# Fourier Theory

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These are notes for a talk I am giving in the Kleine AG on the *Weil conjectures*, organized by Mingjia Zhang and Ben Heuer and taking place in the winter term 2021/2022 in Bonn. I thank both of the organizers for their effort and in particular Mingjia Zhang for helping me with some confusion. References for the material are [FK88] and [KW01]. There are probably some mistakes in these notes (that are all my fault), so use them at your own risk!

The goal of our Kleine AG is to prove the following theorem.

**Theorem 0.1** (Main Theorem, Deligne)**.** *Let $f\colon X \to S$ be a map of algebraic $\kappa$-schemes and let $\mathcal{F}$ be a $\tau$-mixed sheaf on $X$. Then also the direct images with compact support $R^i f_!(\mathcal{F})$ are $\tau$-mixed and we have*

$$w(R^i f_!(\mathcal{F})) \leq w(\mathcal{F}) + i.$$

We have already reduced this to a fairly concrete statement about certain sheaves on the affine line. The goal of this talk is to prove this statement (see Proposition 5.1) and thus finish the proof of the Main Theorem.

## 1 Notation

- $n, m, m'$ will always denote positive integers with $n \mid m \mid m'$. $w$ will always denote a real number.
- We fix a finite field $\kappa$ of characteristic $p$ and cardinality $q$. Let $k$ be a fixed algebraic closure of $\kappa$ and denote the geometric Frobenius $a \mapsto a^{1/q}$ on $k$ by Fr. Also let $\kappa_n$ be the unique degree $n$ extension of $\kappa$ inside $k$ and write $\mathrm{Fr}_n \coloneqq \mathrm{Fr}^n$.
- $X$ always denotes an algebraic $\kappa$-scheme (i.e. a $\kappa$-scheme that is separated and of finite type). We write $|X|$ for the set of closed points of the underlying topological space of $X$. Given $x \in X$ we write $d(x) \coloneqq [\kappa(x) : \kappa]$ and $N(x) \coloneqq |\kappa(x)| = q^{d(x)}$.
- Fix a prime number $\ell \neq p$ and an algebraic closure $\overline{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$. Also fix an isomorphism $\tau\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ of (abstract) fields.
- When we say *sheaf* we really mean *Weil-$\overline{\mathbf{Q}}_\ell$-sheaf.* We denote the derived category of sheaves on $X$ by $D^b_c(X)$.
- We identify sheaves on $\mathrm{Spec}(\kappa_n)$ with finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces with an automorphism $\mathrm{Fr}_n$. Given an abstract finite field extension $\kappa'$ of $\kappa$ of degree $n$ and a sheaf $\mathcal{F}$ on $\mathrm{Spec}(\kappa')$ we can still obtain a $\overline{\mathbf{Q}}_\ell$-vector space with an automorphism $\mathrm{Fr}_n$ but it is only well-defined up to noncanonical isomorphism. Nevertheless we still obtain a well-defined characteristic polynomial $\det\big(1 - t\,\mathrm{Fr}_n \bigm| \mathcal{F}\big)$.
- Given $K \in D^b_c(X)$ and $x \in X(\kappa_n)$ (resp. $x \in |X|$) we write $K_x$ for the stalk of $K$ at $x$, i.e. the pullback of $K$ along $\mathrm{Spec}(\kappa_n) \to X$ (resp. $\mathrm{Spec}(\kappa(x)) \to X$).
  Similarly we write $R\Gamma(X, K)$ (resp. $R\Gamma_c(X, K)$) for the derived pushforward (with compact support) of $K$ along $X \to \mathrm{Spec}(\kappa)$ and $H^i(X, K)$ (resp. $H^i_c(X, K)$) for its cohomology.
- For $b \in \overline{\mathbf{Q}}_\ell^\times$ we denote by $\mathcal{L}_b$ the lisse sheaf of rank 1 on $\mathrm{Spec}(\kappa)$ with associated character $\mathrm{Fr} \mapsto b$.
- We write $\|\bullet\|$ for the standard norm on the space of $\mathbf{C}$-valued functions on some finite set.

## 2 Generalities on $\ell$-adic cohomology

**Theorem 2.1** (Base Change)**.** *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

*be a pullback diagram of algebraic $\kappa$-schemes and let $K \in D^b_c(X)$. Then there is a natural isomorphism*

$$g^* Rf_!(K) \cong Rf'_! g'^*(K).$$

**Theorem 2.2** (Projection Formula)**.** *Let* $f\colon X \to S$ *be a map of algebraic* $\kappa$*-schemes and let* $K \in D_c^b(S)$ *and* $L \in D_c^b(X)$*. Then there is a natural isomorphism*

$$K \otimes^L Rf_!(L) \cong Rf_!\big(f^*(K) \otimes^L L\big).$$

**Theorem 2.3** (Grothendieck Trace Formula)**.** *Let* $K \in D_c^b(X)$*. Then the following identities hold:*

$$\prod_{i\in\mathbf{Z}}\prod_{x\in|X|} \det\big(1 - t^{d(x)}\operatorname{Fr}_{d(x)} \,\big|\, \mathscr{H}^i(K)_x\big)^{(-1)^i} = \prod_{i\in\mathbf{Z}} \det\big(1 - t\operatorname{Fr} \,\big|\, H_c^i(X,K)\big)^{(-1)^i}$$

$$\sum_{i\in\mathbf{Z}}\sum_{x\in X(\kappa_n)} (-1)^i \operatorname{tr}\big(\operatorname{Fr}_n \,\big|\, \mathscr{H}^i(K)_x\big) = \sum_{i\in\mathbf{Z}} (-1)^i \operatorname{tr}\big(\operatorname{Fr}_n \,\big|\, H_c^i(X,K)\big)$$

**Theorem 2.4** (Poincaré Duality)**.** *Suppose that* $X$ *is smooth and let* $K \in D_c^b(X)$*. Then there is a natural isomorphism*

$$R\Gamma(X, K^\vee)(d)[2d] \cong R\Gamma_c(X,K)^\vee.$$

# 3 Pure, Mixed and Real Sheaves

**Definition 3.1.** Let $\mathcal{G}$ be a sheaf on $X$.

- $\mathcal{G}$ is *$\tau$-pure of weight $w$* if we have $|\tau(\alpha)|^2 = N(x)^w$ for all $x \in |X|$ and eigenvalues $\alpha \in \overline{\mathbf{Q}}_\ell^\times$ of $\operatorname{Fr}_{d(x)}\colon \mathcal{G}_x \to \mathcal{G}_x$.
- $\mathcal{G}$ is *$\tau$-mixed* if there exists a filtration of $\mathcal{G}$ whose graded pieces are $\tau$-pure (possibly of varying weight).
- $\mathcal{G}$ is *$\tau$-real* if the characteristic polynomial $\tau\det\big(1 - t\operatorname{Fr}_{d(x)} \,\big|\, \mathcal{G}_x\big) \in \mathbf{C}[t]$ has coefficients in $\mathbf{R}$ for all $x \in |X|$.
- We define
$$w(\mathcal{G}) \coloneqq \sup_{x\in|X|}\max_\alpha \log_{N(x)}\big(|\tau(\alpha)|^2\big) \in \mathbf{R}$$
where $\alpha$ runs through all eigenvalues of $\operatorname{Fr}_{d(x)}\colon \mathcal{G}_x \to \mathcal{G}_x$.
- We define functions $f_n^{\mathcal{G}}\colon X(\kappa_n) \to \mathbf{C}$ (depending on $n$) by setting $f_n^{\mathcal{G}}(x) \coloneqq \tau\operatorname{tr}(\operatorname{Fr}_n \,|\, \mathcal{G}_x)$.
- We define
$$\|\mathcal{G}\| \coloneqq \sup\Big\{\rho \in \mathbf{R} \;\Big|\; \limsup_n q^{-n(\rho+\dim(X))} \cdot \big\|f_n^{\mathcal{G}}\big\|^2 > 0\Big\} \in \mathbf{R}.$$

We also extend the definition of $f_n^{\mathcal{G}}$ to objects $K \in D_c^b(X)$ by setting $f_n^K \coloneqq \sum_{i\in\mathbf{Z}}(-1)^i f_n^{\mathscr{H}^i(K)}$.

**Theorem 3.2.** *Suppose that* $X$ *is smooth of pure dimension* 1 *and let* $\mathcal{G}$ *be a* $\tau$*-mixed sheaf on* $X$ *that does not admit sections with finite support. Then we have* $\|\mathcal{G}\| = w(\mathcal{G})$*.*

**Theorem 3.3.** *Let* $\mathcal{G}$ *be a* $\tau$*-real sheaf on* $X$*. Then* $\mathcal{G}$ *is* $\tau$*-mixed.*

The following lemma doesn't really belong here but I don't know where else to put it.

**Lemma 3.4.** *Let* $V$ *be a finite-dimensional* $\mathbf{C}$*-vector space and let* $\varphi\colon V \to V$ *be a (*$\mathbf{C}$*-linear) automorphism all of whose eigenvalues* $\alpha \in \mathbf{C}^\times$ *satisfy* $|\alpha|^2 = u$ *for some fixed* $u \in \mathbf{R}_{>0}$*. Then the the characteristic polynomial of the automorphism* $\varphi \oplus (u\cdot\varphi^{-1})\colon V \oplus V \to V \oplus V$ *has real coefficients.*

# 4 The $\ell$-adic Fourier transform (after Laumon)

## 4.1 The Fourier Transform for Finite Abelian Groups

Let us first recall the Fourier transform for finite Abelian groups. Let $G$ be a finite Abelian group and let $\psi\colon G \times G \to \mathbf{C}^\times$ be a symmetric nondegenerate pairing.

**Definition 4.1.** Given a function $f\colon G \to \mathbf{C}$ we define its *Fourier transform* $T_\psi f\colon G \to \mathbf{C}$ by the formula

$$(T_\psi f)(h) \coloneqq \sum_{g\in G} f(g)\cdot\psi(g,h)^{-1}.$$

The Fourier transformation has the following properties:

**Lemma 4.2.** *Let* $f\colon G \to \mathbf{C}$ *be a function on* $G$*.*

*(Fourier Inversion): We have* $\big(T_{\psi^{-1}} \circ T_\psi\big)(f) = |G|\cdot f$*.*

*(Plancherel Formula): We have* $\|T_\psi f\| = |G|^{1/2} \cdot \|f\|$.

In the following we will be interested in the finite abelian groups $\kappa_n$.

**Notation 4.3.** We introduce the following notation (where $\psi$ denotes a character of $\kappa_n$).

- We set $\psi_{\kappa_m} \coloneqq \psi \operatorname{Tr}_{\kappa_m/\kappa_n}$. This is now a character of $\kappa_m$.
- Given $b \in \kappa_n$ we write $\psi_b$ for the character $\kappa_n \ni a \mapsto \psi(ba)$.

These constructions are compatible in the sense that $\psi_{\kappa_m,b} = \psi_{b,\kappa_m}$.

Now given a nontrivial character $\psi$ of $\kappa$ we have an associated symmetric nondegenerate pairing on $\kappa_n$ that is given by

$$(a, b) \mapsto \psi\big(\operatorname{Tr}_{\kappa_n/\kappa}(ab)\big) = \psi_{\kappa_n,b}(a).$$

We denote this pairing again by $\psi$.

## 4.2 Artin-Schreier Coverings

**Definition 4.4.** We define the *$n$-th Artin-Schreier map*

$$\wp^{(n)} \colon \mathbf{A}^1_\kappa \to \mathbf{A}^1_\kappa, \qquad x \mapsto x^{q^n} - x.$$

We also define the maps

$$\alpha^{(m,n)} \colon \mathbf{A}^1_\kappa \to \mathbf{A}^1_\kappa, \qquad x \mapsto \sum_{i=0}^{m/n-1} x^{q^{in}}.$$

One immediately checks the identity $\wp^{(n)}\alpha^{(m,n)} = \wp^{(m)}$.

**Lemma 4.5.** *$\wp^{(n)}$ is a finite étale geometrically connected covering of degree $q^n$. After extending scalars to $\kappa_n$ it is Galois (even Abelian) with automorphism group*

$$\kappa_n \cong \operatorname{Aut}(\wp^{(n)}_{\kappa_n}), \qquad a \mapsto (x \mapsto x + a).$$

*Under this isomorphism the map $\operatorname{Aut}(\wp^{(m)}_{\kappa_m}) \to \operatorname{Aut}(\wp^{(n)}_{\kappa_m})$ induced by $\alpha^{(m,n)}$ identifies with $\operatorname{Tr}_{\kappa_m/\kappa_n} \colon \kappa_m \to \kappa_n$.*

**Notation 4.6.** Given a character $\psi \colon \kappa_n \to \overline{\mathbf{Q}}_\ell^\times$ we denote by $\mathcal{L}(\psi)$ the associated lisse sheaf of rank 1 on $\mathbf{A}^1_{\kappa_n}$. Note that we have $\mathcal{L}(\psi_{\kappa_m}) \cong \mathcal{L}(\psi)_{\kappa_m}$.

**Lemma 4.7** (Cohomology and Stalks of $\mathcal{L}(\psi)$)**.** *Let $\psi \colon \kappa_n \to \overline{\mathbf{Q}}_\ell^\times$ be a character. Then we have*

$$H^i_c(\mathbf{A}^1_{\kappa_n}, \mathcal{L}(\psi)) \cong \begin{cases} \overline{\mathbf{Q}}_\ell(-1) & \textit{if } \psi = 1 \textit{ and } i = 2, \\ 0 & \textit{else.} \end{cases}$$

*Moreover, for $a \in \mathbf{A}^1_{\kappa_n}(\kappa_m) = \kappa_m$ we have*

$$\det\big(\operatorname{Fr}_m \,\big|\, \mathcal{L}(\psi)_a\big) = \psi_{\kappa_m}(a)^{-1}.$$

*Proof.* It is a standard result in étale cohomology that $R\Gamma(\mathbf{A}^1_{\kappa_n}, \overline{\mathbf{Q}}_\ell) \cong \overline{\mathbf{Q}}_\ell$. Applying Theorem 2.4 to this yields $R\Gamma_c(\mathbf{A}^1_{\kappa_n}, \overline{\mathbf{Q}}_\ell) \cong \overline{\mathbf{Q}}_\ell(-1)[-2]$. The first part of the lemma now follows because $R\wp^{(n)}_{\kappa_n,*}(\overline{\mathbf{Q}}_\ell) \cong \bigoplus_\psi \mathcal{L}(\psi)$ (where the direct sum is over all characters of $\kappa_n$).

Let us now turn to the second part and suppose we are given $a \in \mathbf{A}^1_{\kappa_n}(\kappa_m) = \kappa_m$. Then we can choose $b \in \mathbf{A}^1_{\kappa_n}(\kappa_{m'})$ for some $m'$ such that $\wp^{(n)}_{\kappa_n}(b) = a$. This gives rise to a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(\kappa_{m'}) & \xrightarrow{\ b\ } & \mathbf{A}^1_{\kappa_n} \\ \big\downarrow & & \big\downarrow \wp^{(n)}_{\kappa_n} \\ \operatorname{Spec}(\kappa_m) & \xrightarrow{\ a\ } & \mathbf{A}^1_{\kappa_n}. \end{array}$$

The induced map $\operatorname{Gal}(\kappa_{m'}/\kappa_m) \to \operatorname{Aut}(\wp^{(n)}_{\kappa_n})$ sends $\operatorname{Fr}_m$ to the automorphism $x \mapsto x - \operatorname{Tr}_{\kappa_m/\kappa_n}(a)$, i.e. the element $-\operatorname{Tr}_{\kappa_m/\kappa_n}(a) \in \kappa_n$. The result now readily follows as $\psi(-\operatorname{Tr}_{\kappa_m/\kappa_n}(a)) = \psi_{\kappa_m}(a)^{-1}$. $\square$

## 4.3 The $\ell$-adic Fourier transform

In this subsection $\psi$ always denotes a nontrivial character $\kappa \to \overline{\mathbf{Q}}_\ell^\times$.

**Definition 4.8** (Fourier Transform)**.** Define the *$\ell$-adic Fourier Transform* $T_\psi\colon D_c^b(\mathbf{A}_\kappa^1) \to D_c^b(\mathbf{A}_\kappa^1)$ by the formula

$$T_\psi(K) \coloneqq R\,\mathrm{pr}_{1,!}\big(\mathrm{pr}_2^* K \otimes m^*\mathcal{L}(\psi)\big)[1].$$

Here $m\colon \mathbf{A}_\kappa^2 \to \mathbf{A}_\kappa^1$ denotes the multiplication map.

**Lemma 4.9** (Stalks of the Fourier Transform)**.** *For $K \in D_c^b(\mathbf{A}_\kappa^1)$ and $b \in \mathbf{A}_\kappa^1(\kappa_n) = \kappa_n$ we have*

$$T_\psi(K)_b \cong R\Gamma_c\big(\mathbf{A}_{\kappa_n}^1, K_{\kappa_n} \otimes \mathcal{L}(\psi_{\kappa_n,b})\big)[1]$$

*in $D_c^b(\mathrm{Spec}(\kappa_n))$.*

*Proof.* Applying Theorem 2.1 yields

$$T_\psi(K)_b \cong R\Gamma_c\big(\mathbf{A}_{\kappa_n}^1, K_{\kappa_n} \otimes \lambda^{b,*}\mathcal{L}(\psi)_{\kappa_n}\big)[1]$$

where the map $\lambda^b\colon \mathbf{A}_{\kappa_n}^1 \to \mathbf{A}_{\kappa_n}^1$ is given by $x \mapsto bx$. Thus we are done if we can show that $\lambda^{b,*}\mathcal{L}(\psi)_{\kappa_n} \cong \mathcal{L}(\psi_{\kappa_n,b})$. To see this note that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{A}_{\kappa_n}^1 & \xrightarrow{\lambda^b} & \mathbf{A}_{\kappa_n}^1 \\ \downarrow{\scriptstyle \wp_{\kappa_n}^{(n)}} & & \downarrow{\scriptstyle \wp_{\kappa_n}^{(n)}} \\ \mathbf{A}_{\kappa_n}^1 & \xrightarrow{\lambda^b} & \mathbf{A}_{\kappa_n}^1 \end{array}$$

inducing the map $\mathrm{Aut}(\wp_{\kappa_n}^{(n)}) \to \mathrm{Aut}(\wp_{\kappa_n}^{(n)})$ that is given by $a \mapsto ba$ under the isomorphism $\kappa_n \cong \mathrm{Aut}(\wp_{\kappa_n}^{(n)})$. $\square$

**Remark 4.10.** Let's try to explain why $T_\psi$ is called a "Fourier transform" by giving the following (incomplete) table of analogies (compare with Section 4.1).

| | |
|---|---|
| $G$ | $\mathbf{A}_\kappa^1$ |
| $f\colon G^n \to \mathbf{C}$ | $K \in D_c^b(\mathbf{A}_\kappa^n)$ |
| $\psi^{-1}$ | $m^*\mathcal{L}(\psi)$ |
| $\sum_{h\in G} f(g,h)$ | $R\,\mathrm{pr}_{1,!}(K)$ |

Using this table one should be able to compare Definition 4.8 with Definition 4.1 (except for possibly the appearing shift).

**Lemma 4.11.** *Let $K \in D_c^b(\mathbf{A}_\kappa^1)$. Then we have*

$$f_n^{T_\psi K} = -T_{\tau\psi}(f_n^K).$$

*In particular we get a Plancherel Formula $\|f_n^{T_\psi K}\| = q^{n/2} \cdot \|f_n^K\|$.*

*Proof.* We make the following computation (for $b \in \mathbf{A}_\kappa^1(\kappa_n) = \kappa_n$).

$$\begin{aligned} f_n^{T_\psi K}(b) &= \tau \sum_{i\in\mathbf{Z}} (-1)^i \operatorname{tr}\big(\mathrm{Fr}_n \,\big|\, \mathscr{H}^i(T_\psi K)_b\big) \\ &\overset{4.9}{=} -\tau \sum_{i\in\mathbf{Z}} (-1)^i \operatorname{tr}\big(\mathrm{Fr}_n \,\big|\, H_c^i(\mathbf{A}_{\kappa_n}^1, K_{\kappa_n} \otimes \mathcal{L}(\psi_{\kappa_n,b}))\big) \\ &\overset{2.3}{=} -\tau \sum_{i\in\mathbf{Z}} \sum_{a\in\kappa_n} (-1)^i \operatorname{tr}\big(\mathrm{Fr}_n \,\big|\, \mathscr{H}^i(K_{\kappa_n})_a \otimes \mathcal{L}(\psi_{\kappa_n,b})_a\big) \\ &= -\tau \sum_{a\in\kappa_n} \sum_{i\in\mathbf{Z}} (-1)^i \operatorname{tr}\big(\mathrm{Fr}_n \,\big|\, \mathscr{H}^i(K)_a\big) \cdot \det\big(\mathrm{Fr}_n \,\big|\, \mathcal{L}(\psi_{\kappa_n,b})_a\big) \\ &\overset{4.7}{=} - \sum_{a\in\kappa_n} f_n^K(a) \cdot \tau\psi_{\kappa_n}(ab)^{-1} \\ &= -T_{\tau\psi}(f_n^K)(b) \end{aligned}$$

$\square$

We now also want to establish a "Fourier Inversion"-result for the $\ell$-adic Fourier Transform. To do so, we need the following key computation that should be compared to the identity

$$\sum_{h\in G} \psi(h,k)^{-1} \cdot \psi(g,h) = \begin{cases} |G| & \text{if } g = k, \\ 0 & \text{else.} \end{cases}$$

**Lemma 4.12.** *We have*

$$R\,\mathrm{pr}_{13,!}\big(\mathrm{pr}_{23}^*\,m^*\mathcal{L}(\psi)\otimes\mathrm{pr}_{12}^*\,m^*\mathcal{L}(\psi^{-1})\big)\cong\Delta_*(\overline{\mathbf{Q}}_\ell)(-1)[-2]$$

*where* $\Delta\colon\mathbf{A}^1_\kappa\to\mathbf{A}^2_\kappa$ *denotes the diagonal map.*

*Proof.* We first claim that we have

$$\mathrm{pr}_{23}^*\,m^*\mathcal{L}(\psi)\otimes\mathrm{pr}_{12}^*\,m^*\mathcal{L}(\psi^{-1})\cong\alpha^*m^*\mathcal{L}(\psi)$$

where $\alpha\colon\mathbf{A}^3_\kappa\to\mathbf{A}^2_\kappa$ is the map $(x,y,z)\mapsto(y,z-x)$. To see this we consider

$$X:=\mathrm{Spec}\big(\kappa[x,y,z][\varepsilon,\eta]/(\varepsilon^q-\varepsilon-xy,\eta^q-\eta-yz)\big)\xrightarrow{\chi}\mathbf{A}^3_\kappa.$$

$\chi$ is a finite étale geometrically connected Abelian covering with automorphism group

$$\kappa\times\kappa\cong\mathrm{Aut}(\chi),\qquad(a_\varepsilon,a_\eta)\mapsto(\varepsilon\mapsto\varepsilon+a_\varepsilon,\eta\mapsto\eta+a_\eta).$$

Moreover we have commutative diagrams

$$\begin{array}{ccc}X & \xrightarrow{(??)} & \mathbf{A}^1_\kappa\\ \big\downarrow{\scriptstyle\chi} & & \big\downarrow{\scriptstyle\wp^{(1)}}\\ \mathbf{A}^3_\kappa & \xrightarrow{(?)} & \mathbf{A}^1_\kappa\end{array}$$

with the maps **(?)** and **(??)** given by $m\circ\mathrm{pr}_{23}$, $m\circ\mathrm{pr}_{12}$, $m\circ\alpha$ and $(x,y,z,\varepsilon,\eta)\mapsto\eta,\varepsilon,\eta-\varepsilon$ respectively. Hence we obtain induced maps on the automorphism groups that are given by

$$\kappa\times\kappa\to\kappa,\qquad(a_\varepsilon,a_\eta)\mapsto a_\eta,a_\varepsilon,a_\eta-a_\varepsilon.$$

Our claim now reduces to the identity

$$\psi(a_\eta)\cdot\psi^{-1}(a_\varepsilon)=\psi(a_\eta-a_\varepsilon).$$

Next we note that we have pullback squares

$$\begin{array}{ccc}\mathbf{A}^3_\kappa & \xrightarrow{\alpha} & \mathbf{A}^2_\kappa\\ \big\downarrow{\scriptstyle\mathrm{pr}_{13}} & & \big\downarrow{\scriptstyle\mathrm{pr}_2}\\ \mathbf{A}^2_\kappa & \xrightarrow{\beta} & \mathbf{A}^1_\kappa\end{array}$$

and

$$\begin{array}{ccc}\mathbf{A}^1_\kappa & \xrightarrow{\mathrm{pr}} & \mathrm{Spec}(\kappa)\\ \big\downarrow{\scriptstyle\Delta} & & \big\downarrow{\scriptstyle i_0}\\ \mathbf{A}^2_\kappa & \xrightarrow{\beta} & \mathbf{A}^1_\kappa\end{array}$$

where the map $\beta$ is given by $(x,y)\mapsto y-x$ and $i_0$ is the inclusion of the origin. Applying Theorem 2.1 yields

$$R\,\mathrm{pr}_{13,!}\circ\alpha^*\cong\beta^*\circ R\,\mathrm{pr}_{2,!}\quad\text{ and }\quad\Delta_*\circ\mathrm{pr}^*\cong\beta^*\circ i_{0,*}.$$

As a final ingredient we note that Lemma 4.7 and Lemma 4.9 together imply that $T_\psi(\overline{\mathbf{Q}}_\ell)\cong i_{0,*}(\overline{\mathbf{Q}}_\ell)(-1)[-1]$, i.e. that $R\,\mathrm{pr}_{2,!}\big(m^*\mathcal{L}(\psi)\big)\cong i_{0,*}(\overline{\mathbf{Q}}_\ell)(-1)[-2]$. Now putting it all together gives

$$\begin{aligned}R\,\mathrm{pr}_{13,!}\big(\mathrm{pr}_{23}^*\,m^*\mathcal{L}(\psi)\otimes\mathrm{pr}_{12}^*\,m^*\mathcal{L}(\psi^{-1})\big)&\cong R\,\mathrm{pr}_{13,!}\,\alpha^*m^*\mathcal{L}(\psi)\\&\cong\beta^*R\,\mathrm{pr}_{2,!}\,m^*\mathcal{L}(\psi)\\&\cong\beta^*i_{0,*}(\overline{\mathbf{Q}}_\ell)(-1)[-2]\\&\cong\Delta_*(\overline{\mathbf{Q}}_\ell)(-1)[-2].\end{aligned}$$

□

**Theorem 4.13** (Fourier Inversion)**.** *For* $K\in D^b_c(\mathbf{A}^1_\kappa)$ *we have a natural isomorphism*

$$(T_{\psi^{-1}}\circ T_\psi)(K)\cong K(-1).$$

*Proof.* We make the following computation.

$$\begin{aligned}(T_{\psi^{-1}}\circ T_\psi)(K) &\cong R\,\mathrm{pr}_{1,!}\Big(\mathrm{pr}_2^*\,R\,\mathrm{pr}_{1,!}\big(\mathrm{pr}_2^*\,K\otimes m^*\mathcal{L}(\psi)\big)\otimes m^*\mathcal{L}(\psi^{-1})\Big)[2]\\ &\overset{2.1}{\cong} R\,\mathrm{pr}_{1,!}\Big(R\,\mathrm{pr}_{12,!}\,\mathrm{pr}_{2,3}^*\big(\mathrm{pr}_2^*\,K\otimes m^*\mathcal{L}(\psi)\big)\otimes m^*\mathcal{L}(\psi^{-1})\Big)[2]\\ &\overset{2.2}{\cong} R\,\mathrm{pr}_{1,!}\big(\mathrm{pr}_3^*\,K\otimes \mathrm{pr}_{23}^*\,m^*\mathcal{L}(\psi)\otimes \mathrm{pr}_{12}^*\,m^*\mathcal{L}(\psi^{-1})\big)[2]\\ &\overset{2.2}{\cong} R\,\mathrm{pr}_{1,!}\Big(\mathrm{pr}_2^*\,K\otimes R\,\mathrm{pr}_{13,!}\big(\mathrm{pr}_{23}^*\,m^*\mathcal{L}(\psi)\otimes \mathrm{pr}_{12}^*\,m^*\mathcal{L}(\psi^{-1})\big)\Big)[2]\\ &\overset{4.12}{\cong} R\,\mathrm{pr}_{1,!}\big(\mathrm{pr}_2^*\,K\otimes \Delta_*(\overline{\mathbf{Q}}_\ell)\big)(-1)\\ &\overset{2.2}{\cong} R\,\mathrm{pr}_{1,!}\big(\Delta_*(K)\big)(-1)\cong K(-1)\end{aligned}$$ □

## 5 End of Proof of the Main Theorem

Let us fix the following notation.

- $\psi\colon \kappa\to\overline{\mathbf{Q}}_\ell^\times$ is a nontrivial character as before.
- $j\colon U\to\mathbf{A}^1_\kappa$ is the inclusion of a nonempty open subscheme. Fix a geometric point $\overline{x}\in U(k)$.
- $\mathcal{F}$ is a lisse $\tau$-pure sheaf of weight $w$ on $U$. Write $V:=\mathcal{F}_{\overline{x}}$ and let $\rho\colon W(U,\overline{x})\to\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ be the associated representation. We assume that the following conditions are satisfied.
  - $\mathcal{F}$ is geometrically irreducible and geometrically nonconstant.
  - $\mathcal{F}$ is unramified at $\infty$, i.e. $\mathcal{F}$ extends to a lisse sheaf on $U\cup\{\infty\}\subseteq\mathbf{P}^1_\kappa$.
- We also set $\mathcal{G}:=j_!(\mathcal{F})$.

What remains to be shown in order to prove Theorem 0.1 is the following proposition.

**Proposition 5.1.** *The eigenvalues $\alpha$ of the Frobenius* $\mathrm{Fr}\colon H^1_c(U,\mathcal{F})\to H^1_c(U,\mathcal{F})$ *satisfy* $|\tau(\alpha)|^2\le q^{w+1}$.

Before giving the proof of the proposition we collect some properties of the Fourier Transform $T_\psi(\mathcal{G})$.

**Lemma 5.2.** *$T_\psi(\mathcal{G})$ is concentrated in degree* 0.

*Proof.* Using Lemma 4.9 we need to show that

$$H^i_c\big(\mathbf{A}^1_{\kappa_n},\mathcal{G}_{\kappa_n}\otimes\mathcal{L}(\psi_{\kappa_n,b})\big)\overset{2.2}{\cong} H^i_c\big(U_{\kappa_n},\mathcal{F}_{\kappa_n}\otimes j^*\mathcal{L}(\psi_{\kappa_n,b})\big)\cong 0$$

for $i=0,2$ and all $b\in\kappa_n$. For $i=0$ this is clear because $U$ is affine of positive dimension and $\mathcal{F}$ is lisse. For $i=2$ we apply Theorem 2.4 to see that

$$H^2_c\big(U_{\kappa_n},\mathcal{F}_{\kappa_n}\otimes j^*\mathcal{L}(\psi_{\kappa_n,b})\big)\cong H^0\big(U_{\kappa_n},(\mathcal{F}_{\kappa_n}\otimes j^*\mathcal{L}(\psi_{\kappa_n,b}))^\vee\big)^\vee(-1)$$

so that we need to show that the $\pi_1(U_k,\overline{x})$-invariants of $V(\psi_{\kappa_n,b})^\vee$ are trivial. $\mathcal{F}$ being geometrically irreducible precisely means that $V$ is an irreducible $\pi_1(U_k,\overline{x})$-representation. Thus we are done if we can show that $\mathcal{F}_k\not\cong\mathcal{L}(\psi_{\kappa_n,b})_k$ for any $b\in\kappa_n$.

- As $\mathcal{F}$ is geometrically nonconstant we can't have $\mathcal{F}_k\cong\mathcal{L}(\psi_0)_k$ (note that $\mathcal{L}(\psi_0)\cong\overline{\mathbf{Q}}_\ell$).
- As $\mathcal{F}$ is unramified at $\infty$ we can't have $\mathcal{F}_k\cong\mathcal{L}(\psi_{\kappa_n,b})_k$ for $b\neq 0$ (using that $\pi_1(\mathbf{P}^1_k,\overline{x})=1$). □

**Lemma 5.3.** *The sheaf $T_\psi(\mathcal{G})$ is $\tau$-mixed.*

*Proof.* Let $b:=\tau^{-1}(q^w)\in\overline{\mathbf{Q}}_\ell^\times$. Then the sheaf

$$\mathcal{H}:=\big(\mathrm{pr}_2^*\,j_!\mathcal{F}\otimes m^*\mathcal{L}(\psi)\big)\oplus\big(\mathrm{pr}_2^*\,j_!\mathcal{F}^\vee\otimes m^*\mathcal{L}(\psi^{-1})\otimes\mathcal{L}_{b,\mathbf{A}^2_\kappa}\big)$$

on $\mathbf{A}^2_\kappa$ is $\tau$-real by Lemma 3.4 (where we use that $\mathcal{F}$ is lisse and $\tau$-pure of weight $w$). Applying Lemma 5.2 and Theorem 2.2 gives that

$$R\,\mathrm{pr}_{1,!}\,\mathcal{H}[1]\cong T_\psi(\mathcal{G})\oplus T_{\psi^{-1}}(j_!\mathcal{F}^\vee)\otimes\mathcal{L}_{b,\mathbf{A}^1_\kappa}$$

is concentrated in degree 0. Applying Theorem 2.1 and Theorem 2.3 now shows that $R\,\mathrm{pr}_{1,!}\,\mathcal{H}[1]$ is again $\tau$-real hence also $\tau$-mixed by Theorem 3.3. Thus also $T_\psi(\mathcal{G})\subseteq R\,\mathrm{pr}_{1,!}\,\mathcal{H}[1]$ is $\tau$-mixed as desired. □

**Lemma 5.4.** $H^0_c(\mathbf{A}^1_\kappa, T_\psi(\mathcal{G})) = 0$.

*Proof.* We make the following calculation.

$$H^0_c\big(\mathbf{A}^1_\kappa, T_\psi(\mathcal{G})\big) \overset{4.9}{\cong} \mathscr{H}^{-1}\big((T_{\psi^{-1}} \circ T_\psi)(\mathcal{G})\big)_0 \overset{4.13}{\cong} \mathscr{H}^{-1}\big(\mathcal{G}(-1)\big)_0 \cong 0 \qquad \square$$

*Proof of Proposition 5.1.* Applying the Plancherel Formula from Lemma 4.11 to $\mathcal{G}$ yields

$$\|T_\psi \mathcal{G}\| = \|\mathcal{G}\| + 1.$$

Now $\mathcal{G}$ and $T_\psi(\mathcal{G})$ are both $\tau$-mixed and we have $H^0_c(\mathbf{A}^1_\kappa, \mathcal{G}) \cong H^0_c(\mathbf{A}^1_\kappa, T_\psi(\mathcal{G})) \cong 0$ (for $T_\psi(\mathcal{G})$ this is the content of Lemma 5.3 and Lemma 5.4) so that we can apply Theorem 3.2 to obtain

$$w(T_\psi(\mathcal{G})) = w + 1.$$

This gives us precisely what we want because $T_\psi(\mathcal{G})_0 \cong H^1_c(\mathbf{A}^1_\kappa, \mathcal{G}) \cong H^1_c(U, \mathcal{F})$ by Lemma 4.9. $\square$

## References


[FK88] Eberhard Freitag and Reinhardt Kiehl. *Etale cohomology and the Weil conjecture.* Springer, 1988.

[KW01] Reinhardt Kiehl and Rainer Weissauer. *Weil conjectures, perverse sheaves and l'adic Fourier transform.* Springer, 2001.