RESEARCH STATMENT

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Summary

My research is in low-dimensional topology, Chern-Simons gauge theory and boundary value problems on manifolds, more generally in differential geometry, geometric topology and global analysis.

Motivated by a conjecture of Hans Boden and Chris Herald in [3] regarding the $SU(3)$ Casson invariant for spliced sums of torus knot complements, I extended results from my thesis in order to obtain a splitting formula for the $su(N)$ spectral flow of the odd signature operator coupled to a path of $SU(N)$ connections. Currently, Hans Boden and I are working on applying it in order to settle the conjecture.

The main theorem in [7] gives a splitting formula for the $su(2)$ spectral flow of the odd signature operator coupled to a path of $SU(2)$ connections. As an application of this I proved that the spectral flow between irreducible flat connections on torus bundles over the circle is 0 mod 4, which partially confirms a conjecture of Lisa Jeffrey in [9], the missing piece in her analysis of the asymptotic behavior of Witten's 3-manifold invariants.

In [8] we established the existence and continuity for the Calderón projector of the perturbed odd signature operator on a 3-manifold. As an application we gave a new proof of a result of Taubes relating the mod 2 spectral flow of a family of operators on a homology 3-sphere to the difference in local intersection numbers of the character varieties coming from a Heegard decomposition.

The $SU(3)$ Casson invariant and spliced sums

For the $SU(3)$ Casson invariant one needs to compute the $su(3)$ spectral flow of the odd signature operator D_{A_t} coupled to a path A_t of $SU(3)$ connections from the trivial connection to other flat ones. The odd signature operator twisted by an $SU(N)$ connection A is given by

$$\begin{aligned} D_A : \Omega^{0+1}(M) \otimes su(N) &\to \Omega^{0+1}(M) \otimes su(N) \\ (\alpha, \beta) &\mapsto (d_A^* \beta, d_A \alpha + * d_A \beta), \end{aligned}$$

where $d_A \omega := d\omega + [A, \omega]$ is the twisted exterior derivative on the 0- and 1-forms $\Omega^{0+1}(M) \otimes su(N) := \Omega^0(M) \otimes su(N) \oplus \Omega^1(M) \otimes su(N)$, $[\cdot, \cdot]$ is the wedge product on forms combined with the Lie bracket on the coefficients, and d_A^* is its dual. The spectral flow of a path of (possibly unbounded) Dirac operators D_t is the algebraic intersection number of the track of the spectrum

$$\{(t, \lambda) \mid t \in [0,1], \lambda \in \operatorname{Spec}(D_t)\}$$

with the line segment from $(0,-\varepsilon)$ to $(1,-\varepsilon)$. The orientation is chosen in such a way, that the spectral flow of D_t ($t\in[0,1]$) with spectrum $\{n+t \mid n\in Z\}$ is 1.

There is a recent conjecture by [3] relating the $SU(3)$ Casson invariant of unions of complements of torus knots $M=T_{2,p}\cup_T T_{2,p'}$ to the $SU(2)$ Casson knot invariants of its pieces: $\lambda_{SU(3)}(M)=4\lambda_{SU(2)}(T_{2,p})\lambda_{SU(2)}(T_{2,p'})$. In order to study this relation, I developed a splitting formula which expresses $\mathrm{SF}(D_{A_t})$ in terms of $\mathrm{SF}(D_{A_t}|_{T_{2,p}};\mathcal{P}_t^+)$, $\mathrm{SF}(D_{A_t}|_{T_{2,p'}};\mathcal{P}_t^-)$ and a few simple correction terms, where $\mathcal{P}_t^\pm$ are paths of Atiyah-Patodi-Singer (APS) boundary conditions. In order to do explicit computations, we may restrict ourselves to connections on M which are cylindrical and flat near T. In fact, we can fix a nice parameter space Λ for such flat connections on T and for the corresponding APS boundary conditions.

In order to apply the splitting formula we can refine it in the following way. It is a simple consequence of the splitting formula that the spectral flow of the odd signature operator coupled to a loop of $SU(N)$ connections on a manifold with boundary only depends on its restriction to the boundary. Let ρ_t be an arbitrary loop in Λ. Let A_t be a loop of $SU(N)$ connections on the solid torus restricting to a_{ρ_t} on the boundary. Let $\mathrm{SF}(\rho_t):=\mathrm{SF}(D_{A_t}|_S;\mathcal{P}^+_{\rho_t})$. This can be computed just like in [7] for $SU(2)$.

Consider two flat connections A_0 and A_1 on $M=X\cup_T Y$. Let B_t and B'_t be paths of $SU(N)$ connections on X and Y respectively with $A_\varepsilon|_X=B_\varepsilon$ and $A_\varepsilon|_Y=B'_\varepsilon$, $\varepsilon=0,1$, with ρ and ρ' the corresponding paths in Λ. Then up to simple correction terms we have

$$\mathrm{SF}(A_t)=\mathrm{SF}(D_{B_t}|_X;\mathcal{P}^+_{\rho_t})+\mathrm{SF}(D_{B'_t}|_Y;\mathcal{P}^-_{\rho'_t})+\mathrm{SF}(\rho_{1-t}*\rho'_t).$$

Since the representation variety of the complement of a torus knot is path connected, we can find for two flat connections A_0 and A_1 paths of flat connections B_t and B'_t on $T_{2,p}$ and $T_{2,p'}$ as above. Now we can compute the $SU(3)$ spectral flow and check the conjecture.

The $su(2)$ spectral flow on torus bundles over the circle [7]

In [12] Edward Witten defines certain 3-manifold invariants

$$Z_k(M)=\int_{A\in\mathcal{A}_M} e^{2\pi ik\,cs(A)}\,d\mathcal{A}_M$$

involving the Feynman path integral over connections $\mathcal{A}_M$ on M and the Chern-Simons function

$$cs(A)=\frac{1}{8\pi^2}\int_M \mathrm{tr}(A\wedge dA+\frac{2}{3}A\wedge A\wedge A).$$

The path integral is not defined in a mathematically rigorous way. However, Witten shows that these integrals should have certain nice properties, particularly their behavior under cutting and pasting of manifolds. These can be axiomatized and used to compute the invariants. Further developments by Atiyah [2] and others generalized them to axioms which are now known as the axioms of *Topological Quantum Field Theory* (TQFT). Witten proposed that one also has an asymptotic expansion of his invariants (as k approaches infinity) using the method of stationary phase by analogy to oscillatory integrals in finite dimensions, which introduces gauge theoretic terms.

Thus the path integral predicts a relationship between an entirely combinatorial description in terms of the axioms of TQFT and gauge theoretic quantities. A confirmation of this relationship is certainly very interesting.

In [9] Lisa Jeffrey verifies the asymptotic behaviour for lens spaces. She also confirms this for torus bundles M over the circle, but leaves some details involving spectral flow unresolved. Based on that she conjectures the value of the spectral flow of the family of odd signature operators coupled to A_t with flat endpoints.

In [10] Paul Kirk and Erik Klassen analyze the spectral flow between two irreducible flat $SU(2)$-connections A_0 and A_1 on a torus bundle M over the circle. They consider a specific splitting of $M = S \cup_T X$ into a solid torus S and its complement X along a torus T, so that there is a path A_t of flat connections on X from $A_0|_X$ to $A_1|_X$. Then they show that the spectral flow is 0 mod 4, which partially confirms Jeffrey's conjecture for irreducible connections, provided the restriction of the path of $SU(2)$-representations $\mathrm{hol}(A_t|_X)$ to T is noncentral. The path of connections being noncentral on T ensures that the dimension of the cohomology twisted by the $\mathrm{hol}(A_t|_X)|_T$ is constant as t varies. Equivalently the kernel of the tangential or boundary Dirac operator has constant dimension. This restriction is well-known to people who work with paths of Dirac operators on manifolds with boundary.

I developed a splitting formula, which does not make any assumption on $\mathrm{hol}(A_t|_X)|_T$. I proved the following theorem, which generalizes the results in [10] and confirms Lisa Jeffrey's conjecture for irreducible connections:

Theorem 1 ([7])**.** *The spectral flow of the odd signature operator between two irreducible flat connections on a torus-bundle over the circle is 0 mod 4.*

The perturbed Calderón projector [8]

In [11] Taubes defines an invariant for homology 3-spheres by extending the Euler characteristic to the moduli space of gauge equivalence classes of $SU(2)$-connections in the spirit of the Poincaré-Hopf theorem. Then he proves that his invariant is actually equal to the topologically defined Casson invariant. This marks the beginning of Floer's instanton homology.

With a detailed proof of the existence and continuity of the Calderón projector for the perturbed Dirac operator Paul Kirk, Matthias Lesch and I can deal with perturbation issues related to the morse theory of the Chern-Simons function, such as the generalization of Liviu Nicolaescu's splitting formula for spectral flow of a path of odd signature operators to a path of perturbed odd signature operators. This theorem is useful in 3-dimensional generalizations of Casson's invariant and is used by [4] to compute $SU(3)$ Casson invariants of Brieskorn spheres.

An application of this result is given in this paper, reproving Taubes' result relating the mod 2 spectral flow with the difference in local intersection numbers of the character varieties coming from the Heegard decomposition. I expect to use it for other spectral flow calculations related to the $SU(3)$ Casson Invariant [4], [5].

Future Research

(1) I would like to completely prove Lisa Jeffrey's conjecture regarding the spectral flow on torus bundles over the circle.
(2) My work allows computations of the $SU(3)$ Casson invariant and η-invariants for Dehn surgeries on figure eight, other two bridge knots and hyperbolic knots.
(3) Related to the previous point it would be interesting to know whether the SU(3) Casson invariant has a surgery formula.
(4) I would like to study quantum invariants of 3-manifolds and their asymptotic expansion, as well as investigate resulting gauge theoretic quantities. E.g. the asymptotic expansion conjecture by J. E. Andersen concerning the Reshetikhin-Turaev invariants would be of interest. This conjecture has already been confirmed for mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least 2 and for any Lie algebra g [1]. Its proof for all Seifert manifolds with orientable base or non-orientable base with even genus in the cases $g = sl_2(\mathbf{C})$ is announced in [6].

Literatur

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