 = (1 - \lambda^{-1}) \cdot T = \operatorname{ad}(1)(T) - \operatorname{ad}\left(e^{-2\pi i \alpha_\varphi}\right) \cdot (T) = T - e^{-4\pi i \alpha_\varphi} T$$

implies $T = 0$ if $\alpha_\varphi \notin \frac{1}{2}\mathbf{Z}$, that is, if $\rho_\varphi(\lambda) \neq \pm 1$, which yields $H^1(M_\gamma; su(2)_{\rho_\varphi}) \cong \mathbf{R}$. On the other hand if $\rho_\varphi(\lambda) = \pm 1$, then T can be arbitrary in $\mathbf{C}j$, which implies $H^1(M_\gamma; su(2)_{\rho_\varphi}) \cong \mathbf{R}^3$.

Consider the case $\varphi \in \left(\frac{1}{2}\mathbf{Z}\right)^2$. Note that in this case $\rho_\varphi(\lambda) = \pm 1$. Then $\zeta([x,y]) = 0$ is automatically true. We can change ζ by a coboundary, so that $T \in \mathbf{R}j$. Since $y^k \cdot X = X$ and $x^k \cdot Y = Y$ we get

$$0 = \zeta(x^a y^c \tau x^{-1} \tau^{-1}) = aX + cY + x^a y^c \tau x^{-1} \cdot X + (x^a y^c - 1) \cdot T = aX + cY - \tau \cdot X.$$

Note that $\tau \cdot X = jX_i i j^{-1} + jX_j j j^{-1} + jX_k k j^{-1} = -X_i i + X_j j - X_k k$. Comparing coefficients gives $(a+1)X_i + cY_i$, $(a-1)X_j + cY_j$ and $(a+1)X_k + cY_k$. Since $c \neq 0$, each of these three equations determines a 1-dimensional subspace of $\mathbf{R}^2$. Thus $H^1(M_\gamma; su(2)_{\rho_\varphi}) \cong \mathbf{R}^4$. □

Before we compute what $\operatorname{Im}(H^*(M_\gamma; su(2)_{\rho_\varphi}) \to H^*(\partial M_\gamma; su(2)_{\rho_\varphi}))$ looks like in detail, let us make the following general observation proved by a standard argument:

LEMMA 6.3.2. *Let X be a compact k-manifold with boundary. Then for the twisted cohomology with V-coefficients and a non-degenerate invariant inner product $V \times V \to$* $\mathbf{R}$

(1) $\dim(\mathrm{Im}(H^n(M_\gamma;V_\rho)\to H^n(\partial M_\gamma;V_\rho)) = \frac{1}{2}\dim H^n(\partial M_\gamma;V_\rho)$

(2) $\dim(\mathrm{Im}(H^*(M_\gamma;V_\rho)\to H^*(\partial M_\gamma;V_\rho)) = \frac{1}{2}\dim H^*(\partial M_\gamma;V_\rho)$

where $H^* = \bigoplus_{i=0}^k H^i$ *and* $k = 2n+1$.

For the proof we need to use Poincaré duality for cohomology with twisted coefficients

$$H^i(M;V_\rho)\cong H_{n-i}(M;V_\rho)$$

and the universal coefficient theorem

$$H^i(M;V_\rho)\cong \mathrm{Hom}(H_i(M;V_\rho),\mathbf{R}),$$

where M is a closed n-manifold and V is equipped with a non-degenerate inner product $<\cdot,\cdot>$, as well as the relative versions for manifolds with boundary. In order to prove Poincaré duality, we will use a triangulation of M and the dual regular cell complex written M^τ and M' respectively. The chains and cochains of interest are $C_i(N,V_\rho) := C_i(\tilde{N})\otimes_{\mathbf{Z}\pi} V$ and $C^i(N,V_\rho) := \mathrm{Hom}_{\mathbf{Z}\pi}(C_i(\tilde{N}),V)$, where N can be M^τ or M'. Let

$$\Psi : C_{n-i}(\tilde{M^\tau})\times C_i(\tilde{M}')\to \mathbf{R}$$

be the (non-degenerate) intersection pairing. Then Poincaré duality can be checked by directly showing that the map

$$\begin{aligned} C_{n-i}(M^\tau,V_\rho) &\to C^i(M',V_\rho)\\ \sigma\otimes v &\mapsto (\tau\mapsto \Psi(\sigma,\tau)v)\end{aligned}$$

is a chain isomorphism. The universal coefficient theorem can be shown by proving that the pairing

$$\begin{aligned} C^i(M^\tau,V_\rho)\times C_i(M^\tau,V_\rho) &\to \mathbf{R}\\ (f,\sigma\otimes v) &\mapsto \langle f(\sigma),v\rangle\end{aligned}$$

is non-degenerate. (For well-definedness note that $f(r\sigma) = rf(\sigma)$ for $r \in \pi$, $f \in C^i(M^\tau, V_\rho)$ and that $\Psi(r\sigma, \tau) = \Psi(\sigma, r^{-1}\tau)$.)

PROOF OF LEMMA 6.3.2. We will omit the coefficients in the notation. Consider the following diagram made up out of the middle part of the long exact sequences in homology and cohomology of the pair $(X, \delta X)$ and the Poincaré-duality isomorphisms:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^n(X) & \xrightarrow{j^*} & H^n(\partial X) & \longrightarrow & H^{n+1}(X,\partial X) & \longrightarrow & \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ \cdots & \longrightarrow & H_{n+1}(X,\partial X) & \xrightarrow{i_*} & H_n(\partial X) & \xrightarrow{j_*} & H_n(X) & \longrightarrow & \cdots \end{array}$$

Then $\dim(\mathrm{Im}j^*) = \dim(\mathrm{Im}i_*) = \dim(\mathrm{Ker}j_*)$.

Now recall the fact, that if $L : V \to W$ is a homomorphism between finite-dimensional $\mathbf{R}$-vector spaces and $L^* : W^* \to V^*$ its induced dual homomorphism, then $\mathrm{Ker}L \cong \mathrm{Coker}L^* = V^*/\mathrm{Im}L^*$. The proof is easy: Extend a basis $\{w_i\}_{i=1,\dots,k}$ of $\mathrm{Im}L$ to a basis $\{w_i\}_{i=1,\dots,m}$ of W and let $v_i \in L^{-1}(\{w_i\})$. One can see and easily check, that the basis of $\mathrm{Ker}L$ will extend $\{v_i\}_{i=1,\dots,k}$ to a basis $\{v_i\}_{i=1,\dots,n}$ of V. As usual an isomorphism $i : V \to V^*$ can be defined by $i(v_i) = v_i^*$ where $v_i^*(v_j) = \delta_{i,j}$. Then $i_* : \mathrm{Ker}L \to \mathrm{Coker}L^*$ defined by $i_*(v) := i(v) + \mathrm{Im}L^*$ is an isomorphism. On the one hand i_* is injective, since given $v = \sum_{i=k+1}^n a_i v_i \in \mathrm{Ker}L$ such that $i(v) \in \mathrm{Im}L^*$, that is, $L^*(w^*) = i(v)$ for some w^* we have:

$$i(v)(v_j) = \begin{cases} i(v)(v_j) = \sum_{i=k+1}^n a_i v_i(v_j) = \sum_{i=k+1}^n a_i \delta_{i,j} = 0 & \text{if } 1 \leqslant j \leqslant k \\ L^*(w^*)(v_j) = w^*(L(v_j)) = w^*(0) = 0 & \text{if } k < j \leqslant n \end{cases}$$

On the other hand i_* is surjective, since given $v^* = \sum_{i=1}^n a_i v_i =\in V^*$ we get $v^* + \mathrm{Im}L^* = \sum_{i=k+1}^n a_i v_i + \mathrm{Im}L^*$ since $L^*(w_j^*) = v_j^*$ by

$$L^*(w_j^*)(v_i) = w_j^*(L(v_i)) = w_j^*(w_i) = \delta_{i,j} = v_j^*(v^i)$$

for $j = 1, \ldots, k$ and for $i = 1, \ldots, n$ we have.

With this fact, $H^i(X) = \mathrm{Hom}(H_i(X), \mathbf{R}) = (H_i(X))^*$ and $H^i(\partial X) = (H_i(\partial X))^*$ we get

$$\dim(\mathrm{Im} j^*) = \dim(\mathrm{Im} i_*) = \dim(\mathrm{Ker} j_*) = \dim(\mathrm{Coker} j^*) = \dim(H^n(\partial X)) - \dim(\mathrm{Im} j^*)$$

which proves $\dim(\mathrm{Im} j^*) = \frac{1}{2}\dim(H^n(\partial X)$.

For the second part of the lemma consider the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^i(X) & \xrightarrow{j_i^*} & H^i(\partial X) & \longrightarrow & H^{i+1}(X, \partial X) & \longrightarrow & \cdots \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\cdots & \longrightarrow & H_{k-i}(X, \partial X) & \xrightarrow{i_*^{k-i}} & H_{k-i}(\partial X) & \xrightarrow{j_*^{k-i}} & H_{k-i-1}(X) & \longrightarrow & \cdots
\end{array}$$

By a similar argument – we only have to be careful with indices – we get

$$\dim(\mathrm{Im} j_i^*) = \dim(H^i(\partial X)) - \dim(\mathrm{Im} j_{2n+1-i}^*)$$

But then

$$\begin{aligned}
\dim(H^*(\partial X)) &= \sum_{i=0}^{2n+1} \dim(H^i(\partial X)) = \sum_{i=0}^{2n+1} (\dim(\mathrm{Im} j_i^*) + \dim(\mathrm{Im} j_{2n+1-i}^*) \\
&= 2 \sum_{i=0}^{2n+1} \dim(\mathrm{Im} j_i^*)
\end{aligned}$$

which proves the second part. □

Given a representation $\rho : \pi_1 T = \langle \mu, \lambda \mid [\mu, \lambda] \rangle \to SU(2)$. Consider the set of 6 derivations of $\pi_1(T)$ determined for $v \in \{i, j, k\}$ and $z, z' \in \{\mu, \lambda\}$ via

$$\zeta_{zv}(z') = \zeta_{zv}^{\rho}(z') = \begin{cases} v & \text{if } z = z' \\ 0 & \text{otherwise.} \end{cases}$$

One can prove the following lemma by applying the De Rham map to the De Rham cohomology on the torus as in Proposition 3.2.1. However, we provide a proof using group cohomology as well, because it is very short.

LEMMA 6.3.3. *We have*

$$H^0(\partial M_\gamma; su(2)_{\rho_\varphi}) = \begin{cases} \mathbf{R}i & \text{if} \quad \rho_\varphi(\lambda) \neq \pm 1 \text{ or } \rho_\varphi(\mu) \neq \pm 1 \\ su(2) & \text{if} \quad \rho_\varphi(\lambda) = \pm 1 \text{ and } \rho_\varphi(\mu) = \pm 1 \end{cases}$$

$$H^1(\partial M_\gamma; su(2)_{\rho_\varphi}) = \begin{cases} \langle \{\zeta_{\mu i}, \zeta_{\lambda i}\} \rangle \cong \mathbf{R}^2 & \text{if} \quad \rho_\varphi(\lambda) \neq \pm 1 \text{ or } \rho_\varphi(\mu) \neq \pm 1 \\ \langle \{\zeta_{zv}\}_{z,v} \rangle \cong \mathbf{R}^6 & \text{if} \quad \rho_\varphi(\lambda) = \pm 1 \text{ and } \rho_\varphi(\mu) = \pm 1 \end{cases}$$

PROOF. The computation of $H^0(\partial M_\gamma; su(2)_{\rho_\varphi})$ works just like Lemma 6.3.1. The matrices corresponding to $\delta(\cdot)(\lambda)$ and $\delta(\cdot)(\mu)$ look very similar to $\delta(\cdot)(x)$ and $\delta(\cdot)(y)$.

Now let ζ be a 1-cocycle and $\zeta(\lambda) = X$ and $\zeta(\mu) = Y$. Then just like in Lemma 6.3.1

$$0 = \zeta([\lambda, \mu]) = (1 - \mu)X + (\lambda - 1)Y$$

and we conclude that we can change ζ by a unique coboundary in the case $\rho_\varphi(\lambda) \neq \pm 1$ or $\rho_\varphi(\mu) \neq \pm 1$ so that $X \in \mathbf{R}i$ and $Y \in \mathbf{R}i$, that is, $H^1(\partial M_\gamma; su(2)_{\rho_\varphi}) \cong \mathbf{R}^2$. If $\rho_\varphi(\lambda) = \pm 1$ and $\rho_\varphi(\mu) = \pm 1$, then automatically $0 = \zeta([\lambda, \mu])$, thus $H^1(\partial M_\gamma; su(2)_{\rho_\varphi}) \cong \mathbf{R}^6$. □

It will be good to know, how these group cohomology computations give you an explicit description of the scattering Lagrangian in $\mathcal{H}^{0+1+2}_{a_{\alpha,\beta}}(T; su(2))$. In particular we need to compute the De Rham map from Definition 2.11.8 for the torus. We already work with a trivialization of P which yields a trivialization of $\pi^*(P)$, where $\pi : \tilde{T} = \mathbf{R}^2 \to T$ is the universal cover. However, the De Rham map does not depend on a particular trivialization of P. We need to trivialize $\pi^*(P)$ using the horizontal lift determined by $a_{\alpha,\beta}$. This induces the correct trivialization of $\Omega^i(\pi^* \mathrm{ad} P_{\mathrm{hol}(a_{\alpha,\beta})})$ and lets us compute the De Rham map.

Fix a base point $m \in T$, say $(1,1) \in S^1 \times S^1 = T$ and a lift $\tilde{m} \in \mathbf{R}^2$, say $\tilde{m} = 0$, to the universal cover of T. Trivialize the fiber of $\tilde{m}$ in $\mathrm{ad}P_{\mathrm{hol}(a_{\alpha,\beta})}$, that is $\tilde{m} \times su(2)$. Let $\gamma(t) = (e^{it}, 1)$ be a curve in the torus $S^1 \times S^1 = T = T \times 0 \subset T \times G = P$ starting at m and consider its lift $\tilde{\gamma} = (t, 0)$ of γ to the universal cover $\mathbf{R}^2 = \mathbf{R}^2 \times 0 \subset \mathbf{R}^2 \times G = \pi^* P$. Using $a_{\alpha,\beta}$ we can trivialize $\Omega^i(\pi^* \mathrm{ad}P_{\mathrm{hol}(a_{\alpha,\beta})})$ as follows. We compute the horizontal lift $(\tilde{\gamma}_t, g_t) \subset \pi^* P$ of $\tilde{\gamma}_t$, then the correct trivialization of $\pi^* \mathrm{ad}P_{\mathrm{hol}(a_{\alpha,\beta})}$ is given by $[((\tilde{\gamma}_t, g_t), v)] \to (\tilde{\gamma}, v)$.

We have $\dot{\gamma}(t) = (ie^{it}, 1)$. It is not hard to find g satisfying $g(0) = 1$ and

$$-\dot{g}(t) g^{-1}(t) = A(\dot{\gamma}(t)) = -i\alpha.$$

We have $\dot{g}_t = i\alpha g_t$ and get $g_t = e^{i\alpha t}$. If $\gamma(t) = (1, e^{it})$, then $g_t = e^{i\beta t}$.

Then the restriction of the map $\Omega^i_{\alpha,\beta}(T; su(2)) \to \Omega^i(\tilde{T}) \otimes su(2)$ to the harmonic forms is given (on the standard basis) by

$$i \mapsto i, \ i\, dm \mapsto i\, dm, \ i\, dl \mapsto i\, dl, \ i\, dm \wedge dl \mapsto i\, dm \wedge dl,$$

when $(\alpha, \beta) \in \mathbf{R}^2 - (\frac{1}{2}\mathbf{Z})^2$. If $(\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2$ we also have

$$e^{i(2\alpha m + 2\beta l)} j \mapsto j, \ e^{i(2\alpha m + 2\beta l)} j\, dm \mapsto j\, dm,$$

$$e^{i(2\alpha m + 2\beta l)} j\, dl \mapsto j\, dl, \ e^{i(2\alpha m + 2\beta l)} j\, dm \wedge dl \mapsto j\, dm \wedge dl,$$

$$e^{i(2\alpha m + 2\beta l)} k \mapsto k, \ e^{i(2\alpha m + 2\beta l)} k\, dm \mapsto k\, dm,$$

$$e^{i(2\alpha m + 2\beta l)} k\, dl \mapsto k\, dl, \ e^{i(2\alpha m + 2\beta l)} k\, dm \wedge dl \mapsto k\, dm \wedge dl.$$

Furthermore integrating these forms over cycles yields cocycles. Taking this a step further, namely to group cohomology, we get the following.

LEMMA 6.3.4. *The isomorphism from the harmonic forms* $\mathcal{H}^{0+1}_{\alpha,\beta}(T; su(2))$ *to group cohomology* $H^{0+1}(\pi_1(T); su(2)_{\rho_{\alpha,\beta}})$ *is given by*

$$i \mapsto i, \; i\,dm \mapsto \zeta_{\mu i}, \; i\,dl \mapsto \zeta_{\lambda i}$$

when $(\alpha, \beta) \in \mathbf{R}^2 - (\frac{1}{2}\mathbf{Z})^2$. *If* $(\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2$ *we also have*

$$e^{i(2\alpha m+2\beta l)} j \mapsto j, \; e^{i(2\alpha m+2\beta l)} j\,dm \mapsto \zeta_{\mu j}, \; e^{i(2\alpha m+2\beta l)} j\,dl \mapsto \zeta_{\lambda j},$$

$$e^{i(2\alpha m+2\beta l)} k \mapsto k, \; e^{i(2\alpha m+2\beta l)} k\,dm \mapsto \zeta_{\mu k}, \; e^{i(2\alpha m+2\beta l)} k\,dl \mapsto \zeta_{\lambda k}.$$

This allows us to go back and forth between harmonic forms and group cohomology. In particular we can do Maslov index computations for paths of pairs of Lagrangians in the harmonic forms on T, while it is a priori not clear how to do this in group cohomology.

LEMMA 6.3.5. *Let* $|\mathrm{tr}B| \neq 2$. *Then we get*

$$\mathrm{Im}(i_0^* : H^0(M_\gamma; su(2)_{\rho_\varphi}) \to H^0(\partial M_\gamma; su(2)_{\rho_\varphi}))$$
$$= \begin{cases} \mathbf{R}j & \text{if } \varphi \in \left(\frac{1}{2}\mathbf{Z}\right)^2 \\ 0 & \text{if } \varphi \notin \left(\frac{1}{2}\mathbf{Z}\right)^2 \end{cases}$$

$$\mathrm{Im}(i_1^* : H^1(M_\gamma; su(2)_{\rho_\varphi}) \to H^1(\partial M_\gamma; su(2)_{\rho_\varphi}))$$
$$= \begin{cases} \langle (\zeta_{\lambda i} + b\zeta_{\mu i}, (d+1)\zeta_{\mu i}) \binom{-c}{a+1} \rangle \cong \mathbf{R} & \text{if } \rho(\lambda) \neq \pm 1 \text{ or } \rho(\mu) \neq \pm 1 \\ \langle \{ (\zeta_{\lambda i} + b\zeta_{\mu i}, (d+1)\zeta_{\mu i}) \binom{-c}{a+1}, \zeta_{\mu j}, \zeta_{\mu k} \} \rangle \cong \mathbf{R}^3 & \text{if } \left\{ \begin{array}{l} \varphi \notin (\frac{1}{2}\mathbf{Z})^2, \\ \rho_\varphi(\lambda) = \pm 1 \\ \text{and } \rho_\varphi(\mu) = \pm 1 \end{array} \right\} \\ \langle \{ (\zeta_{\lambda i} + b\zeta_{\mu i}, (d+1)\zeta_{\mu i}) \binom{-c}{a+1}, \\ (\zeta_{\lambda j} + b\zeta_{\mu j}, (d-1)\zeta_{\mu j}) \binom{-c}{a-1}, \\ (\zeta_{\lambda k} + b\zeta_{\mu k}, (d+1)\zeta_{\mu k}) \binom{-c}{a+1} \} \rangle \cong \mathbf{R}^3 & \text{if } \varphi \in (\frac{1}{2}\mathbf{Z})^2 \end{cases}$$

Proof. By Lemma 6.3.1 we have $H^0(M_\gamma; su(2)_{\rho_\varphi}) = 0$ and thus $\mathrm{Im}(i_0^*) = 0$ if ρ_φ is nonabelian. In the other case i_0^* is injective (e.g. since $H^0(X, A; su(2)) = 0$ for X path-connected and $A \subset X$ nonempty), that is, $\mathrm{Im}(i_0^*) = \mathbf{R}j$.

For the first cohomology note, that $i_1^*(\widehat{\zeta}_{zv})(z') = \widehat{\zeta}(i_*(z')) = \widehat{\zeta}(z')$. First consider the case, when the restriction of ρ_φ to the boundary is noncentral. Using the Fox calculus we get $\widehat{\zeta}_{\tau j}(\lambda) = \widehat{\zeta}_{\tau k}(\lambda) = 0$, $\widehat{\zeta}_{\tau j}(\mu) \in \mathbf{C}j$ and $\widehat{\zeta}_{\tau k}(\mu) \in \mathbf{C}j$. Thus $i^*(\widehat{\zeta})$ is cohomologous to 0 if $\rho_\varphi(\mu) \neq \pm 1$. We also have $\widehat{\zeta}_{xi}(\lambda) = i$, $\widehat{\zeta}_{xi}(\mu) = bi$, $\widehat{\zeta}_{yi}(\lambda) = 0$ and $\widehat{\zeta}_{yi}(\mu) = (d+1)i$. Then we get

$$\mathrm{Im}(i_1^*) = \langle (\zeta_{\lambda i} + b\zeta_{\mu i}, (d+1)\zeta_{\mu i}) \tbinom{-c}{a+1} \rangle \cong \mathbf{R}. \tag{6.3.12}$$

This space is one-dimensional, because $b \neq 0$ and $c \neq 0$. In other words if we identify $H^1(\partial M_\gamma; su(2)_{\rho_\varphi}) = \mathbf{R}^2$ by letting $\zeta_{\lambda i} = (1,0)$ and $\zeta_{\mu i} = (0,1)$, then $\mathrm{Im}(i_1^*) = \{(ub + v(d+1), u) \mid u(a+1) + vc = 0\}$.

If the restriction to the boundary is central, notice that

$$\pm 1 = \rho_\varphi(\mu) = \rho_\varphi(x^b y^d)\rho_\varphi(\tau y^{-1}\tau^{-1}) = \rho_\varphi(x^b y^d)\rho_\varphi(y).$$

In particular $\rho_\varphi(x^b y^d)v = \rho_\varphi(y^{-1})v$ for $v \in su(2)$. If ρ_φ is nonabelian (that is, $\varphi \notin (\frac{1}{2}\mathbf{Z})^2$), then $\rho_\varphi(x) = \rho_\varphi(\lambda) = \pm 1$ implies $\rho_\varphi(y) \neq \pm 1$. Thus $\widehat{\zeta}_{\tau j}(\mu) = (x^b y^d - 1) \cdot j = (y^{-1} - 1) \cdot j \neq 0$. Similarly $\widehat{\zeta}_{\tau k}(\mu) = (e^{-4\pi i \varphi_2} - 1)k \neq 0$. So we get

$$\mathrm{Im}(i_1^*) = \langle \{ (\zeta_{\lambda i} + b\zeta_{\mu i}, (d+1)\zeta_{\mu i}) \tbinom{-c}{a+1}, \zeta_{\mu j}, \zeta_{\mu k} \} \rangle.$$

If ρ_φ is abelian, then analogous to equation (6.3.12) we get

$$\begin{aligned} \mathrm{Im}(i_1^*) \quad = \quad & \langle \{ (\zeta_{\lambda i} + b\zeta_{\mu i}, (d+1)\zeta_{\mu i}) \tbinom{-c}{a+1}, \\ & (\zeta_{\lambda i} + b\zeta_{\mu i}, (d-1)\zeta_{\mu i}) \tbinom{-c}{a-1}, \\ & (\zeta_{\lambda i} + b\zeta_{\mu i}, (d+1)\zeta_{\mu i}) \tbinom{-c}{a+1} \} \rangle \cong \mathbf{R}^3. \end{aligned}$$

□

By Lemmas 6.3.1 and 6.3.5 we get the following useful observation.

Corollary 6.3.6. *If* $\varphi \notin (\frac{1}{2}\mathbf{Z})^2$, *then*

$$\dim(\operatorname{Ker}(i_0^* \oplus i_1^*) = \begin{cases} 2 & \text{if } \rho_\varphi(\lambda) = \pm 1 \text{ and } \rho_\varphi(\mu) \neq \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\varphi \in (\frac{1}{2}\mathbf{Z})^2$, *then* $\dim(\operatorname{Ker}(i_0^* \oplus i_1^*) = 1$.

In particular the non-resonance level is 0, when ρ_φ central when restricted to T, but irreducible.

6.4. Computation of Spectral Flow on M

We want to compute spectral flow on M between irreducible flat connections A_0 and A_1. The idea is to split M into a solid torus $S = \text{nbhd}(\gamma)$ and its complement $X = M_\gamma$, such that there is a path of connections A_t connecting A_0 and A_1 which is flat on X. Kirk and Klassen have computed this spectral flow, whenever the representations $\rho_t = \text{hol}(A_t)|_T$ are noncentral on $T = \partial S$ for all t. Thus we are left with dealing with the situation when the associated path of representations ρ_t goes through a representation which is central when restricted to T, but irreducible on X.

The spectral flow on the solid torus has been completely computed. All that is left is the spectral flow on X. We will also have to deal with the Maslov triple indices in the case that we start or end at representations which are central on the boundary.

Observe that then the scattering Lagrangian on X splits into $\mathbf{R}i$ and $\mathbf{C}j$ part: $\mathscr{L}_X = \widehat{\mathscr{L}_X} \oplus \widecheck{\mathscr{L}_X}$, just like the scattering Lagrangian on S. Thus the Maslov triple indices in Theorem 4.5.5 reduce to $\tau(K^-_{\rho_0}, \widecheck{\mathscr{L}}_{S,0}, \widecheck{\mathscr{L}}_{X,0})$ and $\tau(K^-_{\rho_1}, \widecheck{\mathscr{L}}_{S,1}, \widecheck{\mathscr{L}}_{X,1})$.

LEMMA 6.4.1. *Let A be flat on X with $A|_T = a_{\alpha,\beta}$ with $(\alpha,\beta) \in (\frac{1}{2}\mathbf{Z})^2$. Then*

$$\dim(K^-_{(\alpha,\beta,\theta)} \cap \breve{\mathscr{L}}_{X,A}) = \begin{cases} 2 & \textit{if } \theta = \pm 1, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Let $(\alpha,\beta) \in (\frac{1}{2}\mathbf{Z})^2$ and consider $\varphi \in K^-_{(\alpha,\beta,\theta)} \cap \breve{\mathscr{L}}_{X,A}$. Write

$$\varphi = a_1\psi_1^- j + a_2\psi_2^- j + a_3\psi_1^- k + a_4\psi_2^- k$$

in terms of basis vectors of $K^-_{(\alpha,\beta,\theta)}$ as in Lemma 3.3.2. By Lemma 6.3.5 and the fact that $J\mathscr{L}_{X,A} = \mathscr{L}^{\perp}_{X,A}$ we get

$$\breve{\mathscr{L}}_{X,A} = e^{i(2\alpha m + 2\beta l)}\operatorname{span}\{j\,dm, k\,dm, j\,dm \wedge dl, k\,dm \wedge dl\}.$$

By comparing this with Lemma 3.3.2, we see that we must have $a_1 = a_3 = 0$, because $\varphi \in \breve{\mathscr{L}}_{X,A}$, and thus is orthogonal to j and k. Furthermore φ must be orthogonal to $j\,dl$ and $k\,dl$, thus $a_2 = a_4 = 0$ if $\theta \neq \pm 1$. In particular

$$\dim(K^-_{(\alpha,\beta,\theta)} \cap \breve{\mathscr{L}}_{X,A}) = \begin{cases} 2 & \text{if } \theta = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

LEMMA 6.4.2. *Let A be flat on X and $(\alpha,\beta) \in (\frac{1}{2}\mathbf{Z})^2$. Then for small enough $\varepsilon > 0$ and varying $t \in (-\varepsilon,\varepsilon)$ we have*

$$\operatorname{Mas}(K^-_{(\alpha,\beta,\theta e^{ti})}, \breve{\mathscr{L}}_{X,A}) = \begin{cases} 2 & \textit{if } \theta = \pm 1 \\ 0 & \textit{otherwise} \end{cases}$$

PROOF. First consider the case $\theta = 1$. To make notation simpler assume $(\alpha,\beta) = (0,0)$. Then $e^{-i(2\alpha m + 2\beta l)} = 1$. By Lemmas 6.4.1 and 3.3.2 we know, that $K^-_{(\alpha,\beta,e^{ti}\theta)}$

and $\breve{\mathscr{L}}_{X,A}$ intersect in

$$\text{span}\{j\,dm \wedge dl - k\,dm, k\,dm \wedge dl + j\,dm\}.$$

Consider the constant path $\tilde{L} = \text{span}\{k\,dm, j\,dm \wedge dl\}$ and the path

$$L_t = \text{span}\{j\,dm \wedge dl - k(\cos t\,dm + \sin t\,dl), j - k(\sin t\,dm - \cos t\,dl)\} \subset K^-_{(\alpha,\beta,e^{ti})}$$

of 2-dimensional Lagrangians in the symplectic subspace

$$\text{span}\{j, k\,dm, k\,dl, j\,dm \wedge dl\} \subset \mathcal{H}^{0+1+2}(M; \mathbf{C}j),$$

parametrized by $t \in (-\varepsilon, \varepsilon)$, ε small. These intersect at $t = 0$ in $\text{span}\{j\,dm \wedge dl - k\,dm\}$. We compute

$$\begin{aligned}
&\text{Mas}(L_t, \tilde{L})\\
&= \text{Mas}(\text{span}\{j\,dm \wedge dl - (e^{Jt}k\,dm), j - k(\sin t\,dm - \cos t\,dl))\}, \tilde{L})\\
&= \text{Mas}(\mathscr{L}_1 * e^{Jt}\text{span}\{j\,dm \wedge dl - k\,dm, j - k(\sin t\,dm - \cos t\,dl)\} * \mathscr{L}_2, \tilde{L})\\
&= \text{Mas}(e^{Jt}\text{span}\{j\,dm \wedge dl - k\,dm, j - k(\sin t\,dm - \cos t\,dl)\}, \tilde{L})\\
&= 1,
\end{aligned}$$

where

$$\mathscr{L}_1(t) = \text{span}\{e^{-Jt}j\,dm \wedge dl - e^{-J\varepsilon}k\,dm, e^{-Jt}(j - k(\sin t\,dm - \cos t\,dl)\}$$

and

$$\mathscr{L}_2(t) = \text{span}\{e^{J(\varepsilon-t)}j\,dm \wedge dl - e^{-J\varepsilon}k\,dm, e^{J(\varepsilon-t)}(j - k(\sin t\,dm - \cos t\,dl)\}$$

are parametrized by $t \in (0, \varepsilon)$. Notice that $\text{Mas}(\mathscr{L}_i, \tilde{L}) = 0$ because $\mathscr{L}_i \cap \langle j - k\,dm\rangle = 0$.

We can can do a similar computation for the orthogonal complement of L_t and $\tilde{L}$ in $\mathcal{H}^{0+1+2}(M; \mathbf{C}j)$, and we will get the same Maslov index.

A similar computation proves the case $\theta = -1$. $\square$

Notice that this implies together with a similar result for the solid torus, Lemma 5.3.3, that the triple Maslov indices are either 0 or 4 each.

Now we are ready to analyze the spectral flow on X, when $\tilde{\rho}$ passes through the half integer lattice, but does not cross any horizontal half integer line otherwise. In the following proposition we are investigating a path $\tilde{\rho}$ in $\tilde{\mathbf{R}}^2$, that crosses $(\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2$ exactly once and makes n and a half counterclockwise twists at (α, β).

Proposition 6.4.3. *Suppose $\tilde{\rho}$ is a path in $\tilde{\mathbf{R}}^2$ with*

(1) $\tilde{\rho}(0), \tilde{\rho}(1) \in \mathbf{Z} \times \mathbf{R} - (\frac{1}{2}\mathbf{Z})^2$
(2) $\tilde{\rho}(t) = (\alpha, \beta, \theta e^{4(n+\frac{1}{2})(t-\frac{1}{4})\pi i})$, $(\alpha, \beta) \in (\frac{1}{2}\mathbf{Z})^2$, $\theta \in S^1$, $n \in \mathbf{Z}$ *for* $t \in (\frac{1}{4}, \frac{3}{4})$
(3) $\pi \circ \tilde{\rho}(t) \notin \mathbf{R} \times \frac{1}{2}\mathbf{Z}$ *for* $t \in (0, \frac{1}{4}) \cup (\frac{3}{4}, 1)$.

Then $\mathrm{Mas}(\mathcal{P}^-_{\tilde{\rho}(t)}(\widehat{\mathscr{L}_S}), \Lambda_{X, A_{\tilde{\rho}(t)}}) = 4n + 2$.

Proof. By corollary 6.3.6 and 2.8.7 we have resonance level 0 on X for all t because $\pi \circ \tilde{\rho}(t) \notin \mathbf{R} \times \frac{1}{2}\mathbf{Z}$. Away from $(\frac{1}{2}\mathbf{Z})^2$, $\mathscr{L}_S$ equals $\widehat{\mathscr{L}_S}$ and we get

$$\Lambda_{X,t} \cap (P_t^- \oplus \widehat{\mathscr{L}_{S,t}}) \cong (P_t^+ \oplus \widehat{\mathscr{L}_{X,t}}) \cap (P_t^- \oplus \widehat{\mathscr{L}_{S,t}}) = \mathscr{L}_{X,t} \cap \mathscr{L}_{S,t} = 0.$$

We have $\widehat{\mathscr{L}_{X,\frac{1}{2}}} \cap \widehat{\mathscr{L}_{S,\frac{1}{2}}} = 0$ and thus for $\tilde{\rho}(t) \in (\frac{1}{2}\mathbf{Z})^2 \times S^1$

$$\begin{aligned} \mathcal{P}^-_{\tilde{\rho}(t)}(\widehat{\mathscr{L}_S}) \cap \Lambda_{X, A_{\tilde{\rho}(t)}} &\cong (P^-_{\frac{1}{2}} \oplus K^-_{\tilde{\rho}(t)} \oplus \widehat{\mathscr{L}_{S,\frac{1}{2}}}) \cap (P^+_{\frac{1}{2}} \oplus \mathscr{L}_{X,\frac{1}{2}}) = (K^-_{\tilde{\rho}(t)} \oplus \widehat{\mathscr{L}_{S,\frac{1}{2}}}) \cap \mathscr{L}_{X,\frac{1}{2}} \\ &= K^-_{\tilde{\rho}(t)} \cap (\mathscr{L}_{X,\frac{1}{2}} / \widehat{\mathscr{L}_{X,\frac{1}{2}}}) = K^-_{\tilde{\rho}(t)} \cap \widecheck{\mathscr{L}_{X,\frac{1}{2}}}. \end{aligned}$$

This carries over to the Maslov index as well. Since $\tilde{\rho}(t)$ goes around the circle at $\rho(\frac{1}{2})$ exactly $(n+\frac{1}{2})$ times, Lemma 6.4.2 immediately shows that $\mathrm{Mas}(\mathcal{P}^-_{\tilde{\rho}(t)}(\widehat{\mathscr{L}_S}), \Lambda_{X, A_{\tilde{\rho}(t)}}) = 4n + 2$. $\square$

Together with the results from [**KK2**], where paths were analyzed that do not go through the half integer lattice, this implies the following.

Theorem 6.4.4. *Let* A_t *be a path of* $SU(2)$ *connections on a torus bundle over* S^1*, where* A_0 *and* A_1 *are flat and irreducible. Then* $\mathrm{SF}(D_{A_t}) \equiv 0 \bmod 4$.

Notation

For reference we summarize notational conventions.

$\mathbf{Z}$	The integers
$\mathbf{R}$	The real numbers
$\mathbf{C}$	The complex numbers
G, $\mathfrak{g}$	A Lie group and its Lie algebra
$SU(2)$, $su(2)$	The Lie group of special unitary 2×2-matrices and its Lie algebra $su(2)$ (Section 2.3)
M	A closed 3-manifold (in Chapter 6 M is a torus bundle over S^1)
P	A principal G-bundle over M
X, Y	3-manifolds with boundary
Σ	A closed surface, usually the boundary of X or Y
S	The standard solid torus $D^2 \times S^1$
T	The standard torus $T = \partial S = S^1 \times S^1$
$\mathcal{A}_M$	The space of connections on (a trivialized principal bundle over) M
A, a	Usually connections on 3, 2-manifolds respectively, usually identified with 1-forms. Other differential forms which do not correspond to connections are denoted by α, β and γ.
$A_{(\alpha,\beta)}$, $a_{(\alpha,\beta)}$	A specific family of connections parametrized by $\mathbf{R}^2$ (Definitions 5.2.1 and 3.1.2)

d_A	The exterior derivative twisted by A (Section 2.4)
D_A	The odd-signature operator twisted by a connection A (Definition 2.5.2)
S_a	The tangential operator of the odd-signature operator twisted by a connection a (Definition 2.5.3)
Δ_a	The Laplacian twisted by a connection a (Section 2.6)
$\Omega^{0+1}(M; su(2))$	The direct sum of 0 and 1-forms on M with values in $su(2)$
$L^2(\Omega^{0+1}(M; su(2)))$	Its completion in the L^2 norm
$\mathcal{H}_a^{0+1+2}(\Sigma; su(2))$	The harmonic 0, 1 and 2-forms of Δ_a (Section 2.6)
$E_{a,\mu}$	The μ-eigenspace of S_a
$P^+_{a,\nu}$, $P^-_{a,\nu}$	The eigenspaces of S_a with eigenvalues greater than ν or less than $-\nu$ respectively (Section 2.6)
$E^+_{a,\nu}$, $E^-_{a,\nu}$	The eigenspaces of S_a with eigenvalues $0 < \mu \leqslant \nu$ or $-\nu \leqslant \mu < 0$ respectively (Section 2.6)
J	An almost complex structure on $\Omega^{0+1+2}(\Sigma; su(2))$, where Σ is a surface (Section 2.7)
$\Lambda_{X,A}$	The Cauchy Data space of D_A on X (Section 2.8.1)
$\mathscr{L}_{X,a}$	The scattering Lagrangian or the limiting values of extended L^2 solutions (Section 2.8.1)
$\widehat{\mathscr{L}}_{S,A}$, $\widecheck{\mathscr{L}}_{S,A}$	The $\mathbf{R}i$ and $\mathbf{C}j$ part of the Scattering Lagrangian (Lemma 4.2.1 and Definition 5.3.8)
ρ	Usually a representation
$H^{0+1}(M; su(2)_\rho)$	The direct sum of zeroth and first (singular) cohomology of M with $su(2)$-coefficients twisted by the representation $\rho : \pi_1 M \to SU(2)$ (for example $\rho = \mathrm{hol}(A)$), where $SU(2)$ acts on $su(2)$ by conjugation

$\widetilde{\mathbf{R}}^2$	See Definition 3.3.1 and Figure 3.3.1
h	The homeomorphism between $\widetilde{\mathbf{R}}^2$ and $\dot{\mathbf{R}}^2$ as in Figure 3.3.1 (Definition 3.3.1)
$\mathcal{P}(L)$	A specific family of APS boundary conditions parametrized by $\widetilde{\mathbf{R}}^2$ (Definition 3.3.4)
$K^{\pm}_{\alpha,\beta,\theta}$	See Theorem 3.3.2
SF	The spectral flow (Section 2.2)
Mas	The Maslov index (Section 2.10)
τ_μ	The Maslov triple index (Definition 2.10.4)
ζ	Usually a derivation (Definition 2.11.1, Proposition 6.3.1)
(α,β), (m,l)	Usually coordinates corresponding to the torus boundary of X or S, often the collar of a knot
(φ_1,φ_2), (ψ_1,ψ_2)	Usually coordinates corresponding to the fiber of a torus-bundle over S^1
B	Usually the Monodromy matrix a the torus bundle
γ	Usually a knot in the fiber T of the torus bundle
M_γ	The torus bundle with a collar of a knot γ removed
$\mathfrak{a}$,$\mathfrak{b}$,$\mathfrak{c}$,$\mathfrak{d}$	Specific gauge transformations (Definition 5.3.12)
$\mathcal{G}_{\text{nf}}$	A certain group of gauge transformations (Page 82)

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Employment

2004–2005	**Postdoctoral Research Fellow**, Max Planck Institute for Matheamtics in Bonn, Germany.
2003–2004	**Research Assistant**, for Paul Kirk, Dept. of Mathematics, IUB. *Cut and paste methods for spectral flow of the odd signature operator on closed* 3*-manifolds.*
2002–2003	**Research Assistant**, for Matthias Weber, Dept. of Mathematics, IUB. *Visualization of surfaces, particularly minimal surfaces, using Mathematica and Scalable Vector Graphics (SVG).*
2002	**Associate Instructor**, Dept. of Mathematics, IUB. *Grading topology courses (Topology I, Knot Theory).*
1999–2001	**Associate Instructor**, Dept. of Mathematics, IUB. *Teaching and assisting calculus courses.*
2001	**Research Consultant**, YY Technologies, Mountain View, CA. *Optimization of algorithms for problems in graph theory.*
1997–1998	**Teaching Assistant**, Dept. of Mathematics and Computer Science, FUB. *Assisting undergraduate computer science and mathematics courses (Algorithms and Programming III, Topology I).*

Publications

- *Calderón Projector for the Hessian of the Perturbed Chern-Simons Function on a* 3*-Manifold with Boundary* with Matthias Lesch and Paul Kirk, Proc. London Math. Soc. 89 (2004) 241–272 (arXiv:math.GT/0302234)

Talks

- Invited Talks:
  - *A Splitting Formula for Spectral Flow on Closed* 3*-Manifolds*
    Annual Mini-Conference in Modern Analysis, IUPUI, March 2004
  - *A Splitting Formula for Spectral Flow on Closed* 3*-Manifolds*
    AMS meeting (#993), Phoenix 2004, January 2004
    Special Session on Low-Dimensional Topology
  - *Eine Spaltungsformel für Spektralfluß auf geschlossenen* 3*-Mannigfaltigkeiten*
    Universität Münster, Universität Göttingen and Universität München
    Germany, December 2003
  - *A Splitting Formula for Spectral Flow on Closed* 3*-Manifolds*
    Topology Seminar, International University Bremen, Germany, December 2003
  - *A Gauge Theoretical Discussion of Casson's Invariant for Homology* 3*-Spheres*
    Graduate Student Topology Conference, University of Notre Dame, April 2003
  - *Cassons Invariante für Homologie* 3*-Sphären und Eichtheorie*
    Topology Seminar, Freie Universität Berlin, Germany, December 2002
- Internal Talks:
  - *A Splitting Formula for Spectral Flow on Closed* 3*-Manifolds*
    Topology Seminar, Indiana University, October 2003
  - *A Brief Introduction to Gauge Theory*
    Graduate Student Seminar, Indiana University, November 2002
  - *Morse-theory – Jacobi-fields*
    Topology Seminar, Indiana University, November 2002
  - *Morse-theory – Geodesics, Energy, Variations*
    Topology Seminar, Indiana University, October 2002
  - *Casson's Invariant and Gauge Theory*
    Topology Seminar, Indiana University, April 2002

Conferences and Workshops

- *Clay Mathematics Institute Summer School on Floer Homology, Gauge Theory, and Low Dimensional Topology*
  Alfréd Rényi Institute of Mathematics, Budapest, Hungary, June 2004

- *Conference on Geometry and Topology of Manifolds*
  McMaster University, Hamilton, Ontario, Canada, May 2004
- *Floer Homology for* 3*-manifolds*
  Banff International Research Station, Canada, November 2003
- *Von Neumann Symposium on Complex Geometry, Calibrations, and Special Holonomy*
  Differential Geometry Program, MSRI, CA, August 2003
- *Triangulation of Point Sets*
  Summer Graduate Program in Hyperplane, MSRI, CA, July 2003
- *AMS Meeting (#985)*
  Central Section, Bloomington, IN, April 2003
- *Dirac Operators and Their Neighborhood*
  Indiana University-Purdue University Indianapolis, IN, February, 2003.
- *Geometric Aspects of Spectral Theory*
  Spectral Invariants Program, Mathematical Sciences Research Center (MSRI), CA, March 2001
- *Geometric Scattering Theory and Elliptic Theory on Noncompact and Singular Spaces*
  Spectral Invariants Program, MSRI, CA, May 2001
- *Midwest Opera-Topology Conference*
  Bloomington, IN, October 2000
- *AMS Meeting (#953)*
  Central Section, Notre Dame, IN, April 2000

Honors

2004	**William B. Wilcox Mathematics Award** in recognition of outstanding scholastic achievement in graduate studies and in expectation of continued academic success.
2000–2004	Fellowship from **e-fellows.net**.
1998–2001	Member of the **German National Merit Foundation**.
1999	Third Prize at the **6th International Mathematics Competition for University Students** in Keszthely, Hungary.
1998–1999	**Graduate Student Exchange Fellowship** from IUB and FUB.

Computer Skills

- Familiarity with HTML, Java, C, C++, Lisp, Scheme, Perl and Assembly language.
- Good knowledge of Unix and Windows.
- Good knowledge of Maple and Mathematica.
- Experience in software development.

Professional Society

- American Mathematical Society (AMS).

Additional Skills

- Native language German; fluent in English; certificates of *Latinum* and *Graecum*.
- Jazz saxophone.

References

- Paul A. Kirk, `pkirk@indiana.edu`
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  Department of Mathematics, Indiana University-Purdue University Indianapolis, IN 46202
- Charles Livingston, `livingst@indiana.edu`
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- Alberto Torchinsky, `torchins@indiana.edu` (*Teaching*)
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